Towards a Robust Geometric Multigrid Scheme for Helmholtz Equation

Rafael Lago, Felix Herrmann
Department of Earth and Ocean sciences, University of British Columbia, Vancouver, BC, Canada

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Abstract : We discuss an improvement of existing multigrid techniques for the solution of the time harmonic wave equation targeting application to seismic inversion and imaging, using non-traditional smoothing and coarse correction techniques, namely the CGMN and CRMN methods. We aim at developing a multigrid scheme to be used as a preconditioner for FGMRES showing less sensibility to changes in the discretization of the operator. We compare this multigrid scheme with recent developments in the multigrid field obtaining very satisfactory results. Our numerical experiments using SEG/EAGE Overthrust velocity model showing not only more robustness when switching from a basic 7 points stencil to a more compact 27 points stencil, but also a considerable reduction in the number of preconditioning steps required to attain convergence, a result encouraging further investigation.

Main objectives : devise a more robust geometric multigrid technique for the solution of the time harmonic wave equation for the seismic inversion and imaging, showing less dependence on the discretization characteristics.

New aspects covered : The use of more sophisticated and not yet investigated smoothers and coarse solvers, namely the CGMN and CRMN method, within a multigrid cycle, which in our experiments has brought significant improvements.

Introduction

Multigrid methods are a very popular iterative tool for computing the numerical solution of partial differential equations. It enjoys optimal convergence properties when used either as a solver or a preconditioner specially for elliptic equations, often being able to converge in a a computational time proportional to the number of unknowns. However, for the important time-harmonic wave equation problem

$$-\Delta u - k^2 u = f \tag{1}$$

widely used in frequency domain migration and full waveform inversion in geophysics, this solver suffers from slow convergence or complete stagnation. Several efforts have been made in the literature to tackle this deficiency with considerable success, the most striking perhaps being the use of the *complex shifted Laplacian operator*

$$-\Delta u - \beta k^2 u = f \tag{2}$$

proposed by Erlangga (2008), where the "complex shift" parameter β is usually taken as half of the imaginary unit.

Assume that a finite difference regular Cartesian grid is to be used to discretize the equations, let h denote the chosen grid spacing and Ω_h denote the discrete domain with h spacing. Then, the most basic multigrid scheme, the so called *V-cycle* comprises the following steps:

- Pre-smoothing on Ω_h, aiming at removing the high frequency components of the error of the current approximate solution.
- Restriction, projecting the current approximation residual to the coarser grid Ω_{2h}.
- Coarse correction, solving the problem exactly (or even approximately in more recent approaches: cf. the work of Elman et al. (2001) and also Calandra et al. (2013)) on Ω_{2h}. This aims at removing the low frequency components of the error.
- Prolongation, takes the solution computed during the coarse correction, projects it onto Ω_h and "updates" the current approximation.
- Post-smoothing on Ω_h smooths down the high components of the error once again.

These steps can be repeated iteratively until convergence or even applied recursively, that is, using a multigrid V-cycle as the coarse solver of another multigrid V-cycle, yielding a multigrid method that visits Ω_h, Ω_{2h} and Ω_{4h}. Multigrid methods profit specially when many hierarchies can be used, but unfortunately this is not always possible. In the case of wave equation, stability conditions require the grid spacing h to satisfy a certain ratio to ensure that the wave has a physical significance, meaning that the more multigrid hierarchies used, less the coarse operator represents a physical wave. Nevertheless, carefully built combinations of techniques may yield competitive multigrid schemes with more than 4 levels of hierarchy, see Elman et al. (2001) for two-dimensional examples. The behaviour of the method, however, usually depends dramatically (amongst other things) on the choice of the smoother and the coarse correction technique. Amongst the most common smoothers are Gauss-Seidel and Jacobi, and common coarse solvers are LU when many hierarchies can be used, or preconditioned GMRES when few hierarchies are used and the coarsest problem is still too large to be solved directly.

A Non-Conventional Smoother

In this section we propose the use of another method for smoothing (and coarsening) during multigrid steps, and we compare this choice with the techniques already known. Since many combinations of smoothers and coarse solvers have been proposed in the literature we limit our comparison to the scheme proposed by Calandra et al. (2013) therein called $\mathscr{T}_{2V}$, obtained after a careful Fourier analysis performed for the Helmholtz and the complex shifted Laplacian operator discretized for the standard 7 points finite differences.

The $\mathscr{T}_{2V}$ preconditioner consists of multigrid V-cycle applied on the Helmholtz operator, where the coarse solver is chosen as two cycles of FGMRES(10) preconditioned by another multigrid V-cycle, this time applied on the complex shifted Laplacian operator. One cycle of GMRES(10) preconditioned by two iterations of Jacobi relaxation with $\omega = 0.2$ is used as coarse solver for the coarsest level. The authors use trilinear interpolation and prolongation in all levels. All smoothing steps consist of two iterations of Jacobi relaxation, with ω being carefully chosen depending on which level of the hierarchy the smoothing is being applied. Figure 1 shows a representation of the $\mathscr{T}_{2V}$ preconditioner.

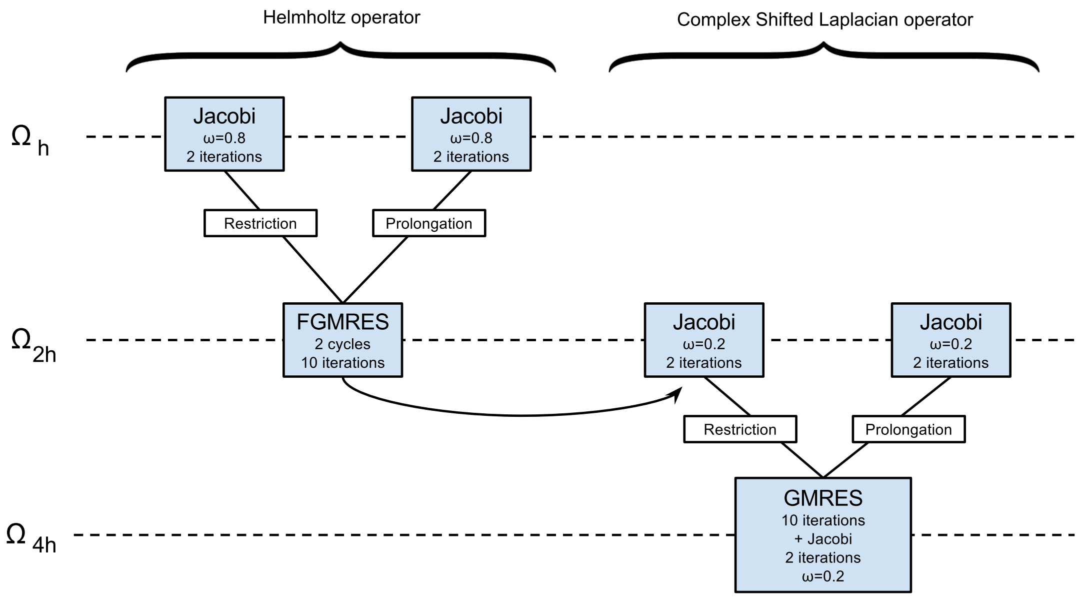

Figure 1 *Illustration of the $\mathscr{T}_{2V}$ preconditioner*

Calandra et al. (2013) compare the $\mathscr{T}_2V$ with other more elaborated combination of multigrid cycles (including the F-cycle) but for their most realistic numerical experiment, the solution of the Helmholtz equation for the SEG/EAGE salt dome model, the $\mathscr{T}_2V$ shows the best performance.

In spite of its impressive performance, the preconditioner $\mathscr{T}_{2V}$ is devised for a rather restrictive scenario: the Helmholtz equation discretized with the standard 7 points finite differences stencil, with 10 points per wavelength. Changing any of these variables might change drastically the behaviour of the preconditioner.

This work is driven by the need to use a more economical discretization scheme, such as the 27 points parsimonious staggered-grid proposed by Operto et al. (2007). This discretization allows stable characterization of the wave equation for as little as 4 points per wavelength, yielding considerably smaller systems to be solved, bringing a key reduction in the memory requirements for solving the problem. Observing that the $\mathscr{T}_{2V}$ preconditioner seems to be *not convergent* for this discretization even at 6 points per wavelength (see comparisons in the next section), we proposed a similar scheme which replaces the smoothing steps and the coarsest solver by CGMN (cf. Björck (1996)) and CRMN (cf. Lago et al. (2014)) respectively. Figure 2 shows a simplified scheme of the resulting preconditioner, which we call $\mathscr{N}_{2V}$ in this extended abstract.

In contrast with $\mathscr{T}_{2V}$, the parameters in the $\mathscr{N}_{2V}$ preconditioner, namely the number of CGMN and CRMN iterations, has been chosen empirically and are not considered by any means to be an optimal

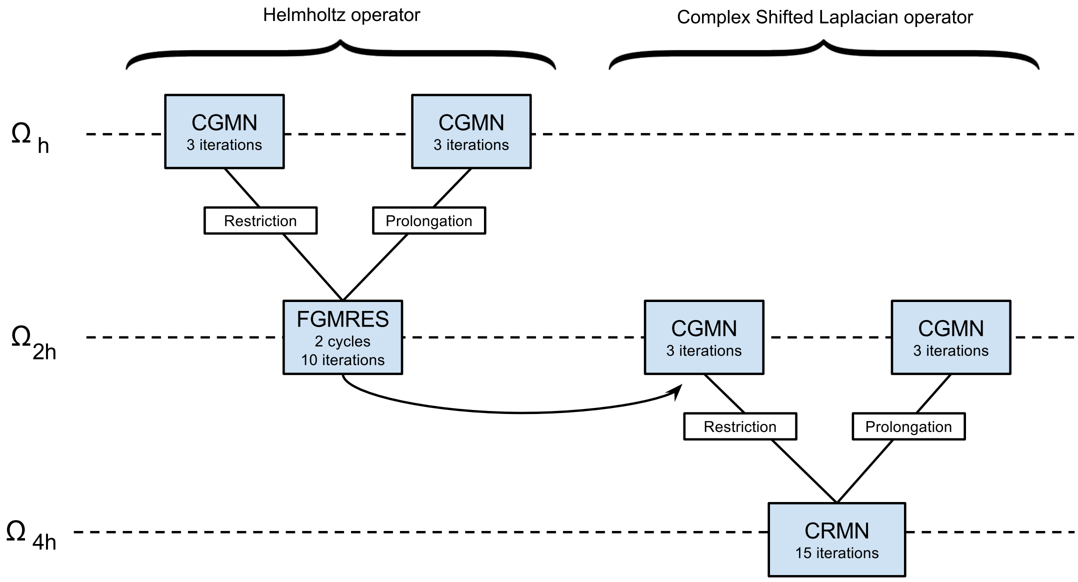

Figure 2 *Illustration of the $\mathcal{N}_{2V}$ preconditioner*

choice, meaning that a more careful analysis could improve the behaviour of this method, being this one of the subjects of future research.

Numerical Experiments

As mentioned in the previous section, the aim of the $\mathcal{N}_{2V}$ preconditioner is to obtain robustness with respect to changes in the characteristics of the discrete operator, reducing memory requirements and improving convergence when possible. We then compare the convergence curve of both $\mathcal{T}_{2V}$ and $\mathcal{N}_{2V}$ in two scenarios. In the first, we solve the Helmholtz equation for the SEG/EAGE Overthrust velocity model at 8 Hz using the standard 7 points finite difference discretization, with a discretization using 10 points per wave length. Both $\mathcal{T}_{2V}$ and $\mathcal{N}_{2V}$ are used as a preconditioner for FGMRES(10) and all experiments are performed using MATLAB 2013a. We show the convergence curve in Figure 3, where the horizontal axis shows the number of iterations of the outer FGMRES(10), that is, effectively how many times the preconditioner was applied. Even though the major goal was to obtain robustness, in this experiment we see that $\mathcal{N}_{2V}$ converges in considerably less iterations than $\mathcal{T}_{2V}$. We would like to remark that $\mathcal{N}_{2V}$ requires more memory than $\mathcal{T}_{2V}$, because Jacobi relaxation can be done “in place”, whereas CGMN requires the storage of 4 model vectors. The total storage for $\mathcal{T}_{2V}$ is 24.5 GB and for $\mathcal{N}_{2V}$ is 29 GB.

In the second scenario, we maintain the SEG/EAGE Overthrust velocity model and the 8Hz frequency, but we change the discretization scheme. In lieu of the standard 7 points stencil, we use the 27 points stencil proposed by Operto et al. (2007) and 6 points per wavelength. This change alone is enough to modify the properties of the Jacobi relaxation used in the $\mathcal{T}_{2V}$ preconditioner and after 100 iterations the method seem to have stagnated, see Figure 4. The $\mathcal{N}_{2V}$ preconditioner however, although affected by the change in the discrete operator, converged after 60 iterations. Using this discretization, the $\mathcal{T}_{2V}$ preconditioner uses 5.3 GB of memory while $\mathcal{N}_{2V}$ uses 6.2 GB, which is much more economical than the previous scenario. However, since $\mathcal{T}_{2V}$ does not converge in this case, only the $\mathcal{N}_{2V}$ may enjoy this advantage of the reduced memory storage.

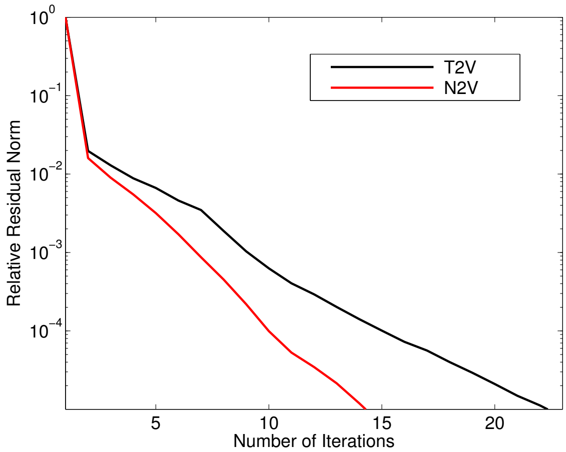

Figure 3 *Second numerical experiment, using 7 points stencil and 10 points per wavelength for SEG/EAGE Overthrust at 8Hz*

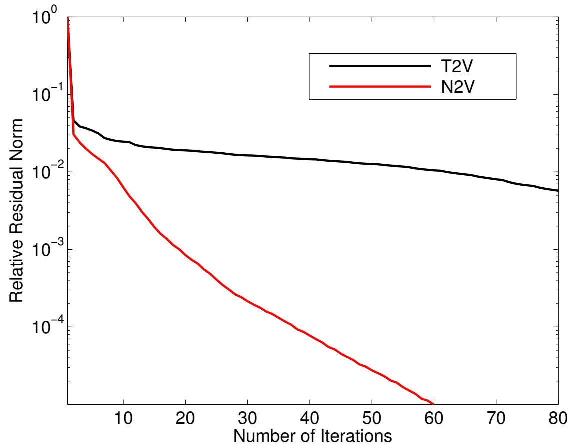

Figure 4 *Second numerical experiment, using 27 points stencil and 6 points per wavelength for SEG/EAGE Overthrust at 8Hz*

Conclusion

We show preliminary results comparing the use of CGMN as smoother and CRMN as coarse solver for multigrid preconditioning for the Helmholtz equation, replacing the use of other more traditional technique. The goal is to obtain a more robust multigrid preconditioner, less sensitive to changes in the discrete operator. In our numerical experiments using the SEG/EAGE Overthrust velocity model at 8Hz, we observe that our proposed method $\mathscr{N}_{2V}$ converges in 14 iterations, 9 less iterations than one of the best reputed multigrid techniques for this problem when using a standard discretization scheme. When using a more economical and compact 27 points scheme, $\mathscr{T}_{2V}$ fails to converge whereas $\mathscr{N}_{2V}$ converges after 60 iterations.

As a future work we plan to further explore the smoothing properties of CGMN for the wave equation, comparing its performance when using high order discretizations and more realistic physics, as well as comparing the performance in terms of computational time when using an optimized implementation in low level language.

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