Multifractional splines: application to seismic imaging

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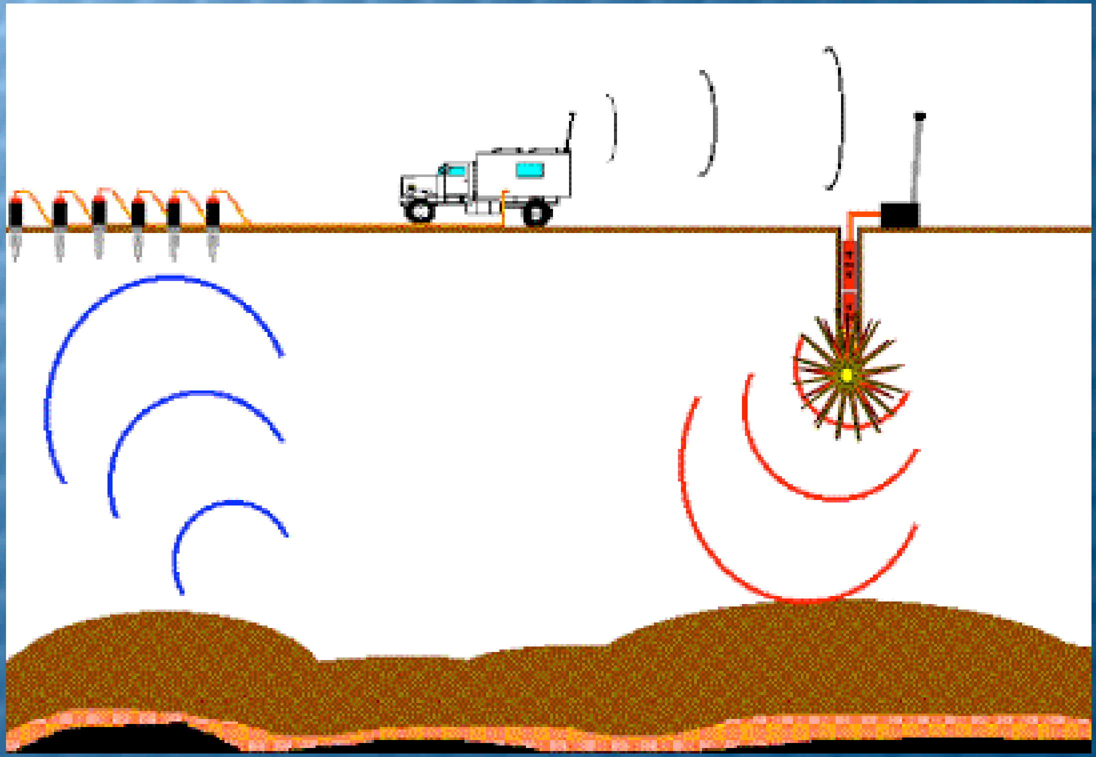
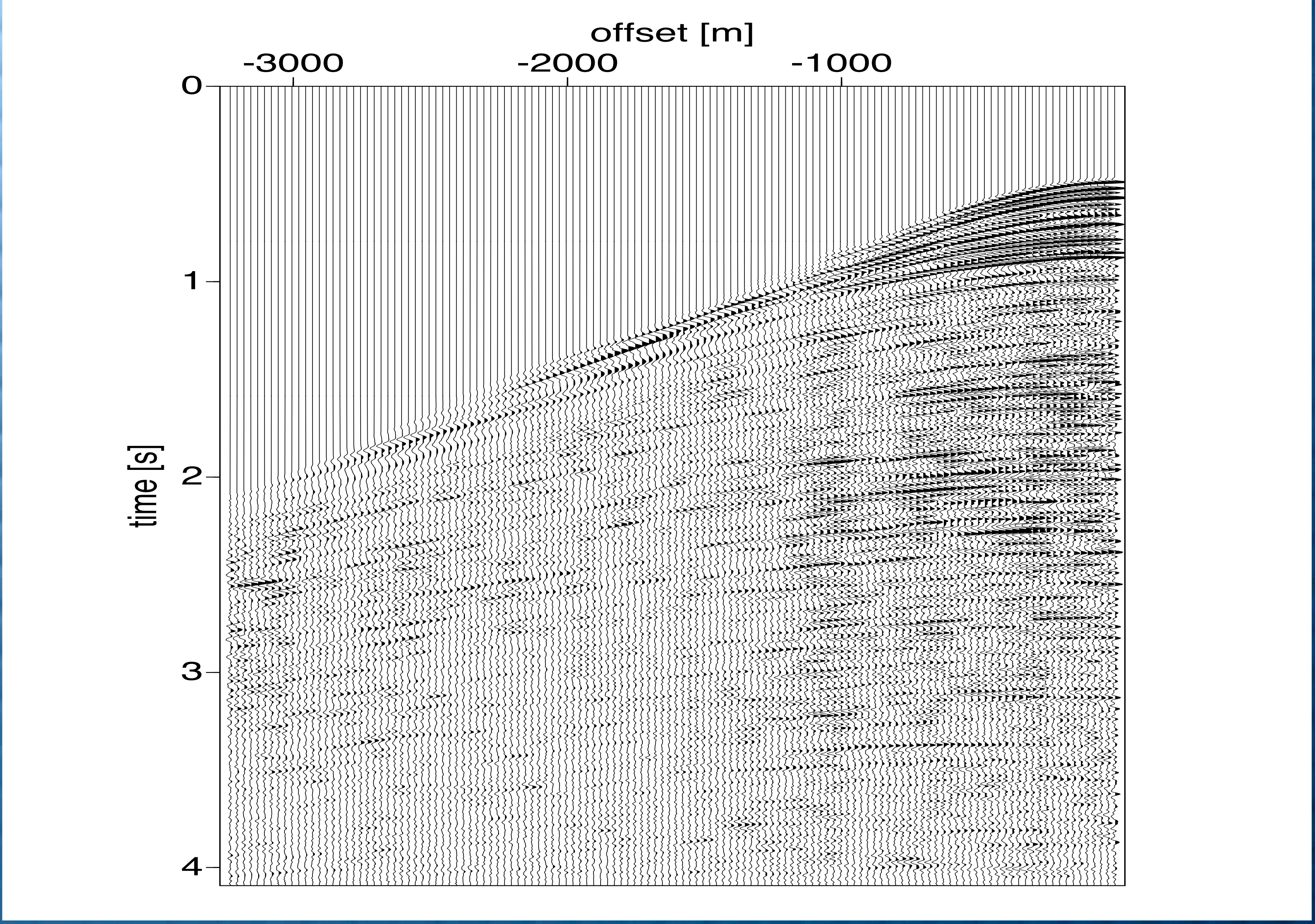
thanks to: Thierry Blu, Minh Do, Jonathan Kane, Mauricio Sacchi, Daniel Trad, Michael Unser, WaveLab

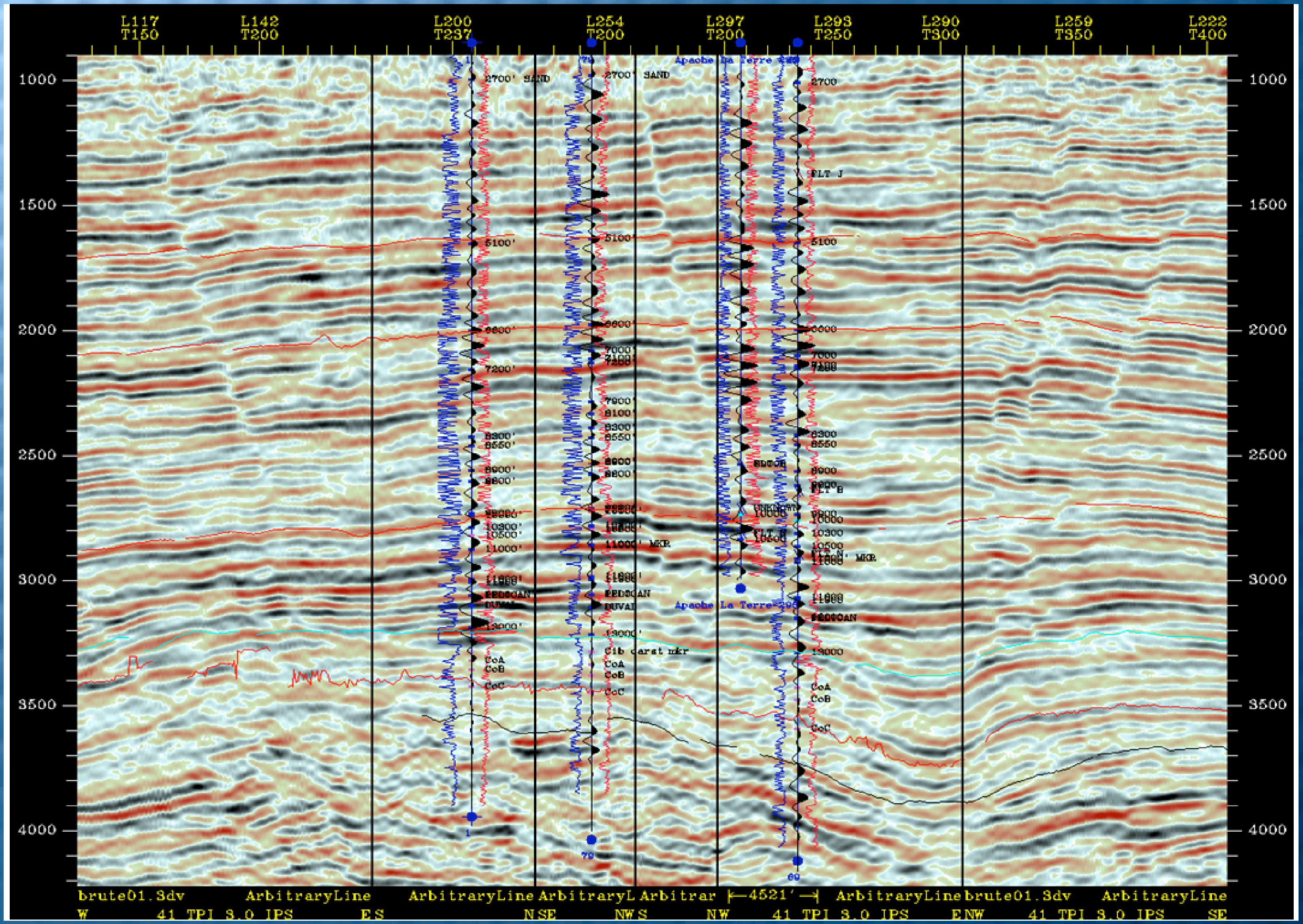
Seismic imaging



Seismic “sinogram”



Seismic imaging



Seismic imaging

We are in the business of

- ***locating the major singularities from bandwidth limited data***
- ***characterization of the major singularities by effective exponents***

In the presence of noise ... Lots of it!

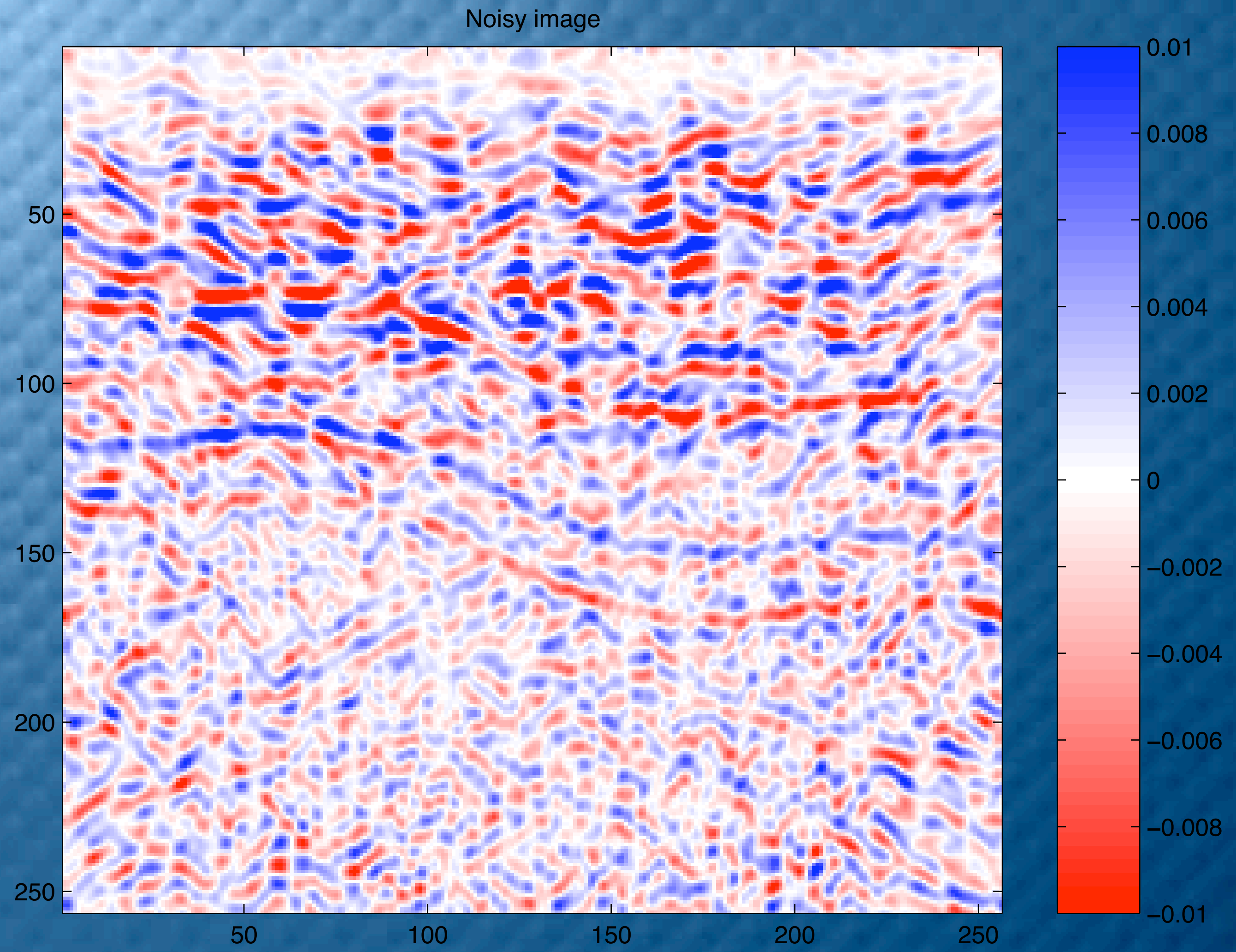
Basic imaging problem

$$\hat{\mathbf{m}}: \quad \min_{\mathbf{m}} \|\mathbf{d} - \mathbf{A}\mathbf{m}\|_2 + \nu J(\mathbf{m})$$

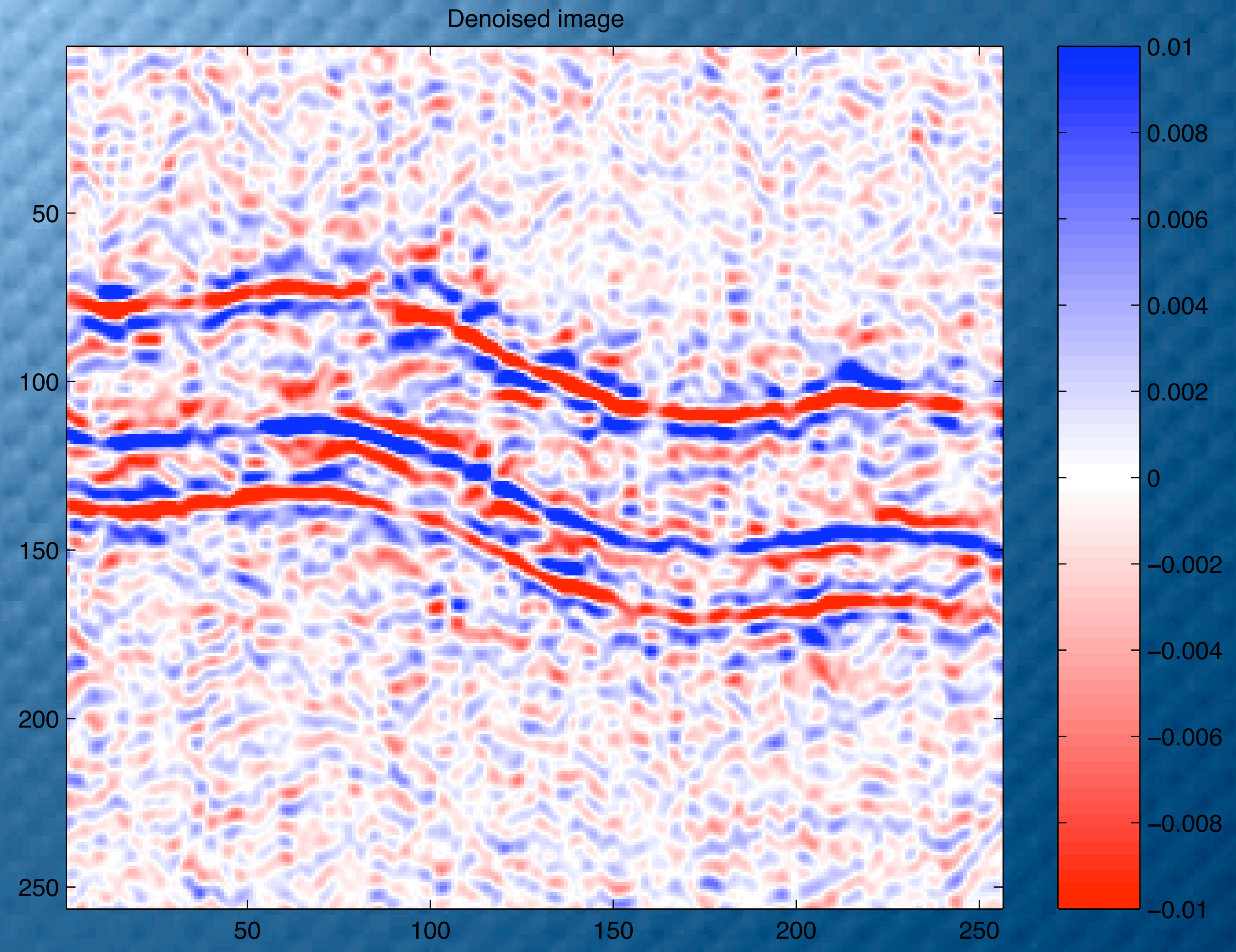
$$\begin{array}{lcl} \mathbf{d} & = & \textbf{measured data.} \\ \mathbf{A} & = & \textbf{linear scattering operator.} \\ \mathbf{m} & = & \textbf{model.} \\ \mathbf{n} & = & \textbf{white Gaussian noise.} \\ J(\mathbf{m}) & = & \textbf{Global penalty function} \end{array}$$

Preserve *singularities* in the presence of noise ...

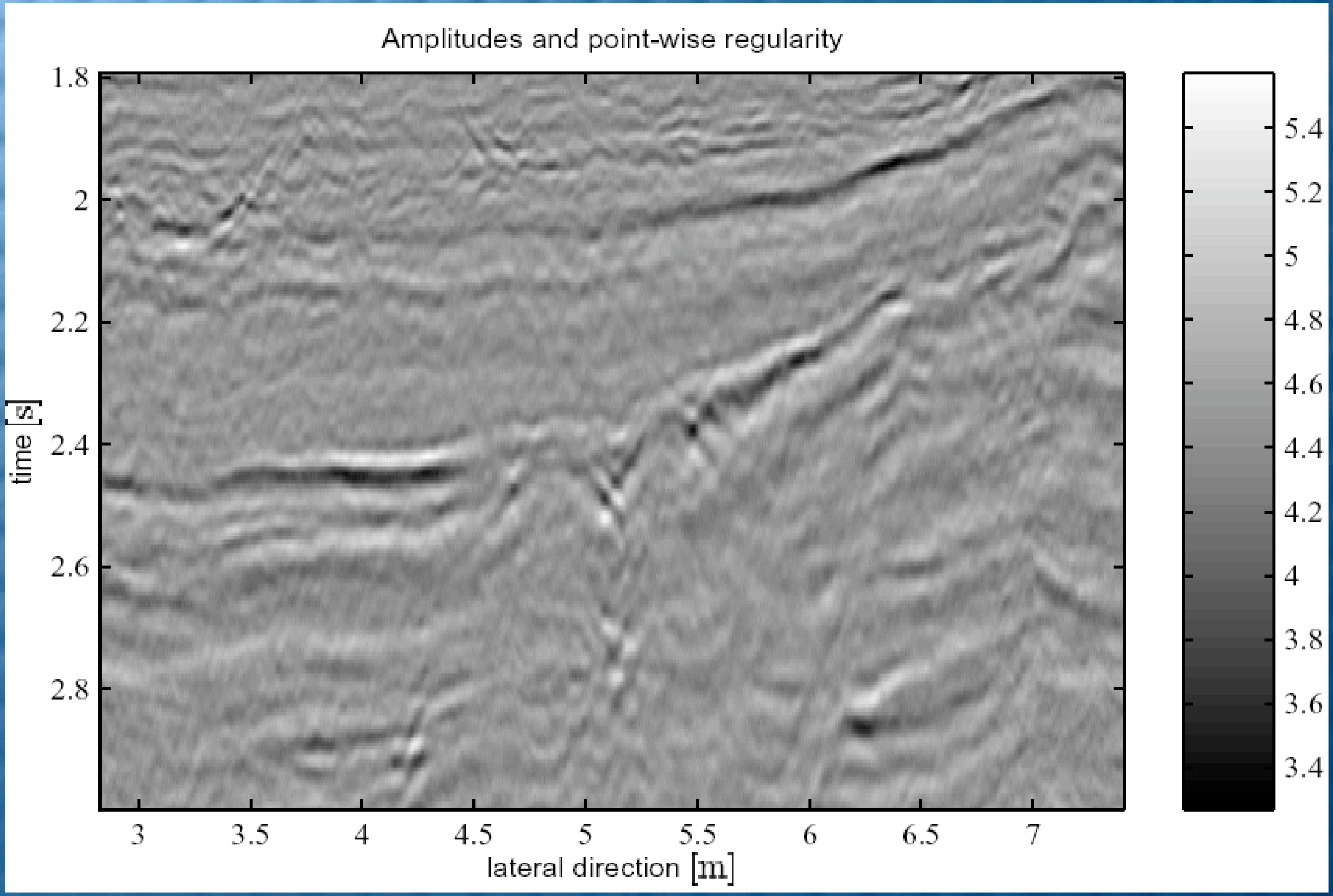
Seismic imaging



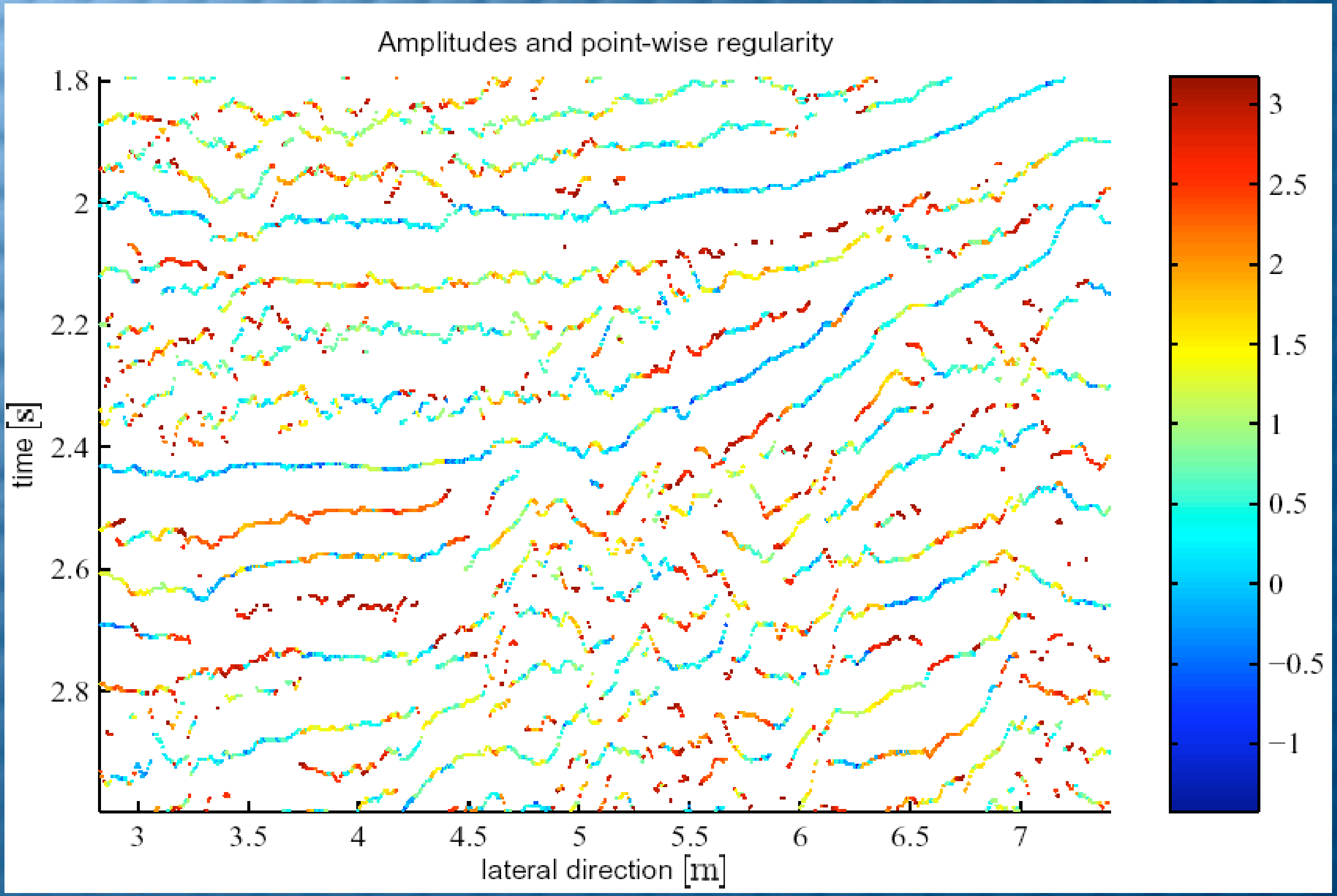
Seismic imaging



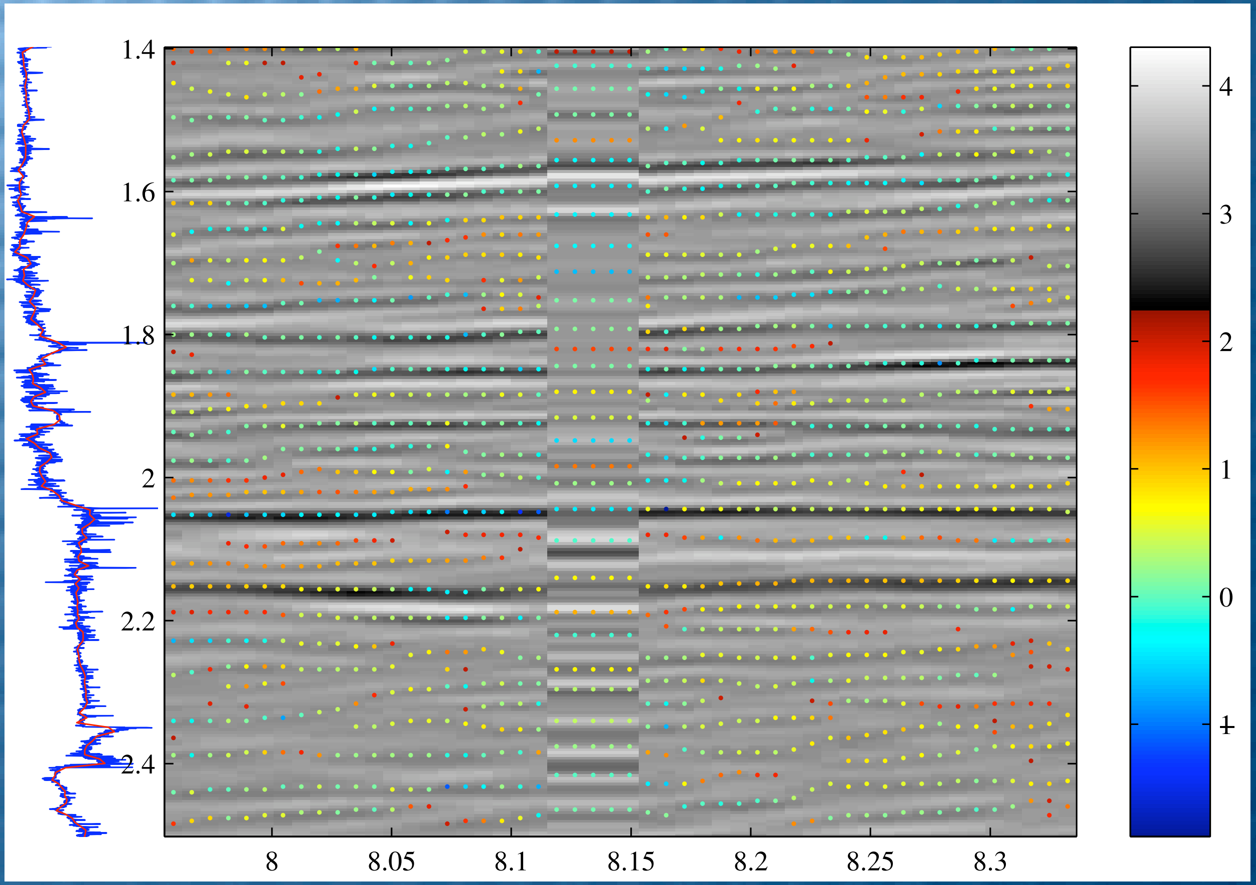
Multifractional Splines



Multifractional Splines



Multifractional Splines



Seismic imaging

Edge-preserved imaging:

- **Exploit redundancy seismic data**
- **Use basis-functions**
  - **local, sparse & well-behaved**
  - **non-lin. estimation by thresholding**
- **Approximate the normal operator**

Seismic characterization

Singularity characterization:

- **Exploit *waveforms* that carry info on the type of *reflectors***
- **Use *adaptive* redundant dictionary**
  - **localize the *knots* (*stratigraphy*)**
  - **estimate local *regularity* (*lithology*)**

Pattern extraction ....

Seismic *vs* medical imaging

	measure	Normal operator	inverse Normal operator	+	-
"medical" Radon	broadband intensity	$\widehat{\mathbf{K}^*\mathbf{K}}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert^{-1}$	$\widehat{\mathbf{K}^*\mathbf{K}}^{-1}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert$	broadband & non-lin. estimation	high. freq. noise
Gen. Radon	bandwidth limited + phase	$\widehat{\mathbf{K}^*\mathbf{K}}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert$	$\widehat{\mathbf{K}^*\mathbf{K}}^{-1}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert^{-1}$	redundancy & normal operator	bandwidth limitation & recon. moduli smooth part

Seismic imaging

Naively regularize (Tikhonov):

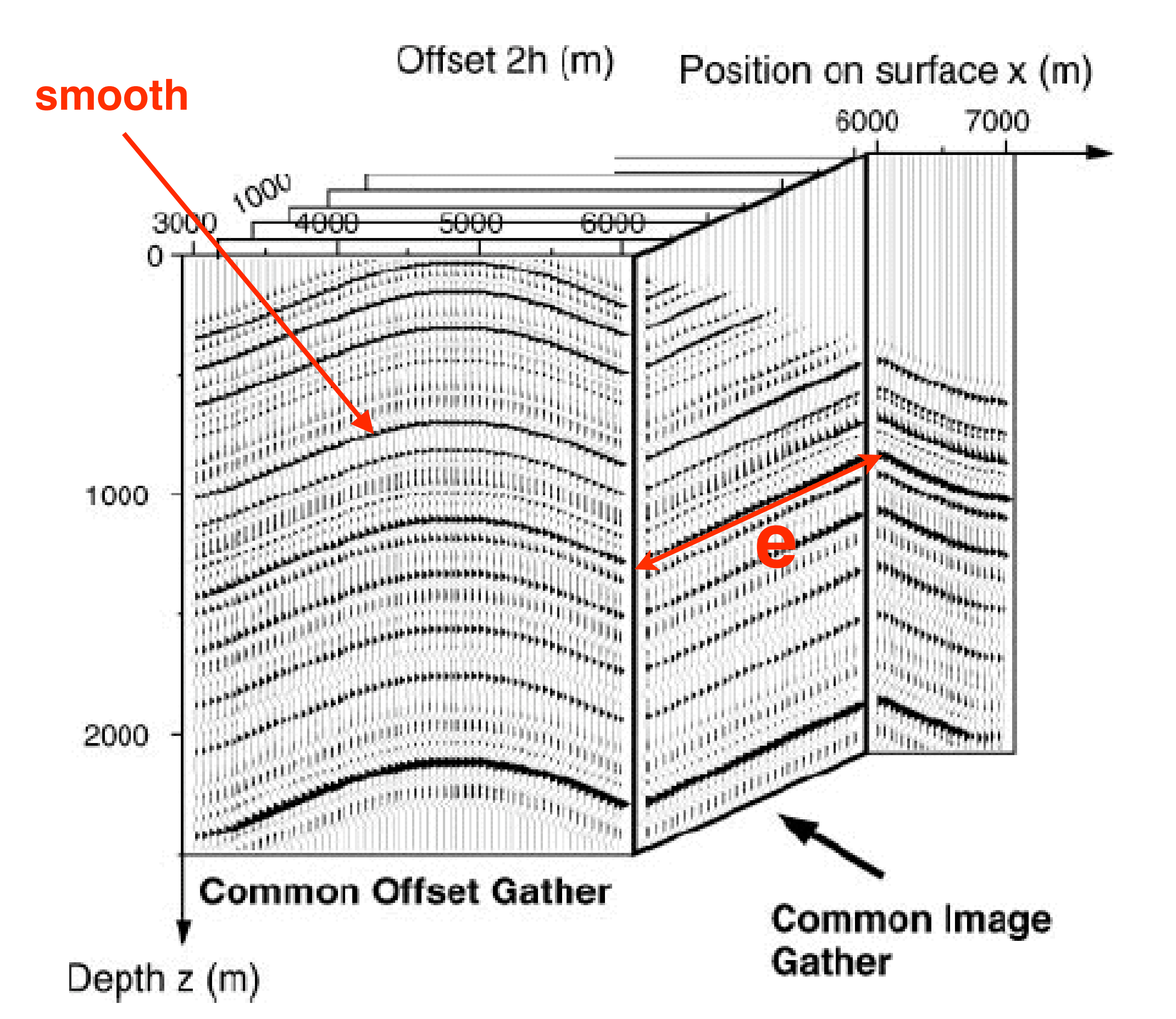
$$\hat{\mathbf{m}} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{d}$$

Instead exploit redundancy (smoothness):

$$\mathbf{A} = \overbrace{\mathbf{S}}^{loc.\,refl\,op.} \qquad \underbrace{\mathbf{K}}_{glob.\,scat.\,op.}$$

- **as function of the off-set**
- **along reflectors**

Seismic imaging



$$\mathbf{u}(\mathbf{x}, z, \mathbf{e}) = \mathbf{K}^* \mathbf{d}$$

Seismic imaging

In noise-free case we can solve for

$$\mathbf{m} = [Z, c_p, \nu]^T$$

From

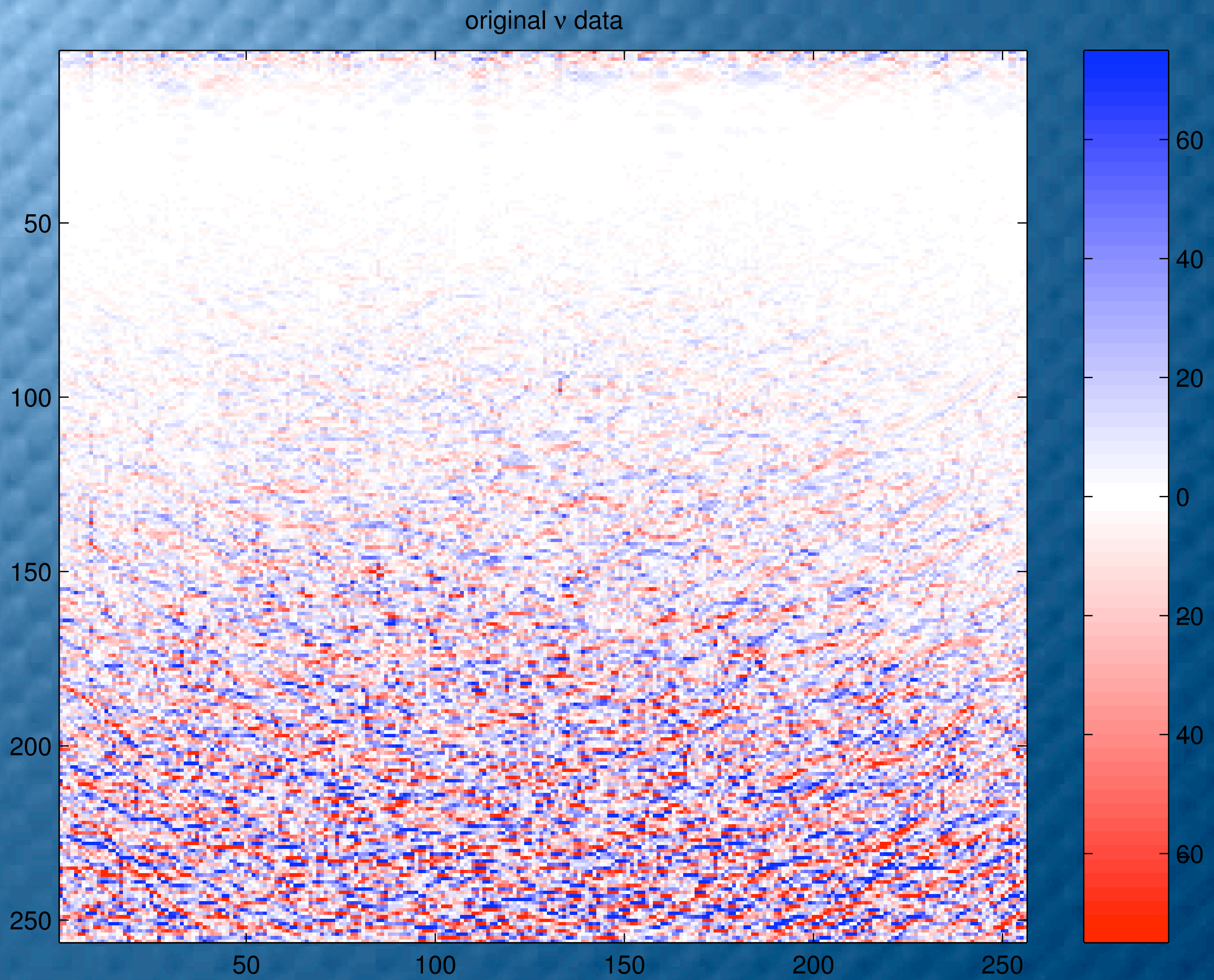
$$\hat{\mathbf{m}} = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*\mathbf{d}$$

without regularization.

When noisy CG fits noise ...

- **no image**
- **no reliable estimates for $\mathbf{m}$**

Seismic imaging



Seismic imaging

$\mathbf{S}, \mathbf{S}^*$ are **smooth.**

$\mathbf{K}, \mathbf{K}^*$ are Fourier Integral operators:

$$Tf(x) = \int e^{i\Phi(z,\zeta)} a(x,\zeta)\hat{f}(\zeta)\mathrm{d}\zeta$$

- ★ transports singularities

$\mathbf{K}^*\mathbf{K}$ is a pseudo differential operator:

$$Tf(x) = \int e^{i2\pi x\cdot\zeta} a(x,\zeta)\hat{f}(\zeta)\mathrm{d}\zeta$$

- ★ preserves singularities

Beylkin 1984. Stolk 2002, ten Kroode 1997, de Hoop 2000, 2003

Seismic imaging

$$\hat{m} = \underbrace{(K^*K)^{-1}}_{\square\mathrm{DO}} \overbrace{K^*d}^{\mathrm{FIO}}$$

	□DO	FIO	d & m	**e**
Wavelets	✔	✗	✗	✔
Curvelets	✔	✔	✔	✔

Candes & Demanet

Seismic imaging

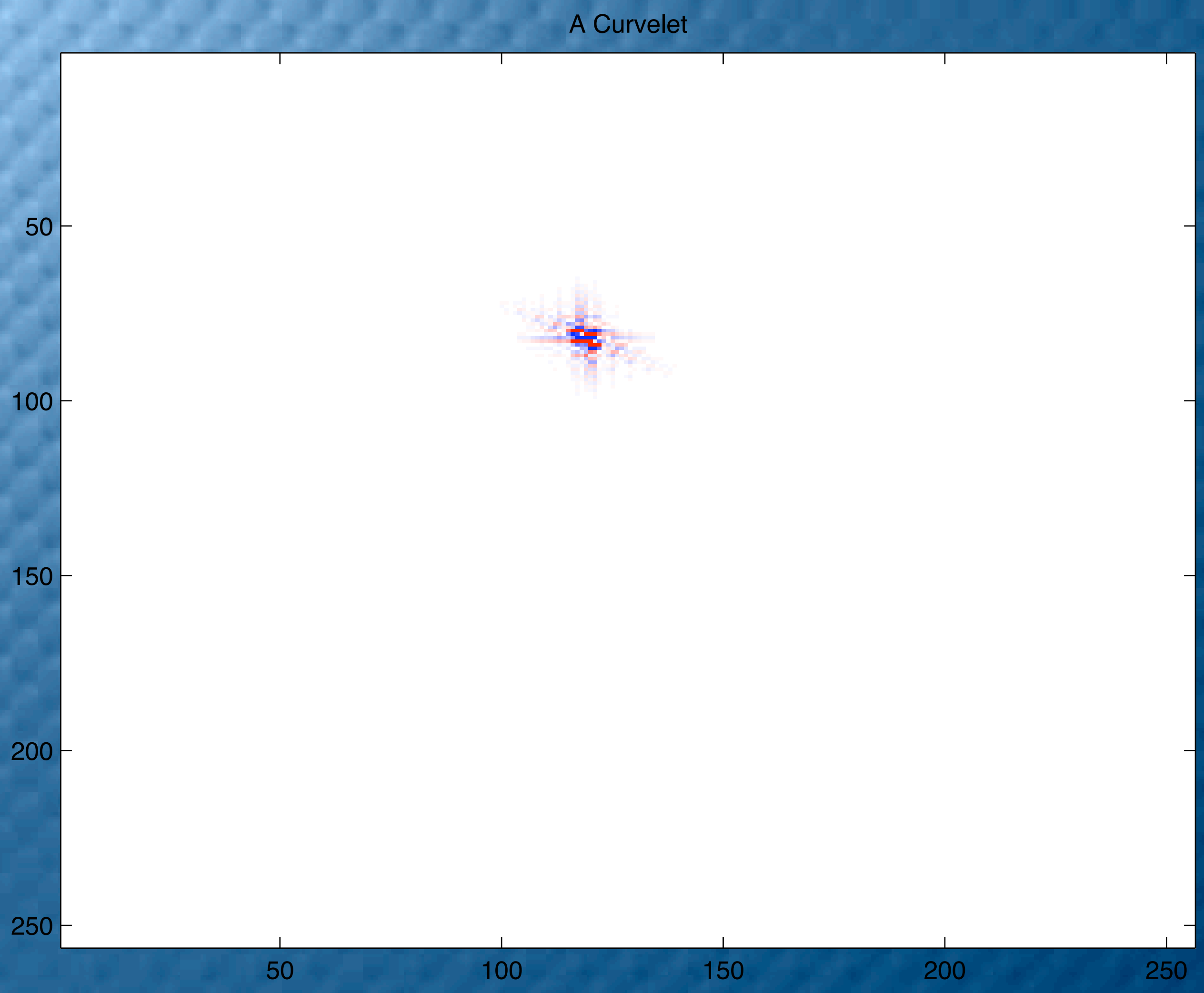
Appropriate domain for estimation:

$$\mathbf{u}(\mathbf{x}, z, \mathbf{e}) = \mathbf{K}^* \mathbf{d}$$

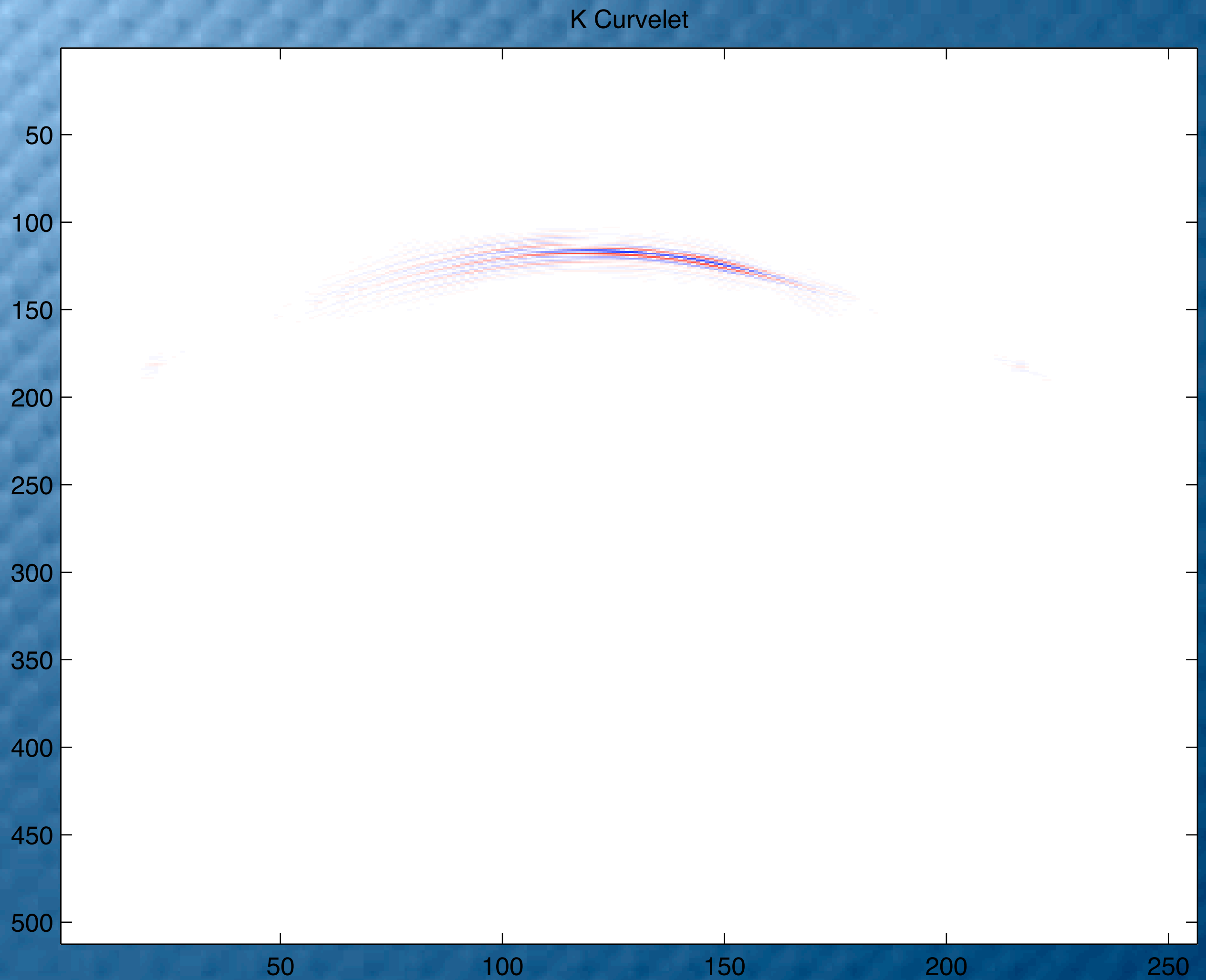
Choose basis functions that:

- ***sparse & local* model space**
- **almost diagonalize *normal* operator**
- **correct for coloring of n**

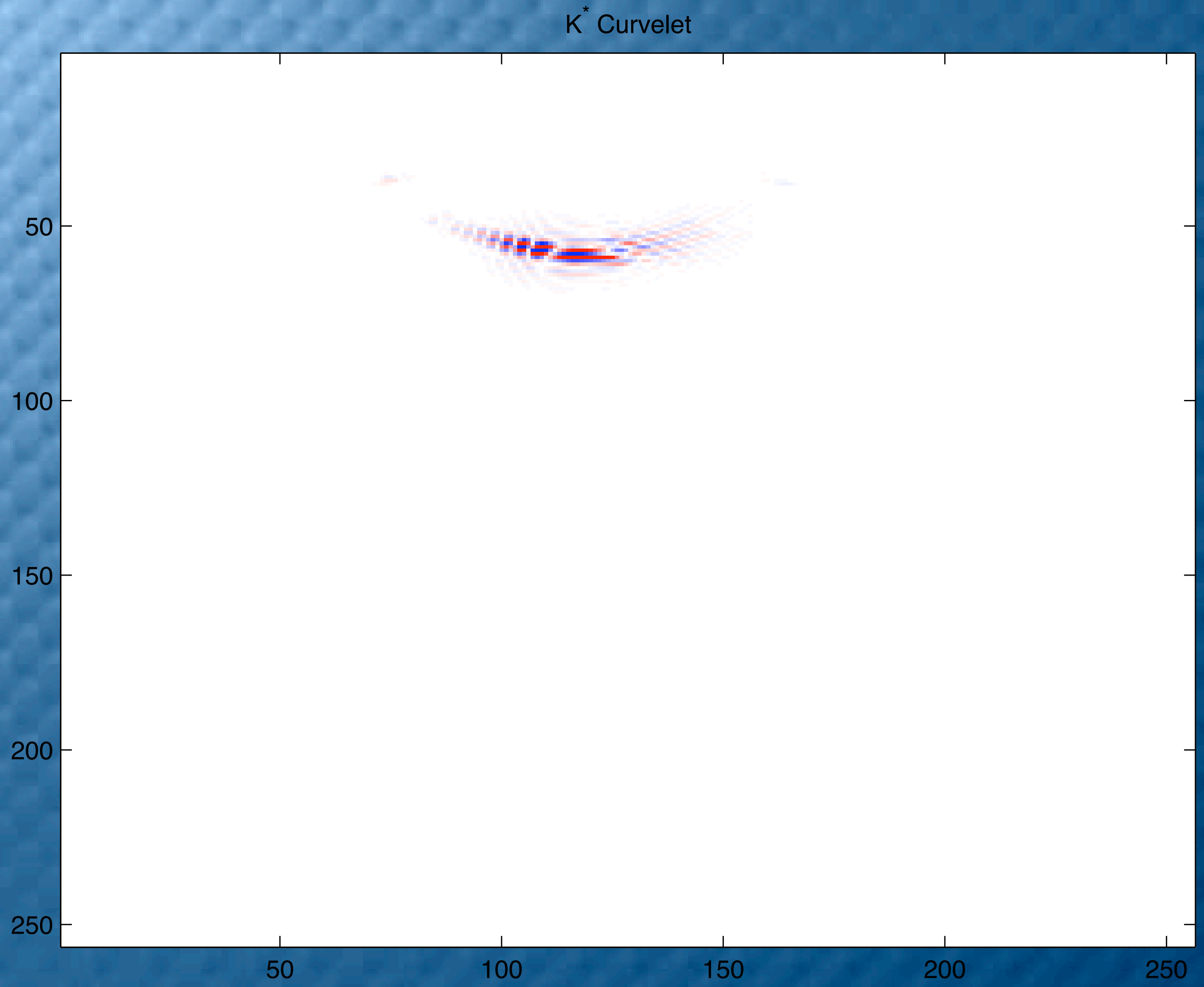
Seismic imaging



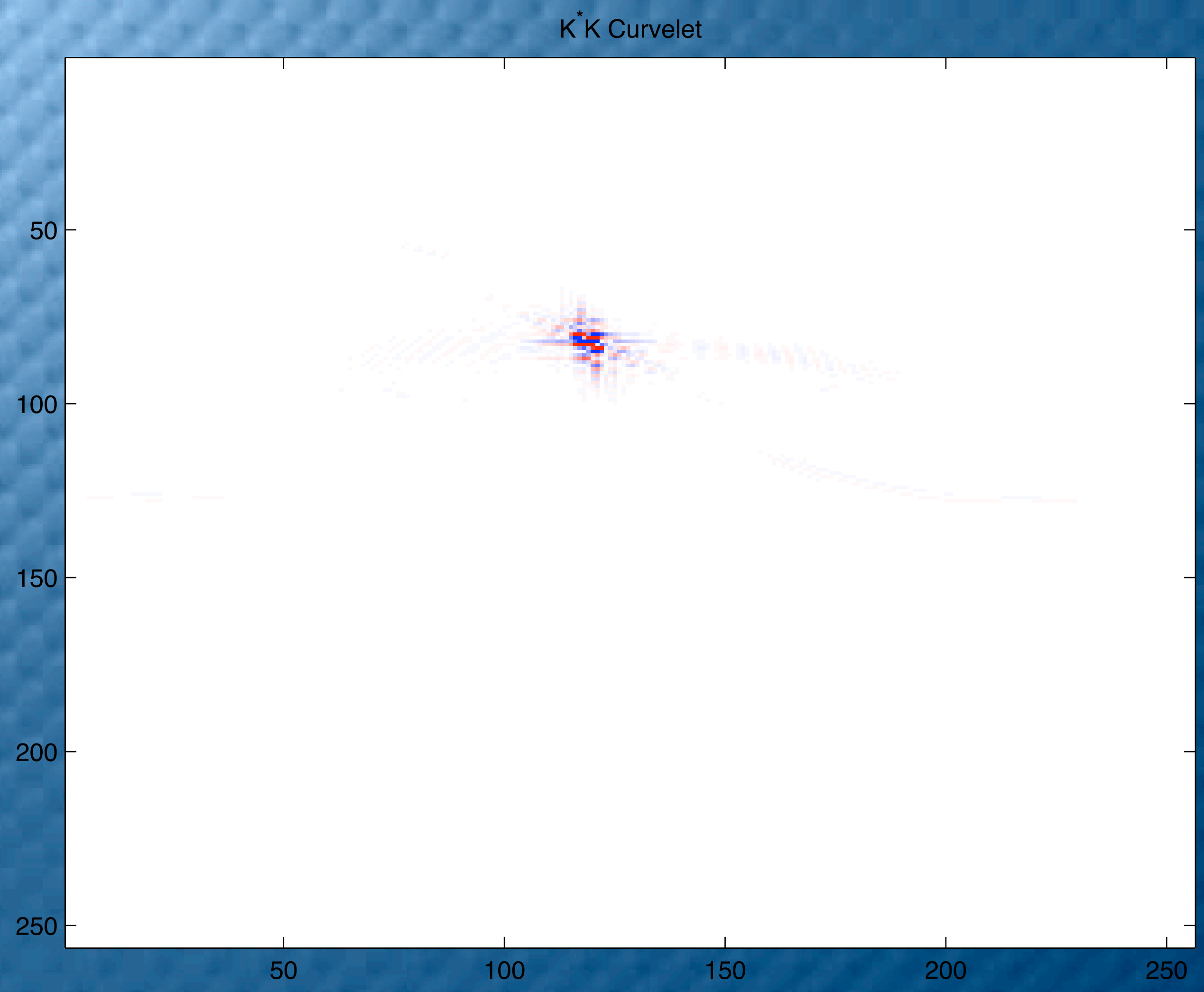
Seismic imaging



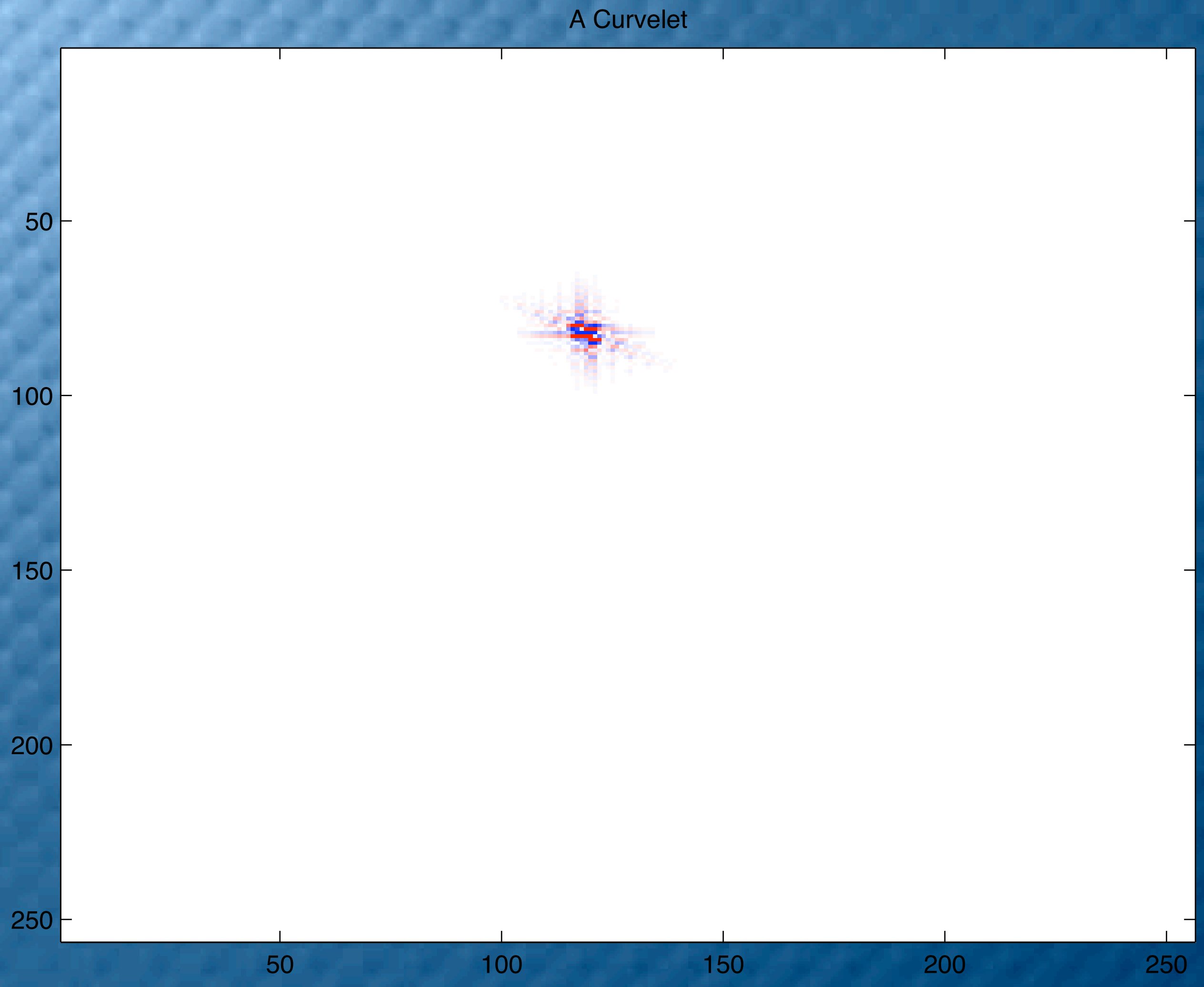
Seismic imaging



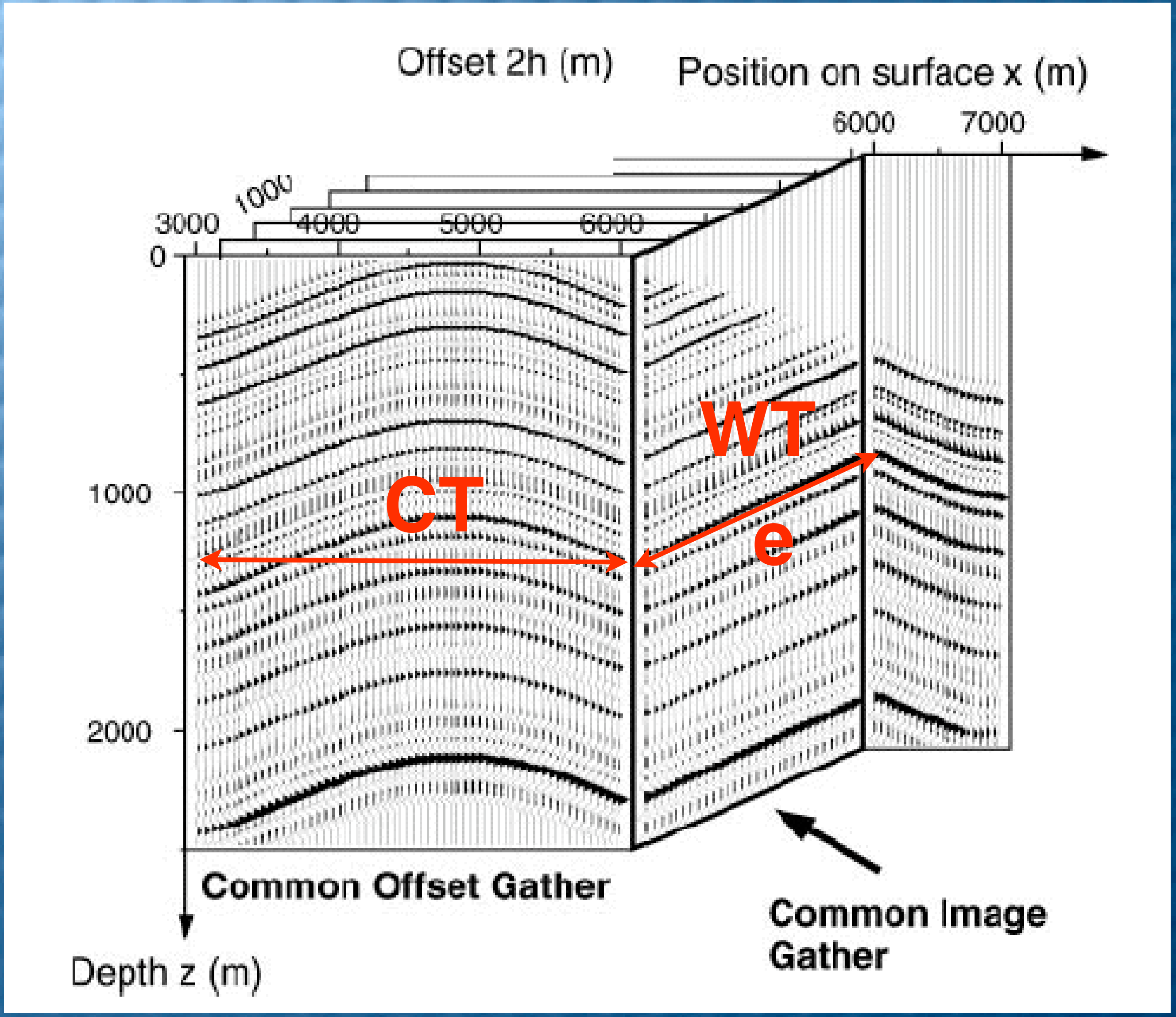
Seismic imaging



Seismic imaging



Seismic imaging



Seismic imaging

- Use Curvelets for each off-set
- Use wavelet in off-set direction

$$\mathbf{B}: \quad f(\cdot,\cdot;\mathbf{e}) \mapsto \overbrace{\mathbf{C}f(\cdots;\mathbf{e})}^{curvelet/contourlet} \mapsto \underbrace{\mathbf{WC}f(\cdots,\cdot)}_{wavelet}$$

$$\mathbf{B}^{-1}: \quad f(\cdots;\cdot) \mapsto \overbrace{\mathbf{W}^* f(\cdots;\mathbf{e})}^{wavelet} \mapsto \underbrace{\mathbf{C}^{-1}\mathbf{W}^* f(x,z,\mathbf{e})}_{curvelet/contourlet}$$

Seismic imaging

Estimate with WV/Quasi-SVD

$$\hat{\mathbf{u}}(\mathbf{x}, z, \mathbf{e}) = \mathbf{B}^{-1}\Gamma^{\dagger}\theta_{t(\Gamma)}\left(\mathbf{B}\mathbf{K}^{*}\mathbf{d}\right)$$

- $\mathbf{K}^{*}\mathbf{K}$ **is a homogeneous/scale-inv. operator**

Curvelets almost diagonalize $\mathbf{K}^{*}\mathbf{K}$

$$\mathbf{K}^{*}\mathbf{K}\cdot \approx \mathbf{B}^{-1}\Gamma\mathbf{B}\cdot$$

- **correct for coloring n via Γ**
- **approximate action** $\left(\mathbf{K}^{*}\mathbf{K}\right)^{-1}$

Donoho,Candes, Lucier, Kane, H.

Seismic imaging

Direct computation of

$$\mathbf{K}\mathbf{B}^* = \mathbf{V}^*\square$$

$$\mathbf{K}\mathbf{U}^* = \mathbf{B}^*\square$$

- is ***prohibitively*** expensive
- requires 1 (de)-migration per basis-function

Donoho,Candes, Lucier, Kane, H.

Seismic imaging

Use MC-sampling:

$$\Gamma\cdot = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{BSK}^*\mathbf{n}_i\right)^2 = \operatorname{diag}\left\{\mathbf{BSK}^*\mathbf{KS}^*\mathbf{B}^{-1}\right\}\cdot$$

Set the threshold:

$$t(\Gamma) = \begin{cases} 3\sigma\Gamma & \text{coarsest scales} \\ 4\sigma\Gamma & \text{finest scales} \end{cases}$$

★ **corrects for coloring of n in threshold**

Candes, Do

Seismic imaging

Approximate the normal operator:

$$\mathbf{K}^*\mathbf{K}\cdot \approx \mathbf{B}^{-1}\Gamma\mathbf{B}\cdot$$

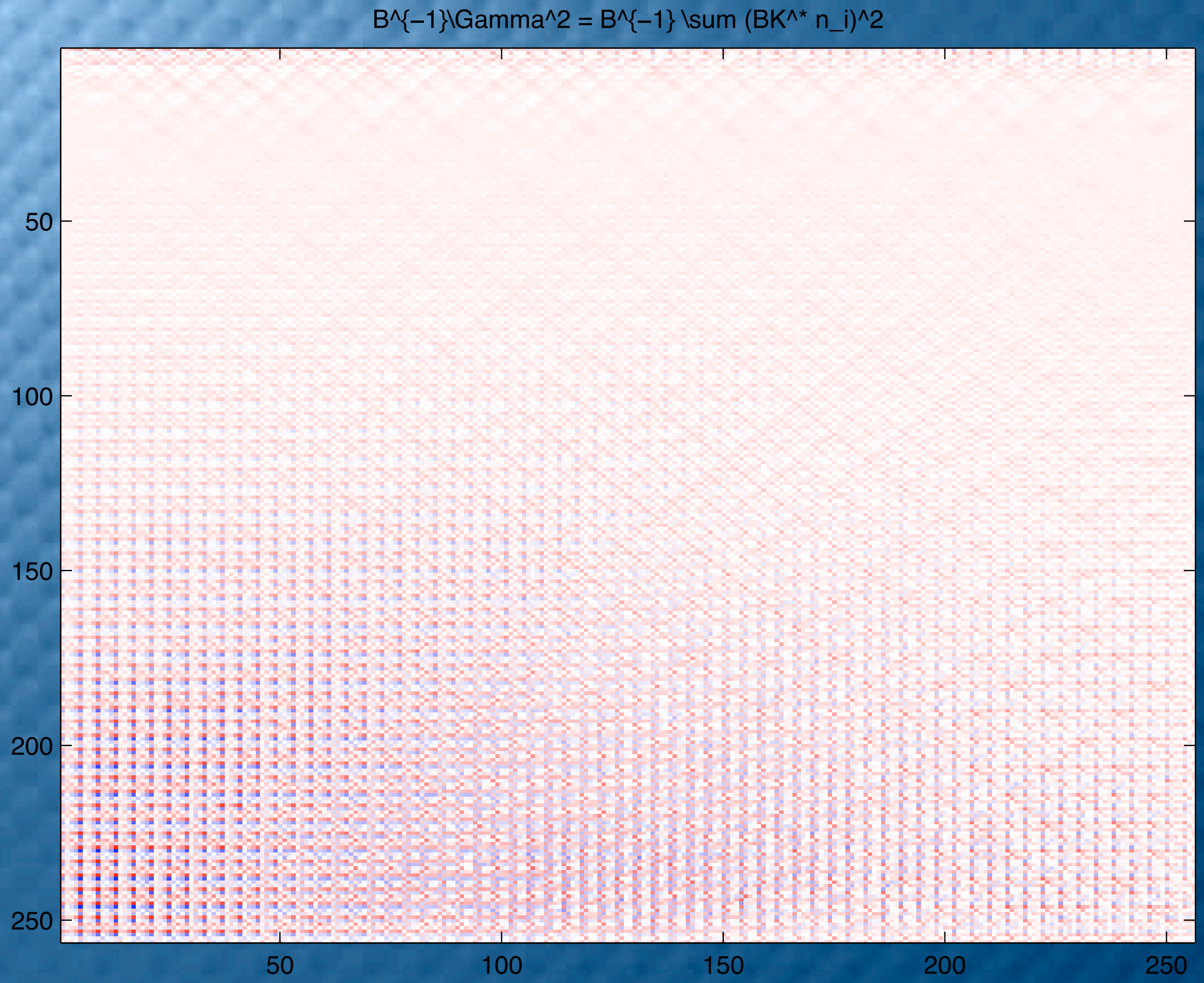
with the "pseudo-inverse"

$$\Gamma^\dagger\cdot = \left(\frac{\frac{1}{N}\sum_{i=1}^{N}(\mathbf{CK}^*\mathbf{n}_i)^2}{\frac{1}{N}\sum_{i=1}^{N}(\mathbf{Cn}_i)^2} + \epsilon\right)^{-1}\cdot$$

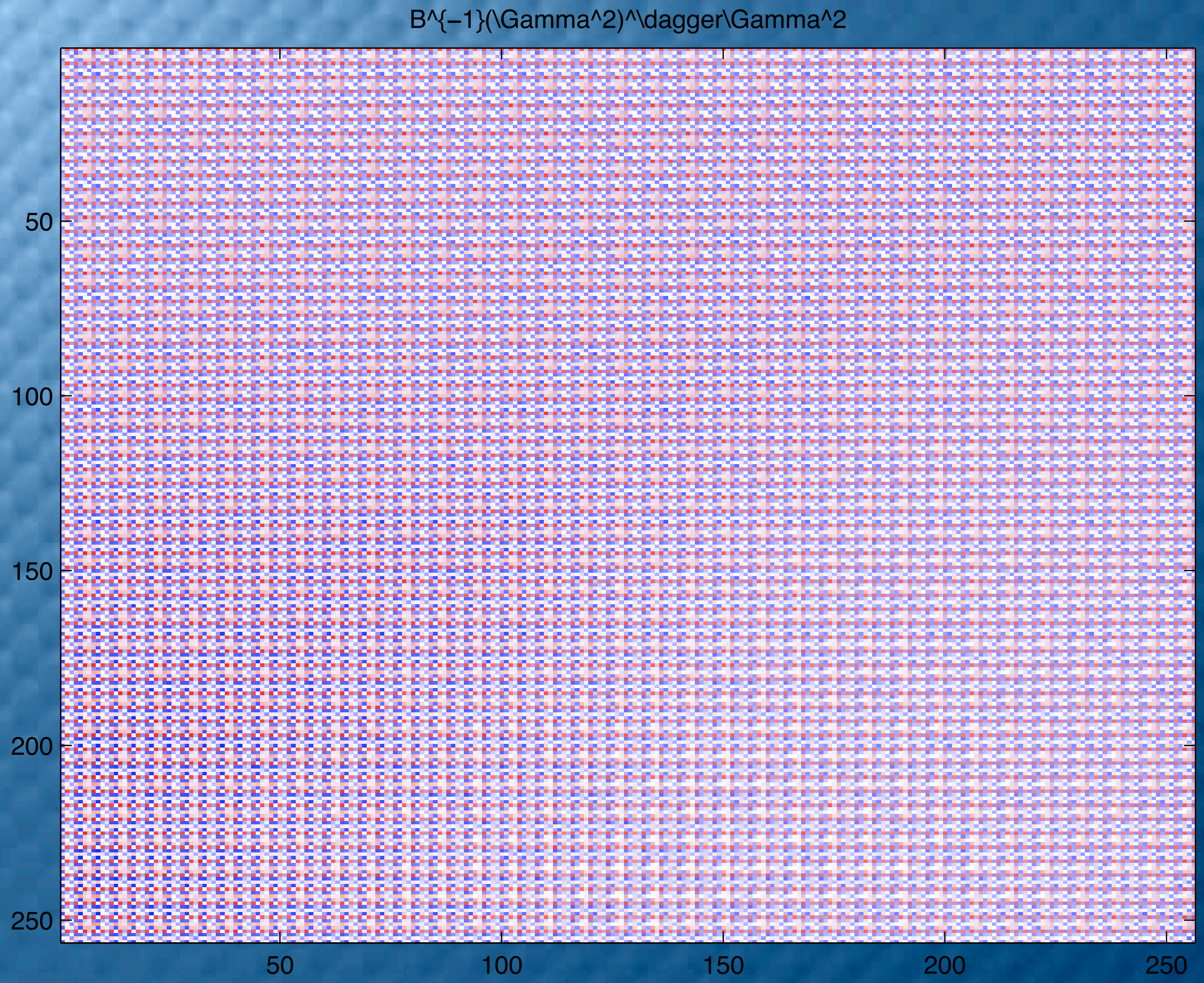
yielding

$$\hat{\mathbf{u}}(\mathbf{x}, z, \mathbf{e}) = \mathbf{B}^{-1}\Gamma^\dagger\theta_{t(\Gamma)}\left(\mathbf{BK}^*\mathbf{d}\right)$$

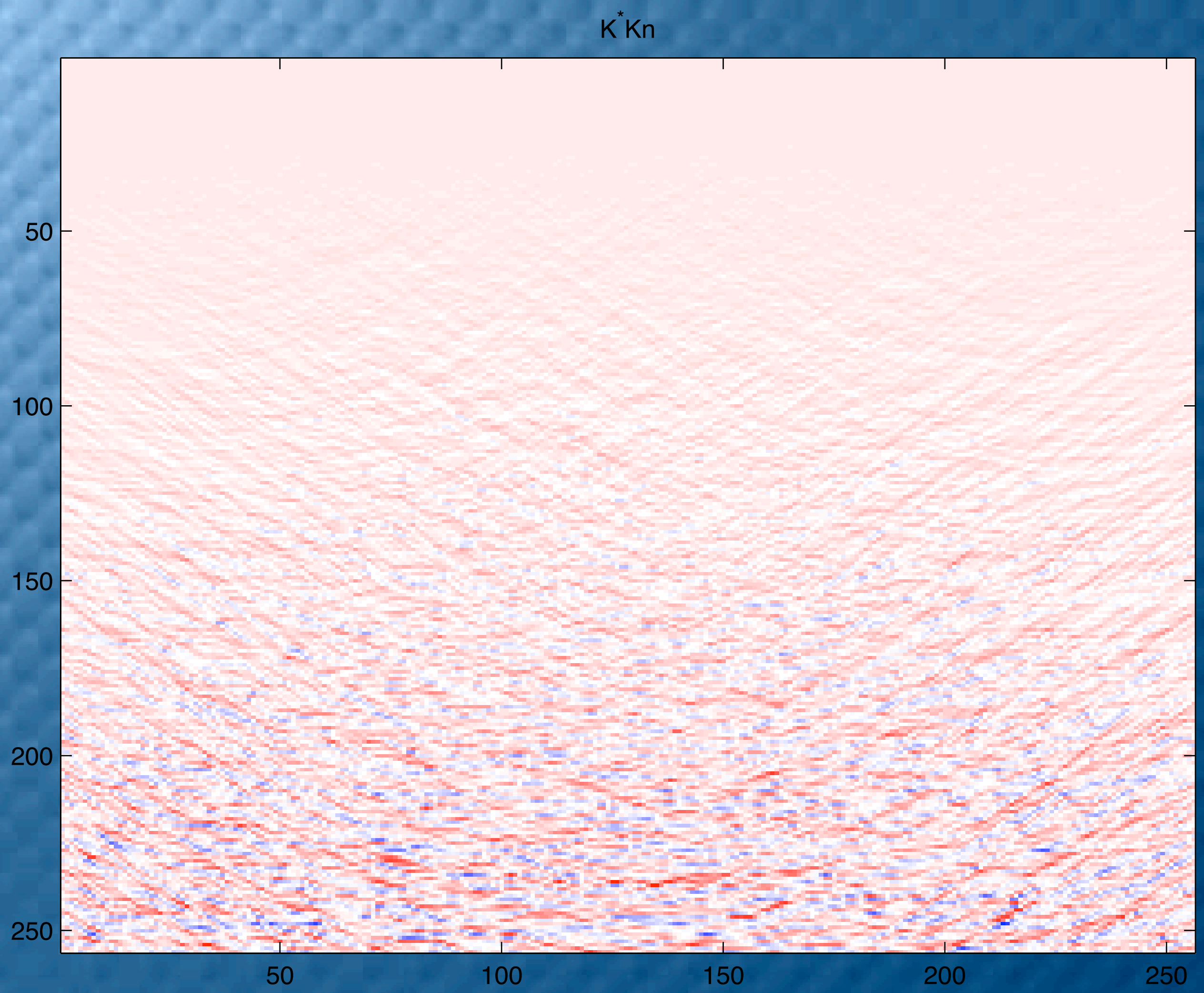
Seismic imaging



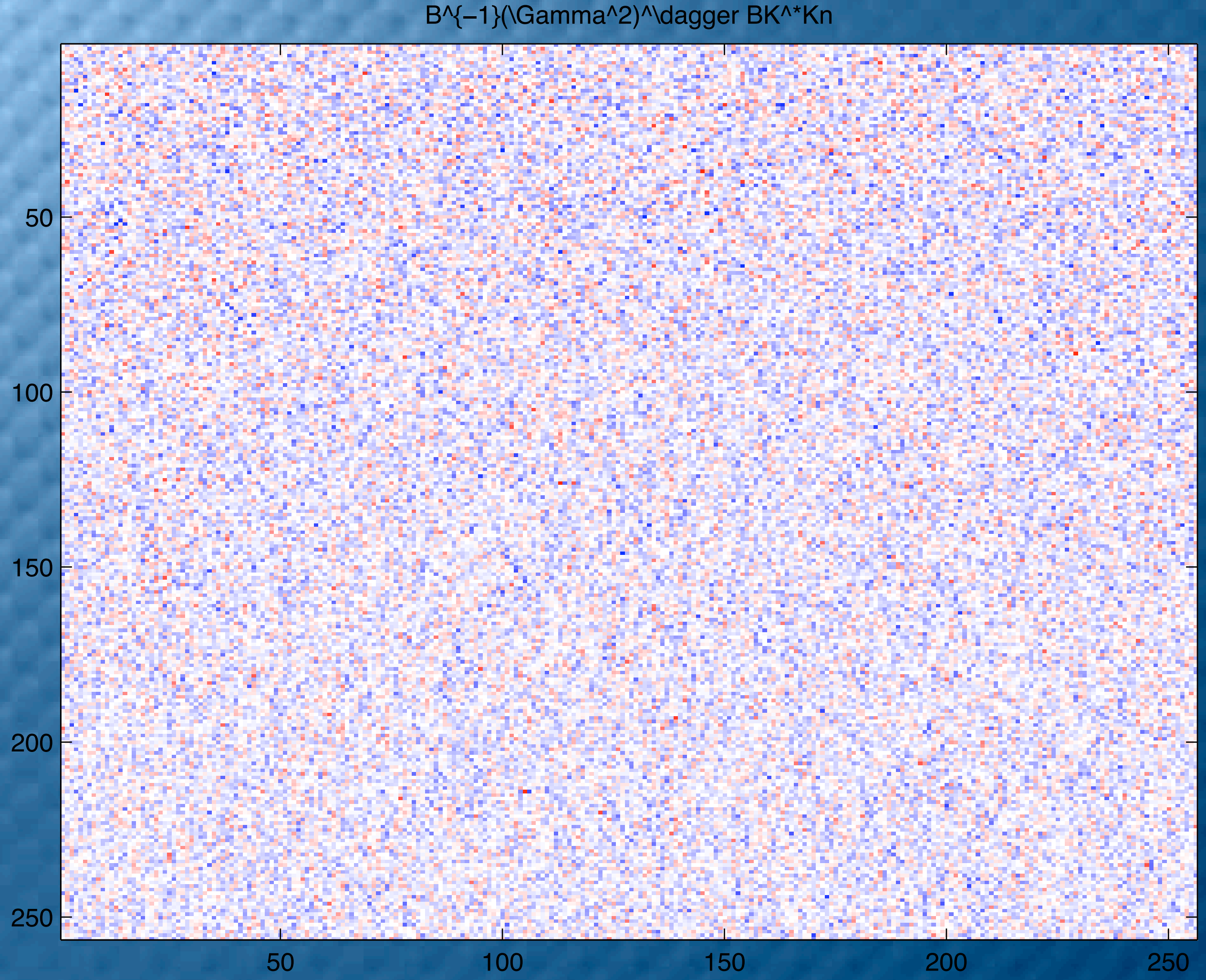
Seismic imaging



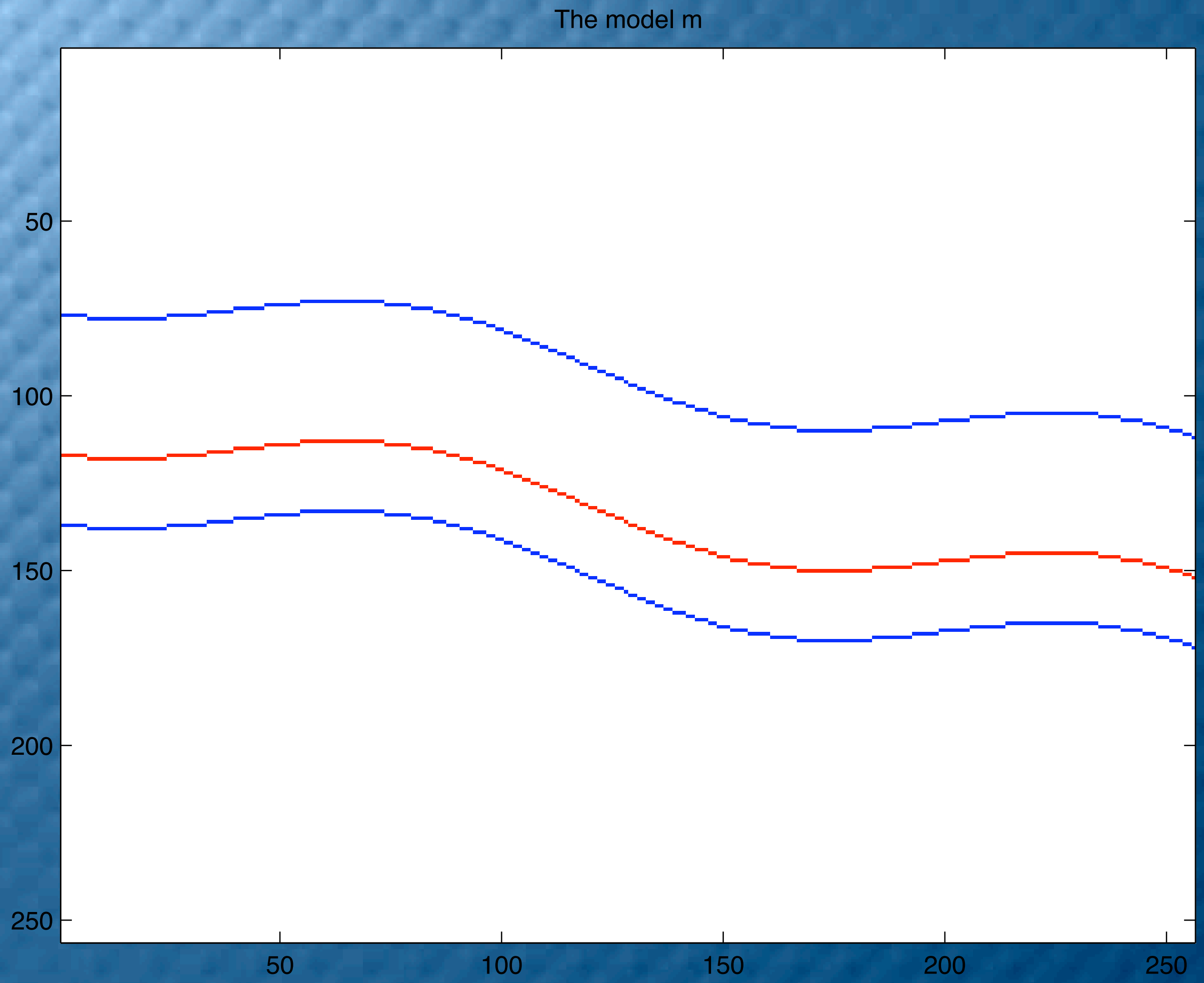
Seismic imaging



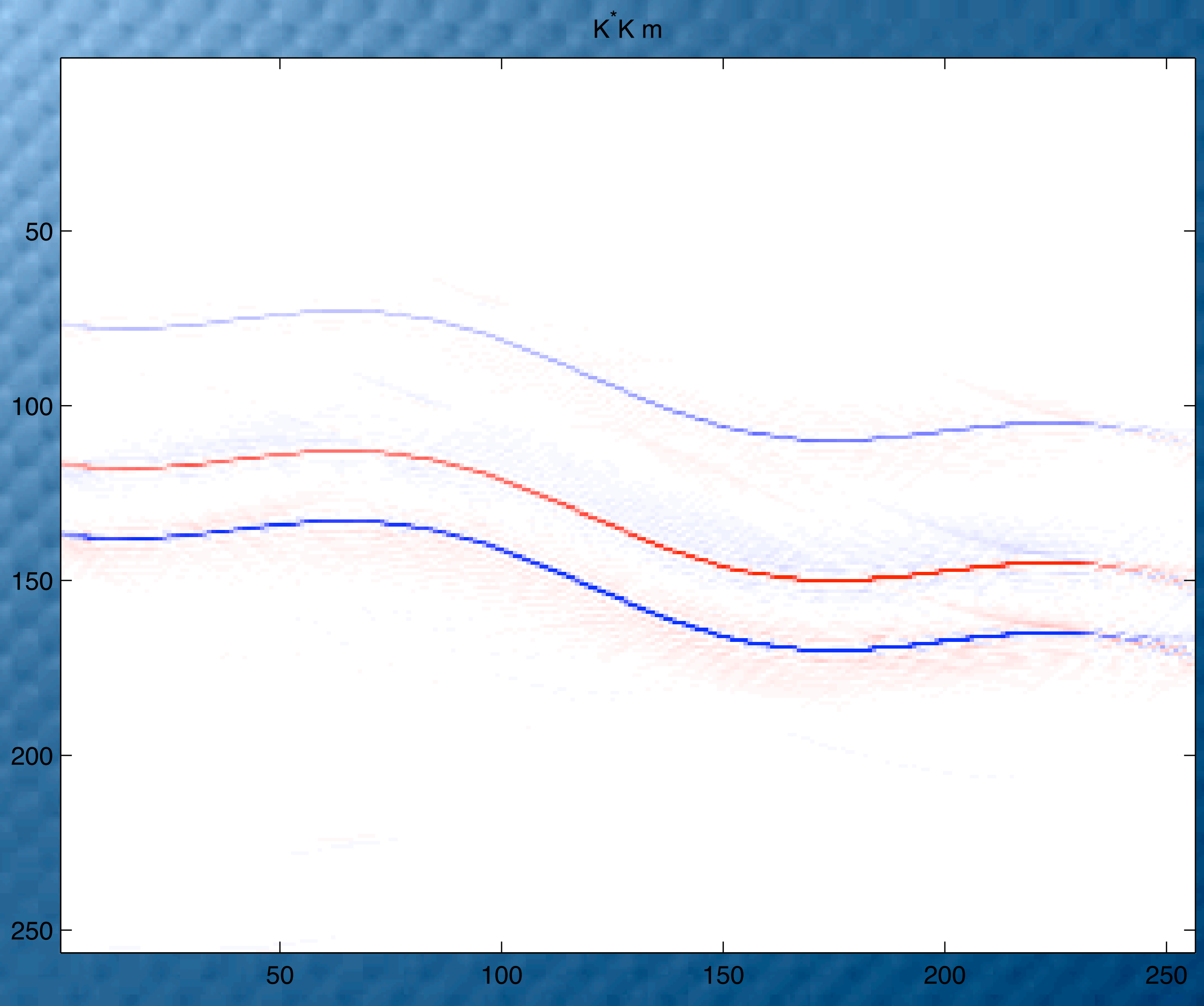
Seismic imaging



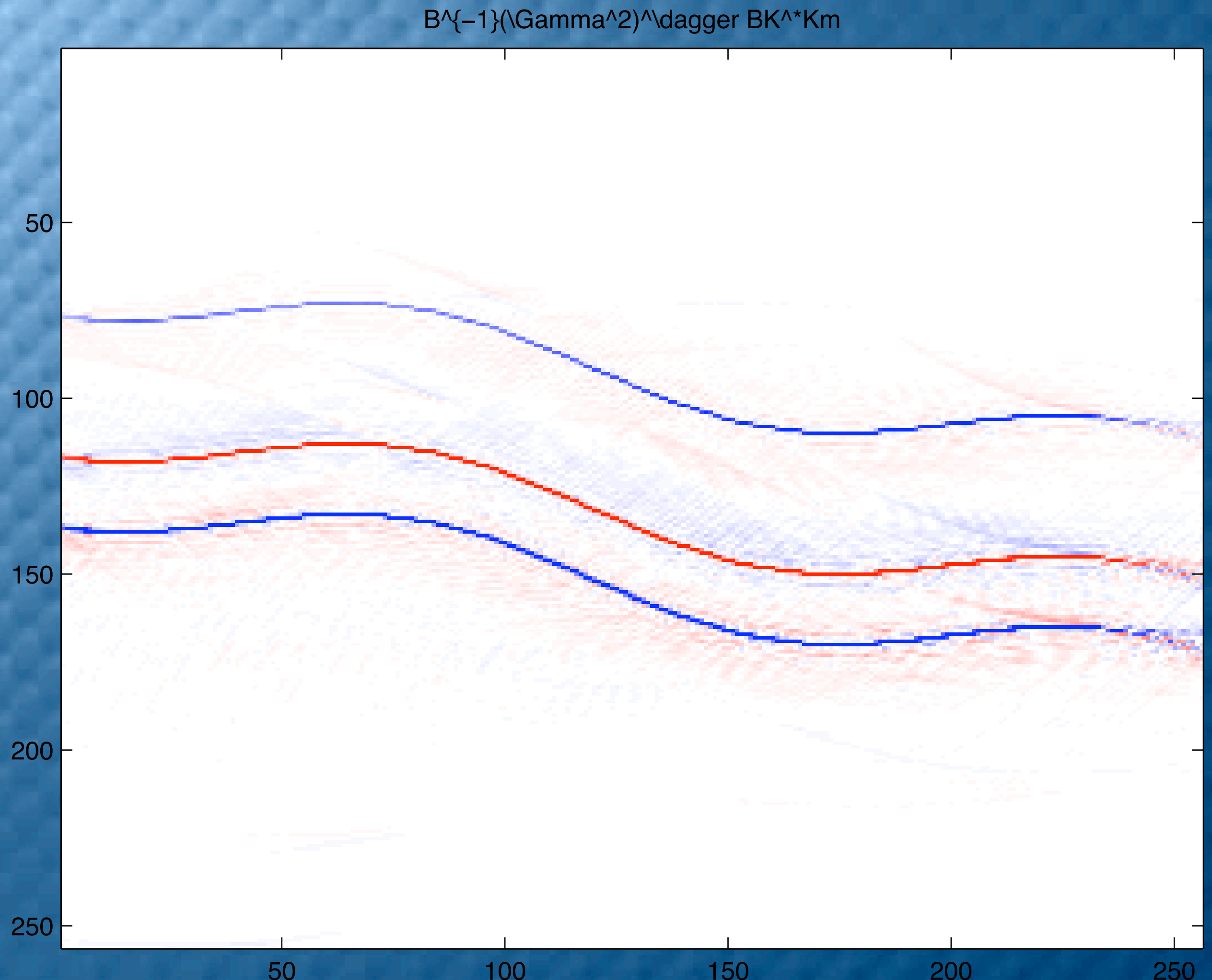
Seismic imaging



Seismic imaging



Seismic imaging



Seismic imaging

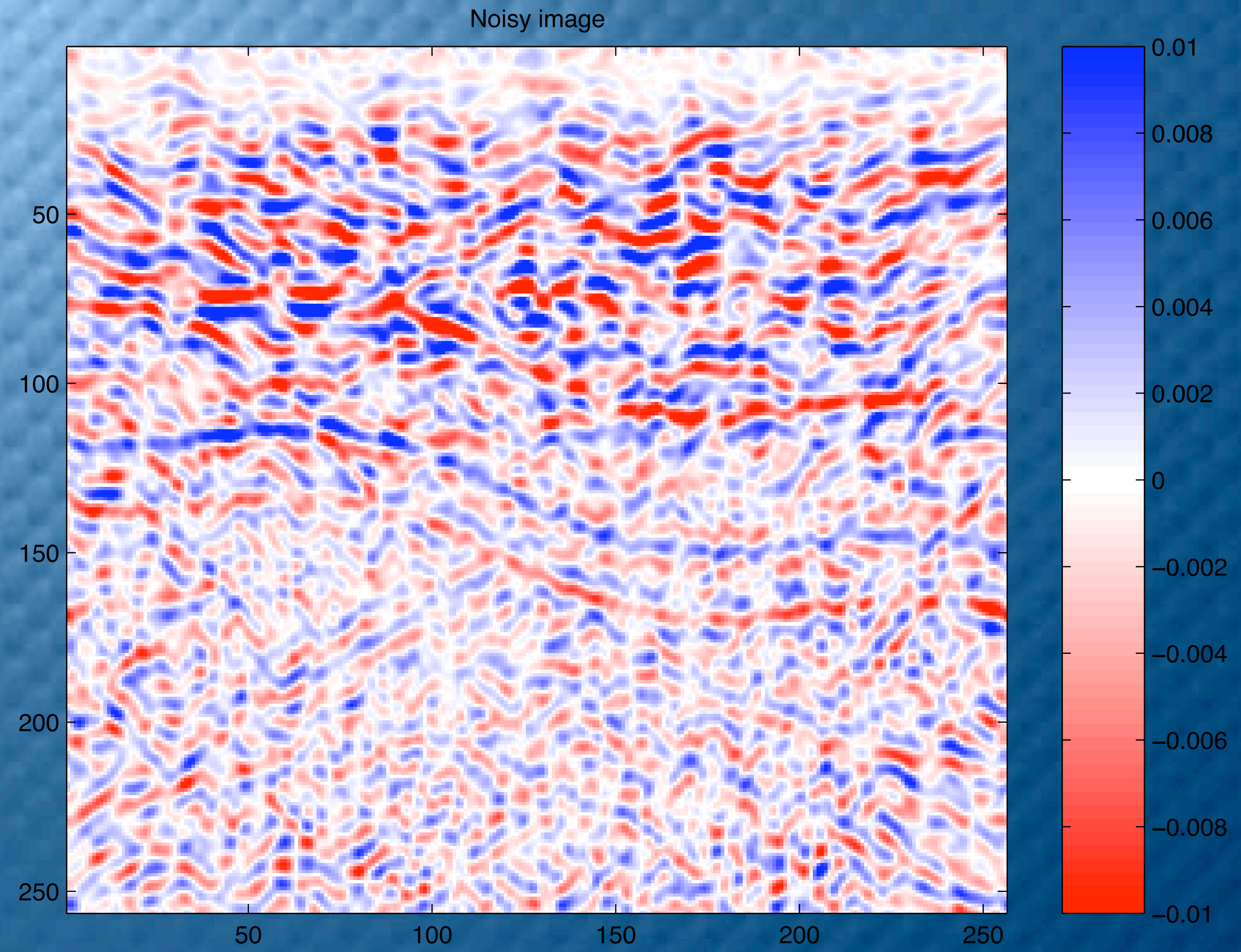
Use denoised image to estimate the model:

$$\hat{\mathbf{m}} = (\mathbf{S}^*\mathbf{S})^{-1}\,\mathbf{S}^*\hat{\mathbf{u}}$$

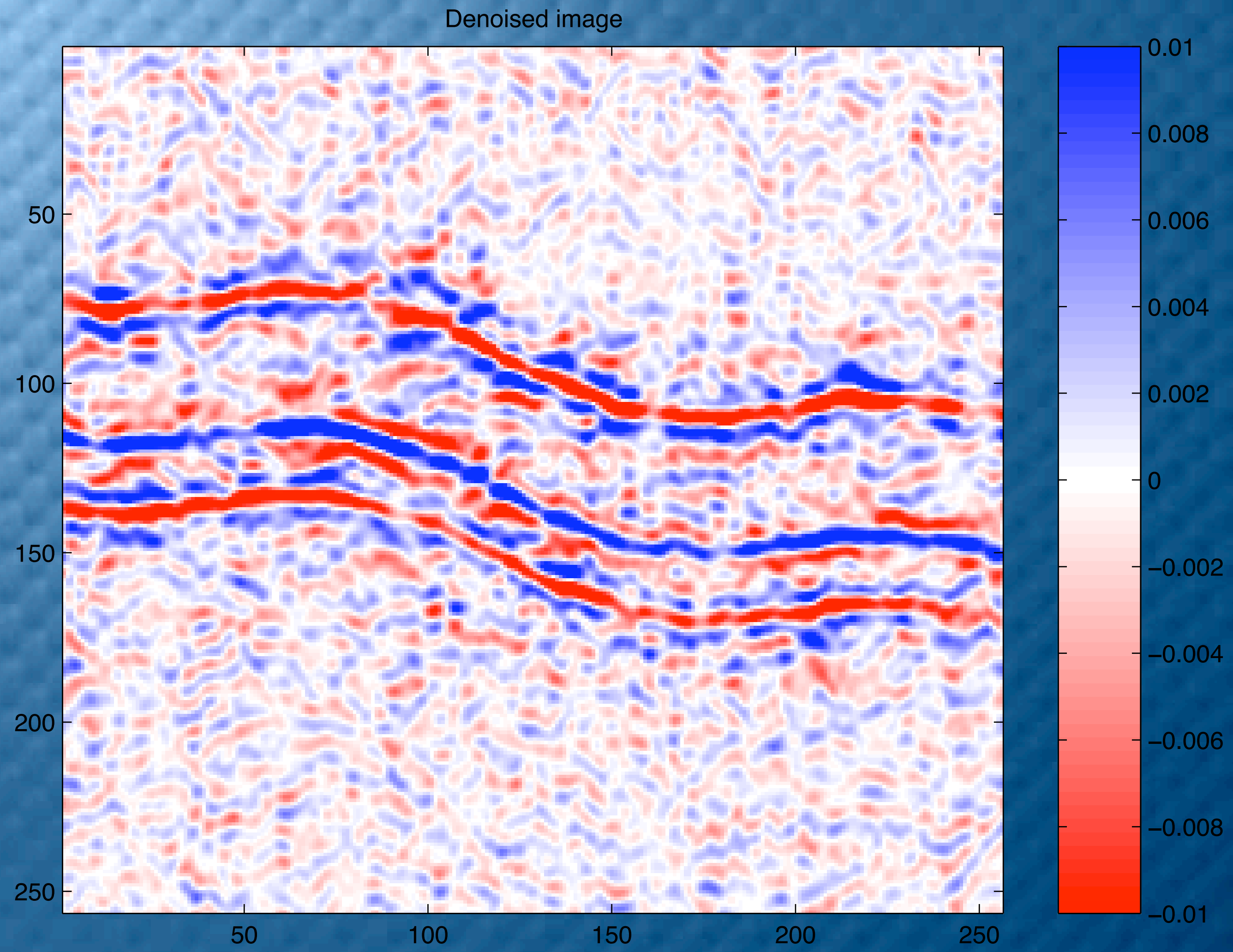
- **corrected for the normal operator**
- **denoised**
- **used optimal non-lin. app. rate curvelets/contourlets**

Singularities in m preserved!

Seismic imaging



Seismic imaging



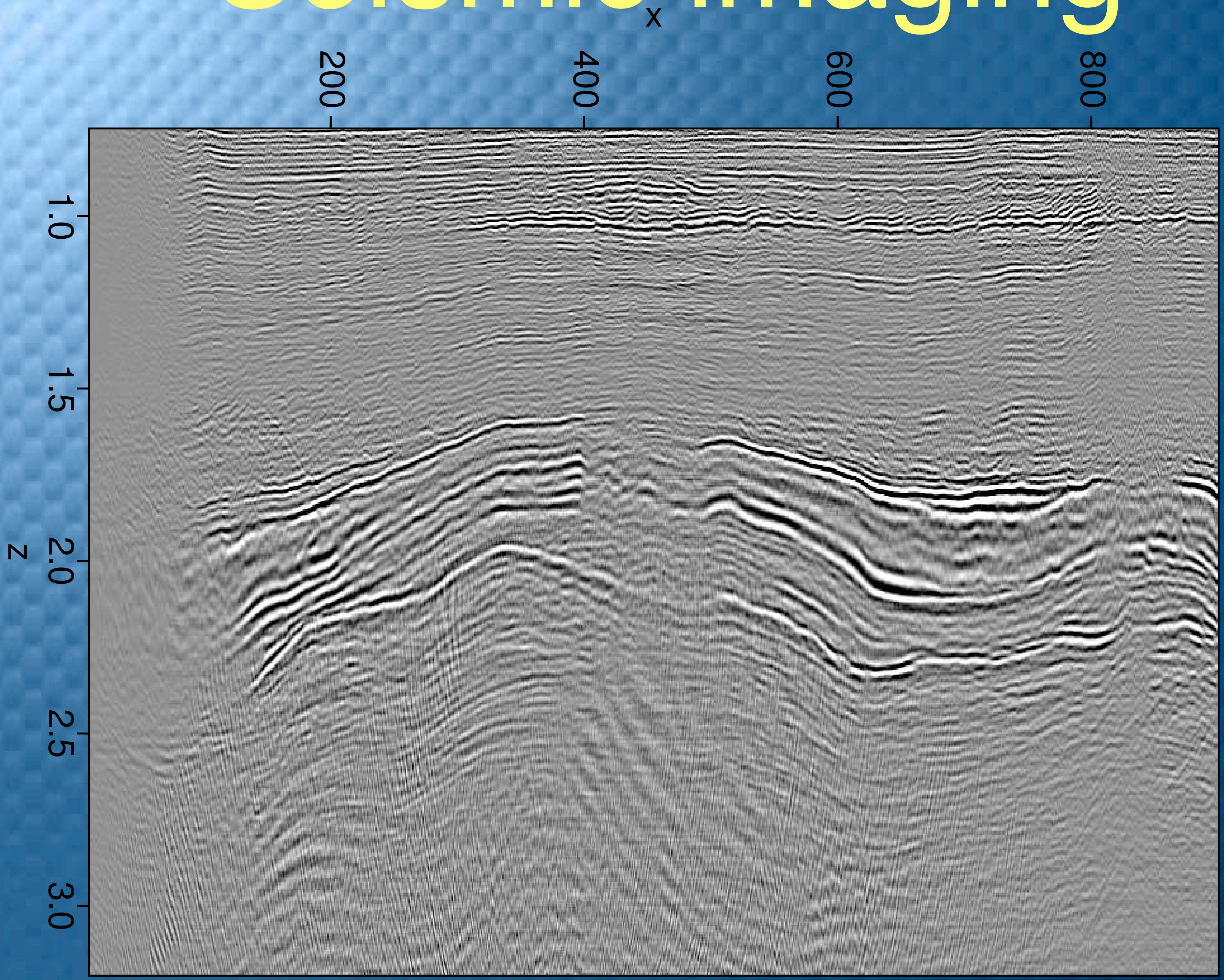
Seismic imaging

Obtained "image" of the singularities

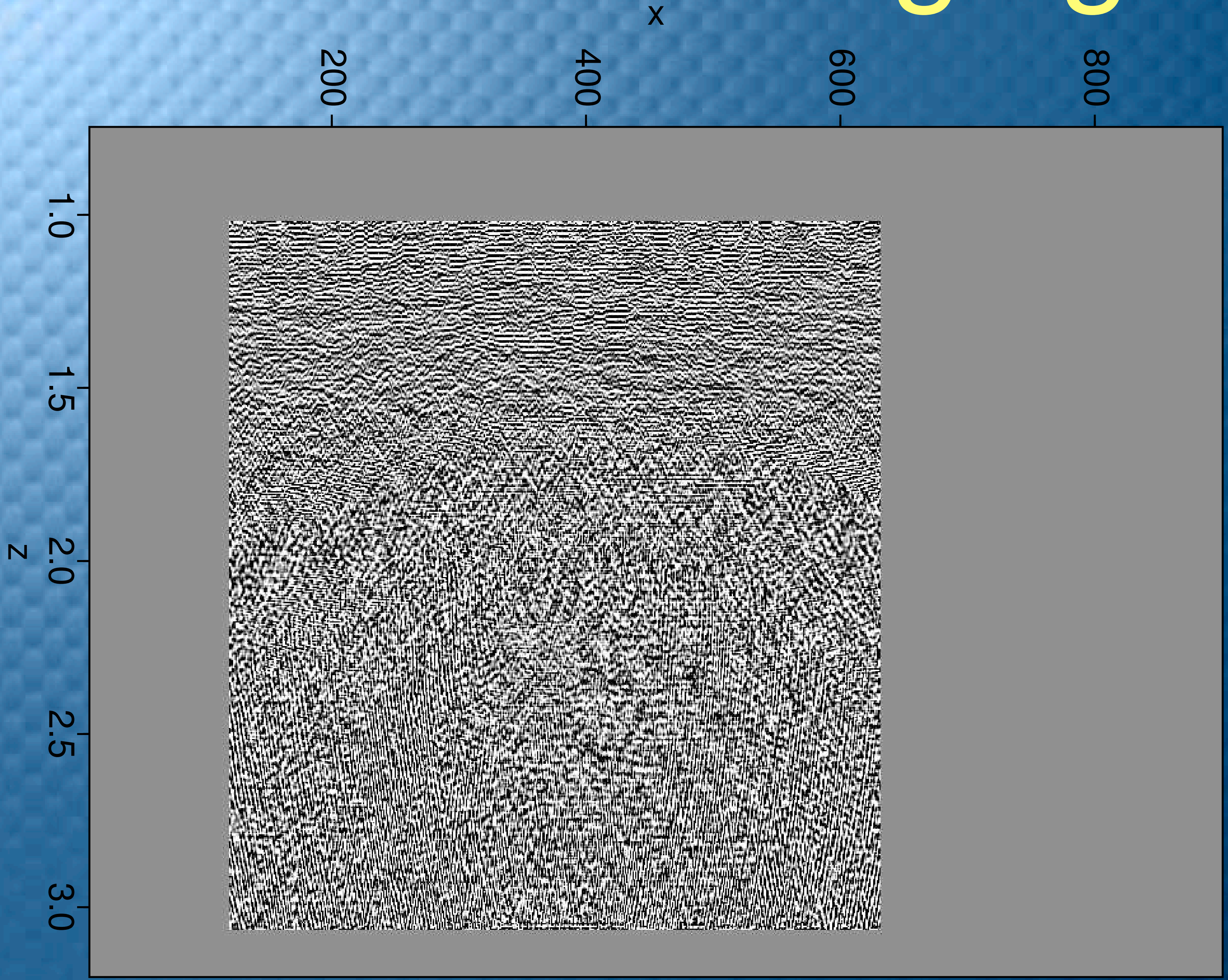
- **band-with limited**
- **singular across the reflector**
- **waveforms contain info on the singularity**

Seismic imaging reconstructs only singular part ...

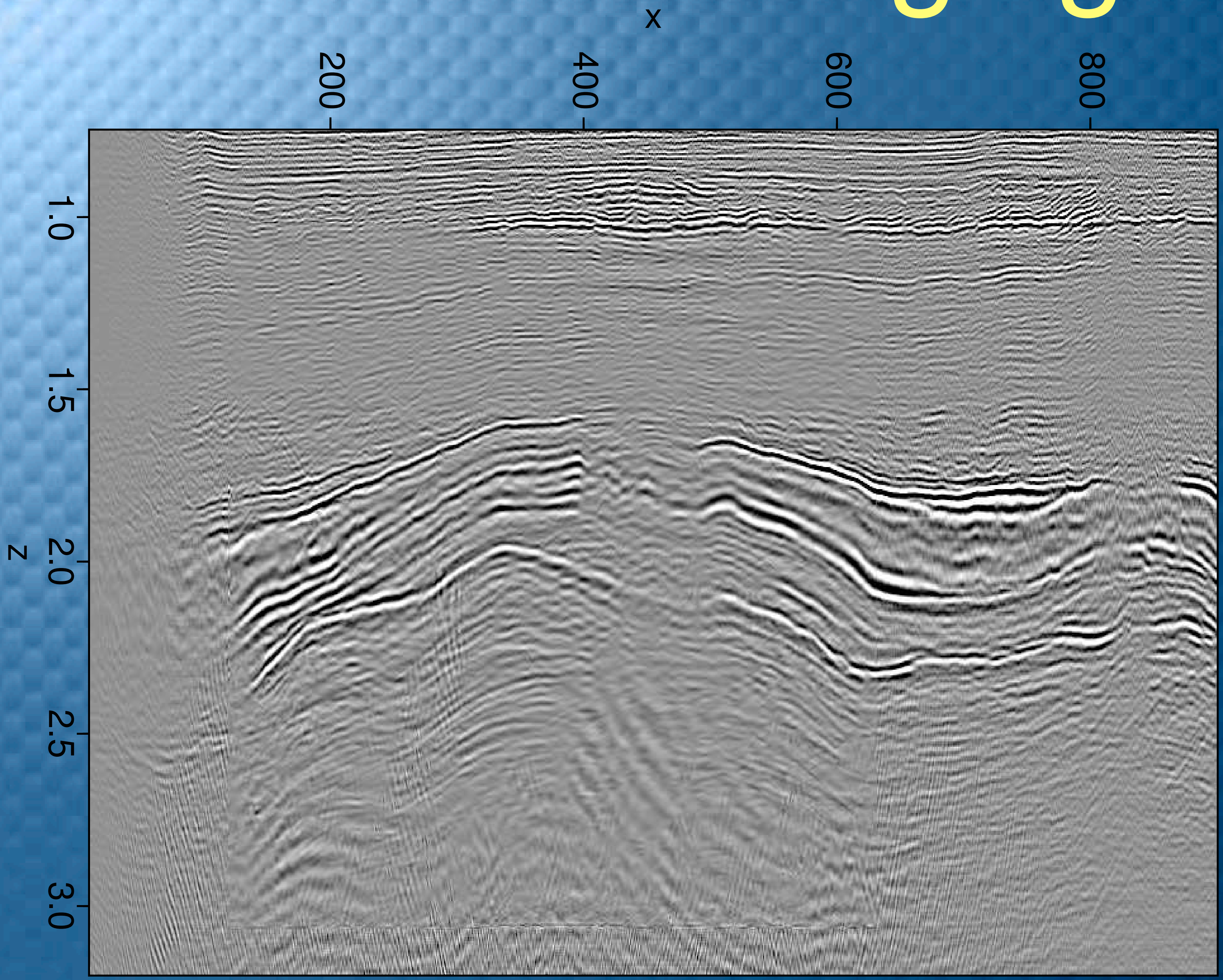
Seismic imaging



Seismic imaging



Seismic imaging



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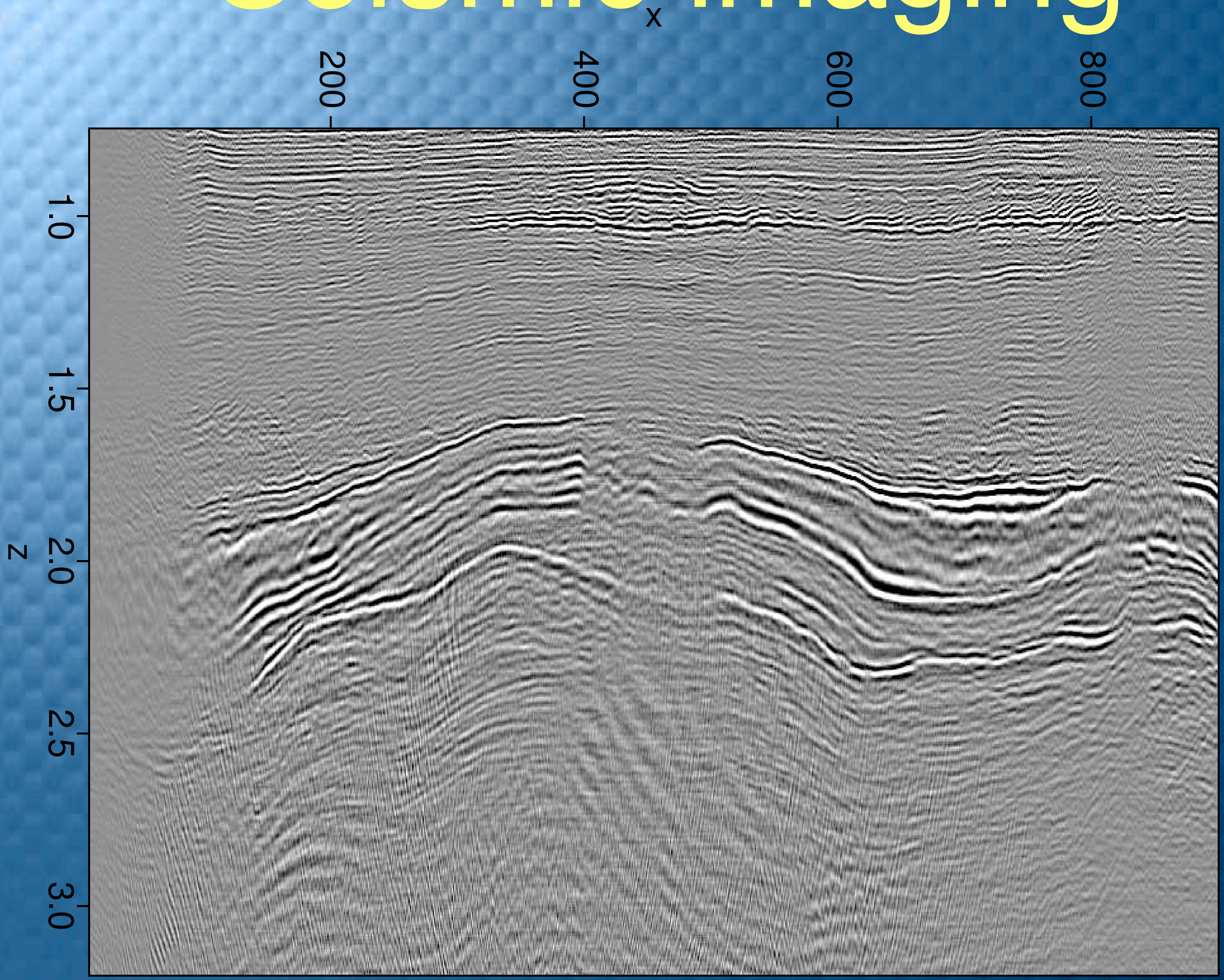

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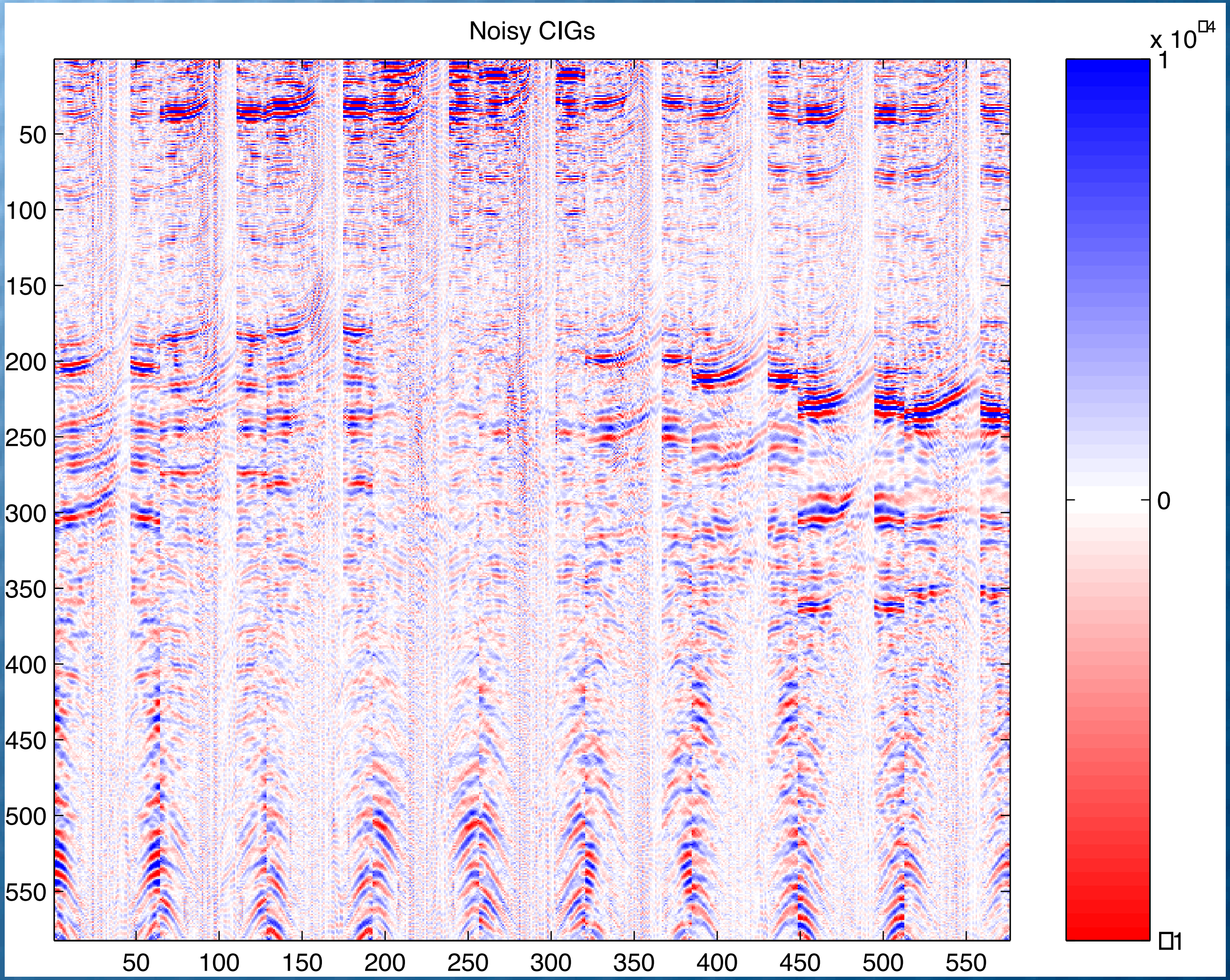

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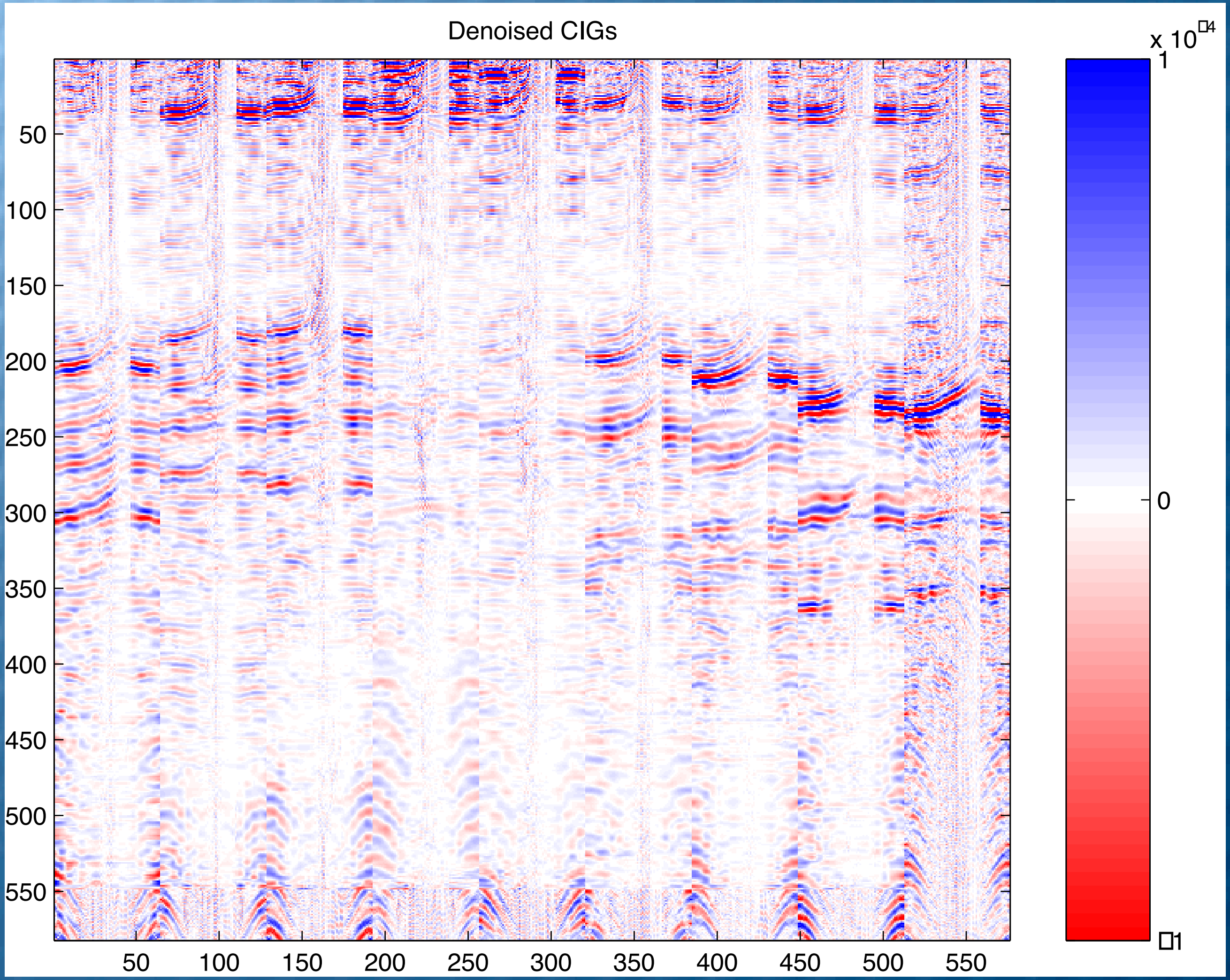

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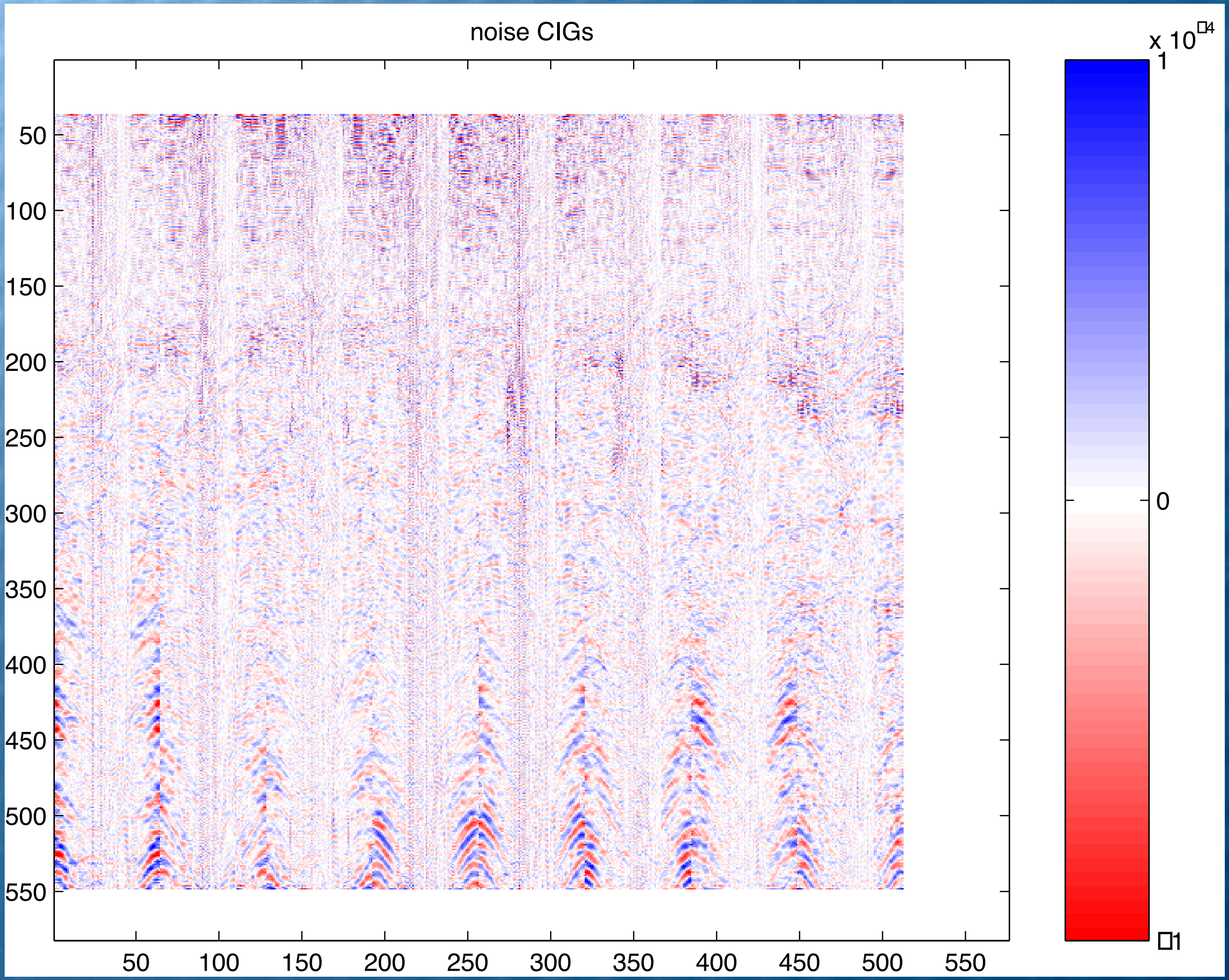


Characterization

Non-adaptive decomposition for estimation.

Data-adaptive decomp. for characterization/ feature extraction:

- **"vertical" earth looks *multifractal***
- **seismic images are like smoothed *derivatives* ⇔ only singularities reconstructed ...**
- **"phase"associated to reflectors**

Characterization

Parametric reflector representation:

$$\chi_{\pm}^{\alpha}(z) = \begin{cases} 0 & z \lesseqgtr 0 \\ \frac{z^{\alpha}}{\Gamma(\alpha+1)} & z \gtrless 0 \end{cases}$$

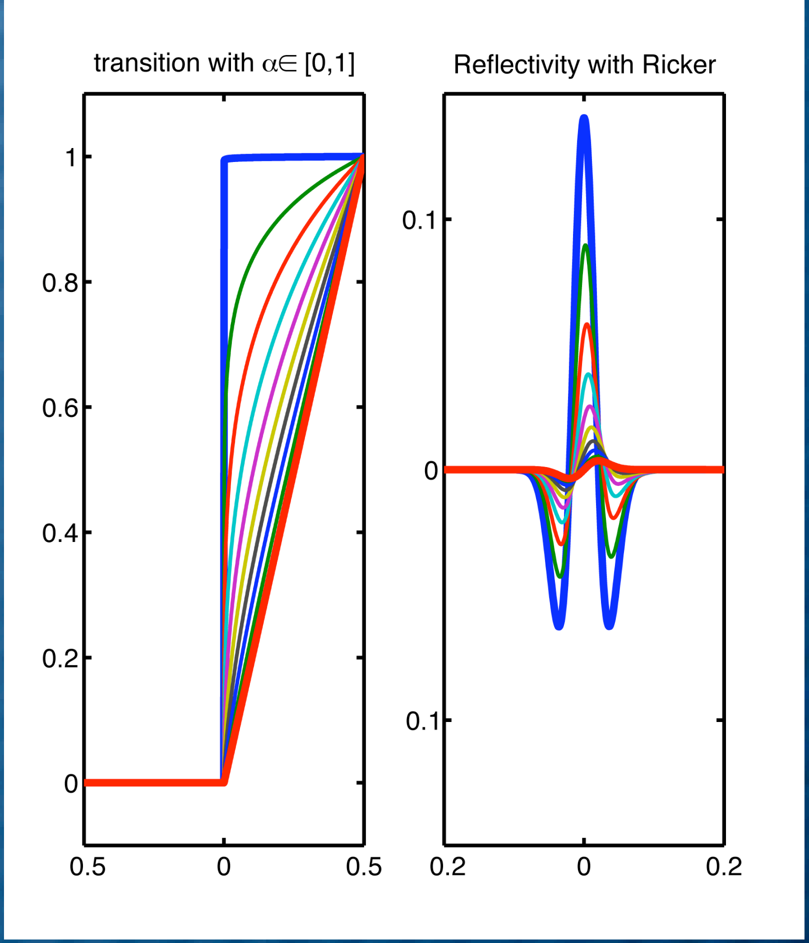
Multifractional spline representation:

$$m(z) = \sum_{n \in N} c_{\pm}^{n} \chi_{\pm}^{\alpha_n}(z - z_n).$$

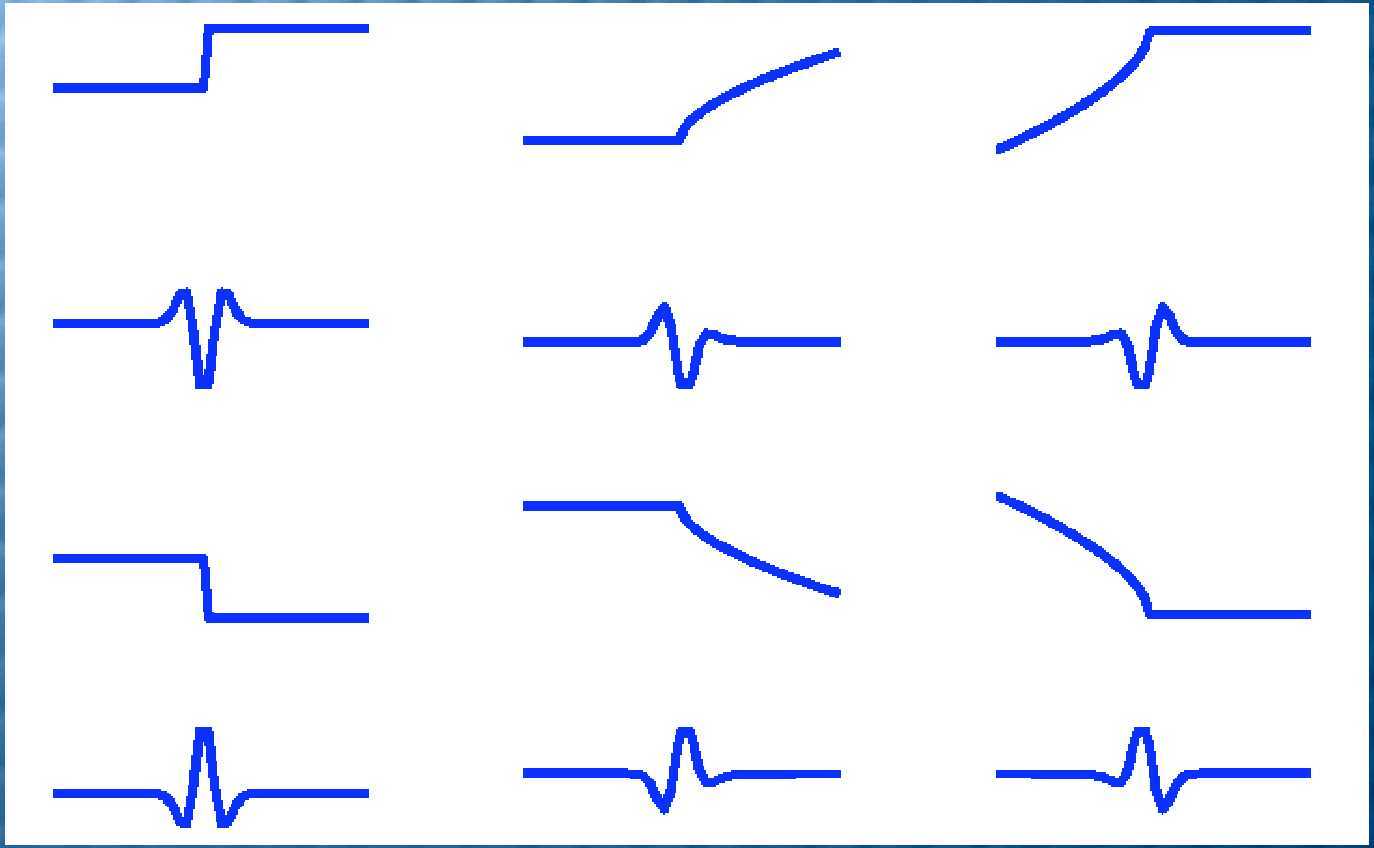
H.

Characterization

- between **step - ramp**
- controls ***abruptness***
- amplitude
- waveform
- fractional splines
- geological meaning



Characterization



Characterization

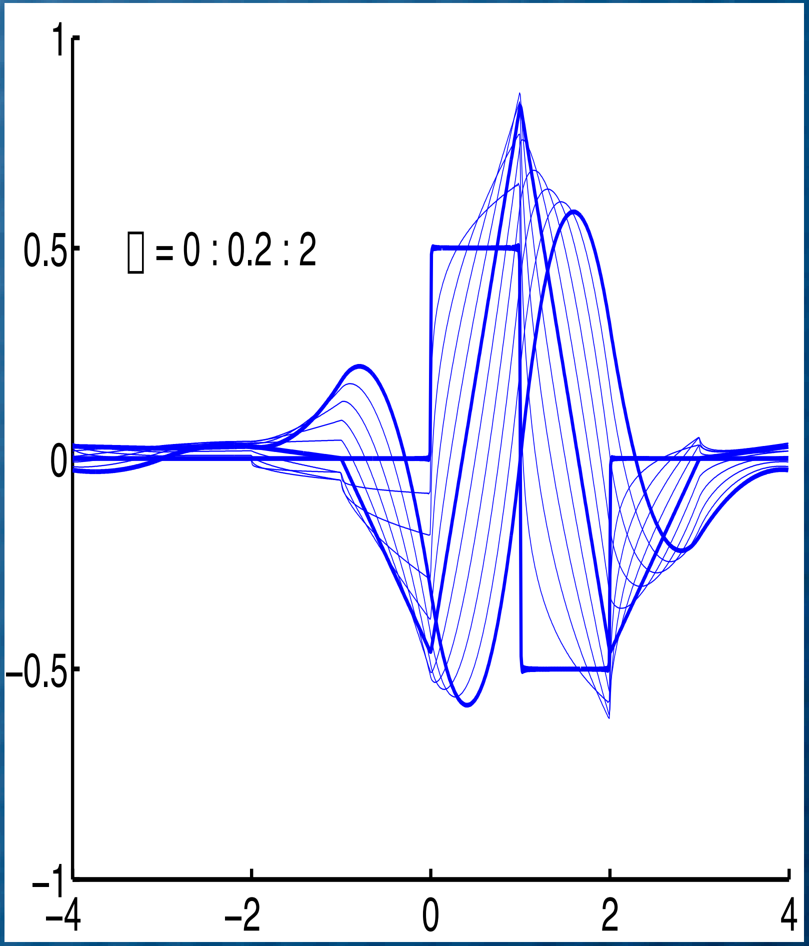
- dictionary non-down sampled fractional spline wavelets

$$\{g_\gamma\}: \quad 2^{j/2}\psi^{\alpha}_{-,*,+}(2^{j/2}z-k) \quad \alpha_0 \leq \alpha < \alpha_1$$

- parameterized by location, scale, order, “direction”

- fractional derivatives:

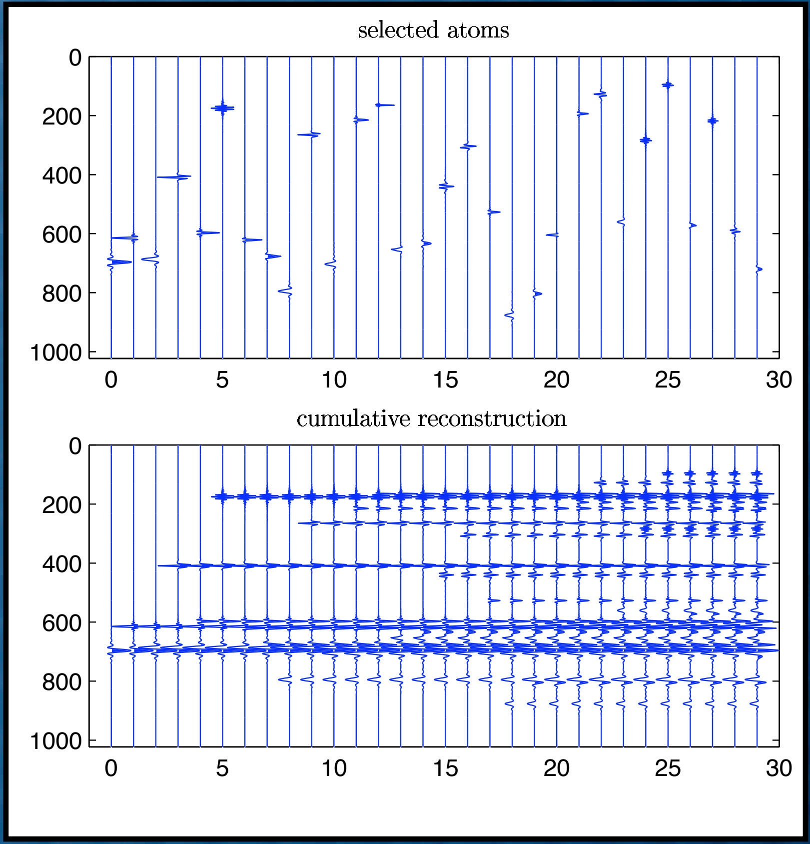
$$\hat{\psi}^{\alpha}_{-,+}(\omega) \sim (\mp j\omega)^{\alpha}$$



Unser, Blu

Characterization

- decomposition into parameterized atoms
- data-adaptive non-linear approximation
- non-linear estimation by greedy matching pursuit:

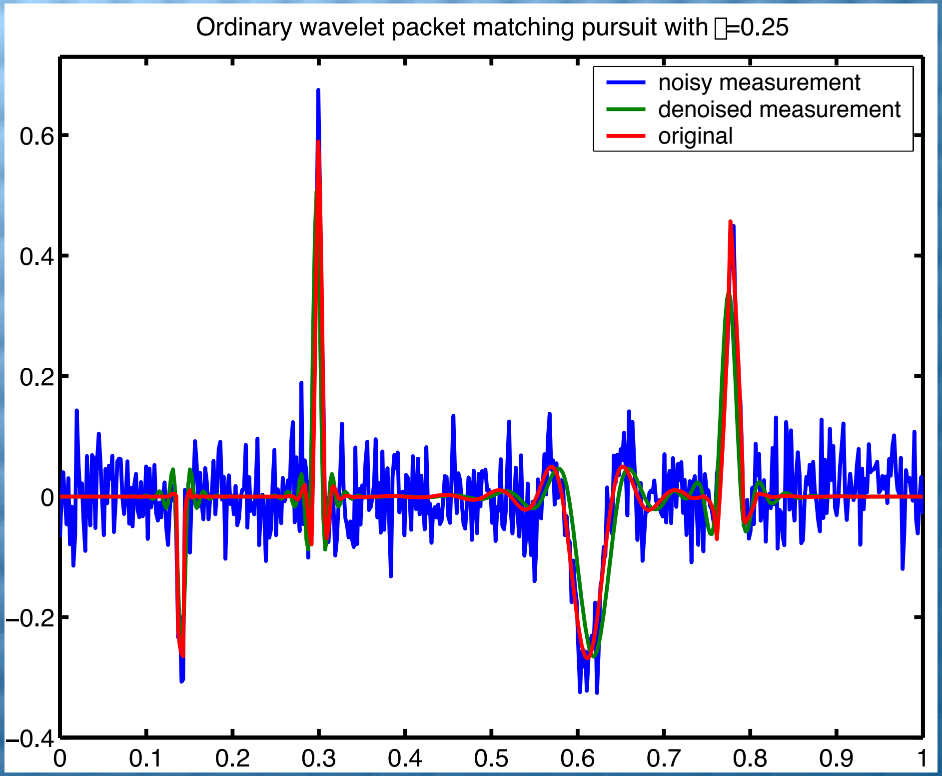


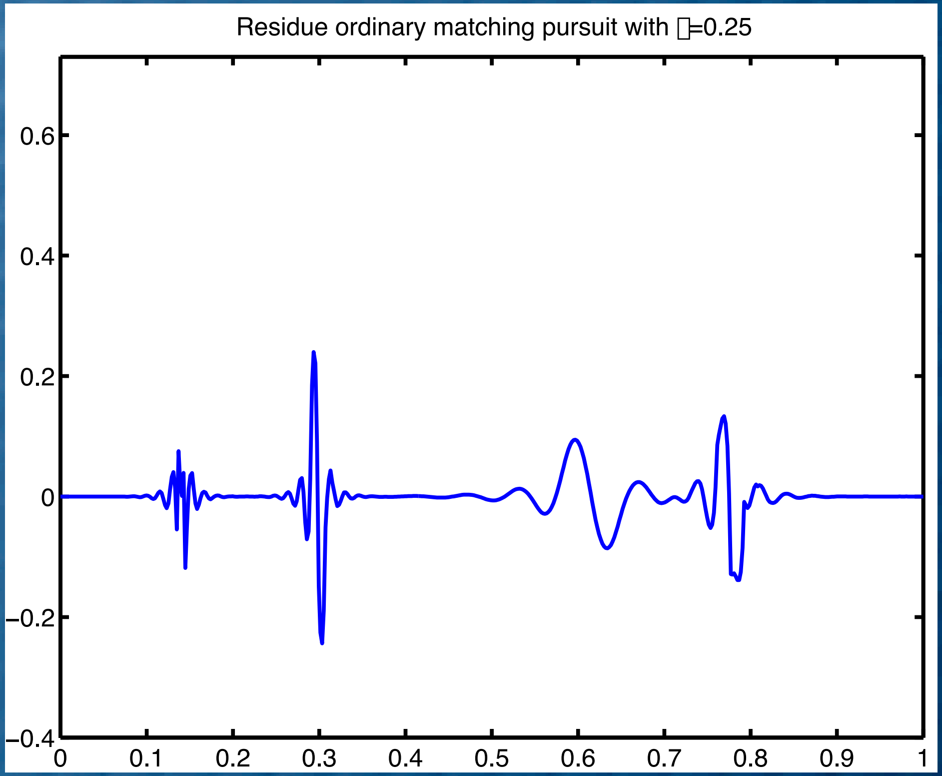
$$|\langle R^k m, g_{\gamma_k}\rangle| \geq \sup_{\gamma\in\Gamma} |\langle R^k m, g_\gamma\rangle|$$

$$\langle R^{k+1} m, g_\gamma\rangle = \langle R^k m, g_\gamma\rangle - \langle R^k m, g_{\gamma_k}\rangle\langle g_{\gamma_k}, g_\gamma\rangle$$

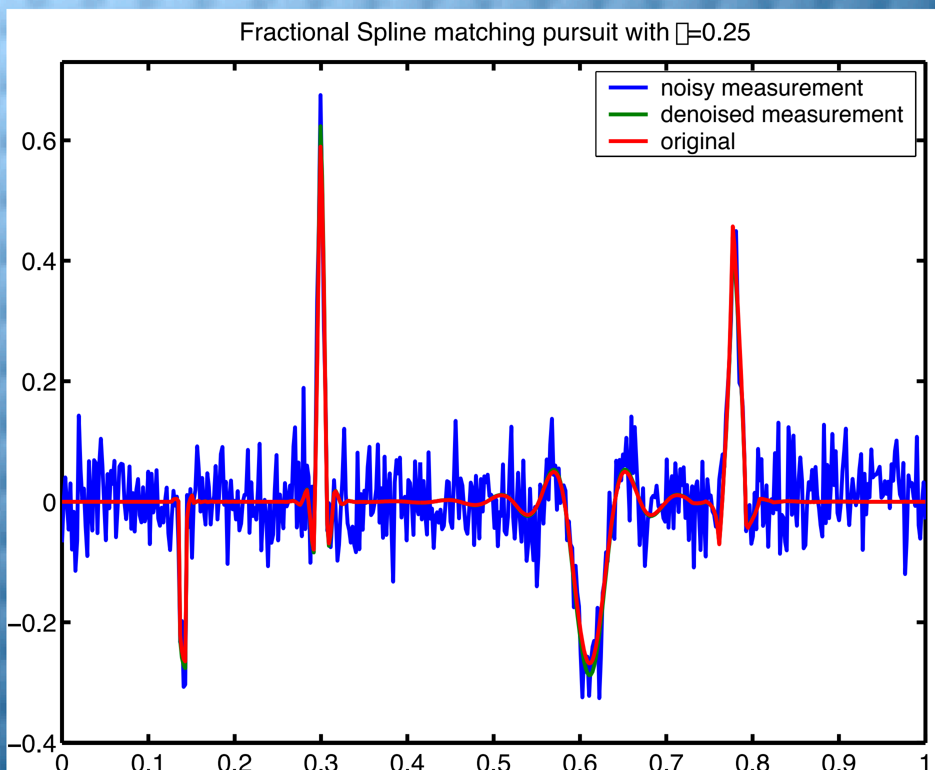
Mallat

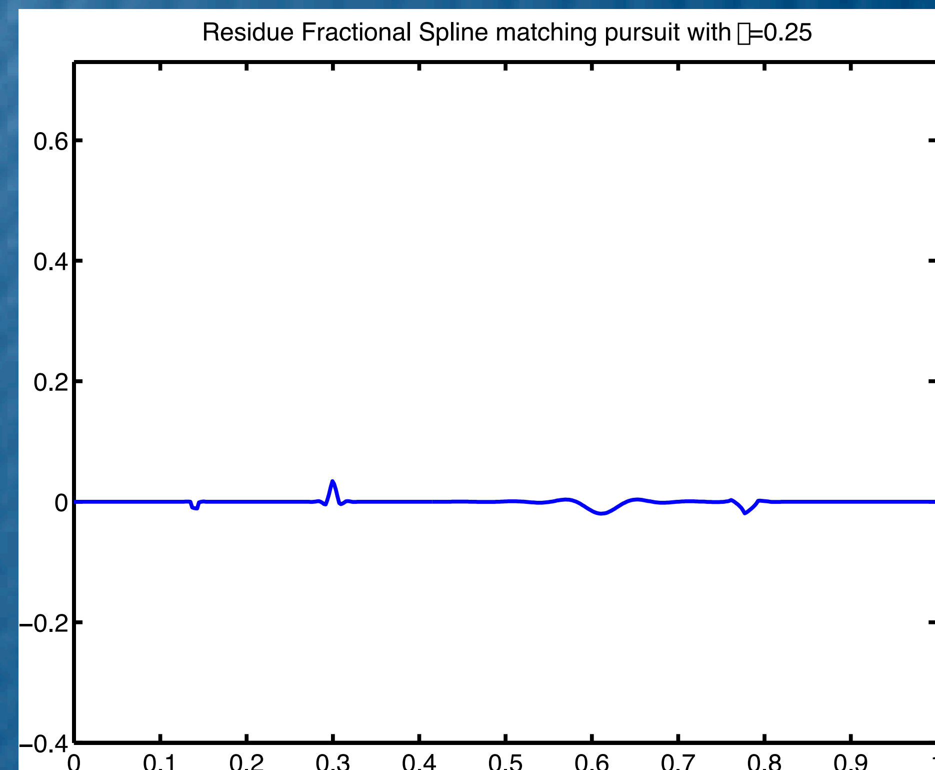
Characterization





Characterization





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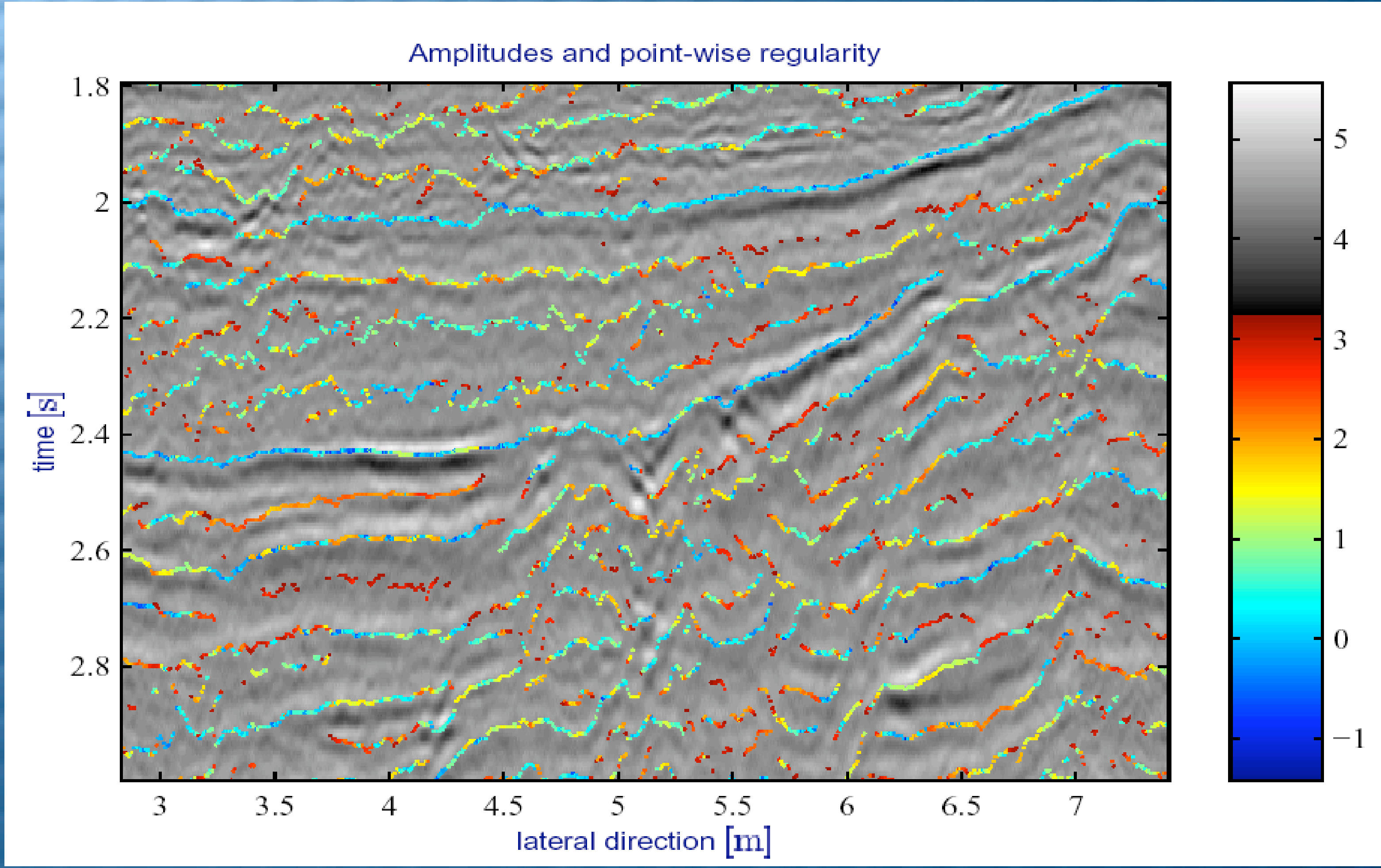

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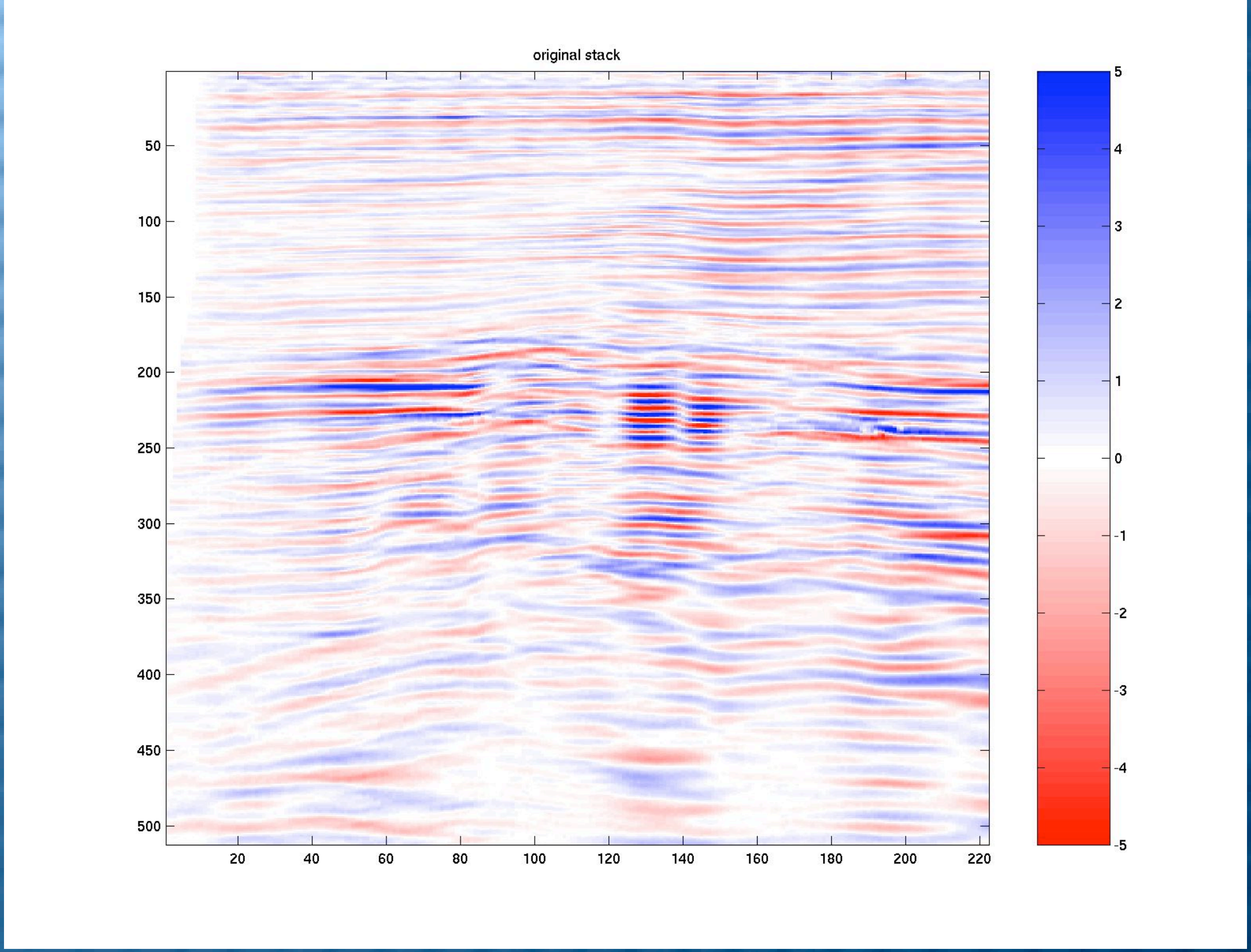

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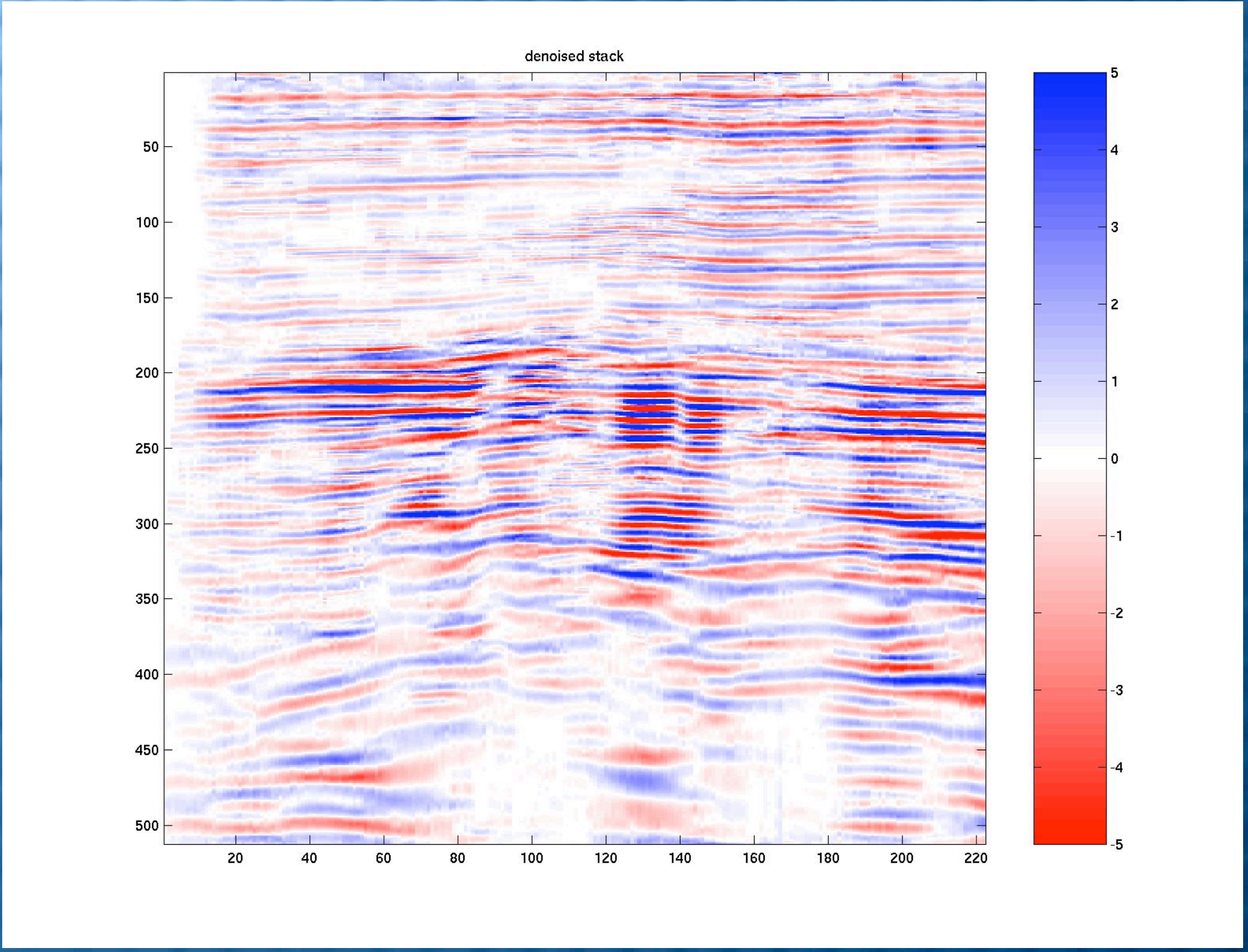

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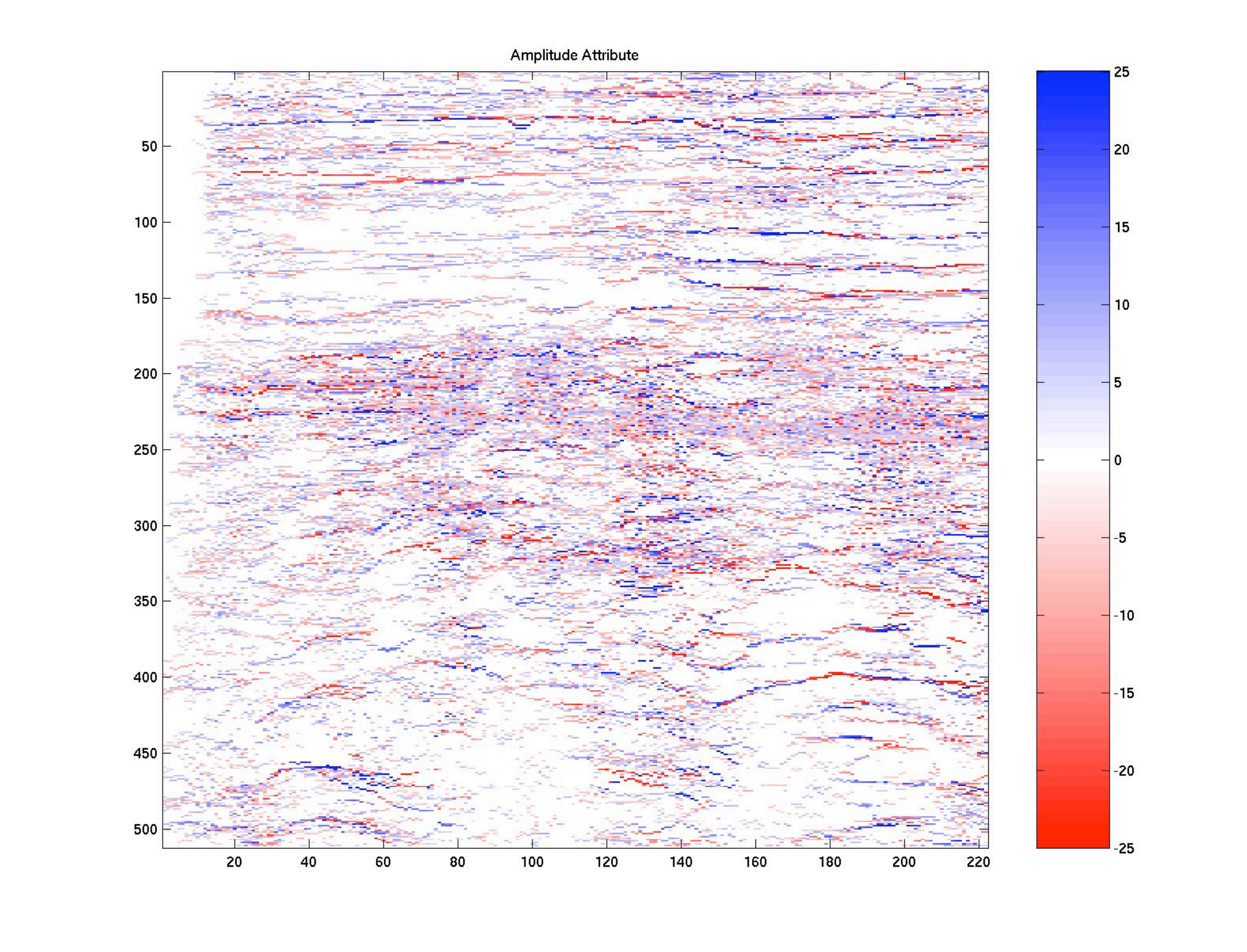

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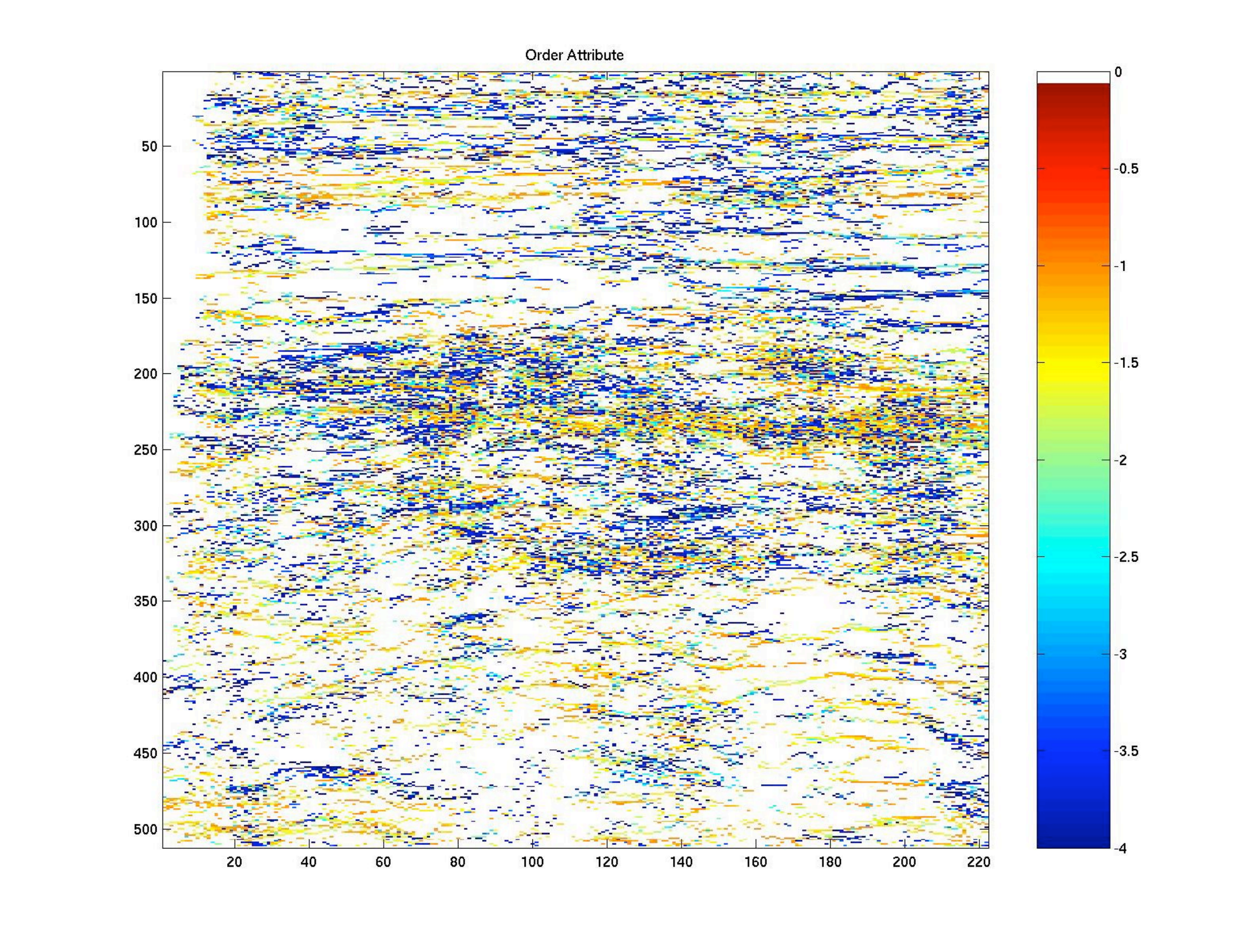

Characterization

Fractional Splines Matching Pursuit:

- **locates the nodes of the splines**
- **their magnitude (”spiky” decon}**
- **their fractional order**
- **their “phase”**

Conclusions

- ***non-adaptive*** basis function expansion & non-linear estimation are the way to go
- Diagonal (symbol) normal operator in Curvelet domain suffices to whiten n!
- Indications it works for m too!
- Exploiting redundancy pays off:
  - *smoothness* along reflectors
  - *smoothess* in e-direction
- **Improved the SNR!**

Conclusions

- many problems remain:
  - variational problem (norm, modulo smooth part ...)
  - direct computation operators in basis-function domain is a challenge

Conclusions

- Fractional splines wavelets are good adaptive dictionaries
  - provide info on type of singularities
  - geologically relevant
  - natural for “multifractal” behavior
- Lack directional selectivity
- Dictionary is prohibitively large ....