A Guide For Successful Low Rank Matrix Recovery In Seismic Applications

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Motivation

- acquisition challenges
  - missing data (subsampled or coverage holes)
  - need efficient acquisition design
- exploit *low-rank* structure of seismic data
  - matrix completion for trace interpolation
  - industrial-scale implementations available
- need analytical tools
  - how should we subsample?
  - results in literature are not applicable

Contributions

- Quantification of subsampling
  - "generalized spectral gap" (GSP)
  - computationally cheap

$$\text{GSP} = \frac{\sqrt{nm}}{|\Omega|}\sigma_1\left(A - \frac{|\Omega|}{nm}\mathbf{1}_{n\times m}\right)$$

- Applications to seismic data acquisition
  - multi-use tool for acquisition design

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Outline

- Matrix Completion
  - literature
  - seismic trace interpolation

- Universal Matrix Completion
  - generalized spectral gap
  - uses in seismic data acquisition

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Matrix Completion Literature

Goal is to approximate $\mathbf{X} \in \mathbb{R}^{n \times m}$, given

noisy observed entries $\Omega \subset \{1, 2, ..., n\} \times \{1, 2, ..., m\}$

via

$$\mathbf{b}_{i,j} = P_{\Omega}(\mathbf{X})_{i,j} = \begin{cases} \mathbf{X}_{i,j} & \text{if } (i,j) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Matrix Completion Literature

If rank $\approx r \ll \min(n,m)$, we attempt to recover unobserved or noisy entries via

$$\left.\underset{\mathbf{Y}}{\text{minimize}} \|\mathbf{Y}\|_* \text{ subject to } \|P_\Omega(\mathbf{Y}) - \mathbf{b}\|_F \le \epsilon, \quad \right\} \quad \mathbf{NN}(b,\epsilon)$$

where $\|\mathbf{Y}\|_* = \sum_k \sigma_k(\mathbf{Y})$.

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Matrix Completion Literature

Typical Assumptions:

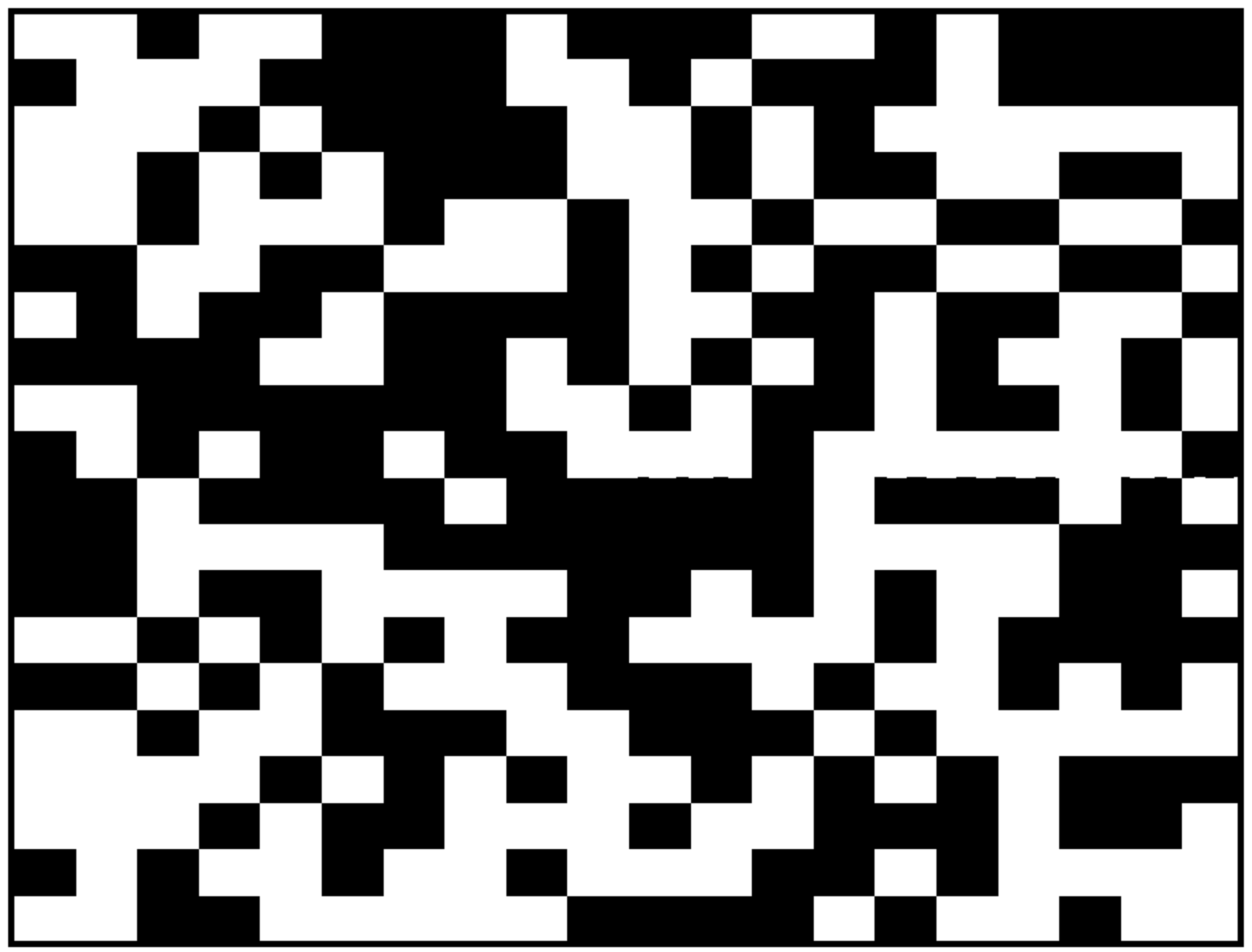
1. Construct Ω by observing entries uniformly at random.

2. $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ is μ-"incoherent".

$$\|\mathbf{U}(k,:)\|_2^2 \leq \frac{\mu r}{n} \qquad \|\mathbf{V}(\ell,:)\|_2^2 \leq \frac{\mu r}{m}$$

Assumption 1: Uniform Random Sampling

Sampling Mask: $A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \Omega \\ 0 & \text{otherwise} \end{cases}$

$A =$ 

$\blacksquare = 0$

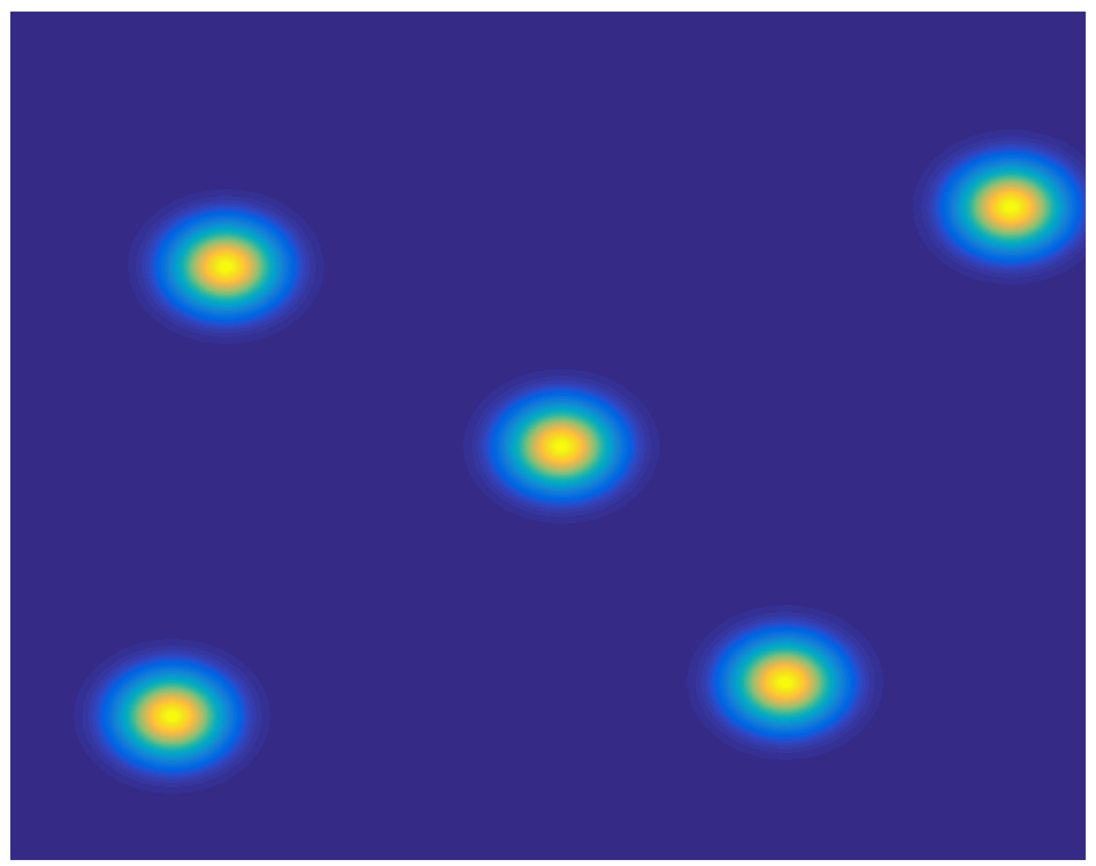
$\square = 1$

Assumption 2: μ-Incoherence

$$\|\mathbf{U}(k,:)\|_2^2 \leq \frac{\mu r}{n} \qquad \|\mathbf{V}(\ell,:)\|_2^2 \leq \frac{\mu r}{m}$$

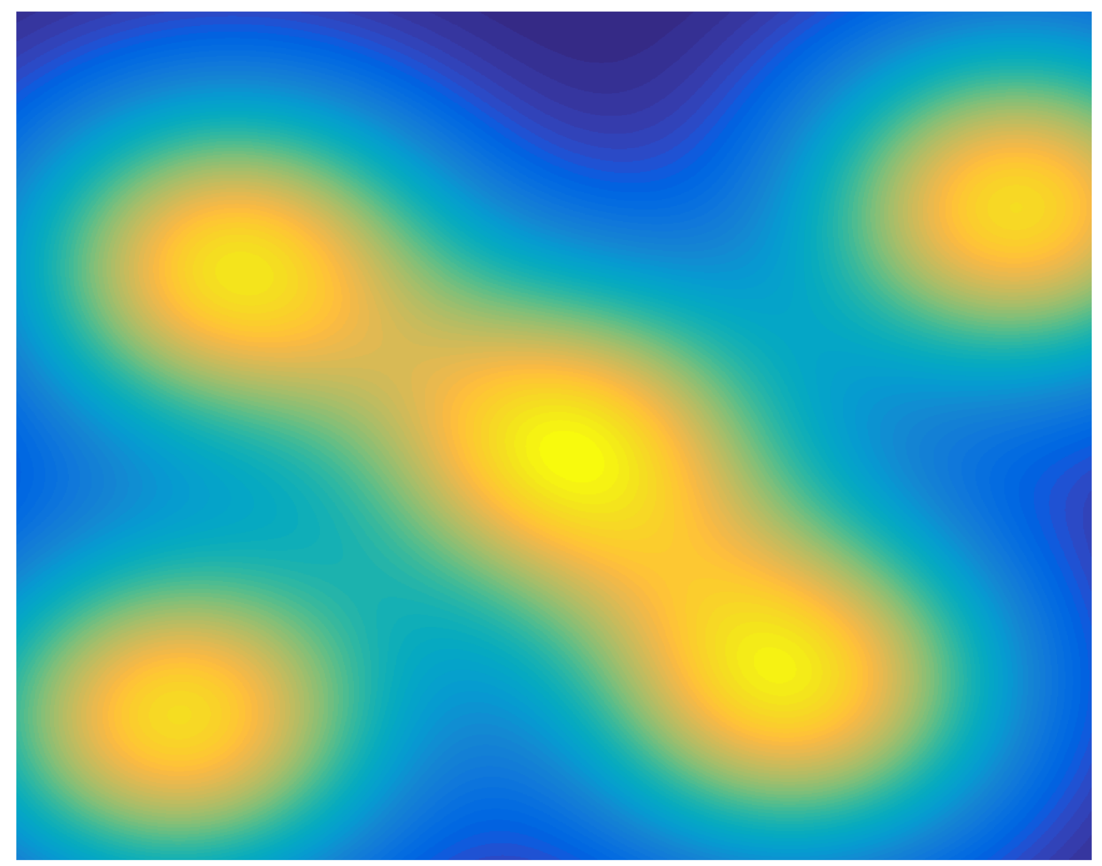
Parameter μ measures how "spread" the energy of data matrix is.

large $\mu \sim \mathcal{O}\left(\frac{n}{r}\right)$



bad for matrix completion

small $\mu \sim \mathcal{O}\left(1\right)$



good for matrix completion

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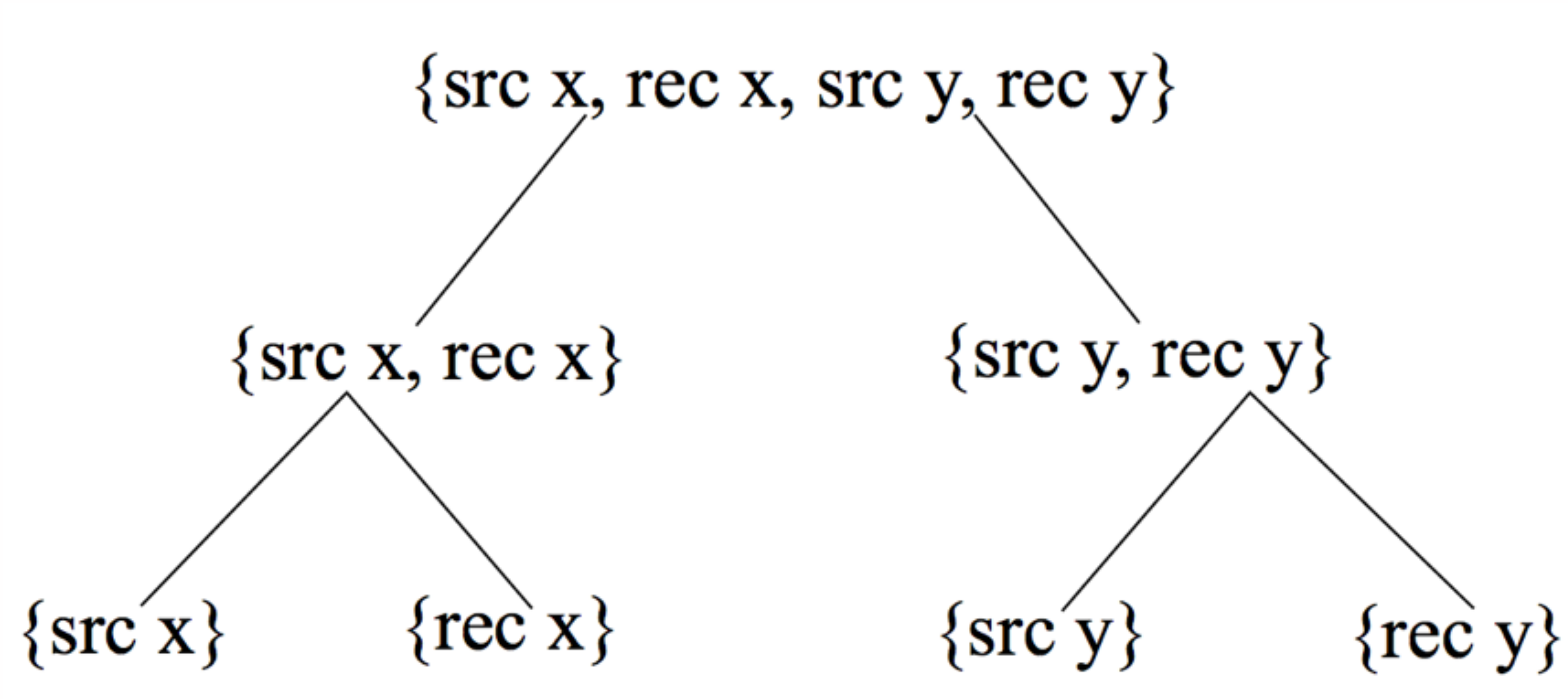
3D Seismic Data Interpolation

- Consider a 3D seismic survey with coordinates (src x, src y, rec x, rec y, time)
- Take a Fourier transform in time and restrict ourselves to a single frequency slice.

3D Seismic Data Interpolation

- Consider a 3D seismic survey with coordinates (src x, src y, rec x, rec y, time)
- Take a Fourier transform in time and restrict ourselves to a single frequency slice.
- Unfold into matrix to apply matrix completion (i.e., "matricize")

3D Data: Matricized

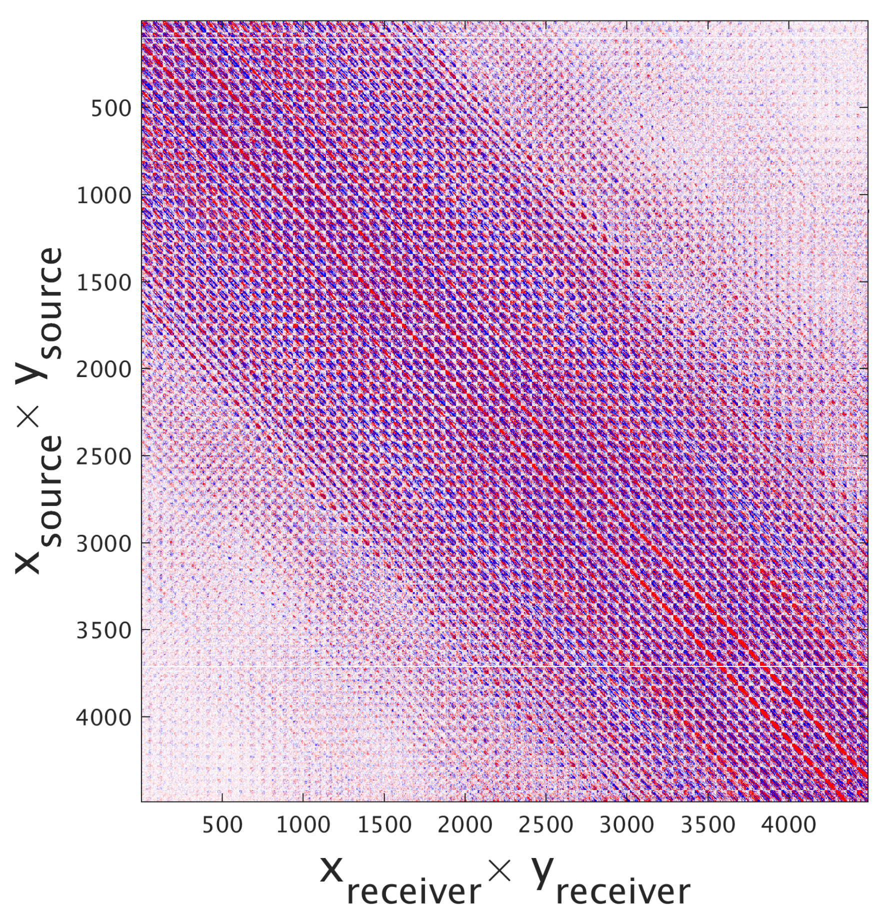


Many options on how to matricize

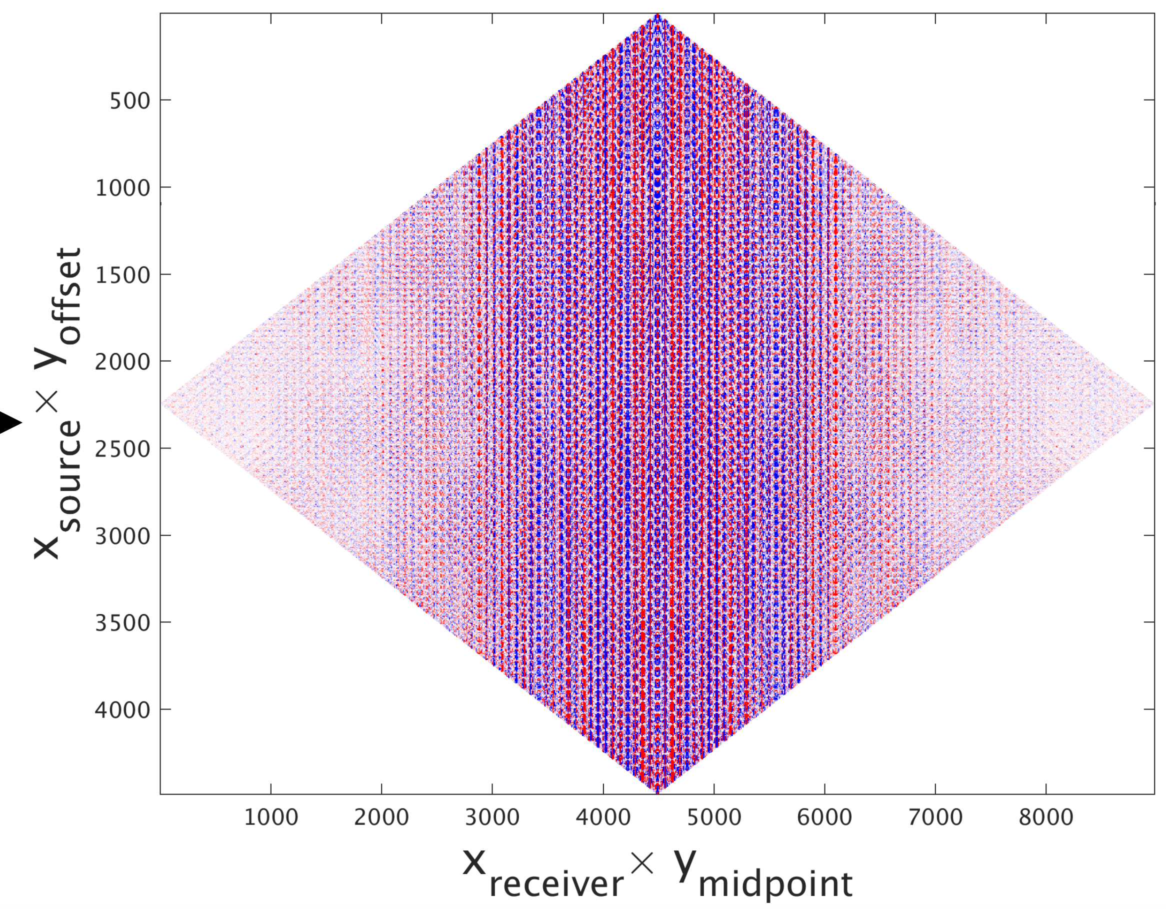
3D Data: Matricized

Canonical

(src,rec)-domain

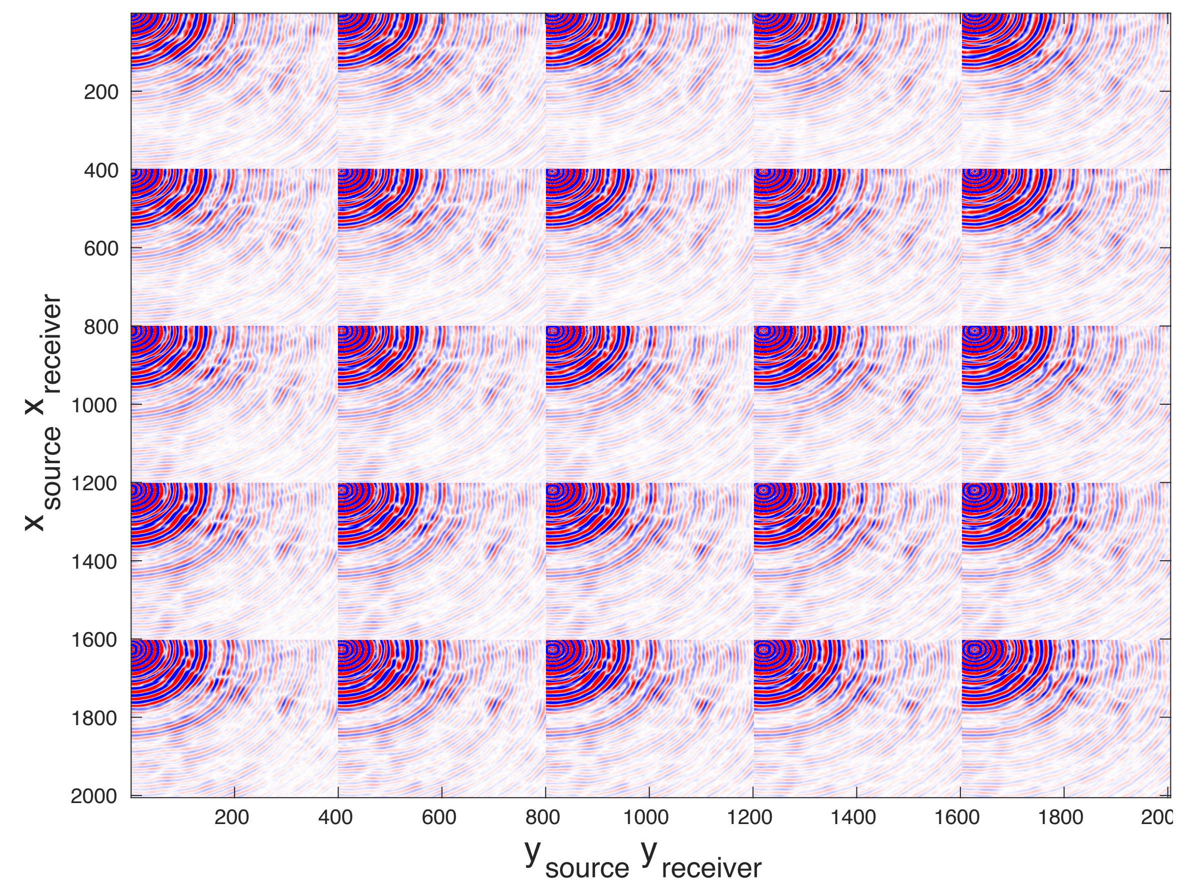


(mid,off)-domain

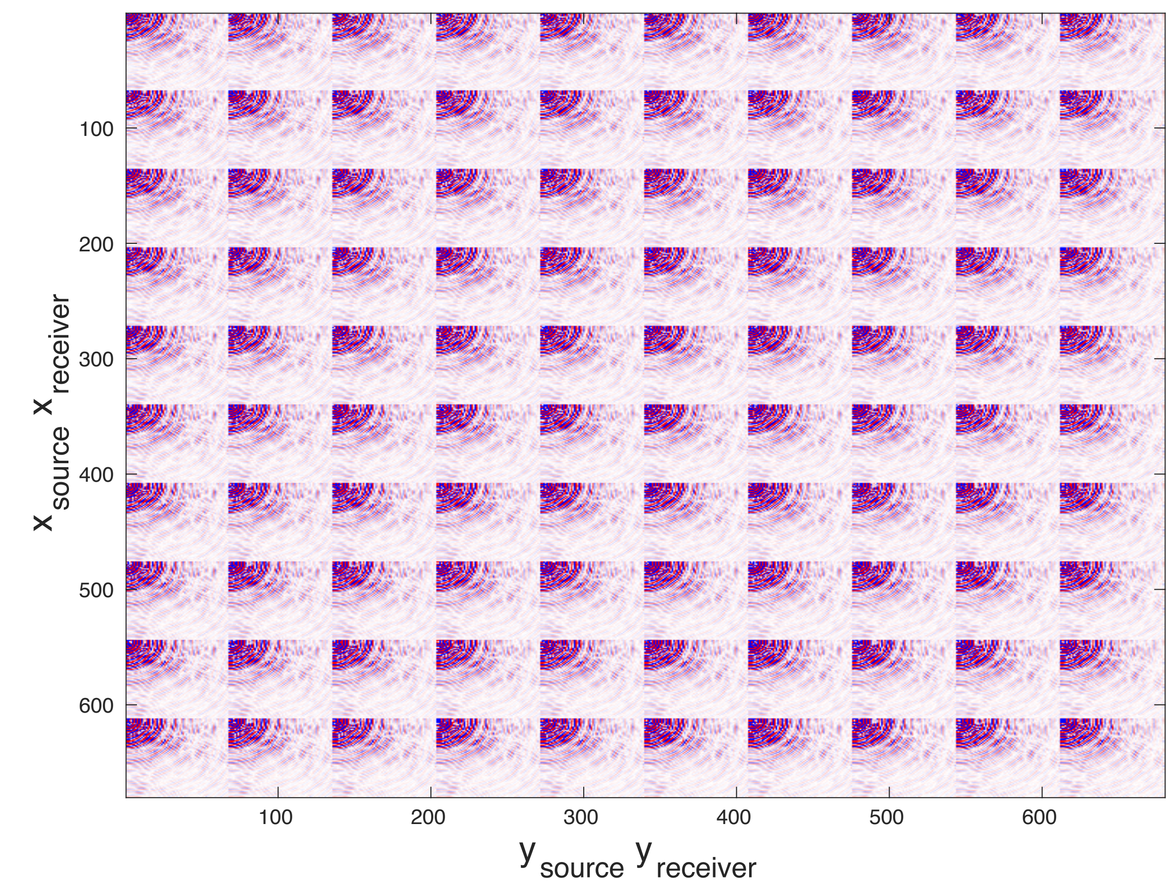


Non-Canonical

(rec,rec)-form

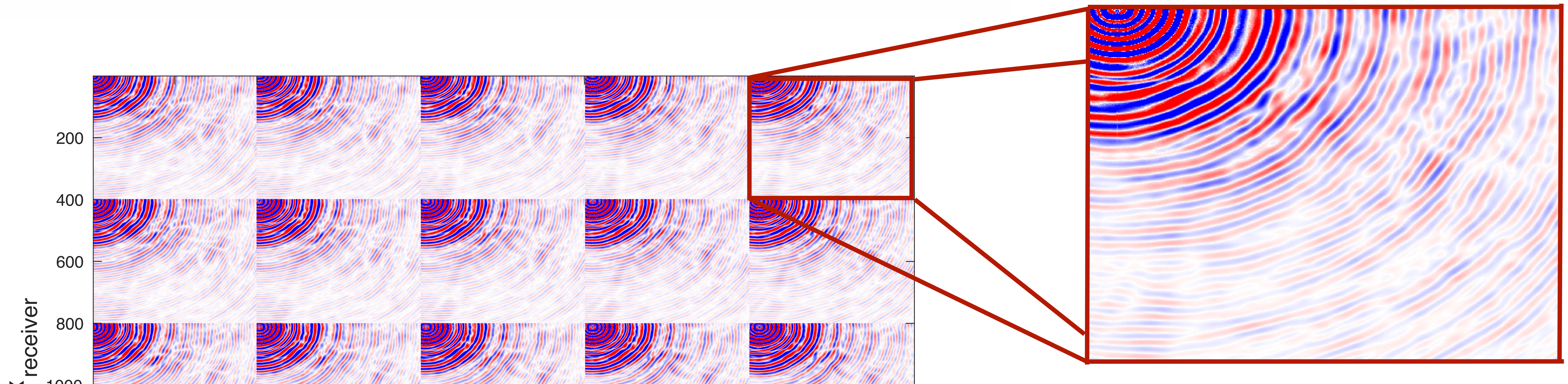


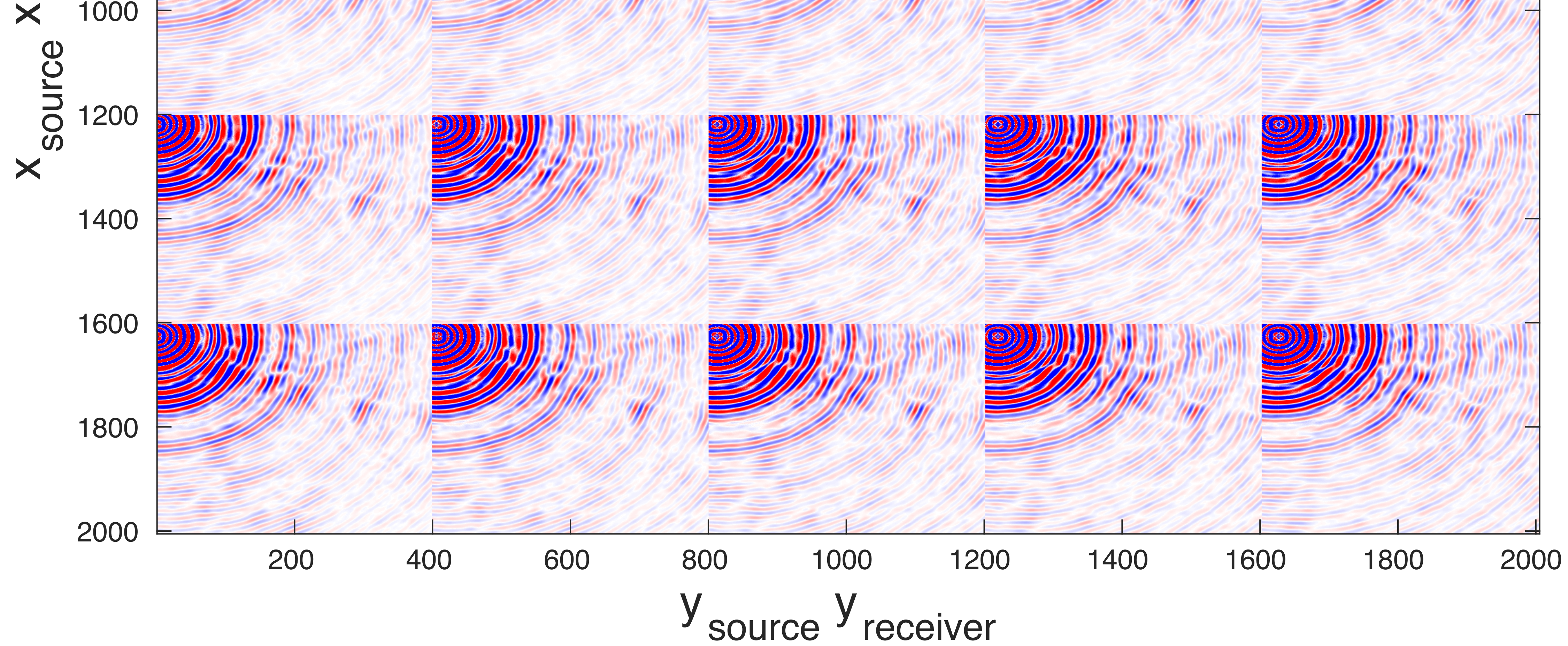
(src,src)-form

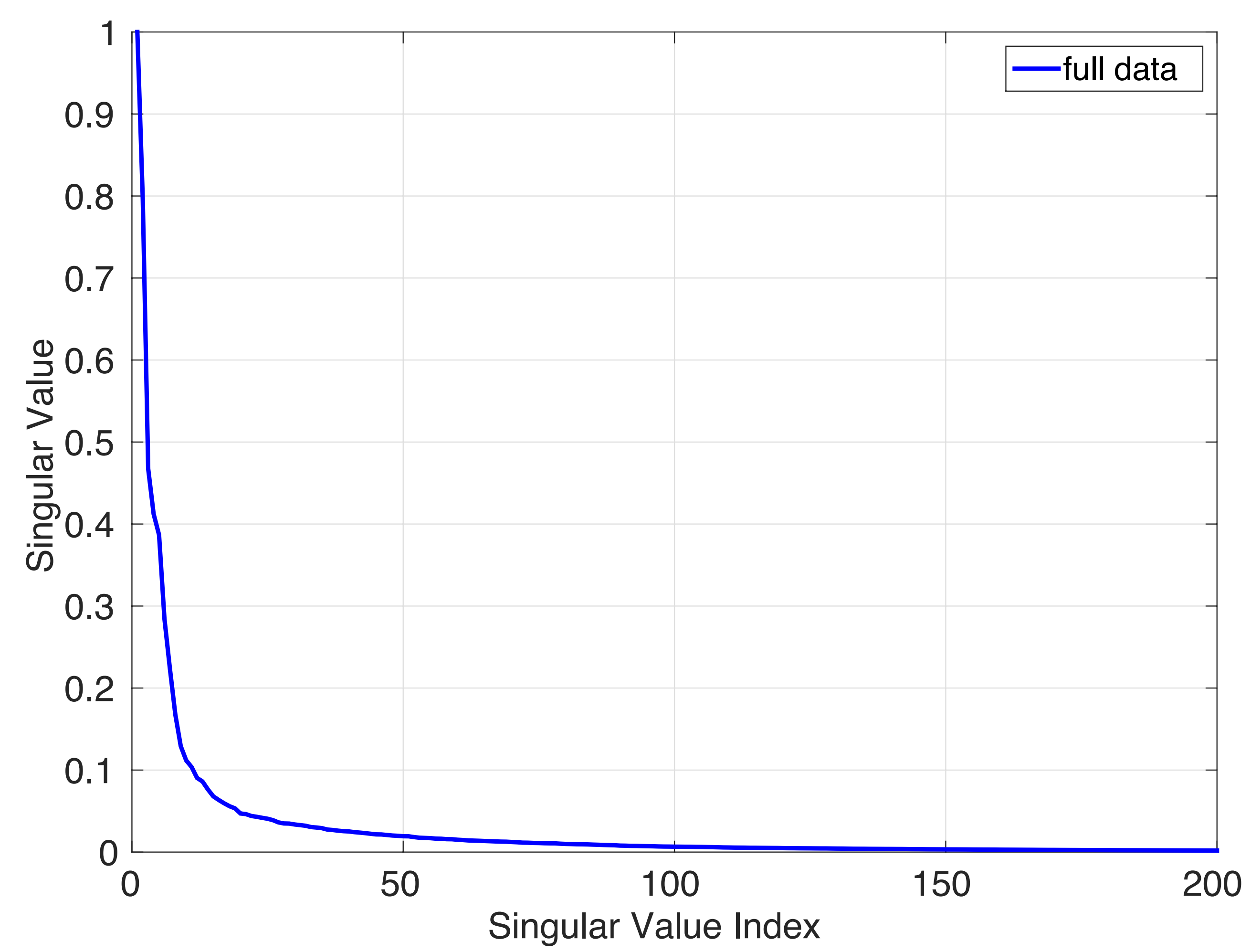


Incoherence: energy of matrices is evenly distributed

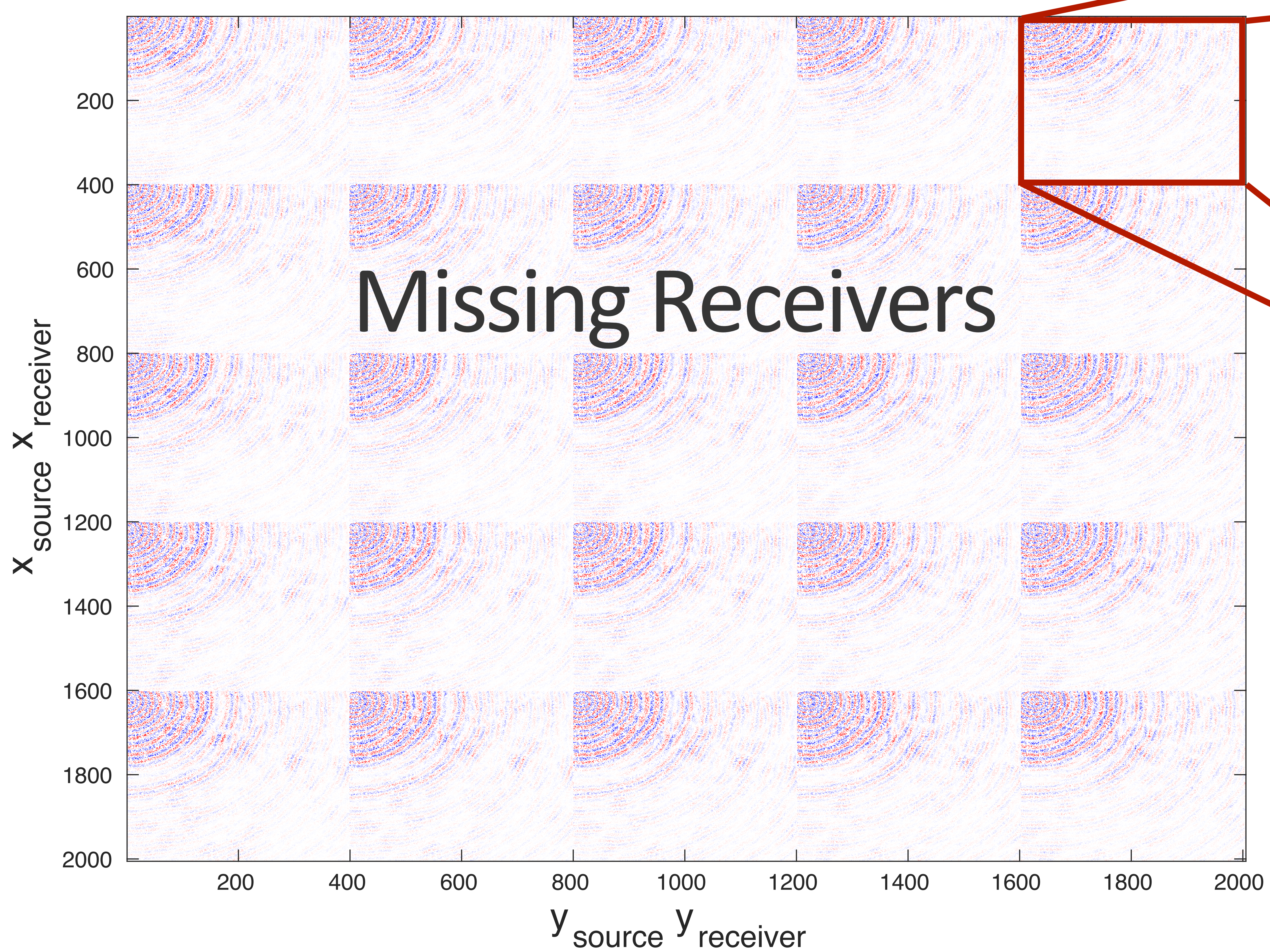
3D Data: Matricized - (rec,rec) form

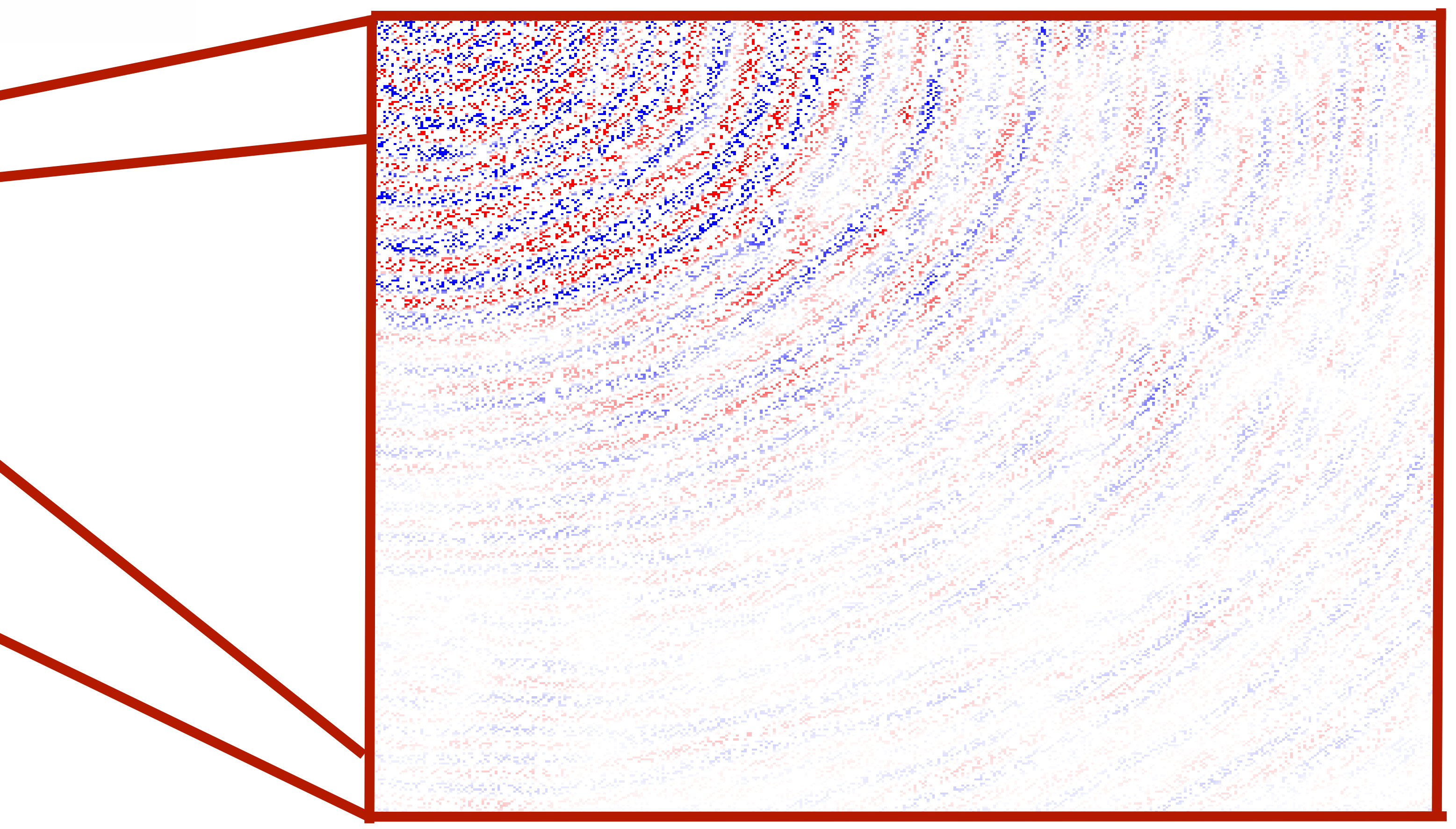


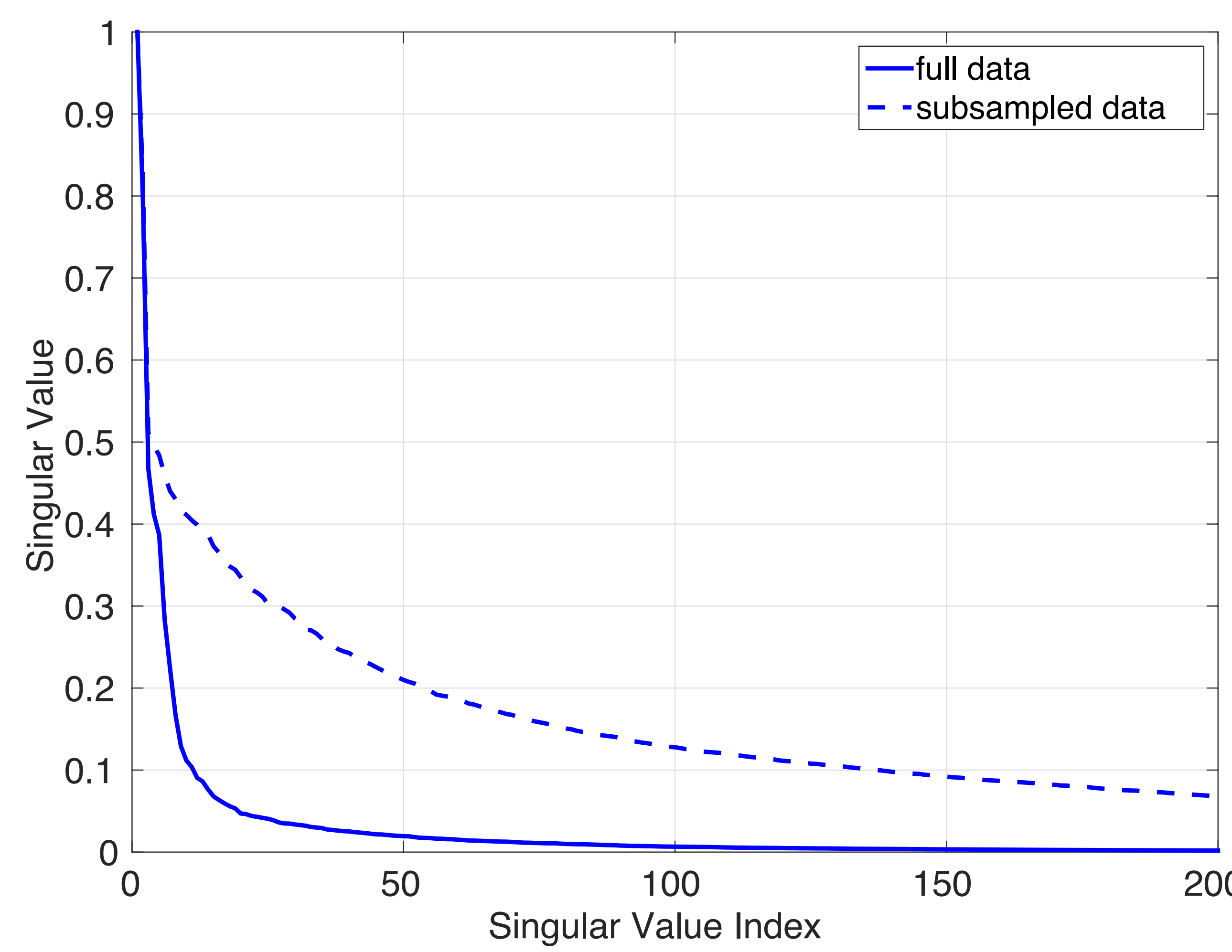




3D Data Matricized - (rec,rec) form



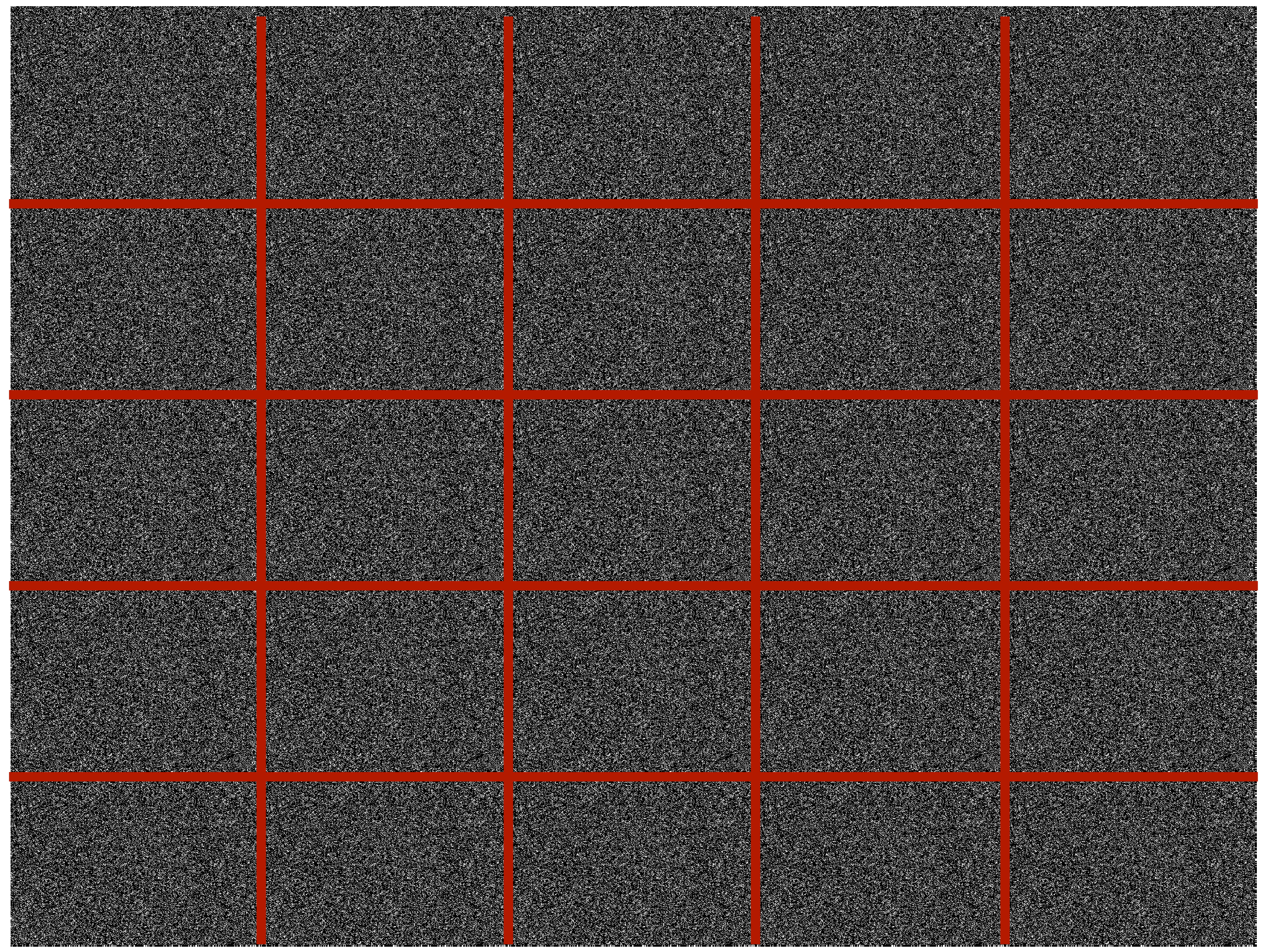




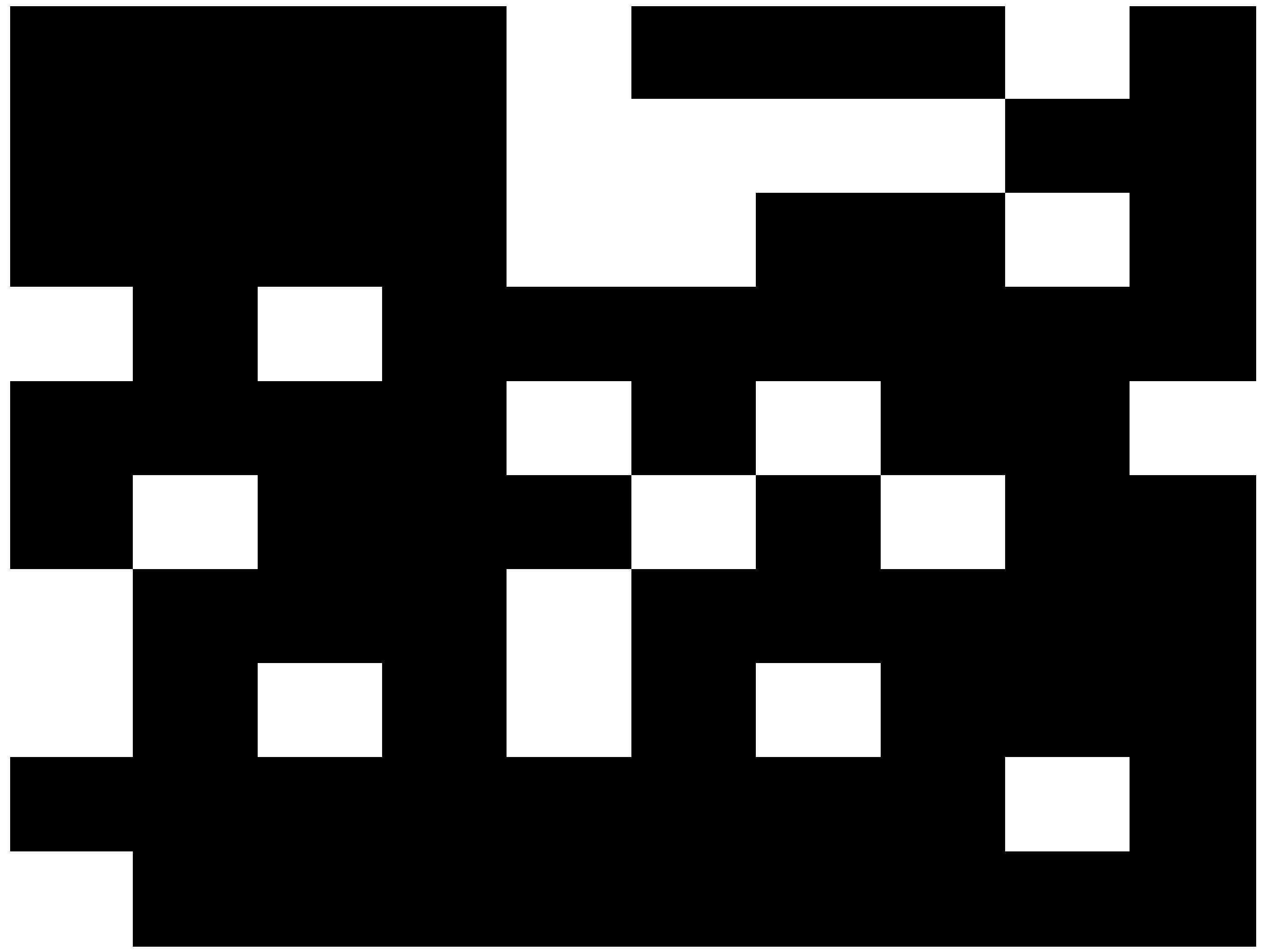
similar low-rank structure for other matricizations

3D Seismic Masks: Missing Receivers

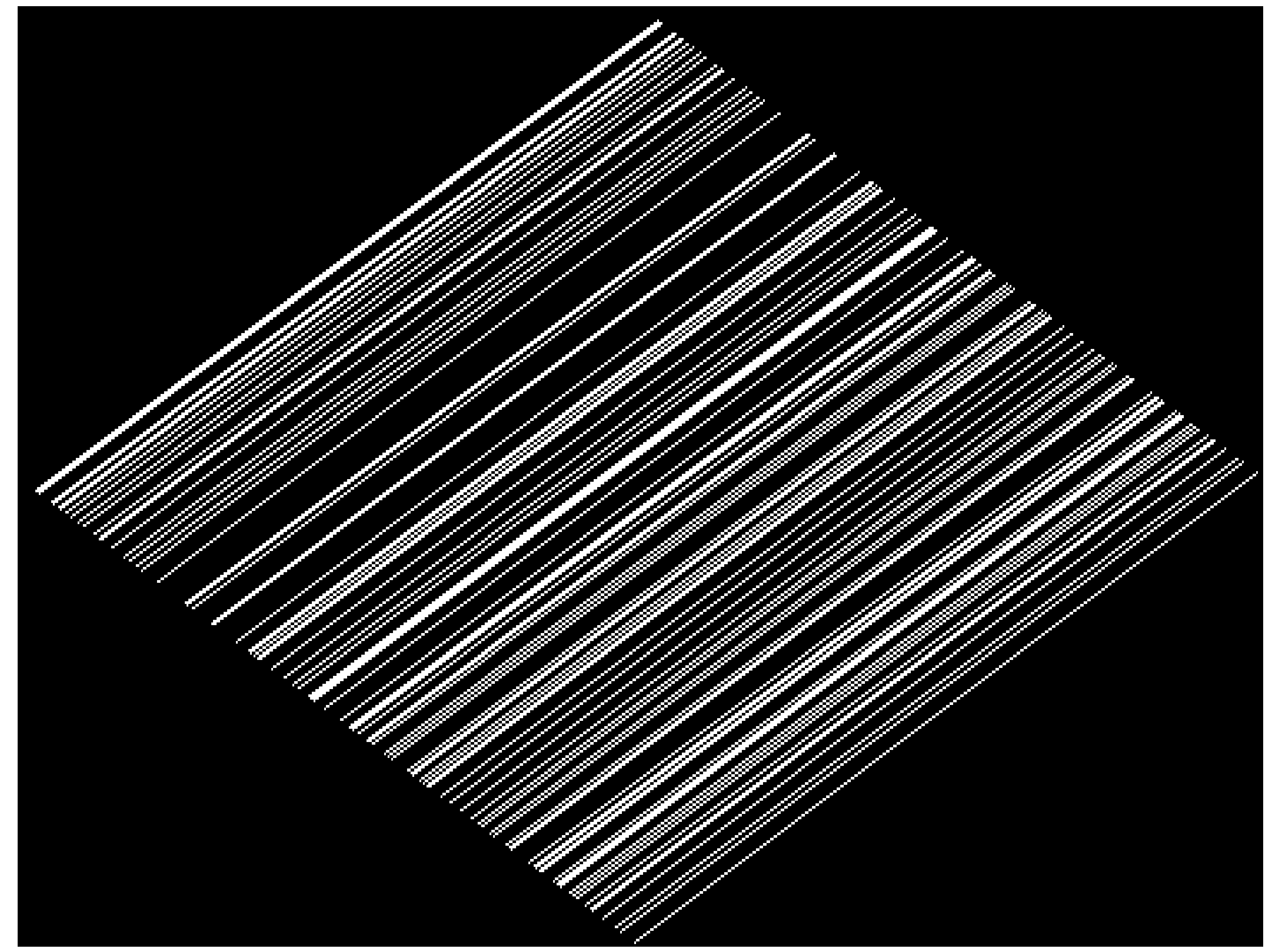
(rec,rec)-form



(src,src)-form



(mid,off)-domain



Sampling mask depends on acquisition: uniform random?

3D Seismic Data Interpolation

- Low-rank structure ✅
- Incoherence (small μ) ✅
- Uniform random sampling ❌

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How should we subsample?

Consider our sampling mask

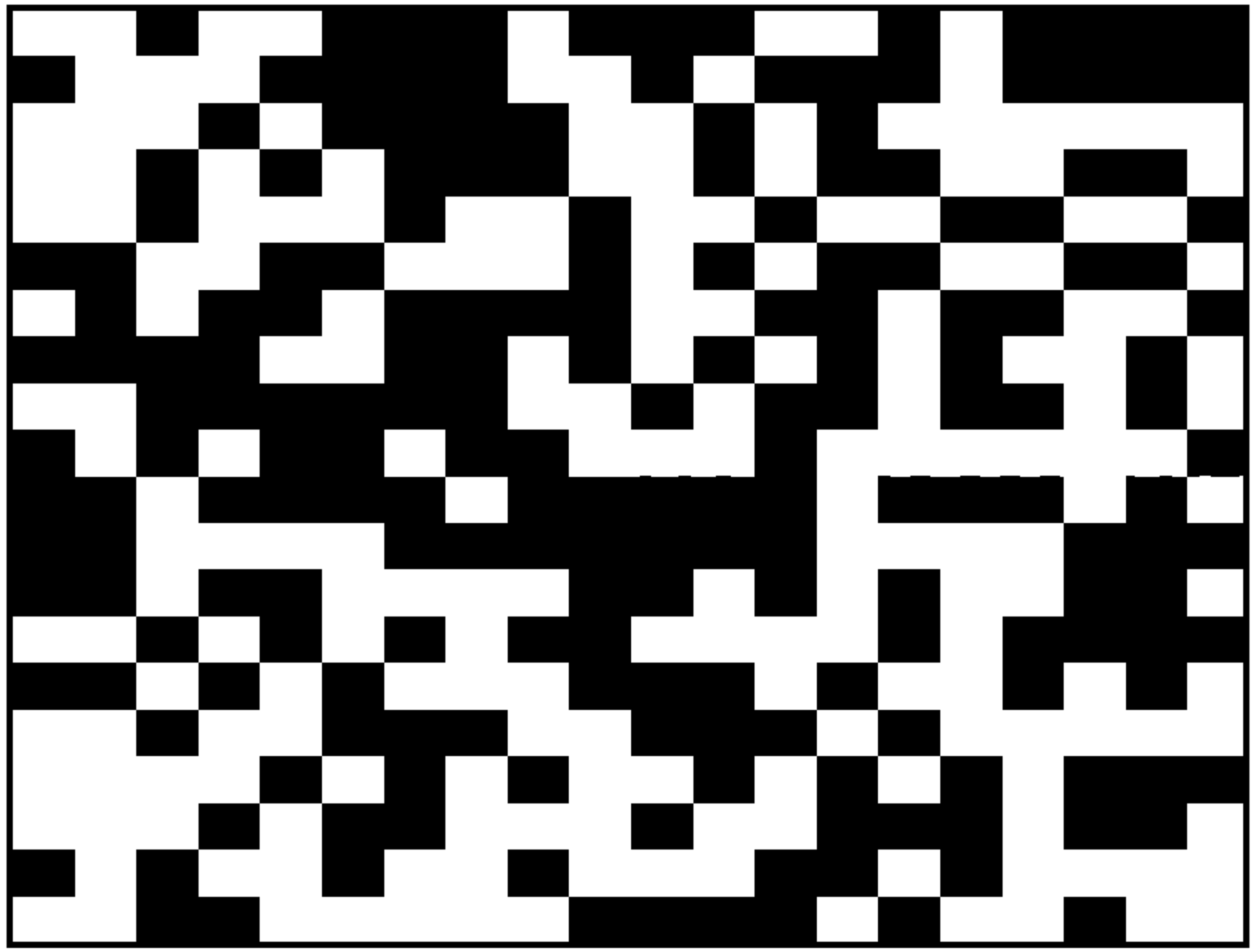
$$A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

What determines if A is good for matrix completion?

Example: Ideal Mask

Samples chosen uniformly at random

$$A =$$



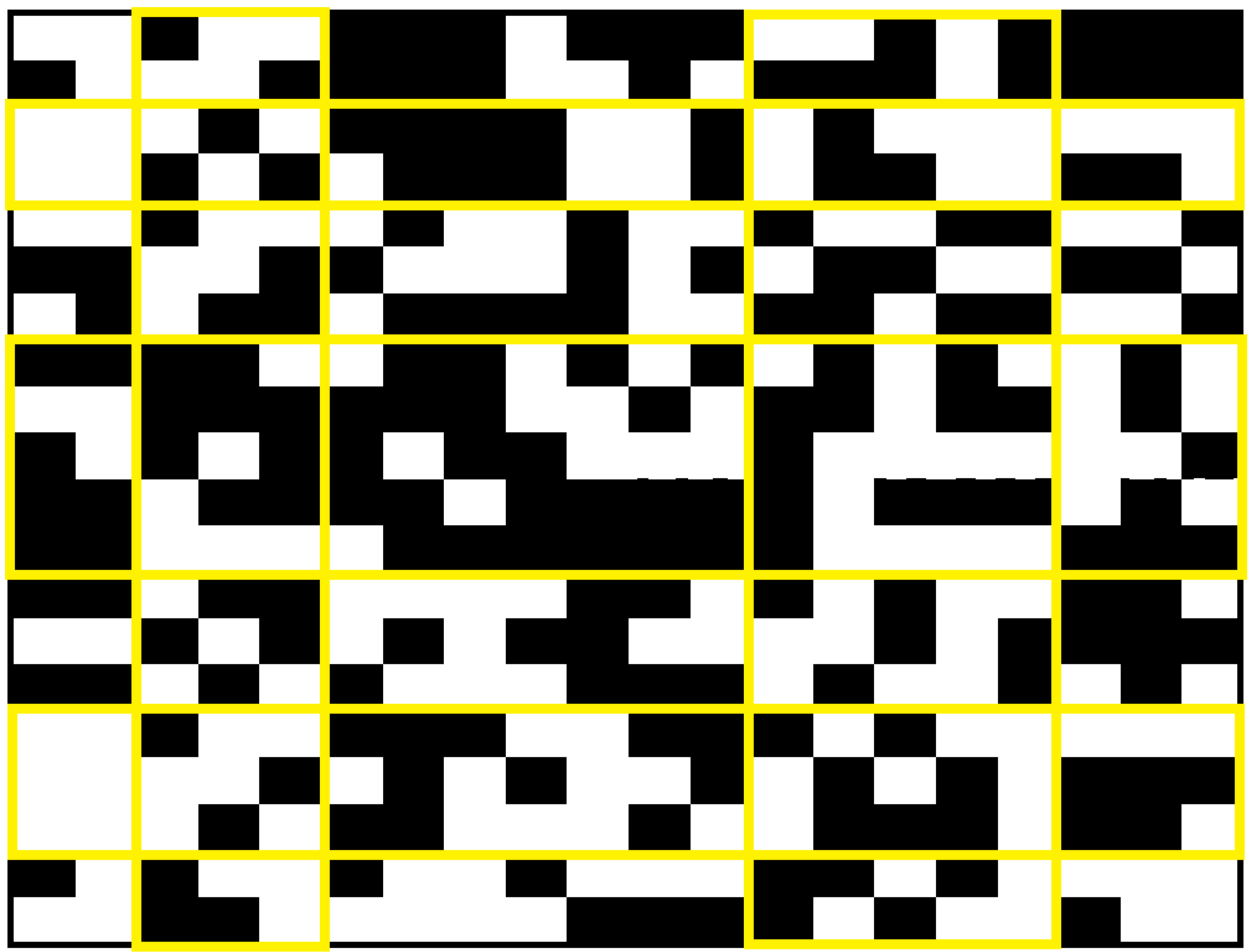
■ $= 0$

□ $= 1$

Example: Ideal Mask

Choose any sub matrix

$$A =$$



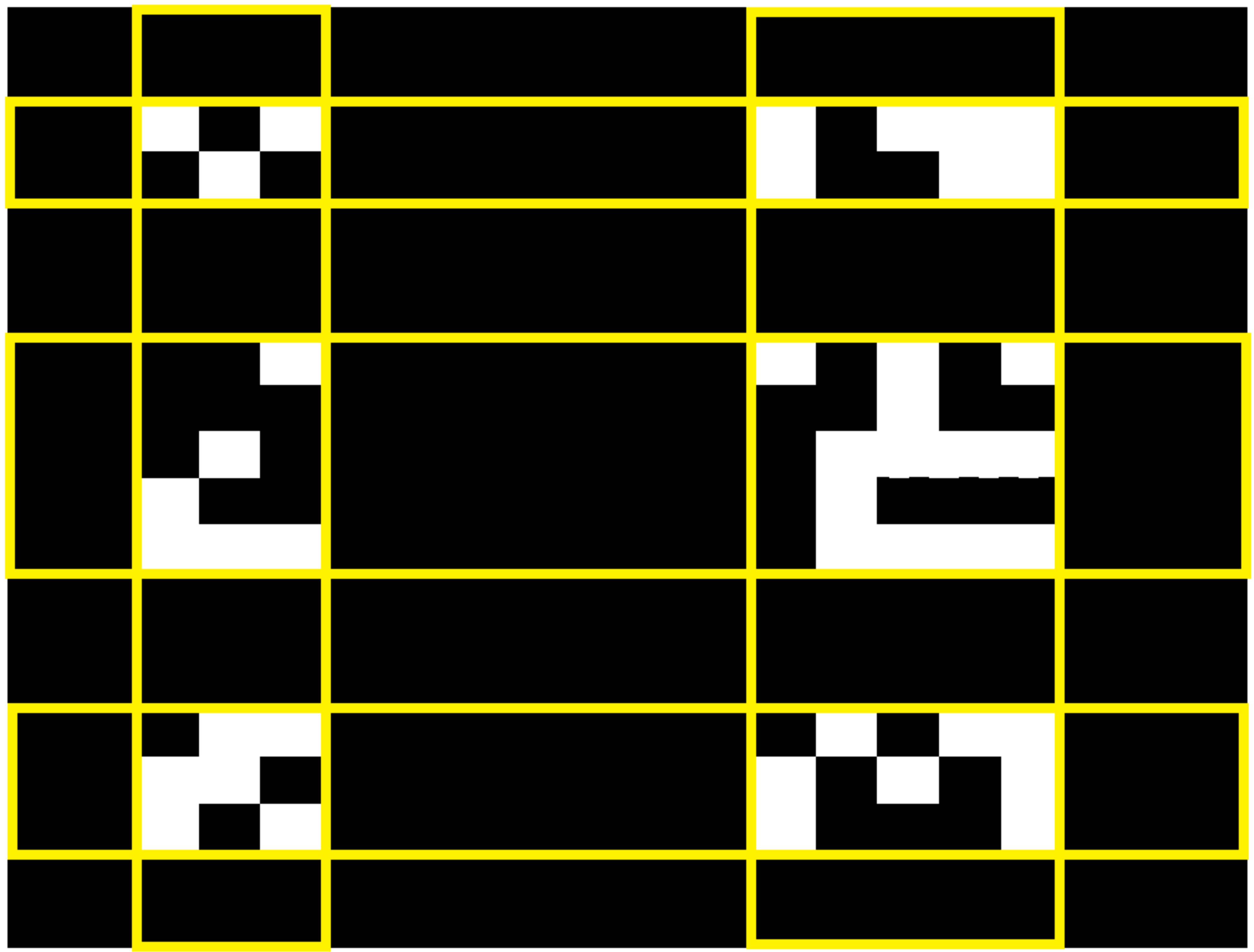
■ $= 0$

□ $= 1$

Example: Ideal Mask

All sub matrices are nicely sampled!

$$A =$$



$\blacksquare = 0$

$\square = 1$

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Spectral Gap

Bhojanapalli, Jain. “Universal Matrix Completion” ICML 2014.

Consider the gap between the two largest singular values of A

$$\frac{\sigma_2(A)}{\sigma_1(A)} = \begin{cases} \approx 1 & \text{small spectral gap} \\ \ll 1 & \text{large spectral gap} \end{cases}$$

Spectral Gap

Bhojanapalli, Jain. "Universal Matrix Completion" ICML 2014.

$$\frac{\sigma_2(A)}{\sigma_1(A)} = \begin{cases} \approx 1 & \text{small spectral gap} \\ \ll 1 & \text{large spectral gap} \end{cases}$$

From graph theory:

A with Large Spectral Gap $\Longrightarrow$ all "sub matrices" are nicely sampled

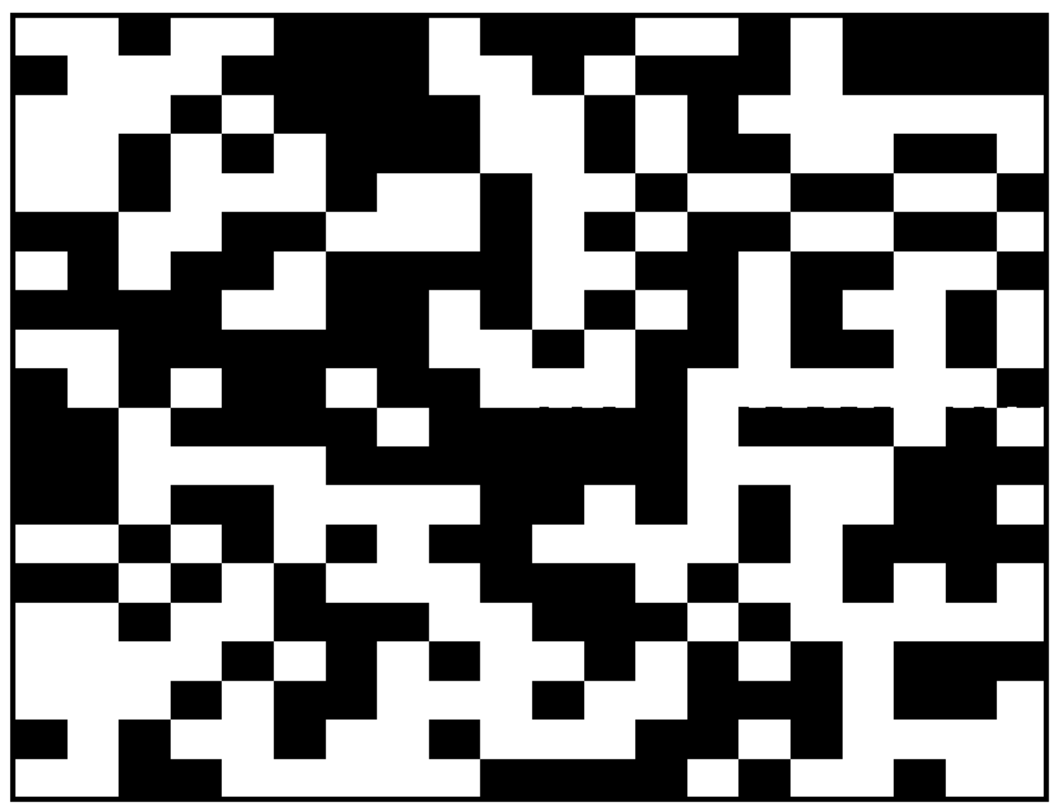
$\Longrightarrow$ better results for matrix completion

Spectral Gap

Bhojanapalli, Jain. "Universal Matrix Completion" ICML 2014.

$$\frac{\sigma_2(A)}{\sigma_1(A)} = \begin{cases} \approx 1 & \text{small spectral gap} \\ \ll 1 & \text{large spectral gap} \end{cases}$$

Restriction: Results only apply to regular graphs



equivalent to having same number of samples in each row and column

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Generalized Spectral Gap

Extend the results by introducing “generalized spectral gap”:

$$\text{GSP} = \frac{\sqrt{nm}}{|\Omega|}\sigma_1\left(A - \frac{|\Omega|}{nm}\mathbf{1}_{n\times m}\right) \approx \frac{\sigma_2(A)}{\sigma_1(A)}$$

As before, small GSP is better for matrix completion.

Only assume that each row/column is sampled.

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Generalized Spectral Gap

Theorem:

Let $\mathbf{X} \in \mathbb{R}^{n \times m}$ be rank-r, μ-incoherent and “strongly incoherent” matrix.

Let Ω contain at least one entry from each row and column.

Then $\mathbf{X}$ is the unique solution of $\mathbf{NN}(P_\Omega(\mathbf{X}), 0)$ given that

$$\text{GSP} \leq \frac{1}{6\mu r}$$

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Then $\mathbf{X}$ is the unique solution of $\mathbf{NN}(P_\Omega(\mathbf{X}), 0)$ given that

$$\text{GSP} \leq \frac{1}{6\mu r} \iff |\Omega| \geq 6\mu r\sqrt{nm}\,\sigma_1\left(A - \frac{|\Omega|}{nm}\mathbf{1}_{n\times m}\right)$$

Generalized Spectral Gap

Bad Case (e.g., periodic sampling):

GSP ≈ 1 $\Longrightarrow$ exact recovery not possible

Good Case (e.g., random sampling):

GSP $\approx \sqrt{\sqrt{nm}/|\Omega|}$ $\Longrightarrow$ $|\Omega| \geq 36\mu^2 r^2 \sqrt{nm}$

Generalized Spectral Gap

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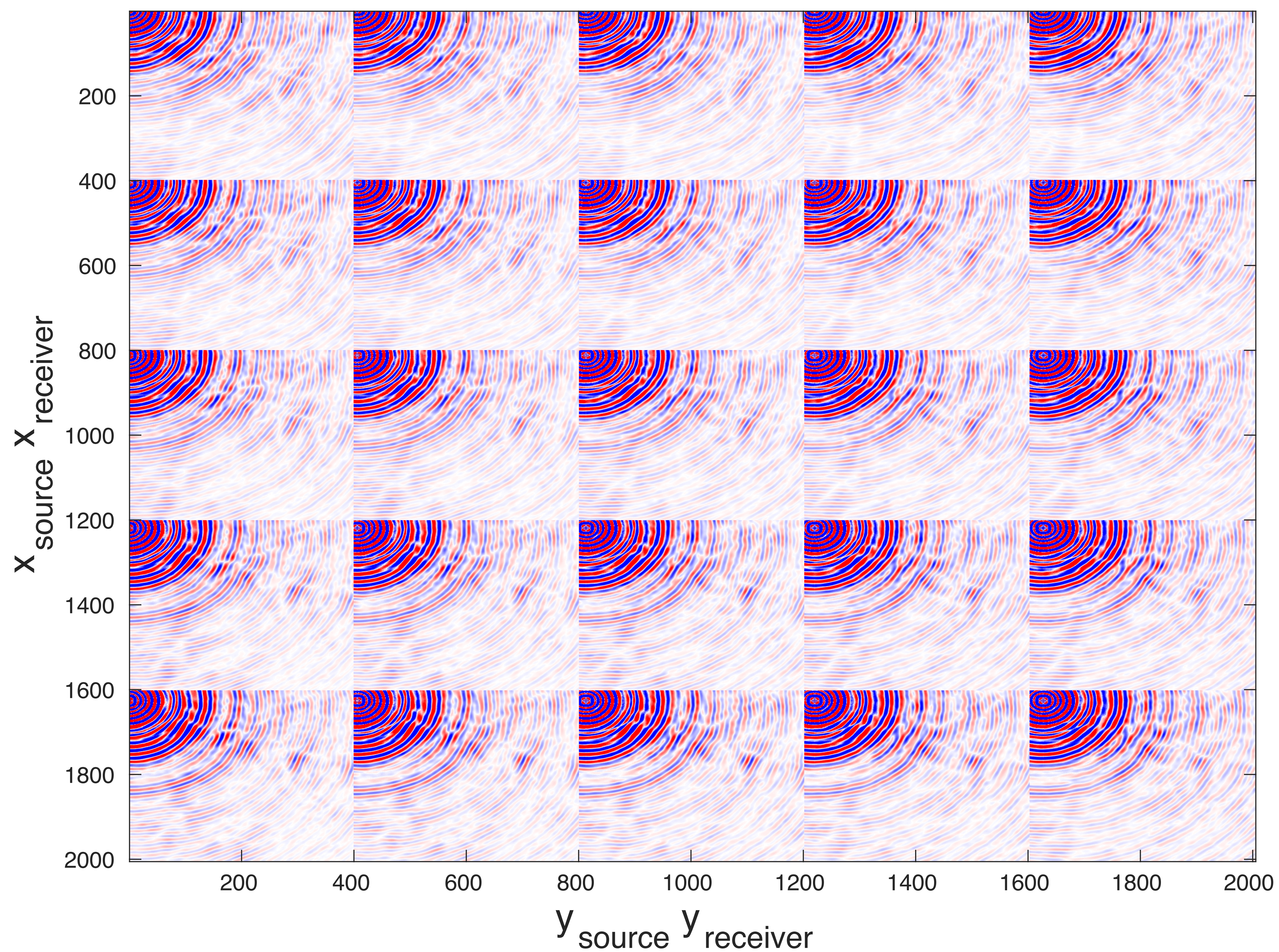
Good Case (e.g., random sampling):

GSP $\approx \sqrt{\sqrt{nm}/|\Omega|}$ $\Longrightarrow$ $|\Omega| \geq 36\mu^2 r^2 \sqrt{nm}$

compare to $|\Omega| \geq C\mu^2 rn\log^2(n)$ from literature (non-determinitic)

3D Interpolation Example

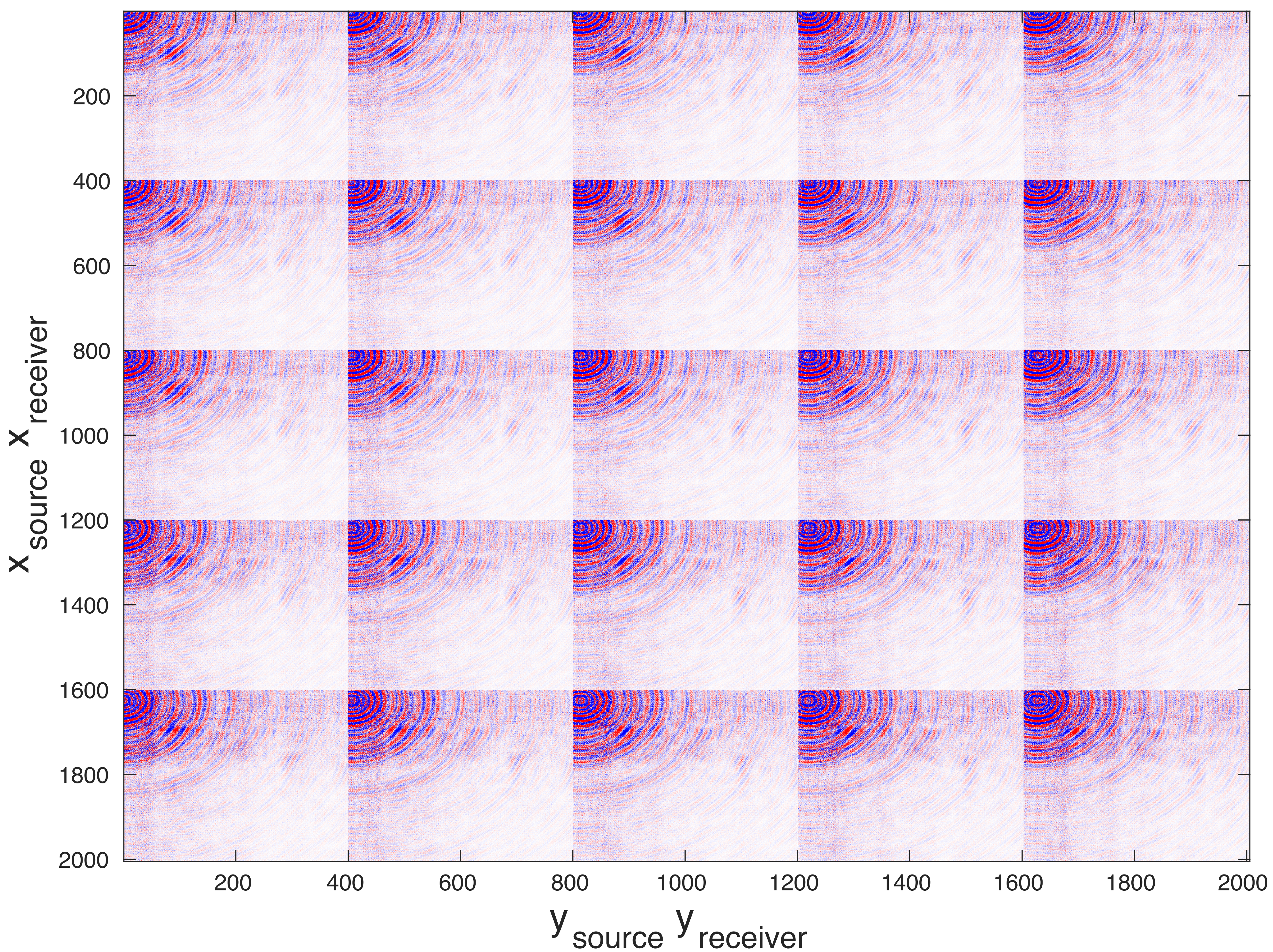


Size: 2005 x 2005

Remove 75 % of Receivers

Compare small and large GSP recovery

3D Interpolation Example: Bad Recovery

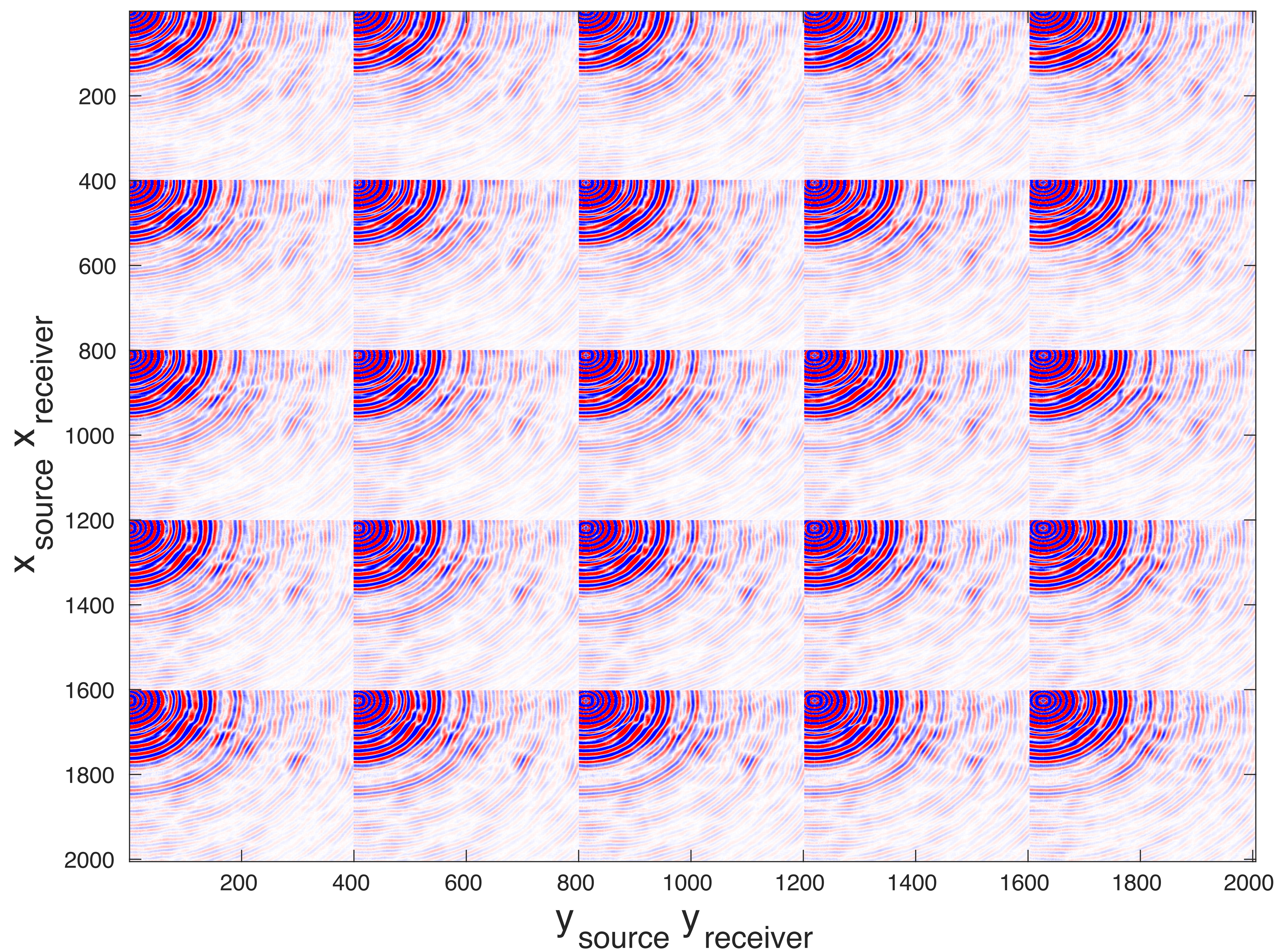


Reconstruction

SNR: 3.5 dB

GSP: .9828

3D Interpolation Example: Good Recovery



Reconstruction

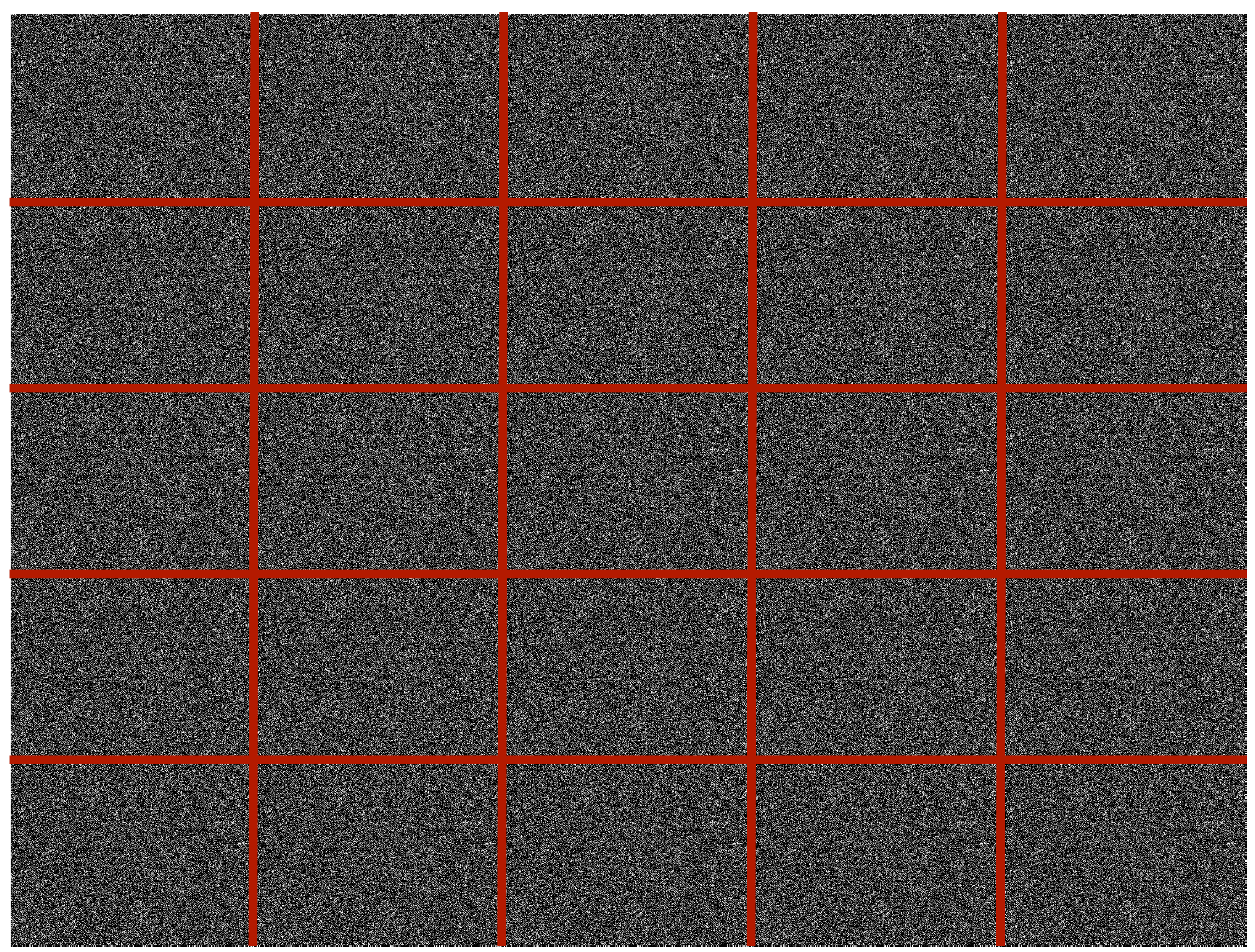
SNR: 20.7 dB

GSP: .1796

3D Interpolation Experiments

Generate 3D seismic Masks with increasing GSP

Plot correlation with reconstruction SNR



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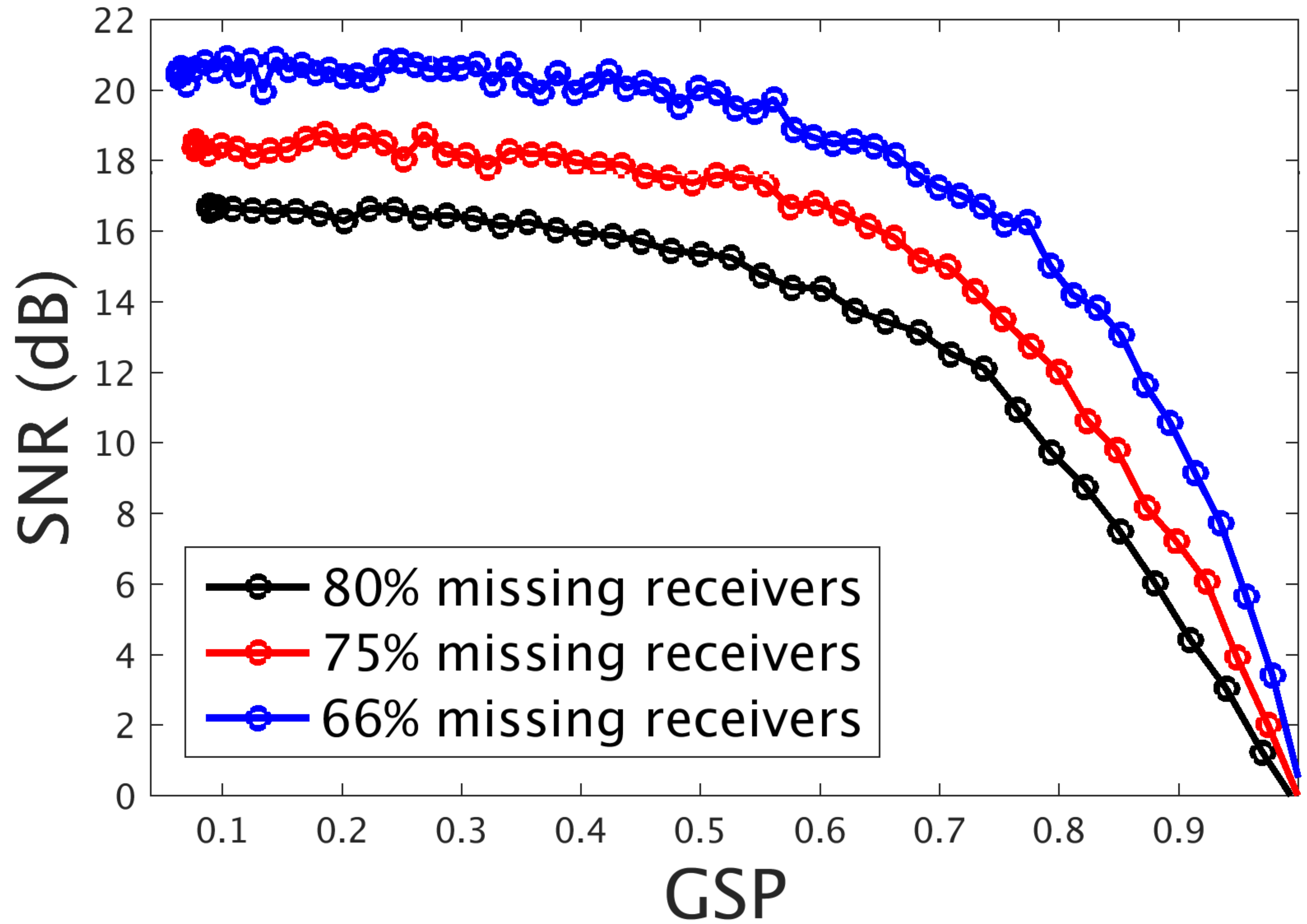

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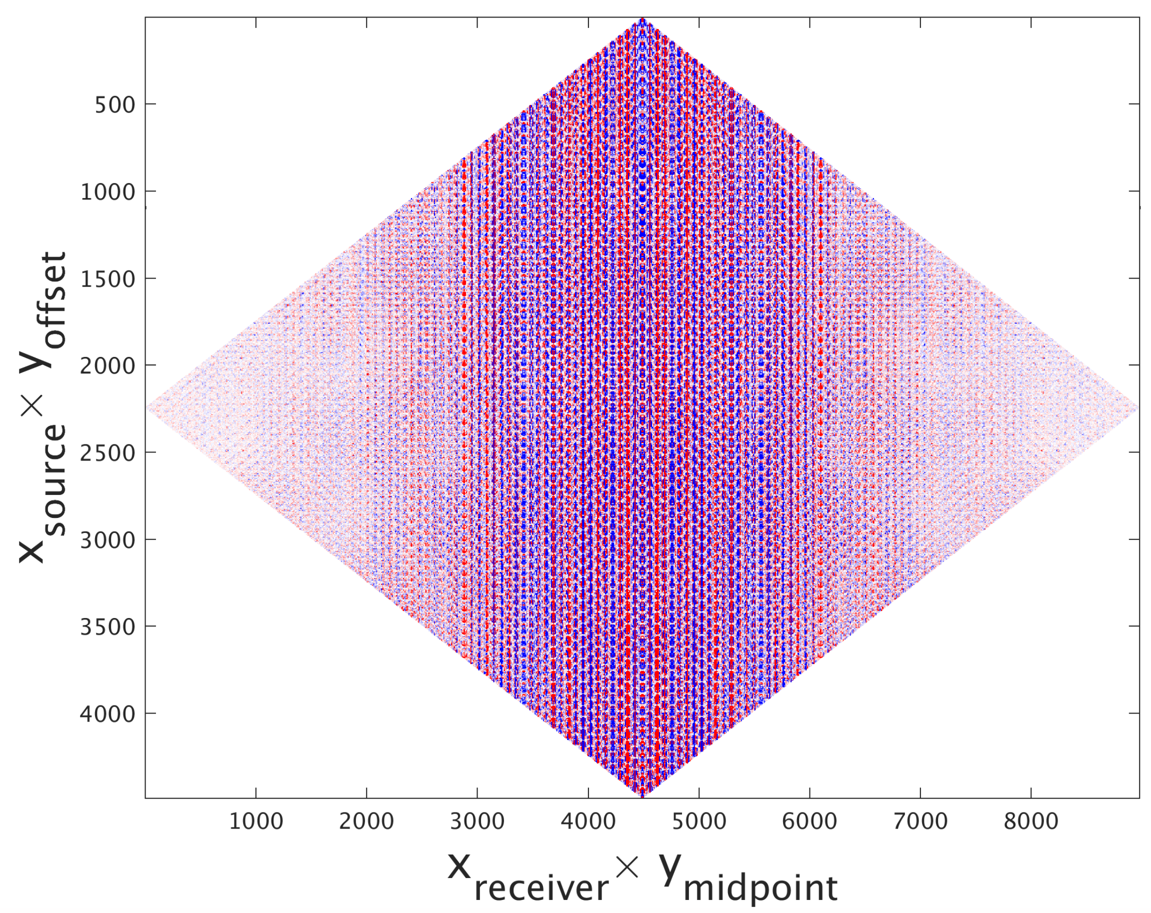
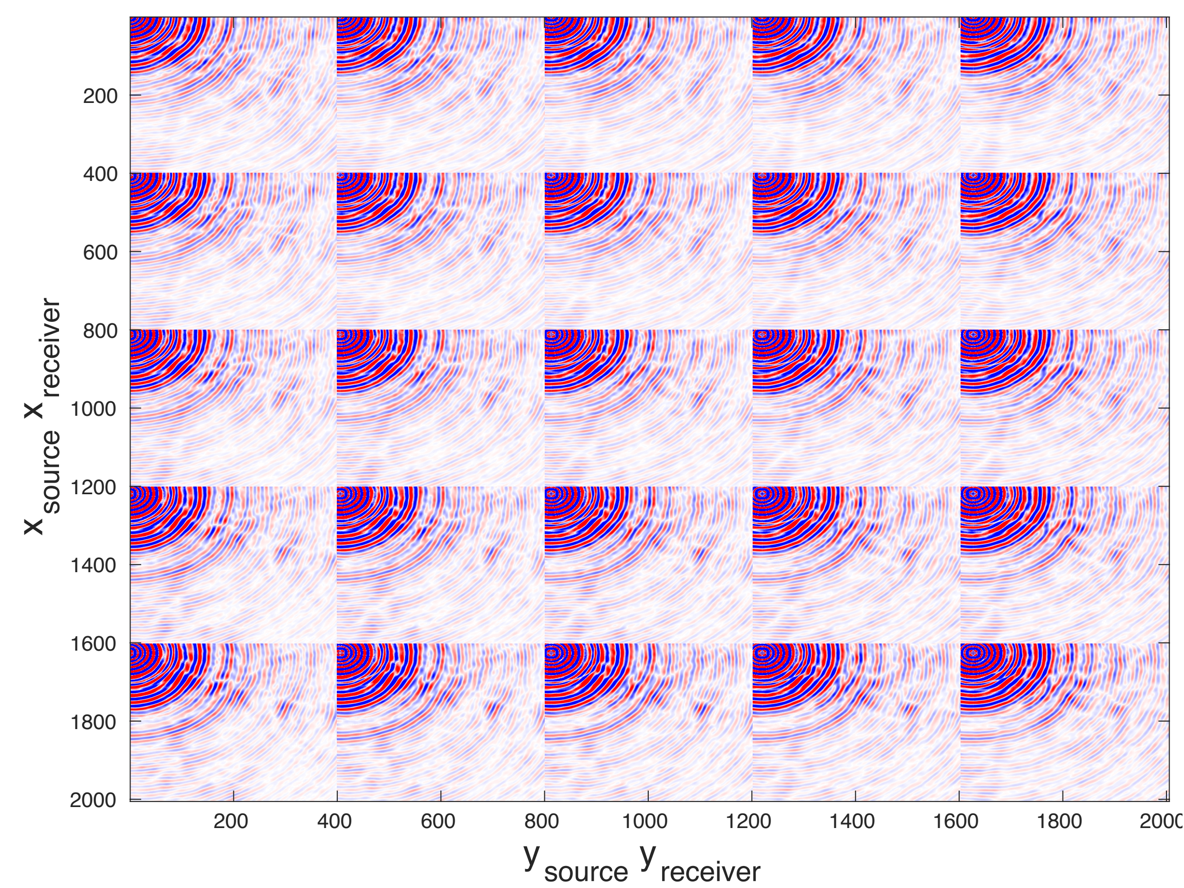
Example 1: How to Matricize?

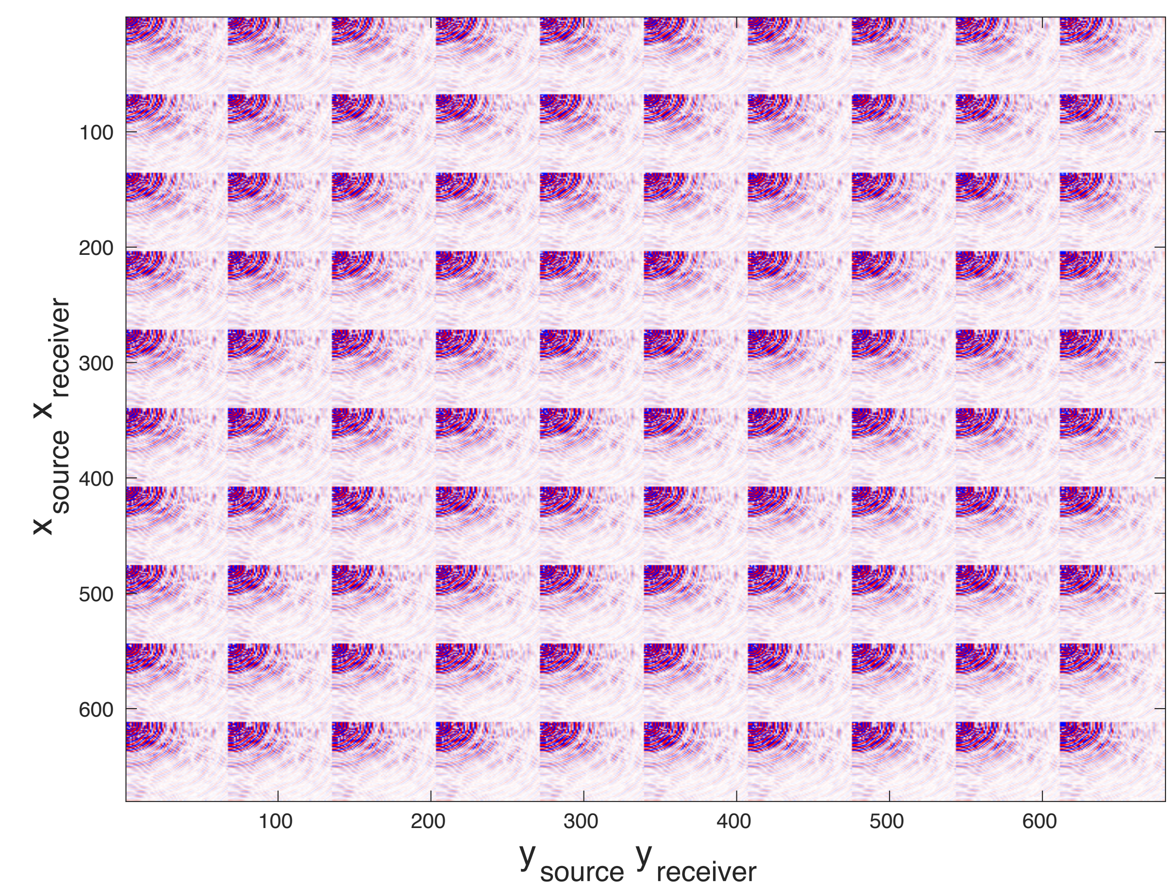
(mid,off)-domain



(rec,rec)-form



(src,src)-form



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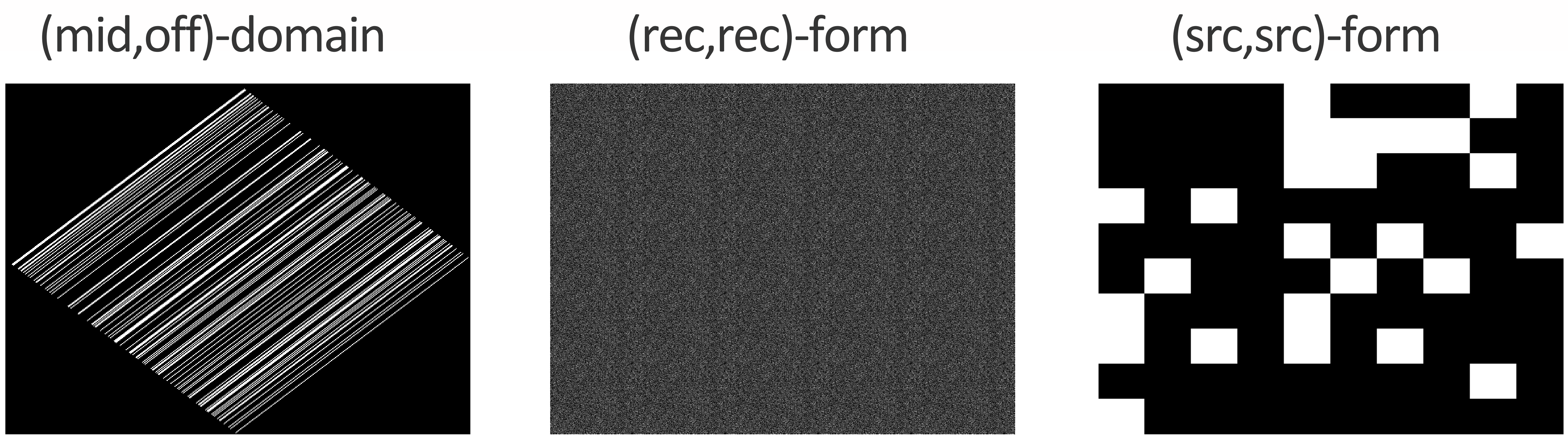
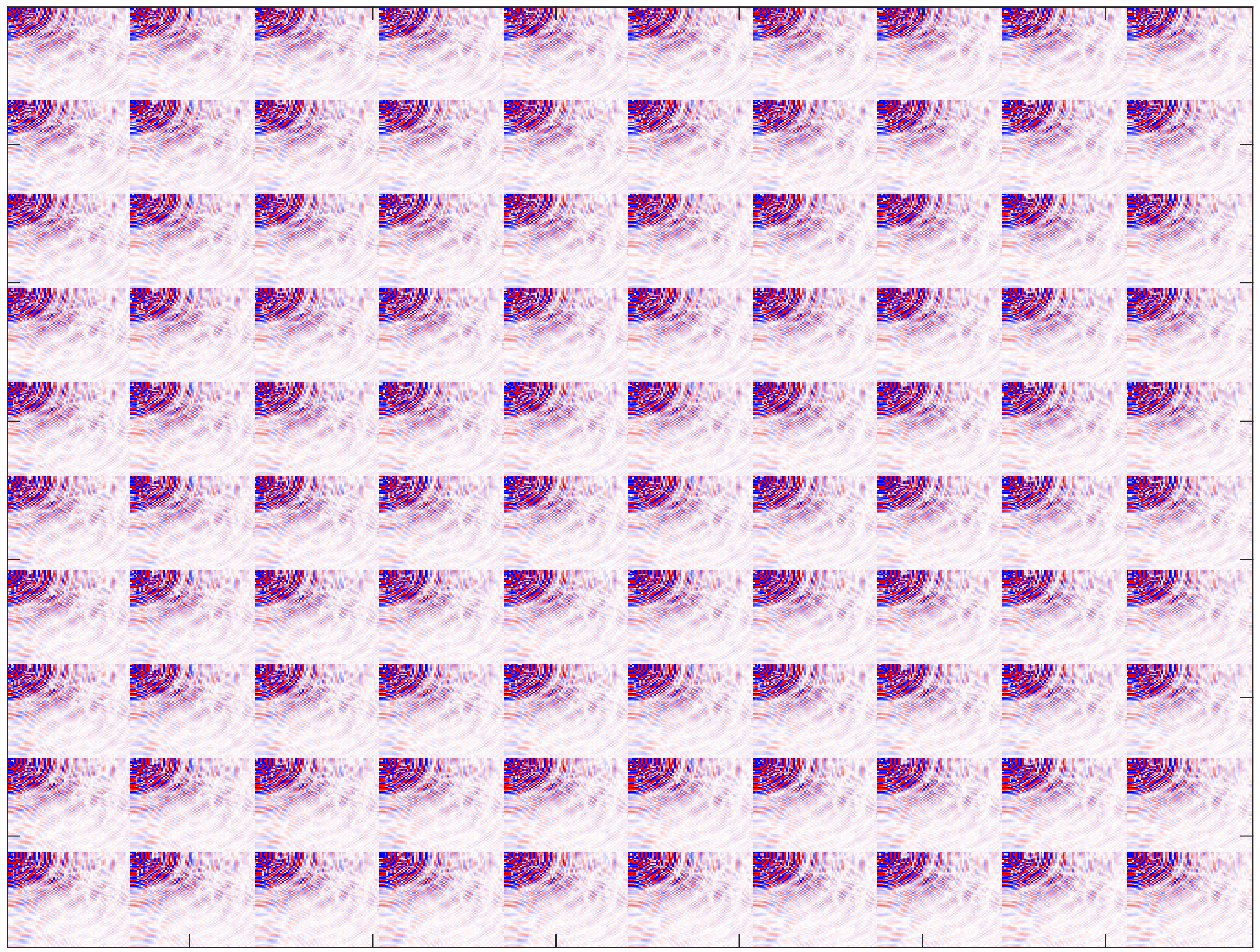


Use GSP to decide which matricization works best for given subsampling

R. Kumar, et al. "Efficient matrix completion for seismic data reconstruction" Geophysics 2014.

Example 2: To window or not to window?

- Full matrix is too large



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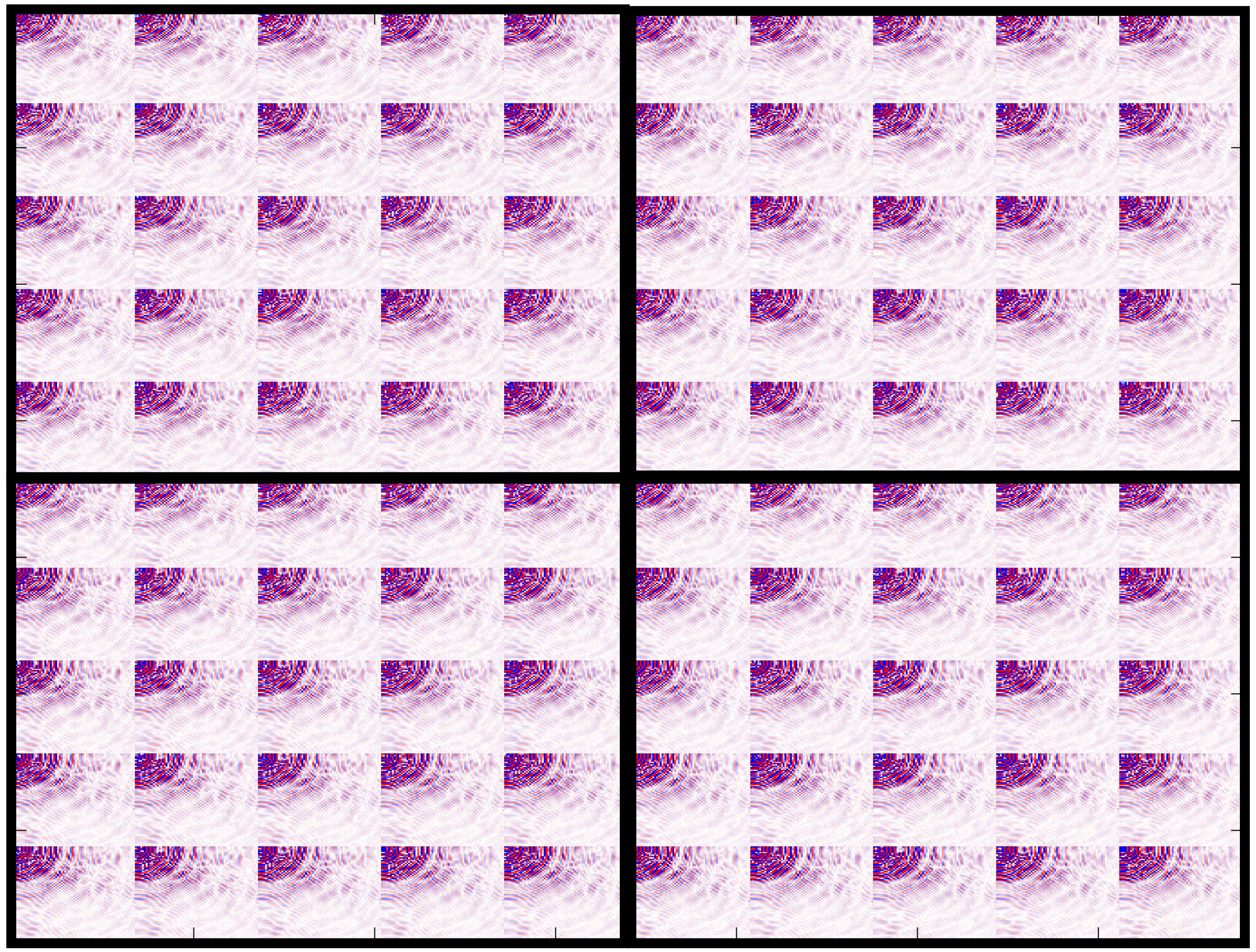
Example 2: To window or not to window?

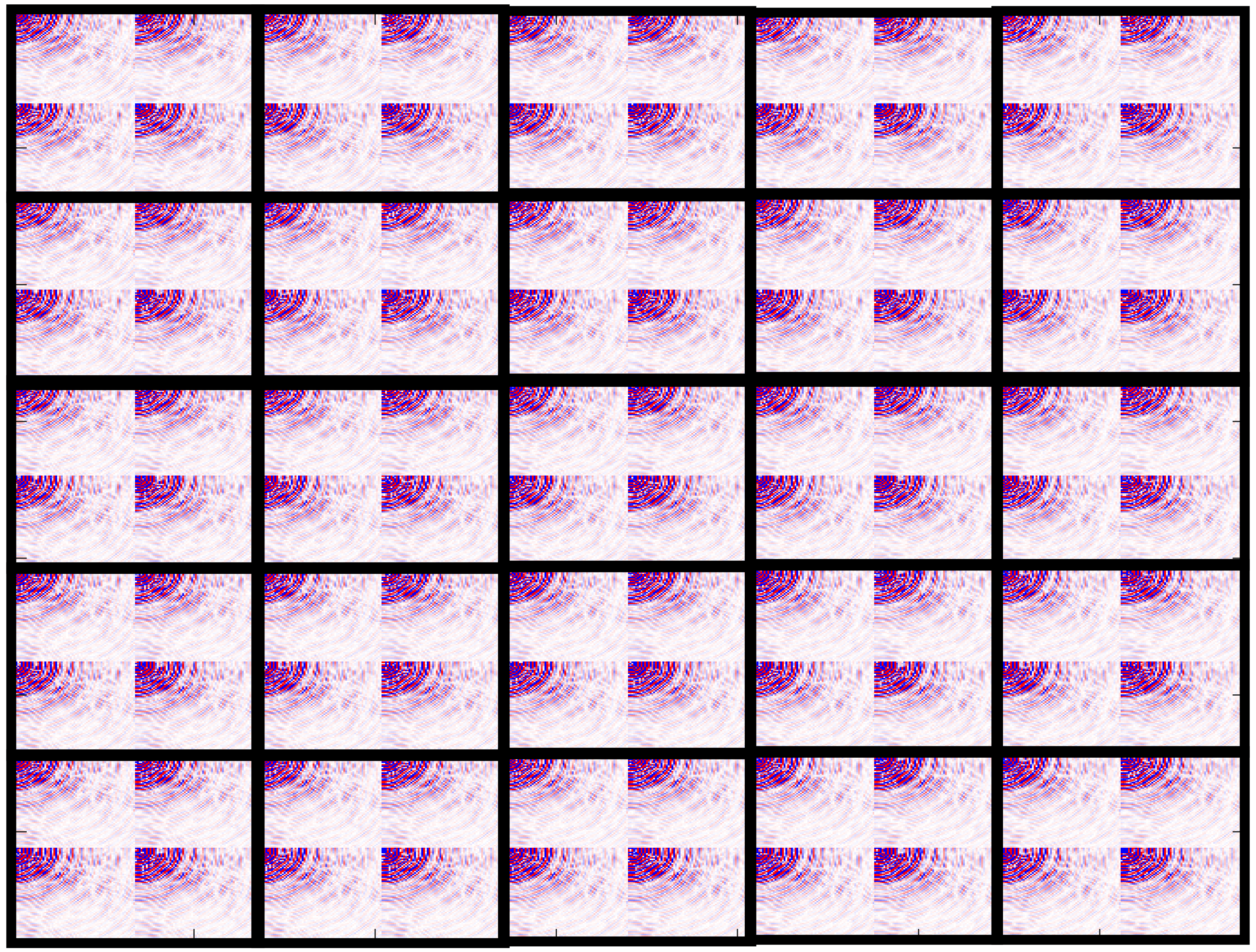
- Full matrix is too large
- Split volume into smaller windows
- Solve in parallel



R. Kumar, et al. "Efficient matrix completion for seismic data reconstruction" Geophysics 2014.

Example 2: To window or not to window?

- Which matricization?
- What size windows?



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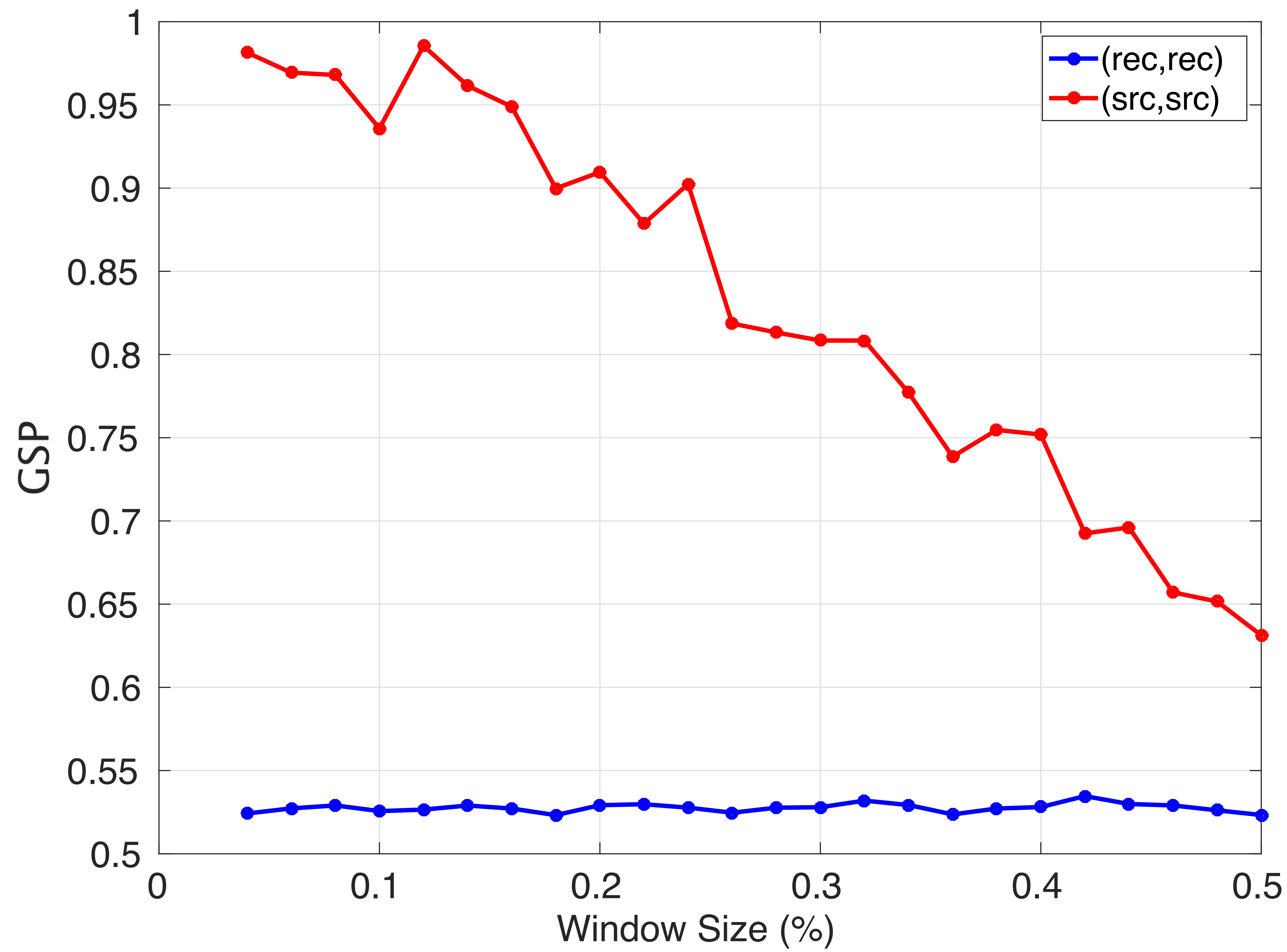
Example 2: To window or not to window?

Average GSP for various window sizes



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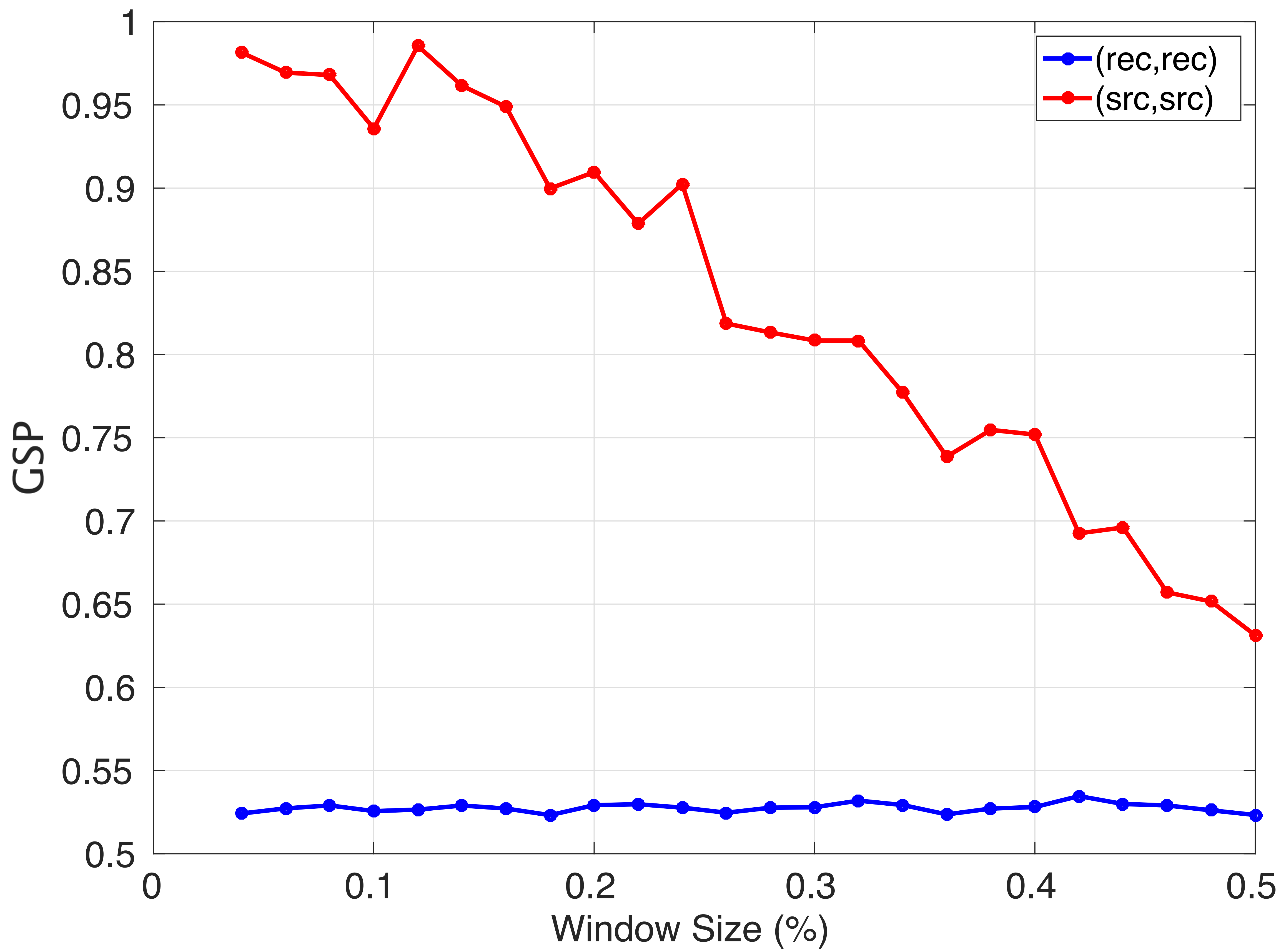
Example 2: To window or not to window?

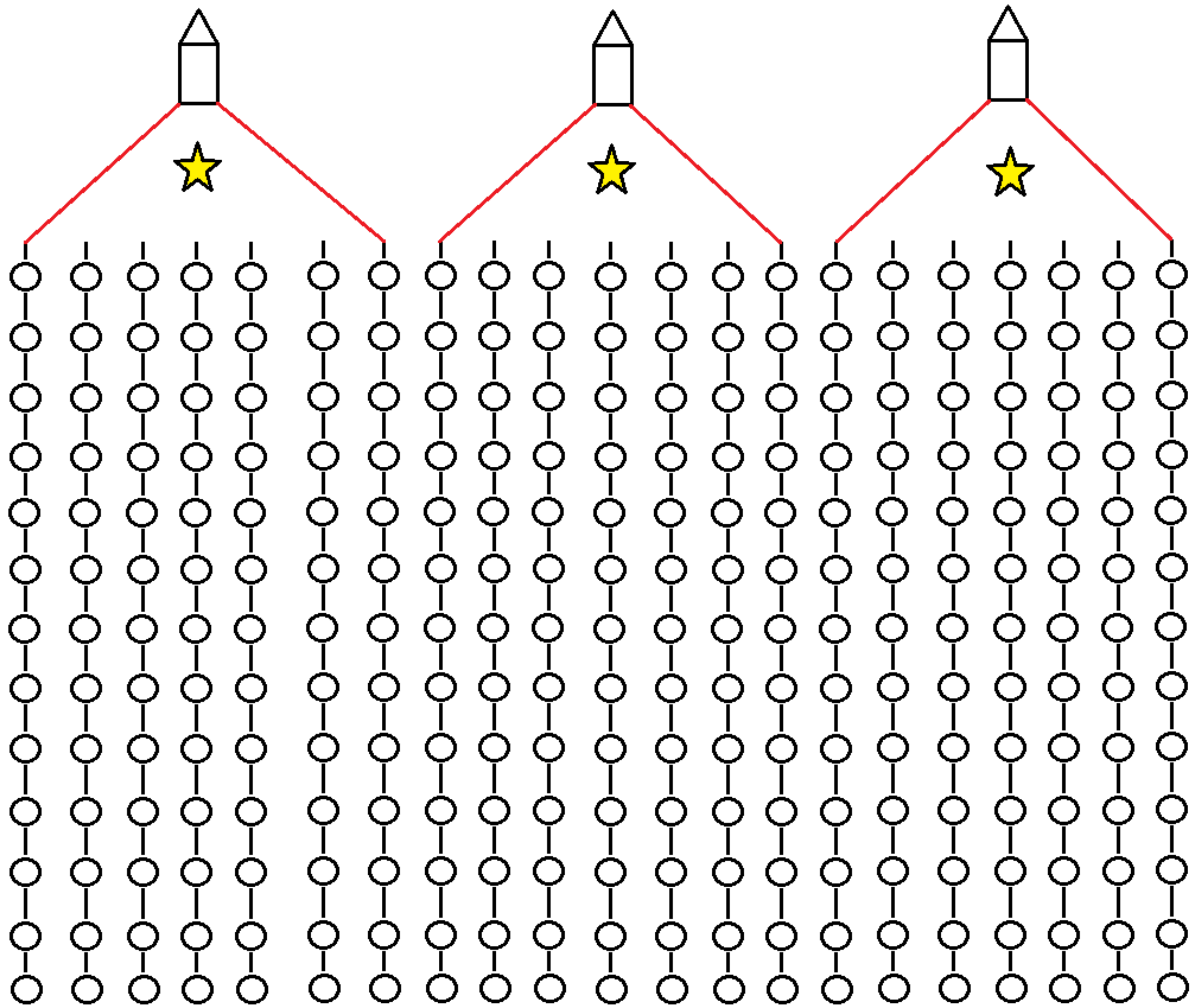
Average GSP for various window sizes

- (rec,rec) is best in this example
- GSP is stable as window size decreases



Example 3: Infill Management

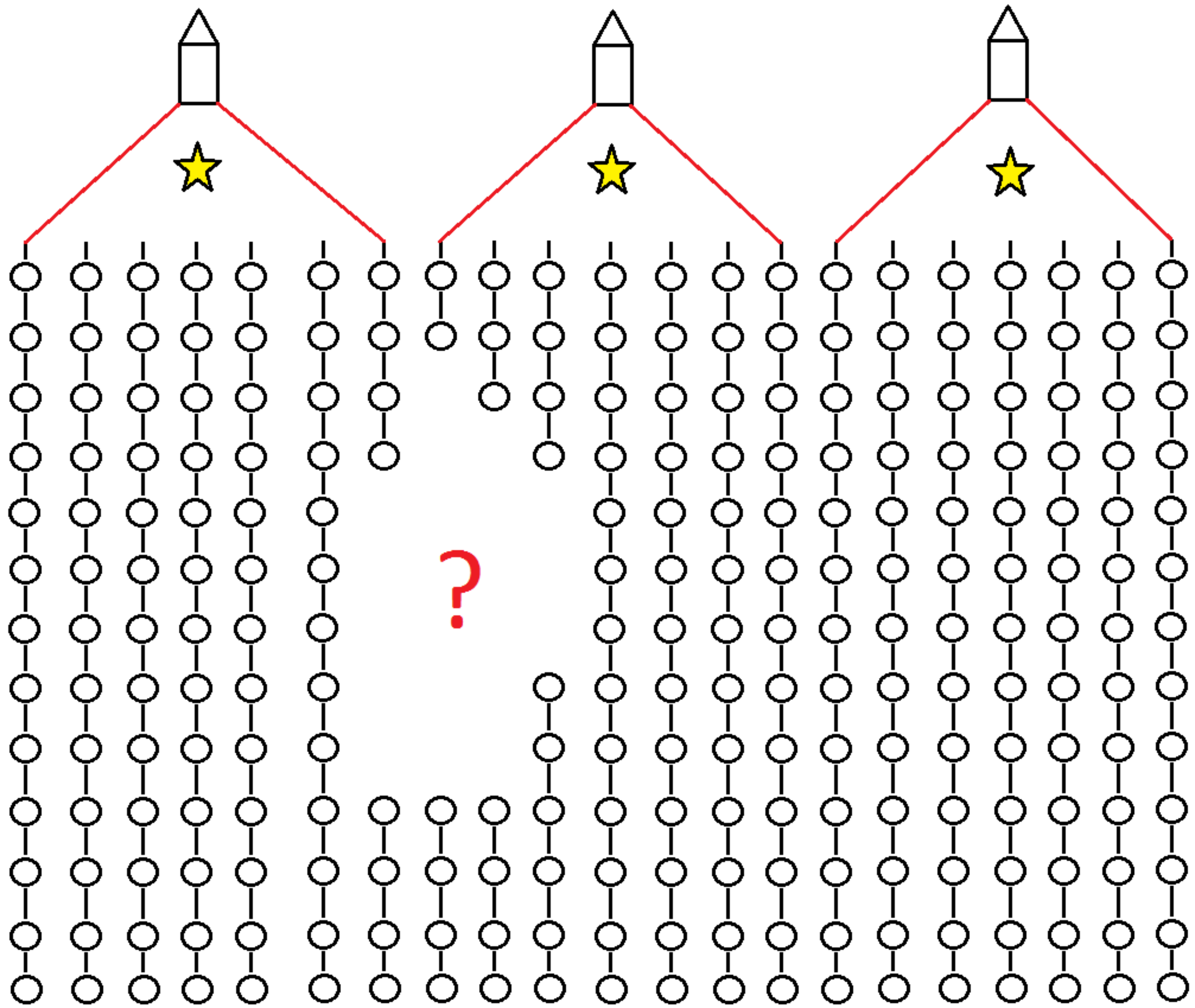
Desired Sampling



Example 3: Infill Management

Coverage Hole:

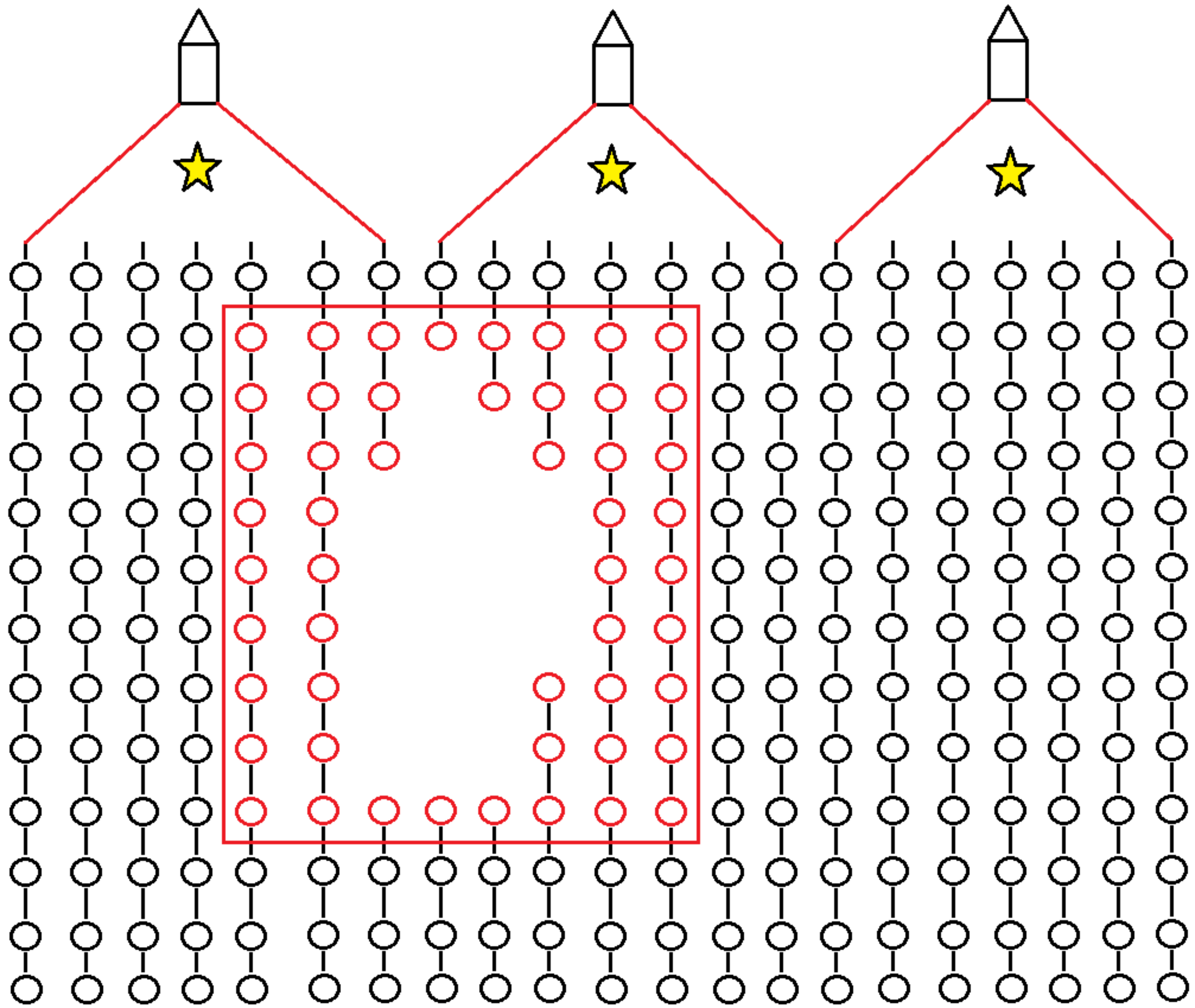
additional infill acquisition?



Example 3: Infill Management

Form local mask:

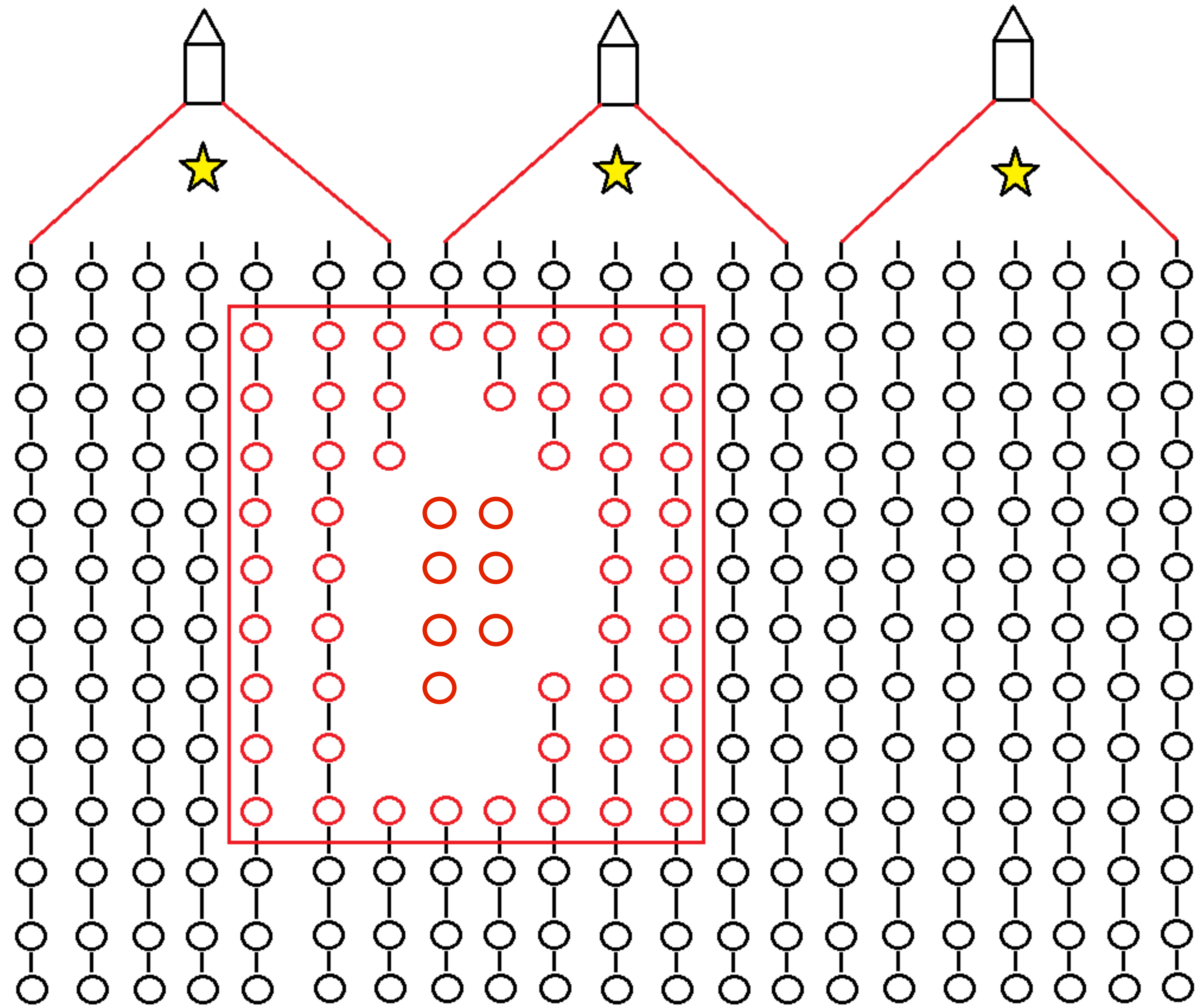
use GSP as decision tool



Example 3: Infill Management

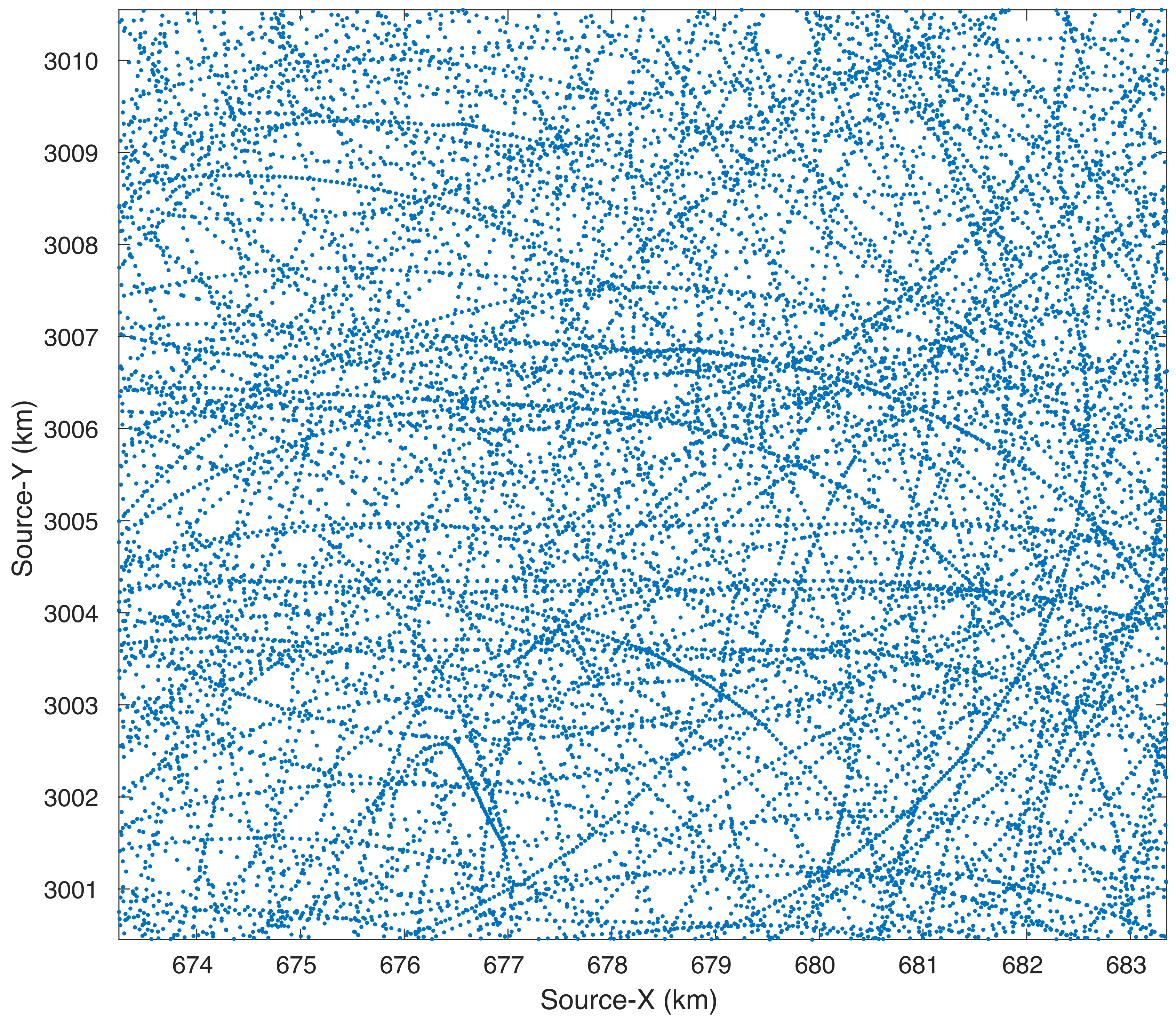
Additional infill acquisition:

use GSP to minimize sampling

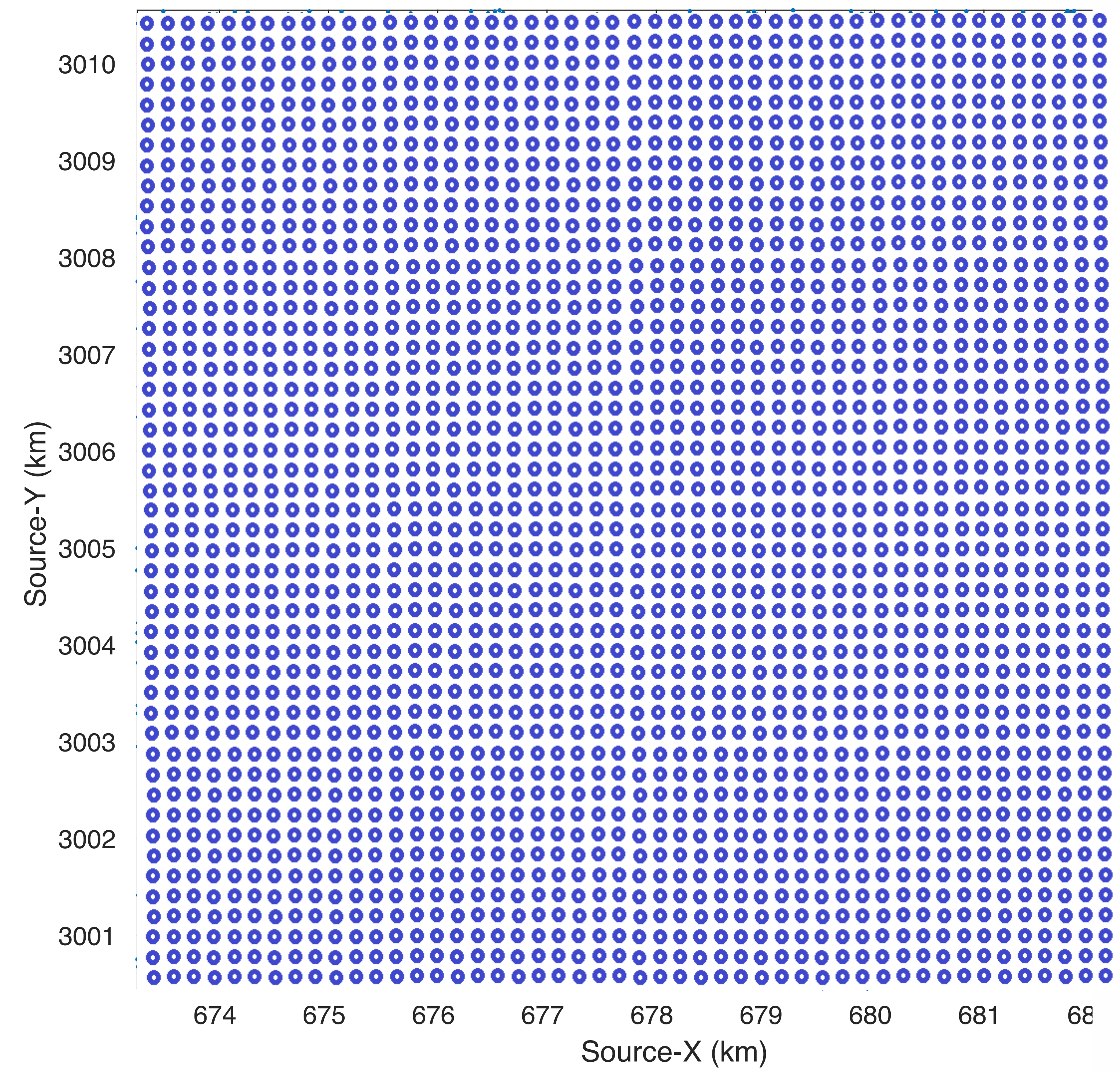


Example 4: Coil Sampling Interpolation

coil sampling source locations



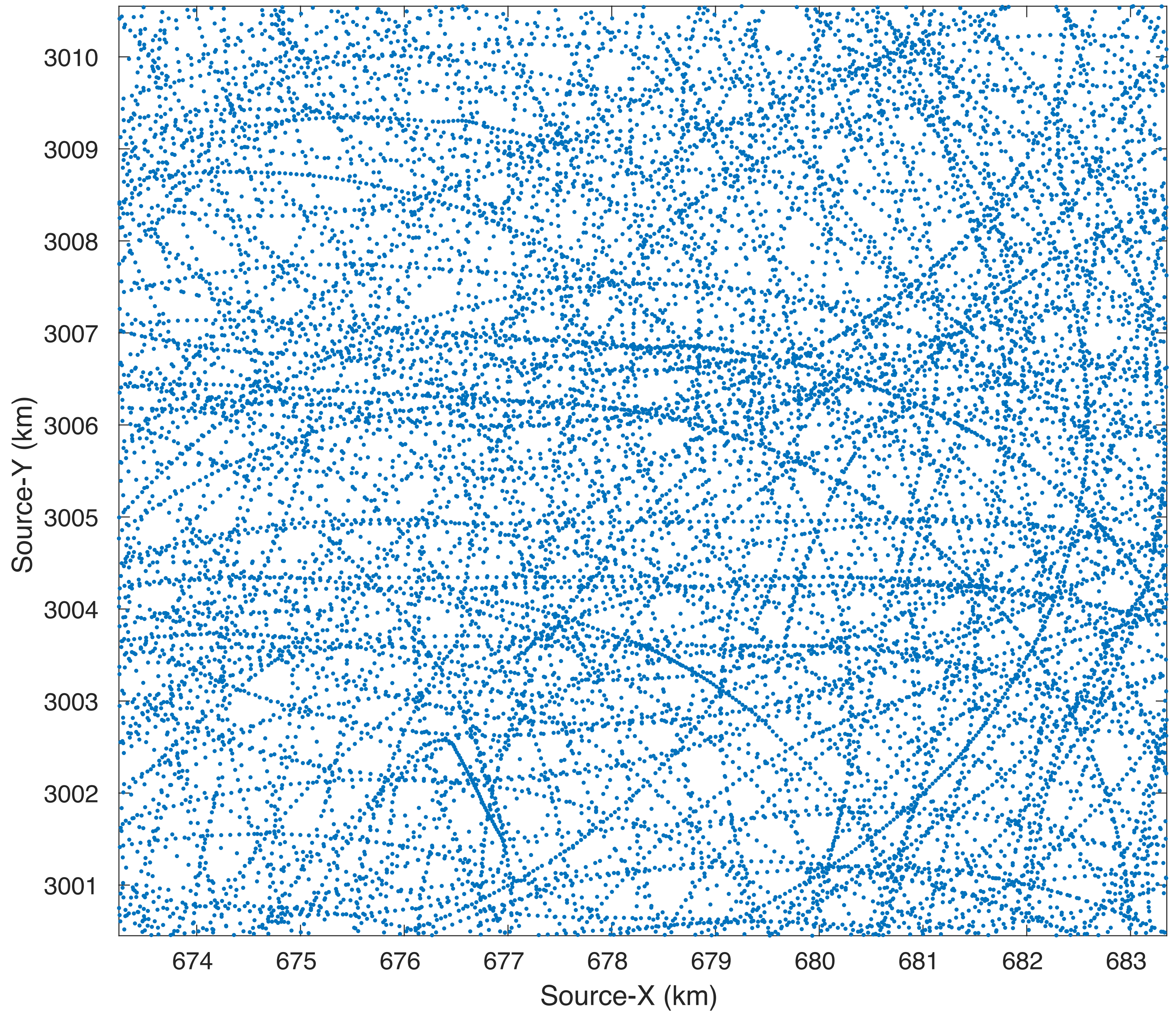
equispaced sources



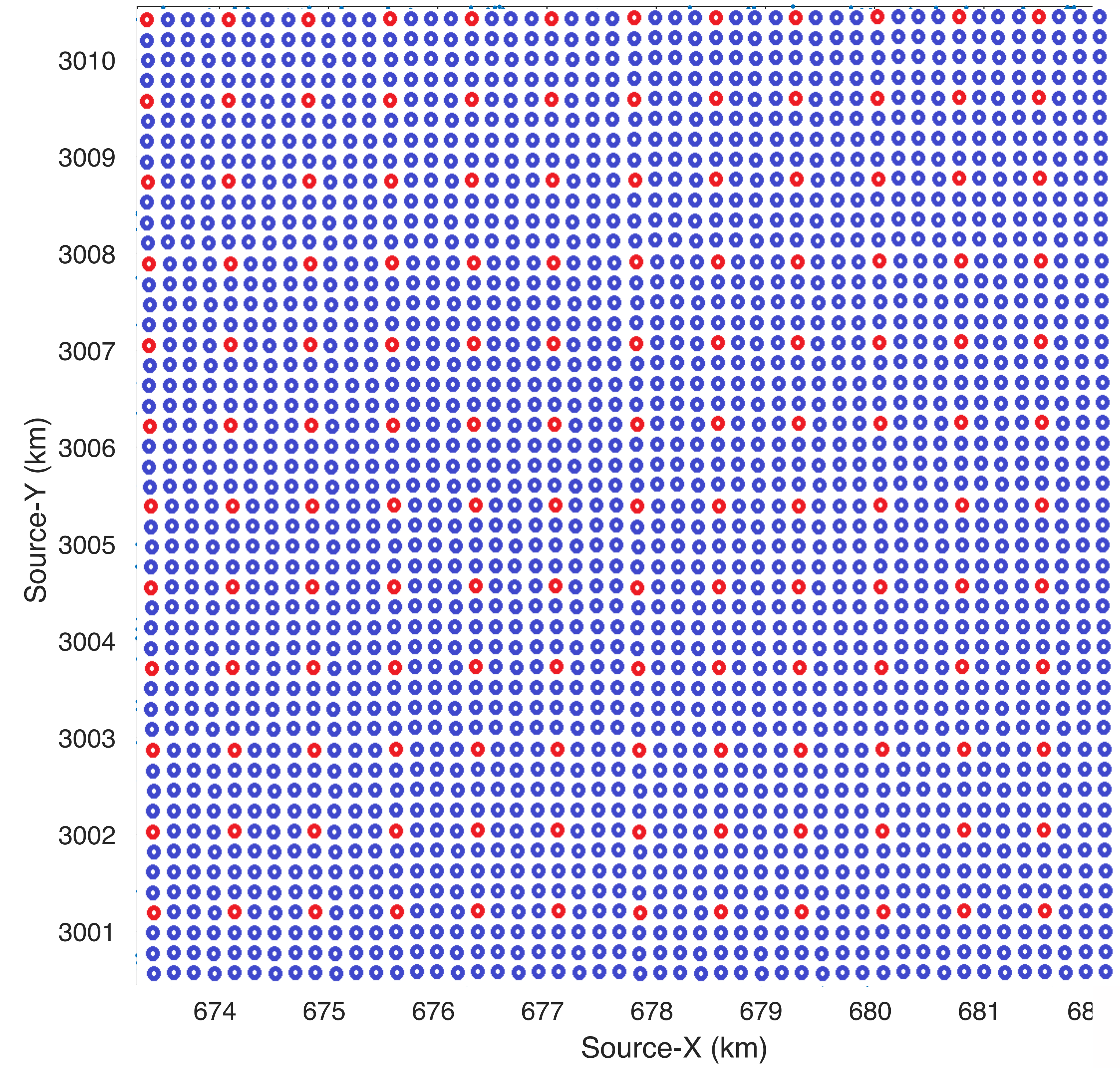
Goal: interpolate coil data onto equispaced grid

Example 4: Coil Sampling Interpolation

coil sampling source locations



equispaced sources



What grid density for sources and receivers?

Example 4: Coil Sampling Interpolation

Receiver Spacing	Source Spacing	Mask GSP
100 m	200 m	0.77
50 m	100 m	0.4
50 m	50 m	0.88

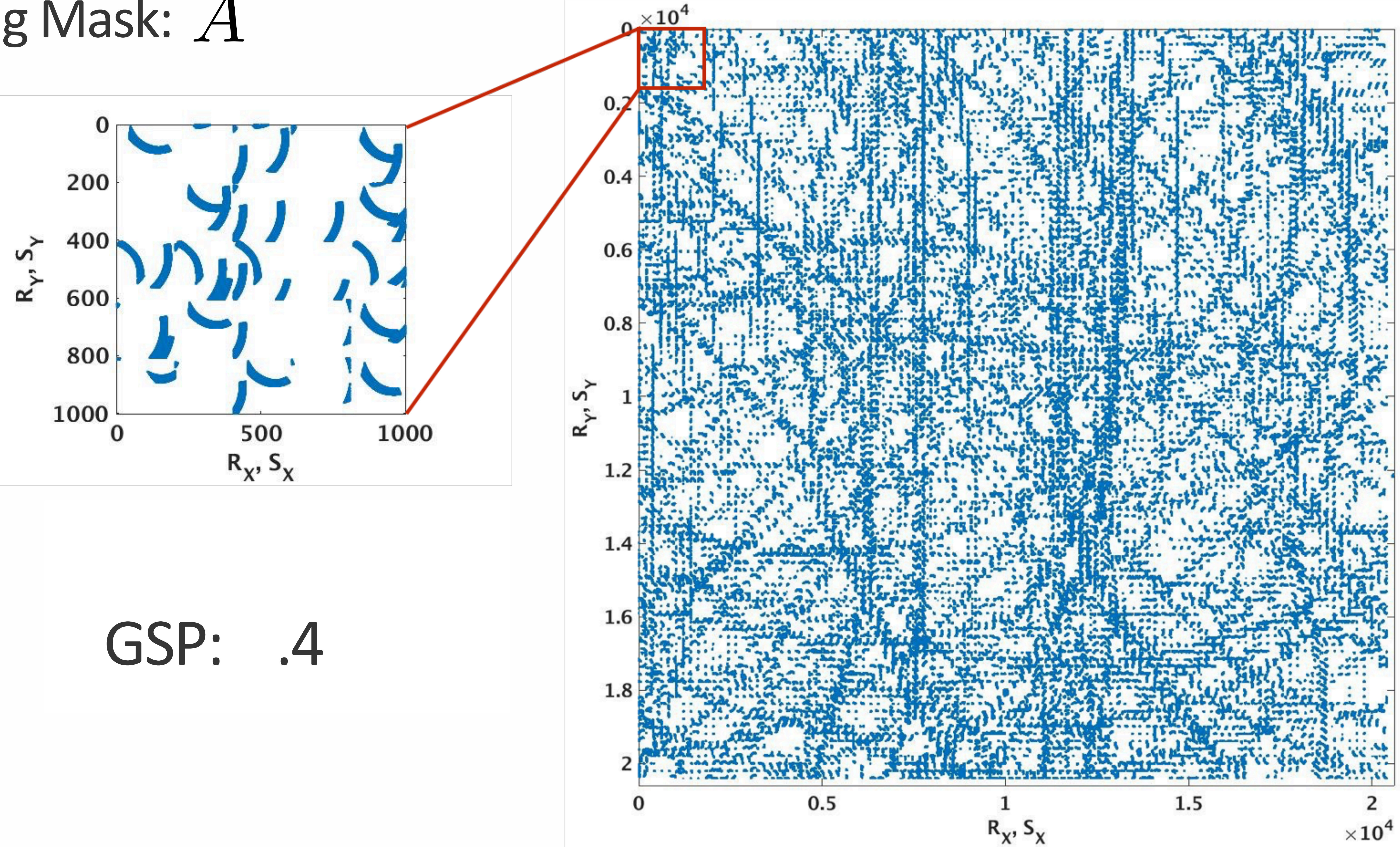
not dense enough: too many sources map to the same grid locations

sweet spot?

too dense: poor sampling ratio

Example 4: Coil Sampling Design

Sampling Mask: A



GSP: .4

Conclusion

- Good understanding of how to subsample

- Simple procedure to quantify acquisition design
  - compute only largest singular value of $A - \frac{|\Omega|}{nm}\mathbf{1}_{n\times m}$
  - useful tool for acquisition design

Future Work

- Generalize theorem to full rank and noisy case: $\mathbf{NN}\left(P_\Omega(X), \epsilon\right)$

- Suggestions? Ideas?

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Thank you for your attention!