Low-rank representation of omnidirectional subsurface extended image volumes

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Seismic imaging

- Forward propagate source wavefields
- Back propagate receiver wavefields
- Cross-correlate wavefields at subsurface locations
Seismic imaging w/ extensions

- Conventional imaging extracts zero-offset section only
- Extension/lifting corresponds to new experiment w/ sources/receivers anywhere in subsurface
- Near isometry
Seismic imaging w/ extensions

- Parametrized by subsurface horizontal offset or angles
- Computed & stored for small subsets of offsets/angles
- Do not explore underlying low-rank structure
Motivation and applications

Form subsurface offset image volumes

Wave-equation migration velocity analysis & continuation

Targeted imaging

Image gather for QC
Extended images in 2D

Marmousi model

Common image point

Source / Receiver location
Extended images in 2D

Common image point gather, 3-30 Hz

$\Delta x$: Horizontal offset

$\Delta z$: Vertical offset
Extended images in 3D

3D BG Compass model

Experimental details
- 1200 source (75 m spacing)
- 2500 receivers (50 m spacing)
- 5-12 Hz
- OBN acquisition
- peak frequency 15 Hz
- One probing vector
- 1500 times faster than conventional method
Extended images in 3D

3D BG Compass model

Common-image point gather
Extended images in 3D

Cross section across common-image point gather
Extended images: difficulties

- use all subsurface offsets (6D volume for 3D model)
- 2-way wave-equation

but…. we can never hope to compute or store such an image volume!

Can we work with the volume *implicitly*?
When the dream comes true

Computation of full-subsurface offset volumes is prohibitively expensive in 3D (storage & computation time)

Past

Can not form full $E$ but action on (random) vectors allows us to get information from all or subsets of subsurface points
When the dream comes true

Computation of full-subsurface offset volumes is prohibitively expensive in 3D (storage & computation time)

Past

Can not form full $E$ but action on (random) vectors allows us to get information from all or subsets of subsurface points

Present

Can not form full $E$ using action on (random) vectors allows us to get information from all or subsets of subsurface points

Efficient ways to extract information from highly compressed image volumes
Extended images via probing
Given two-way wave equations, source and receiver wavefields are defined as

\[ H(\textbf{m}) \textbf{U} = P_s^T \textbf{Q} \]
\[ H(\textbf{m})^* \textbf{V} = P_r^T \textbf{D} \]

where

- \( H(\textbf{m}) \): discretization of the Helmholtz operator
- \( \textbf{Q} \): source
- \( \textbf{D} \): data matrix
- \( P_s, P_r \): samples the wavefield at the source and receiver positions
- \( \textbf{m} \): slowness
Extended images

Organize wavefields in monochromatic data matrices where each column represents a common shot gather

Express image volume tensor for single frequency as a matrix

$$E = VU^*$$
Extended images

In 3D, $E$ is a 6D tensor for each monochromatic slice.

The 4D image volume can be represented as a matrix $n_x \times n_z$. 

### Example

- **Sources**
- **Gridpoints**

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*Thursday, October 5, 2017*
Extended images (Past)

Too expensive to compute (storage and computational time)

Instead, probe volume with tall matrix \( W = [\mathbf{w}_1, \ldots, \mathbf{w}_\ell] \)

\[
\tilde{E} = EW = H^{-*} P_r^T D Q^* P_s H^{-*} W
\]

where \( \mathbf{w}_i = [0, \ldots, 0, 1, 0, \ldots, 0] \) represents single scattering points
Extended images (Present)

Too expensive to compute \((\text{storage and computational time})\)

Instead, probe volume with tall matrix \(W = [w_1, \ldots, w_\ell]\)

\[ \tilde{E} = EW = H^{-*}P_r^T DQ^* P_s H^{-*}W \]

where \(w_i = [0, \ldots, 0, 1, 0, \ldots, 0]\) represents single scattering points

Other choice for \(W\)? And how many vectors are needed?

- random (Gaussian or Rademacher) vectors
- singular vectors from (randomized) SVD
Low-rank representation (5 Hz)

SVD on the monochromatic extended image volume

Model (101x101)  Image Volume (IV)  Singular Values of IV
Rank of the extended image volume

From the formula

\[ \tilde{E} = EW = H^{-*} P_r^\top D Q^* P_s H^{-*} W \]

the rank of \( E \) is given by the rank of the data matrix \( D \).

So, we take \( r \) probing vector \( W = [w_1, \ldots, w_r] \)

— random \(+1/-1\) with probability 0.5
— Gaussian random with 0 mean and variance 1
— our contribution: orthogonal basis of the range of \( E \)
Orthogonal basis of the range of $E$

**Algorithm:**

1. Let $W = [w_1, \ldots, w_r]$ be $r$ Gaussian random vectors
2. Compute $Z = E^*W$ (take only the $r$ first columns of $Q$)
3. Compute $[Q, R] = \text{qr}(Z)$ (orthogonal probing vectors)
4. $E$ is fully described by $Q$ (action of $E$ on $Q$)

**Notation:** $[Q, EQ]$
Randomized SVD algorithm

Algorithm from [1]:

1. \( Y = EW \)  
   probe full extended image volume with virtual sources
2. \([Q, R] = \text{qr}(Y)\)  
   QR factorization
3. \( Z = Q^* E \)  
   probe again with new virtual sources
4. \([U, S, V] = \text{svd}(Z)\)  
   SVD factorization (first few singular values)
5. \( U \leftarrow QU \)  
   update left singular vectors
Randomized SVD algorithm

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   update left singular vectors

Steps 1 to 3 are given by \([Q, EQ]\) (probing only from the right)  
if doing so, step 5 becomes an update of right singular vectors:  
\( V \leftarrow QV \)

Finally  
\[ \vec{E} = EW = USV^* \]
Randomized diagonal extraction

Formula from [2]:

$$\text{diag}(E) \approx \left( \sum_{i=1}^{\ell} w_i \odot (Ew_i) \right) \odot \left( \sum_{i=1}^{\ell} w_i \odot w_i \right)$$

for $W = [w_1, \ldots, w_\ell]$, $+1/-1$ with probability 0.5 random vectors
and $\ell \gg N$ (too expensive)
Randomized diagonal extraction

**Formula from [2]:**

\[
\text{diag}(E) \approx \left( \sum_{i=1}^{\ell} w_i \odot (Ew_i) \right) \odot \left( \sum_{i=1}^{\ell} w_i \odot w_i \right)
\]

for \( W = [w_1, \ldots, w_\ell] \), +1/-1 with probability 0.5 random vectors and \( \ell \gg N \) (too expensive)

With an orthogonal basis \( Q \):

\[
\text{diag}(E) = \sum_{i=1}^{r} q_i \odot (Eq_i)
\]

**Our contribution:** take only \( r \) vectors spanning an orthogonal basis of the range of \( E \)

(exact if \( r \) is the rank of \( E \))

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Orthogonal basis vs random basis

Diagonal extraction of the EIV for different representation (5 Hz, \( r = 15 \))

- full EIV \([E]\)
- with orthogonal basis \([Q, EQ]\)
- with Gaussian basis \([W, EW]\)
Invariance formula for EIVs
Invariance formulation for EIVs...

For monochromatic data and sources

\[ E = H[m]^* P^\top r D Q^* P_s H[m]^* \]

then for two models \( m_1 \) and \( m_2 \)

\[ H[m_1]^* E_1 H[m_1]^* = H[m_2]^* E_2 H[m_2]^* \]
Invariance formulation for EIVs...

For monochromatic data and sources

\[ E = H[m]^* \underbrace{P_r^T DQ^* P_s}_\text{invariant} H[m]^* \]

then for two models \( m_1 \) and \( m_2 \)

\[ H[m_1]^* E_1 H[m_1]^* = H[m_2]^* E_2 H[m_2]^* \]

we deduce \( E_2 \) from \( E_1 \)

\[ E_2 = H[m_2]^* H[m_1]^* E_1 H[m_1]^* H[m_2]^* \]

**Only 2r PDEs solves!**
...from Low-Rank representation

From \([Q_1, E_1 Q_1]\), we get a low-rank formulation for \(E_1\)

\[ E_1 = L_1 R_1^* \]

with \(L_1\) and \(R_1\) two \(N \times r\) matrices given by

\[ L_1 = U_1 \sqrt{S_1} \]
\[ R_1 = V_1 \sqrt{S_1} \]

\([U_1, S_1, V_1]\) from randomized SVD
New extended image

Now we deduce

\[ L_2 = H[m_2]^{-*} H[m_1] L_1 \]

\[ R_2 = H[m_2]^{-1} H[m_1] R_1 \]

to compute

\[ E_2 = L_2 R_2^* \]

with only 2\(r\) extra PDEs solves!
Invariance formula for EIVs (example 1)
Invariance formula for EIVs (example 1)

background model 1  
(correct)

background model 2  
(incorrect)
Invariance formula for EIVs (example 1)

Diagonal extraction of the low-rank EIV (5-30 Hz, step 0.5 Hz, \( r = 15-45 \))

direct reconstruction                     direct reconstruction                   using invariance formula
model 1                                      model 2                            from model 2 to get model 1
from wrong to correct!!!
Invariance formula for EIVs (example 2)
Invariance formula for EIVs (example 2)

background model 1
(correct)

background model 2
(incorrect)
Invariance formula for EIVs (example 2)

Diagonal extraction of the low-rank EIV (5-30 Hz, step 0.5 Hz, r = 15-45)

direct reconstruction                     direct reconstruction                   using invariance formula
model 1                                      model 2                            from model 2 to get model 1

from wrong to correct!!!

Thursday, October 5, 2017
Low-rank formulation for least-squares EIVs
Least-squares extended image volume

**Aim:** build an EIV that fits the data

\[
\min_E \frac{1}{2} \| D - \mathcal{F}(E) \|^2_F
\]

with

\[
\mathcal{F}(E) = P_r H^{-1} E H^{-1} P_s^T Q
\]

**Difficulty:** image volume \( E \) is too large (storage & computational time)

**Our solution:** low-rank factorization of \( E = LR^* \)

with \( L \) and \( R \) two \( N \times r \) matrices
Low-rank least-squares image volume

Least-squares problem for the LR factorization

\[
\min_{L,R} \frac{1}{2} \Phi(L, R) = \frac{1}{2} \|D - \mathcal{F}(LR^*)\|_F^2
\]

for \( E = LR^* \)

Gradients for least-squares formulation

\[
\frac{\partial \Phi}{\partial L}(L, R) = H^{-*} P_r^T (D - \mathcal{F}(LR^*)) Q^* P_s H^{-*} R
\]

\[
\frac{\partial \Phi}{\partial R}(L, R) = H^{-1} P_s^T Q(D - \mathcal{F}(LR^*))^* P_r H^{-1} L
\]

Solution by alternating least-squares on \( L \) and on \( R \)
Full EIV vs low-rank LS image volume

Diagonal extraction of the EIV for frequencies 5-30 Hz, with steps of 0.5 Hz
Full EIV vs low-rank LS image volume

Common image point gather of the EIV for 5-30 Hz, with steps of 0.5 Hz
# Complexity analysis

**Full subsurface offset extended images:**

<table>
<thead>
<tr>
<th></th>
<th># of PDE solves</th>
<th>size of EIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>conventional $E$</td>
<td>2Ns</td>
<td>N x N</td>
</tr>
<tr>
<td>mat-vec $\tilde{E} = EW$</td>
<td>2Nx</td>
<td>N x Nx</td>
</tr>
<tr>
<td>low-rank $L, R$</td>
<td>4r</td>
<td>2N x r</td>
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Ns = # sources  
N = # grid points  
Nx = # probing points  
r = # estimated rank
Complexity analysis

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$Ns = \# \text{ sources}$

$Nx = \# \text{ probing points}$

$N = \# \text{ grid points}$

$r = \# \text{ estimated rank}$

we win when $Nx << Ns$

but usually $Nx \sim N$

( Dirac probing vectors)
## Complexity analysis

*Full subsurface offset extended images:*

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$Ns = \# \text{ sources}$  
$N = \# \text{ grid points}$  
$N_x = \# \text{ probing points}$  
$r = \# \text{ estimated rank}$

We win when $r \ll Ns$  
Okay from low-rank approx. of data matrix!
Observations & Conclusions

Full-offset image volumes can be formed via probing

Form orthonormal basis that spans its range
— low-rank approximation via randomized SVD
— extract (off)diagonals from image volumes

Form least-squares extended images
— via alternating least-squares on low-rank factors

Natural “parametrization” from linear algebra
Acknowledgements

This research was carried out as part of the SINBAD project with the support of the member organizations of the SINBAD Consortium.
The authors wish to acknowledge the SENAI CIMATEC Supercomputing Center for Industrial Innovation, with support from BG Brasil, Shell, and the Brazilian Authority for Oil, Gas and Biofuels (ANP), for the provision and operation of computational facilities and the commitment to invest in Research & Development.
The speaker wishes to acknowledge the Swiss National Science Foundation, which partly funded this work.
Thank you for your attention