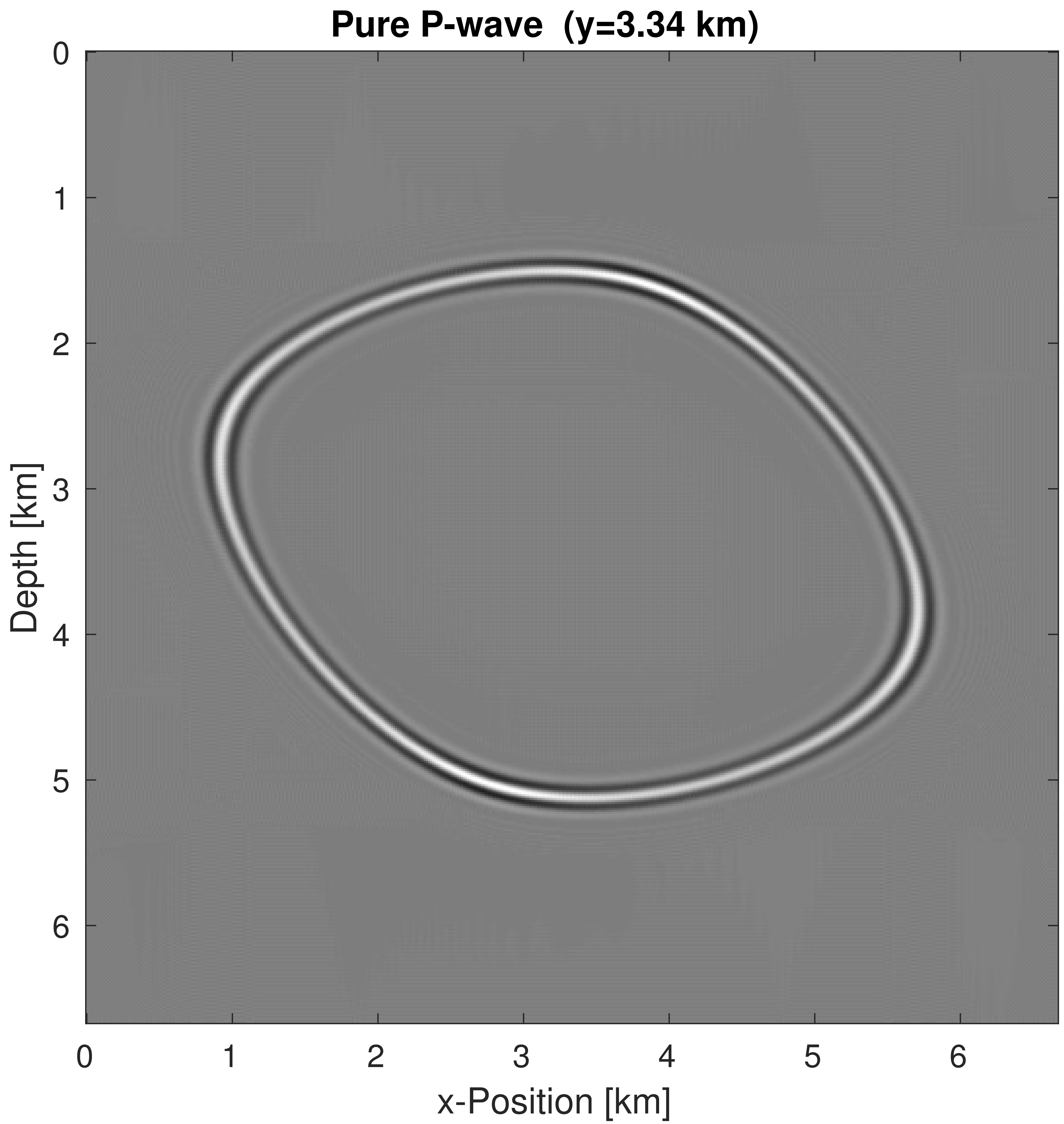
Phase velocity error minimizing scheme for the anisotropic pure P-wave equation

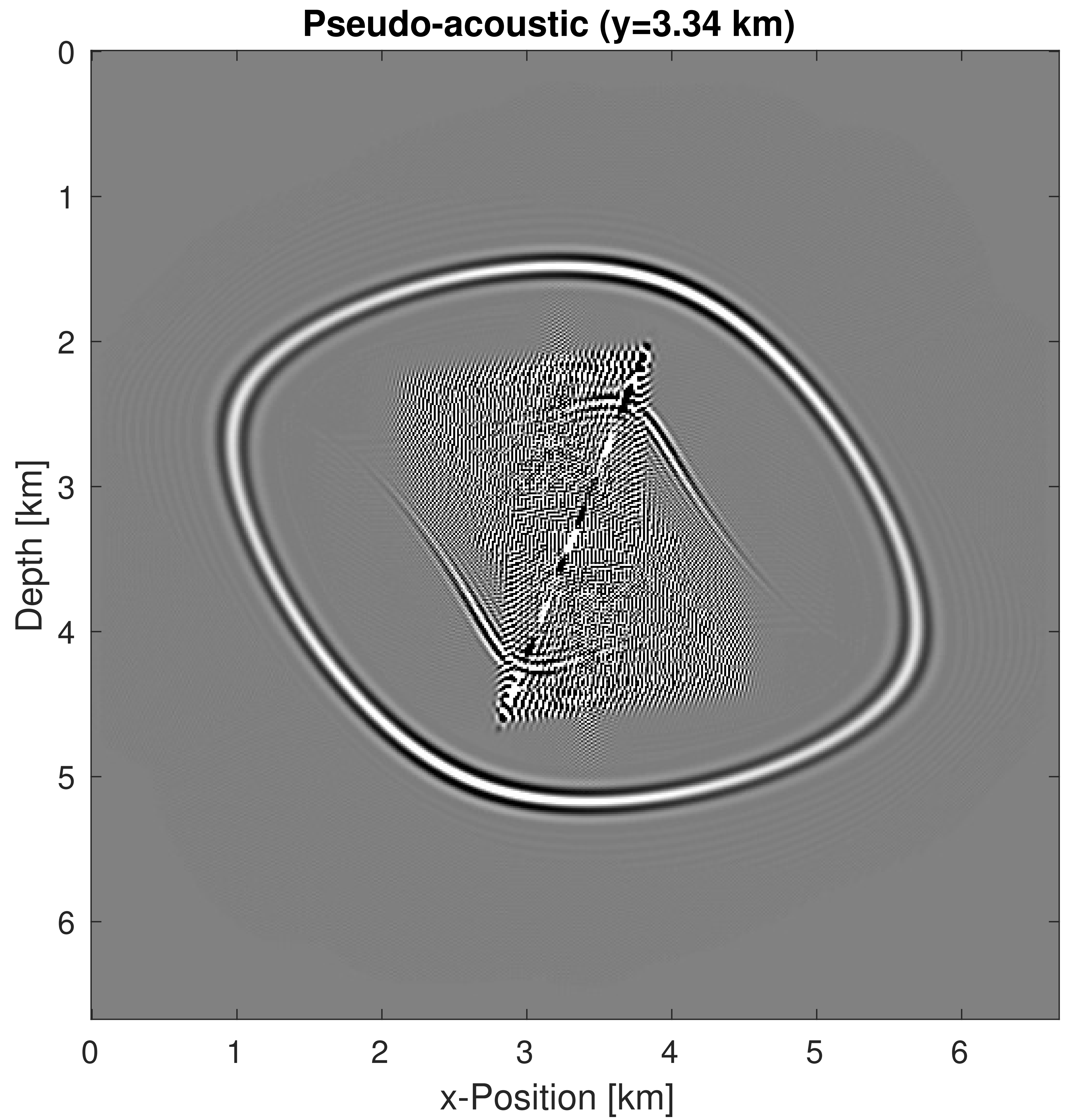
Philipp A. Witte, Christiaan C. Stolk and Felix J. Herrmann

SLIM
University of British Columbia

Motivation



Pure P-wave (y=3.34 km)
Depth [km]
x-Position [km]
0
1
2
3
4
5
6



Pseudo-acoustic (y=3.34 km)
Depth [km]
x-Position [km]
0
1
2
3
4
5
6

Motivation

Modeling in anisotropic acoustic media:

- pseudo-acoustic wave equation
- pure p-wave equation

Pure p-wave equation:

- derived from dispersion relation
- captures kinematics correctly (in theory), but not dynamics
- contains square root of differential operator
- approximation of square root → phase velocity errors

Goal: Reduce phase velocity errors + develop fast modeling scheme

Background

Pseudo-acoustic dispersion relation:

$$-\omega^4 = -\Big[v_{px}^2(\hat{k}_x^2 + \hat{k}_y^2) + v_{pz}^2\hat{k}_z^2\Big]\omega^2 - v_{pz}^2(v_{pn}^2 - v_{px}^2)(\hat{k}_x^2 + \hat{k}_y^2)\hat{k}_z^2$$

Various versions as coupled 2nd order equations (Du et al., 2008; Fowler, 2009; Hestholm et al., 2010)

- Kinematics exact, differ dynamically
- requires $\epsilon \geq \delta$ and contains shear wave artifacts

Alternatively, factorize into pure P- and pure S-wave part

$$-\frac{\omega^2}{v_{pz}^2} = \pm\,\frac{1}{2}\Big[(1+2\epsilon)(\hat{k}_x^2 + \hat{k}_y^2) + \hat{k}_z^2\Big] - \frac{1}{2}\sqrt{\Big[(1+2\epsilon)(\hat{k}_x^2 + \hat{k}_y^2) + \hat{k}_z^2\Big]^2 + 8(\delta - \epsilon)(\hat{k}_x^2 + \hat{k}_y^2)\hat{k}_z^2}$$

Scheme for the pure P-wave dispersion relation

1st order Taylor expansion of square root

$$-\frac{\omega^2}{v_{pz}^2} \approx -\left[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\right] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2}$$

Cannot be turned into a modeling scheme, so simplify further

$$-\frac{\omega^2}{v_{pz}^2} \approx -\left[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\right] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{k^2}$$

This is the most popular pure P-wave equation and used by many authors

(Etgen and Brandsberg-Dahl, 2009; Crawley et al., 2010; Chu et al., 2011; Zhan et al., 2013; etc.)

Scheme for the pure P-wave dispersion relation

Improved versions:

1st order Taylor approximation + geometric series (Chu et al., 2013)

$$-\frac{\omega^2}{v_{pz}^2} \approx -\Big[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\Big] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{k^2}\sum_{m=0}^{M}\left(-2\epsilon\frac{\hat{k}_x^2+\hat{k}_y^2}{k^2}\right)^m$$

2nd order Taylor (with truncation of higher order ϵ, δ terms) (Pestana et al., 2012)

$$-\frac{\omega^2}{v_{pz}^2} \approx -\Big[(1+2\epsilon)(k_x^2+k_y^2)+k_z^2\Big] + \frac{2(\epsilon-\delta)(k_x^2+k_y^2)k_z^2(k_x^2+k_y^2+k_z^2)}{k_z^4+2(1+\frac{\epsilon+\delta}{f})(k_x^2+k_y^2)k_z^2+(1+\frac{4\epsilon}{f})(k_x^2+k_y^2)^2}$$

(VTI only and ϵ, δ constant in denominator)

Scheme for the pure P-wave dispersion relation

Full 2nd order Taylor expansion (just for comparison, cannot be implemented)

$$-\frac{\omega^2}{v_{pz}^2} \approx -\left[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\right] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2} + \frac{\left[2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2\right]^2}{\left[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\right]^3}$$

Other works

- Padé approximations of square root (Schleicher and Costa, 2015)
- Low rank approximations of extrapolation operator (Song et al., 2013; Fomel et al., 2013)
- Optimized low rank approximations (Wu and Alkhalifah, 2014)

Phase velocity error minimizing scheme

Instead of Taylor expansion, expand as generic polynomial series

$$-\frac{\omega^2}{v_{pz}^2} \approx a_1 k^2 + a_2\left((\hat{k}_x^2 + \hat{k}_y^2) - \hat{k}_z^2\right) + a_3 \frac{\left((\hat{k}_x^2 + \hat{k}_y^2) - \hat{k}_z^2\right)^2}{k^2} + a_4 \frac{\left((\hat{k}_x^2 + \hat{k}_y^2) - \hat{k}_z^2\right)^3}{(k^2)^2}$$

with $k^2 = \hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2$

→ Determine (spatially dependent) coefficients $a_j(x, y, z)$ such that the equation has a minimal relative phase velocity error

Phase velocity error minimizing scheme

Define: $\hat{k}_z^2(\alpha) = \cos^2\alpha$ and $\hat{k}_x^2(\alpha) + \hat{k}_y^2(\alpha) = \hat{k}_r^2(\alpha) = \sin^2\alpha$

True phase velocity as function of phase angle $\alpha \in [0, \frac{\pi}{2}]$ and $\epsilon, \delta = \text{const.}$

$$v_{true}^2(\alpha) = \frac{1}{2}\left[(1+2\epsilon)\hat{k}_r^2(\alpha) + \hat{k}_z^2(\alpha)\right] + \frac{1}{2}\sqrt{\left[(1+2\epsilon)\hat{k}_r^2(\alpha) + \hat{k}_z^2(\alpha)\right]^2 + 8(\delta-\epsilon)\hat{k}_r^2(\alpha)\hat{k}_z^2(\alpha)}$$

Approximate phase velocity

$$v_{approx}^2(\alpha, a_1, a_2, a_3, a_4) = a_1 + a_2\left[\hat{k}_r^2(\alpha) - \hat{k}_z^2(\alpha)\right] + a_3\left[\hat{k}_r^2(\alpha) - \hat{k}_z^2(\alpha)\right]^2 + a_4\left[\hat{k}_r^2(\alpha) - \hat{k}_z^2(\alpha)\right]^3$$

Phase velocity error minimizing scheme

Objective function: relative phase velocity error

$$E(a_1, a_2, a_3, a_4) = \sqrt{\frac{v^2_{approx}(\alpha, a_1, a_2, a_3, a_4)}{v^2_{true}(\alpha)}} - 1$$

Approximate $\sqrt{(x)} - 1 \approx \frac{1}{2}(x - 1)$ and solve linear LS problem

$$a_j = \underset{a_j}{\operatorname{argmin}} \int_0^{\frac{\pi}{2}} \left\| \frac{1}{2}\left(\frac{v^2_{approx}(\alpha, a_1, a_2, a_3, a_4)}{v^2_{true}(\alpha)} - 1\right) \right\| d\alpha$$

Phase velocity error minimizing scheme

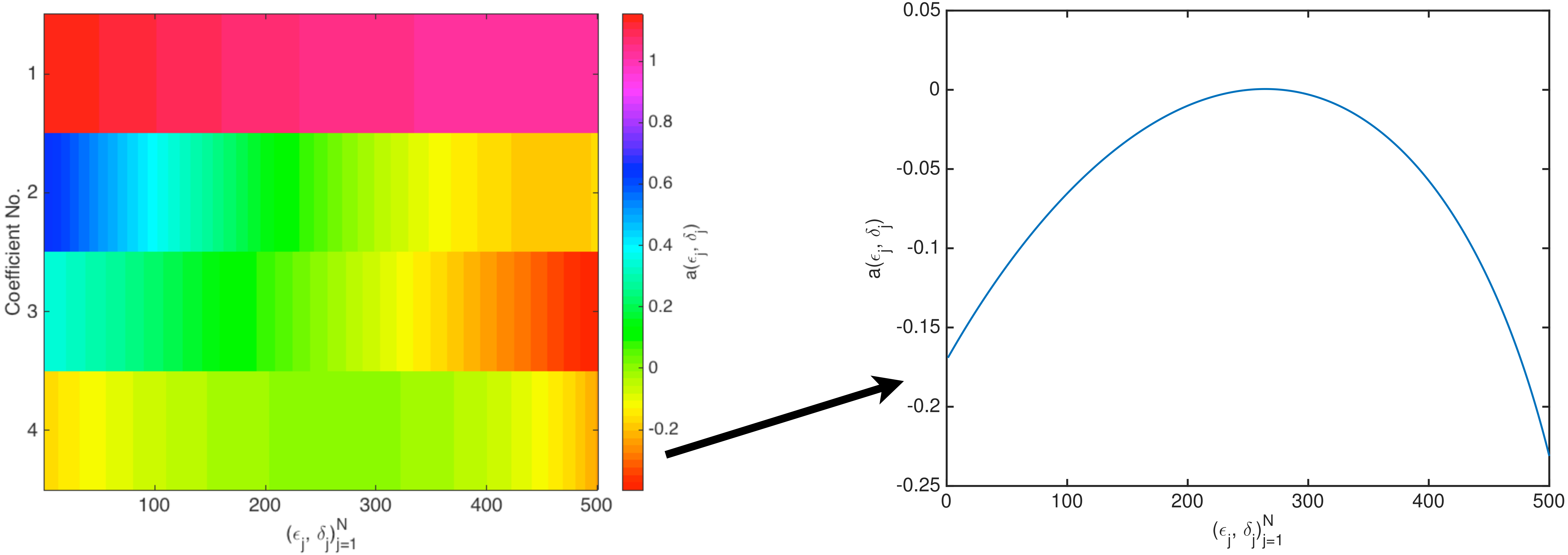
Coefficients are function of Thomsen parameters:

- so far $\epsilon = \text{const.}, \ \delta = \text{const.}$ for one set of $a_j(\epsilon, \delta)$
- requires recalculation of $a_j(\epsilon, \delta)$ for every new ϵ, δ combination

Assumption: the coefficients $a_j(\epsilon, \delta)$ vary smoothly

- plot coefficients for a range ϵ, δ values

Phase slowness error minimizing scheme



Interpolate $a_j(\epsilon, \delta)$ using Legendre polynomials up to order n

$$a_j(\epsilon, \delta) = \sum_{k=1}^{n} \sum_{l=1}^{n} p_{jkl} L_k(\epsilon) L_l(\delta)$$

Phase velocity error minimizing scheme

Replace $a_j(\epsilon, \delta)$ in the phase velocity expression

$$
\begin{aligned}
v^2_{approx}(\alpha, \epsilon, \delta, p_{jkl}) = & p_{1kl} L_k(\epsilon) L_l(\delta) k^2 + p_{2kl} L_k(\epsilon) L_l(\delta) \left[\hat{k}_r^2(\alpha) - \hat{k_z^2}(\alpha) \right] + \\
& p_{3kl} L_k(\epsilon) L_l(\delta) \left[\hat{k}_r^2(\alpha) - \hat{k_z^2}(\alpha) \right]^2 + p_{4kl} L_k(\epsilon) L_l(\delta) \left[\hat{k}_r^2(\alpha) - \hat{k_z^2}(\alpha) \right]^3
\end{aligned}
$$

and solve

$$
p_{jkl} = \underset{p_{jkl}}{\operatorname{argmin}} \int_{\epsilon_{min}}^{\epsilon_{max}} \int_{\delta_{min}}^{\delta_{max}} \int_{0}^{\frac{\pi}{2}} \left\| \frac{1}{2} \left(\frac{v^2_{approx}(\alpha, \epsilon, \delta, p_{jkl})}{v^2_{true}(\alpha, \epsilon, \delta)} - 1 \right) \right\| d\alpha \, d\delta \, d\epsilon
$$

Phase velocity error analysis

Calculate coefficients of optimized scheme:

- Define range of Thomson parameters and number of samples
- E.g. 20 values of $\epsilon \in [0, 0.5]$ and $\delta \in [-0.1, 0.4]$
- Define range and number of samples of phase angle
- E.g. 20 values of $\alpha \in [0, \pi/2]$
- Sampling can be uniform or from any type of other distribution
- Set up linear system and solve directly

Size of linear system is small

- Legendre polynomials up to order 3 → 64 unknowns p_{jkl}
- 20 values of each ϵ, δ, α → 20^3 observations

Phase velocity error analysis

- Plot relative error as function of phase angle for arbitrary combination of ϵ, δ

Phase velocity error analysis

Phase errors of different schemes

1. Optimized scheme

$$-\frac{\omega^2}{v_{pz}^2} \approx a_1 k^2 + a_2\Big((\hat{k}_x^2+\hat{k}_y^2)-\hat{k}_z^2\Big) + a_3\frac{\Big((\hat{k}_x^2+\hat{k}_y^2)-\hat{k}_z^2\Big)^2}{k^2} + a_4\frac{\Big((\hat{k}_x^2+\hat{k}_y^2)-\hat{k}_z^2\Big)^3}{(k^2)^2} \qquad a_j = p_{jkl}L_k(\epsilon)L_l(\delta)$$

2. 1st order Taylor for M=0,1,2

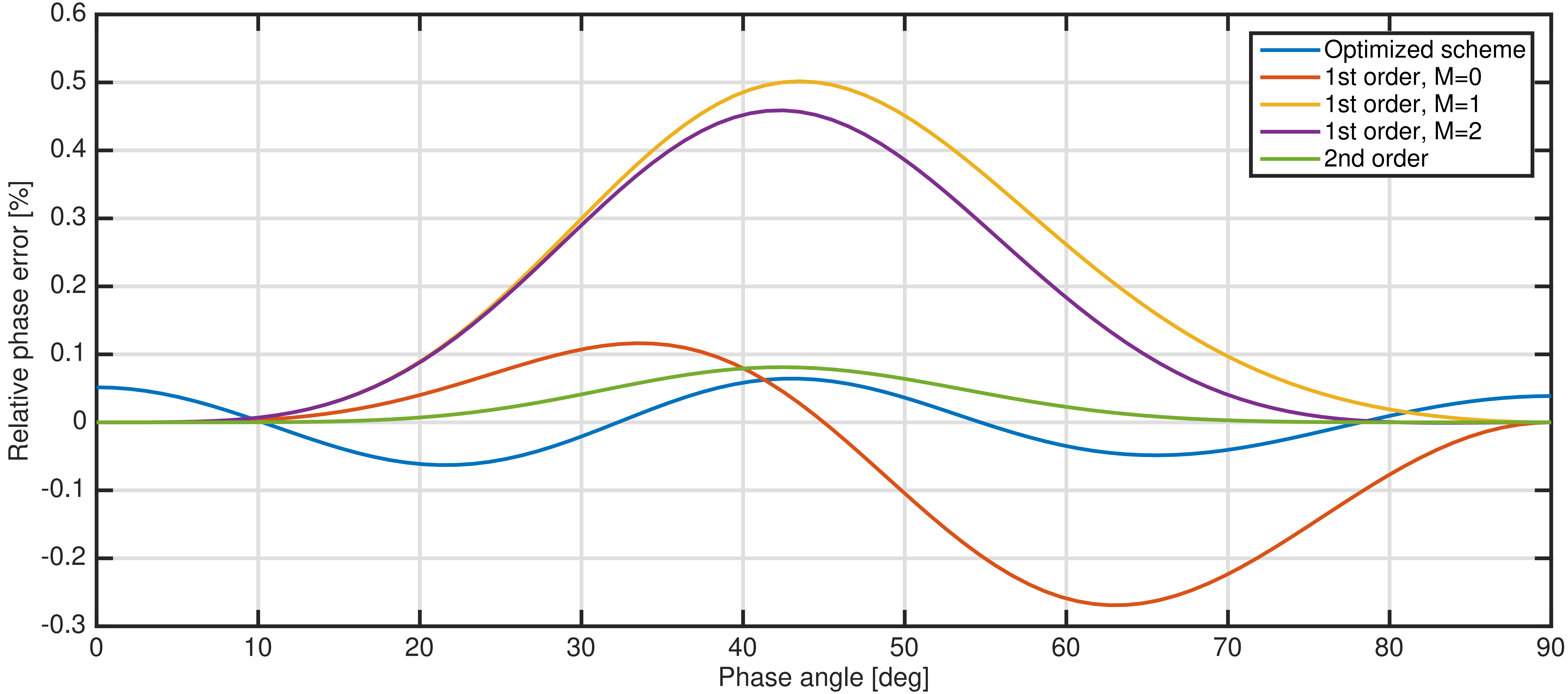
$$-\frac{\omega^2}{v_{pz}^2} \approx -\Big[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\Big] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{k^2}\sum_{m=0}^{M}\Big(-2\epsilon\frac{\hat{k}_x^2+\hat{k}_y^2}{k^2}\Big)^m$$

3. 2nd order Taylor

$$-\frac{\omega^2}{v_{pz}^2} \approx -\Big[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\Big] - \frac{2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2}{(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2} + \frac{\Big[2(\delta-\epsilon)(\hat{k}_x^2+\hat{k}_y^2)\hat{k}_z^2\Big]^2}{\Big[(1+2\epsilon)(\hat{k}_x^2+\hat{k}_y^2)+\hat{k}_z^2\Big]^3}$$

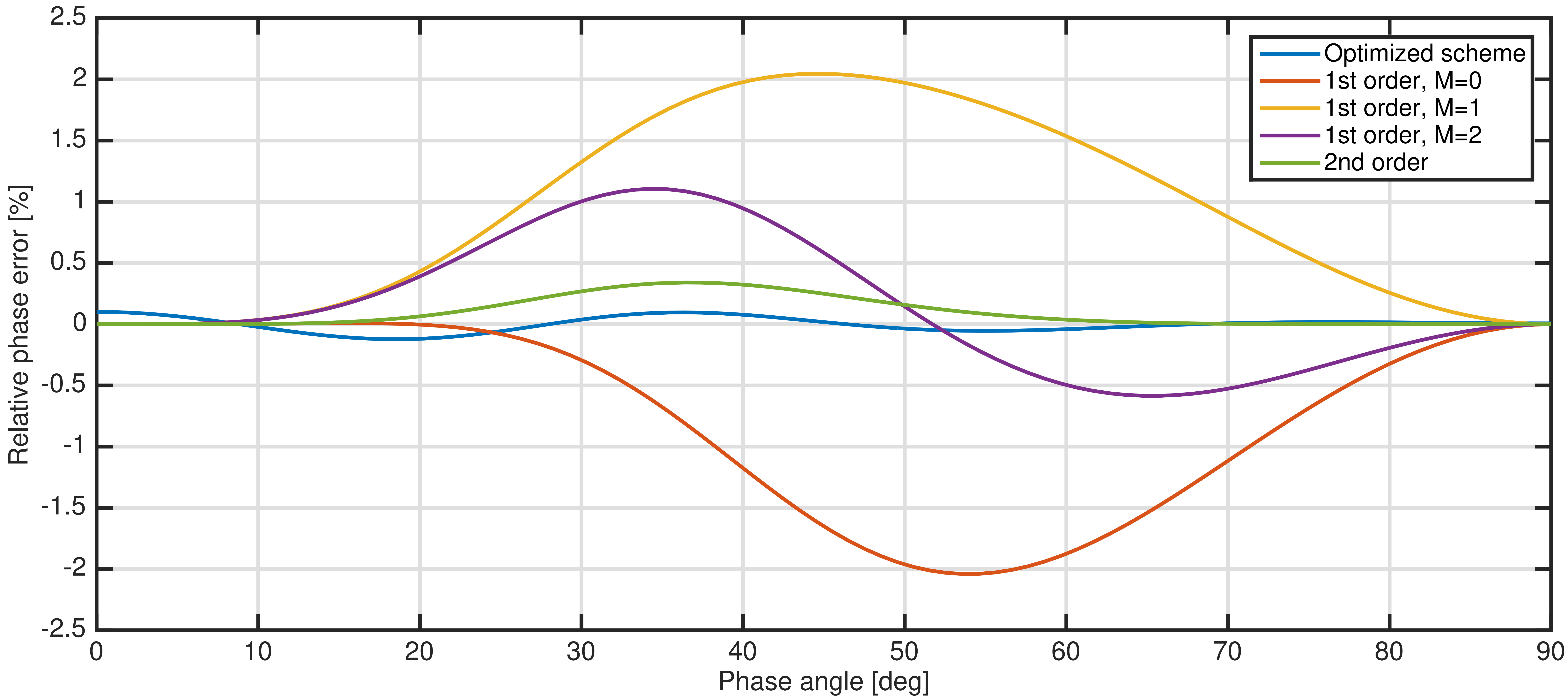
Phase velocity error analysis

$$\epsilon = 0.1, \delta = -0.1$$



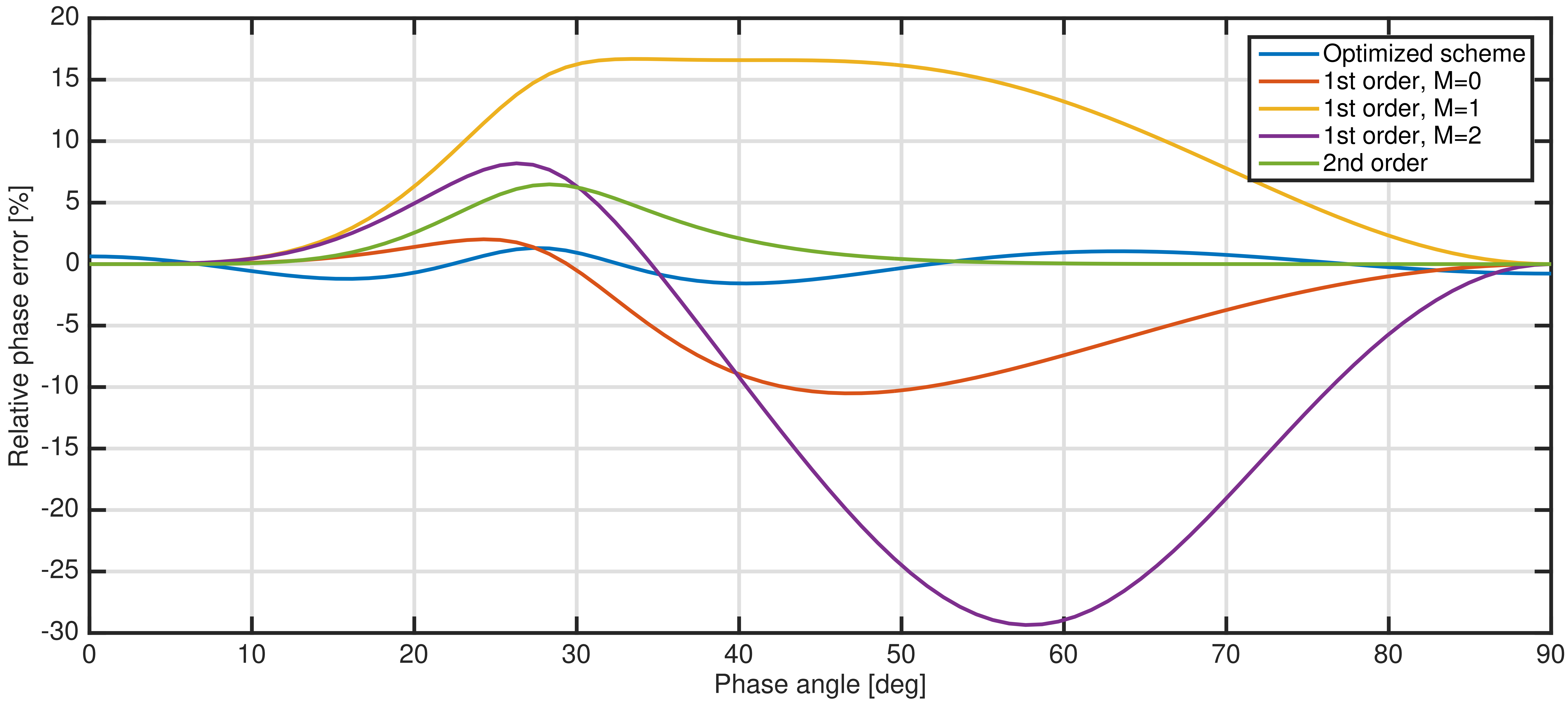
Phase velocity error analysis

$$\epsilon = 0.4, \delta = -0.05$$



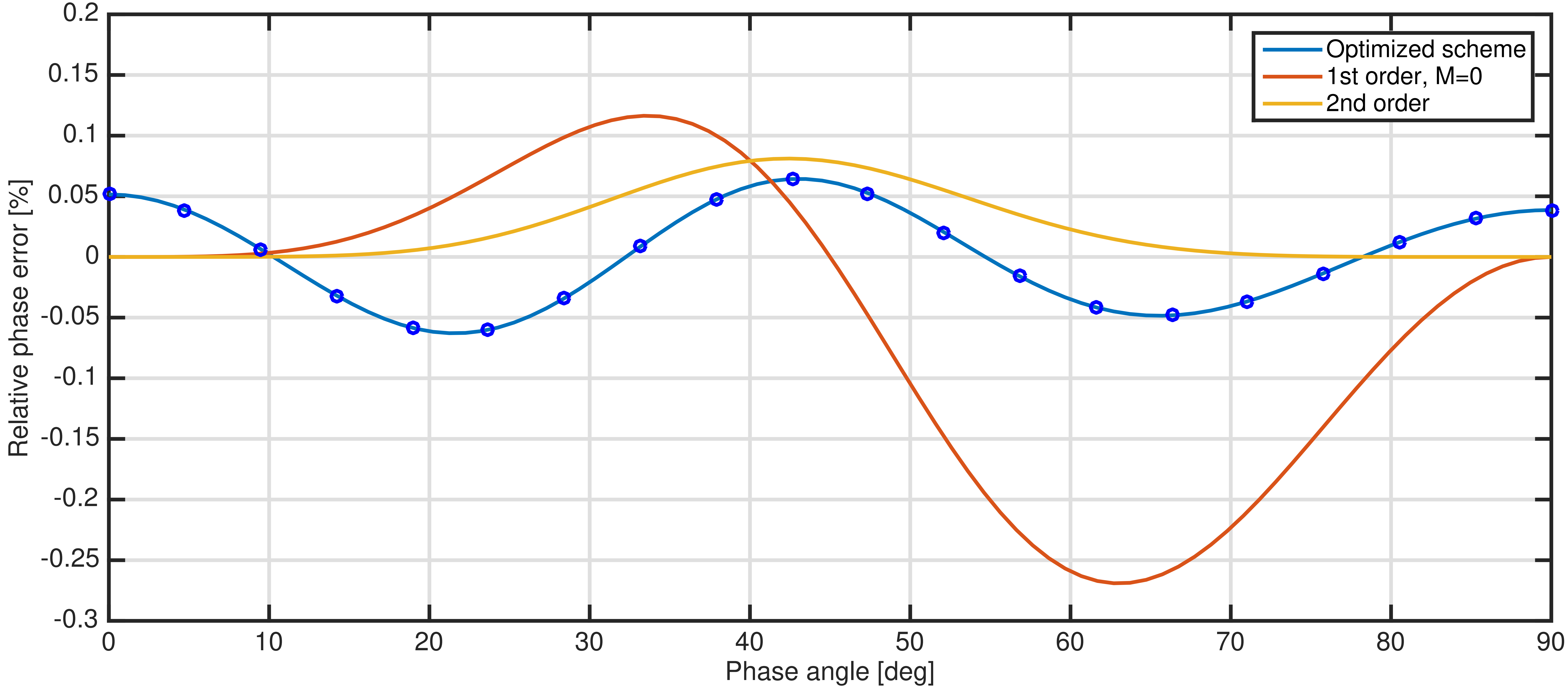
Phase velocity error analysis

$\epsilon = 1.22, \delta = -0.388$ (Biotite Crystal)



Phase velocity error analysis

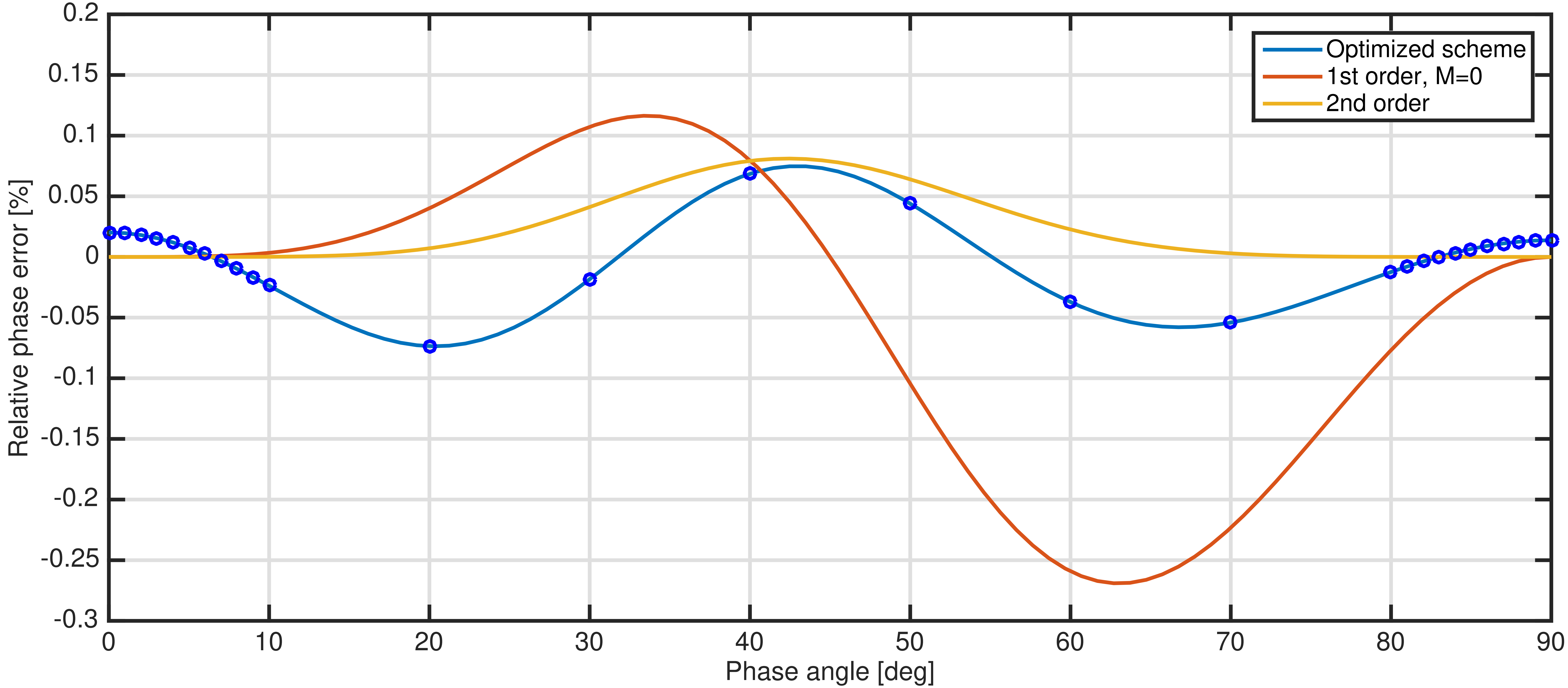
$$\epsilon = 0.1, \delta = -0.1$$



Uniform sampling of phase angles

Phase velocity error analysis

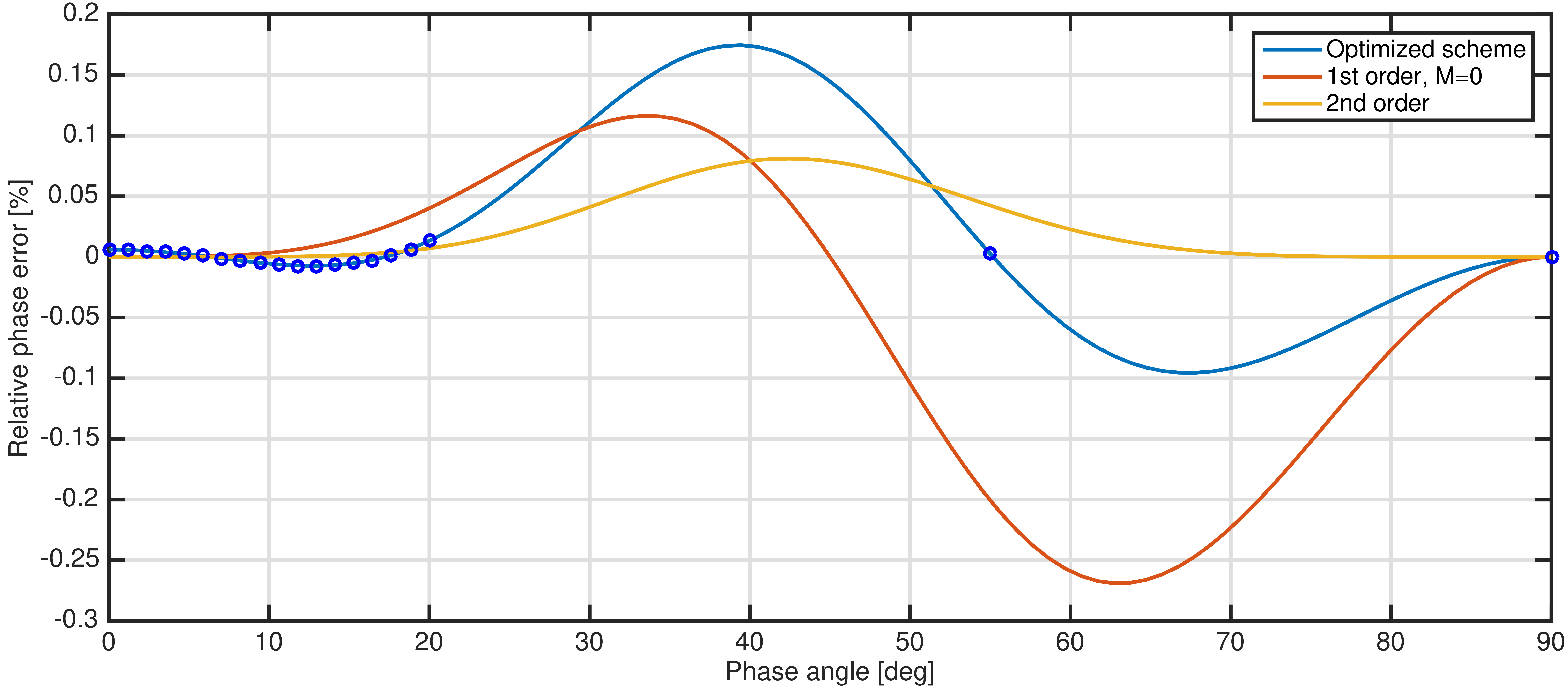
$$\epsilon = 0.1, \delta = -0.1$$



Non-uniform sampling of phase angles

Phase velocity error analysis

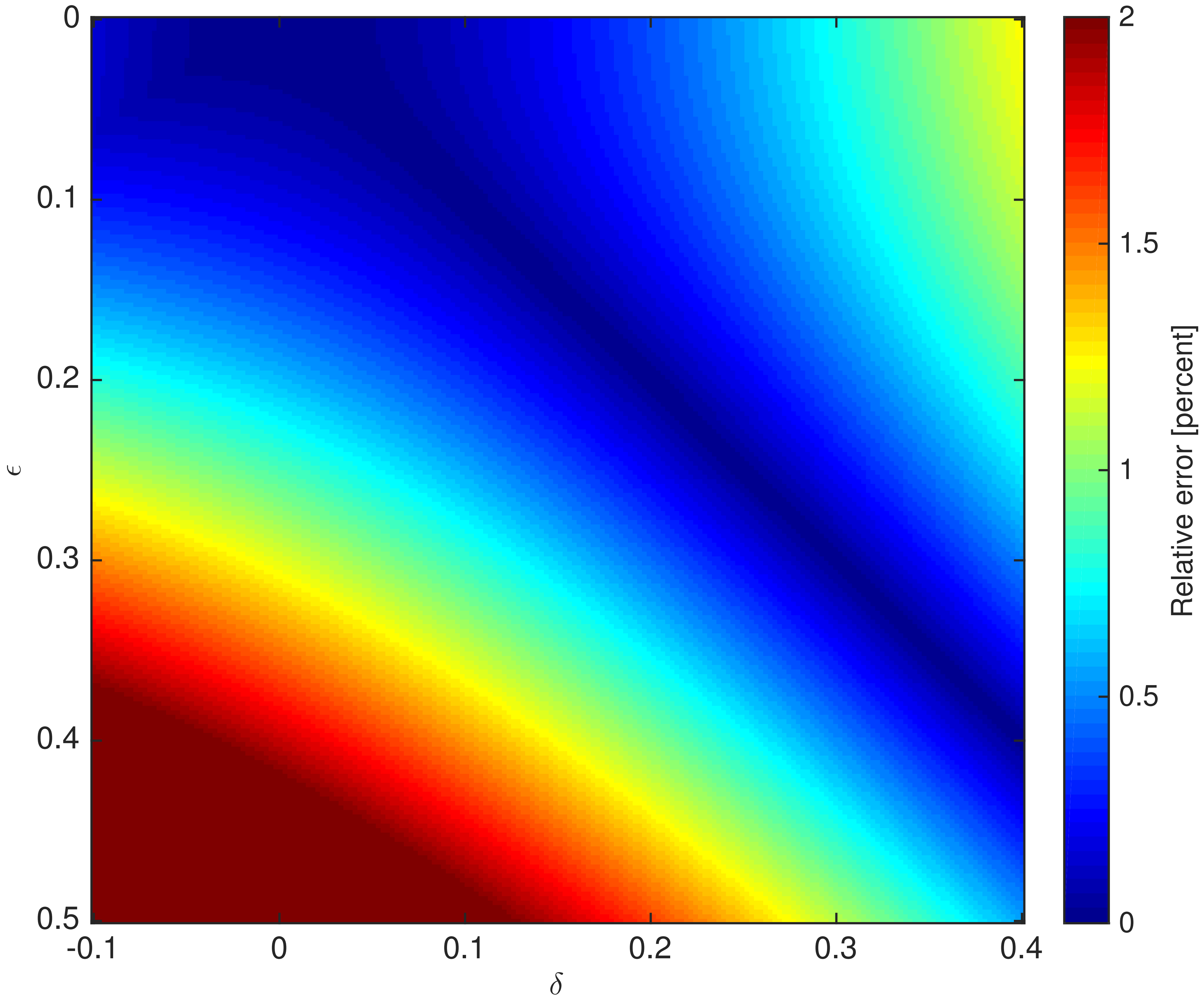
$$\epsilon = 0.1, \delta = -0.1$$



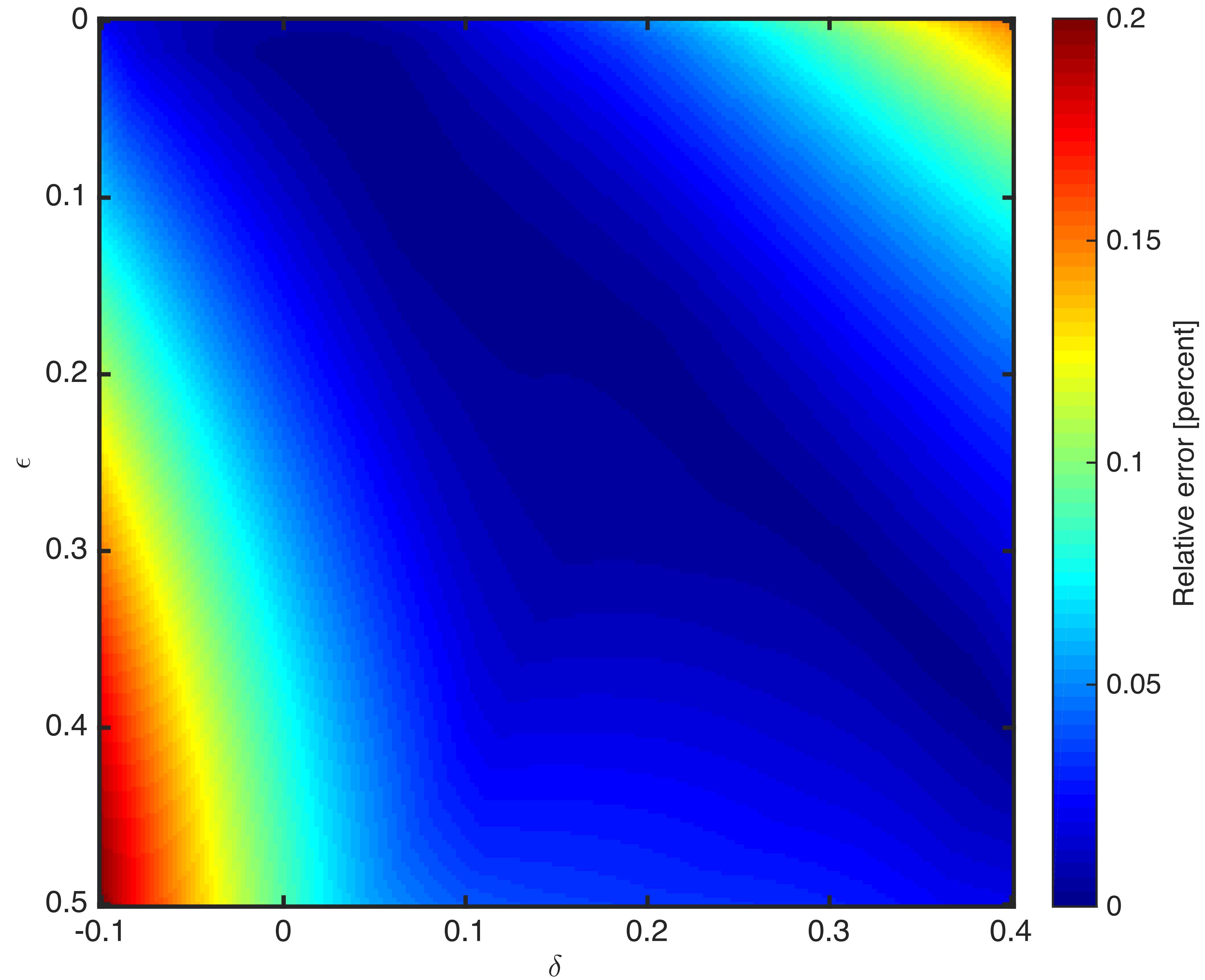
Non-uniform sampling of phase angles

Phase velocity error analysis

Absolute maximum error over a range of Thomsen parameters: $E(\epsilon, \delta) = \max\left(|E(\epsilon, \delta, \alpha)|\right)$



Standard pure P-wave equation (1st order Taylor)



Optimized pure P-wave equation

Forward modeling scheme VTI

VTI medium: direct translation from dispersion relation to modeling scheme

$$
\begin{aligned}
\frac{1}{v_{pz}^2}\frac{\partial^2 P}{\partial t^2} =& p_{1kl} L_k(\epsilon) L_l(\delta) \mathcal{F}^{-1}\left\{k^2 \bar{P}\right\} \\
&+ p_{2kl} L_k(\epsilon) L_l(\delta) \mathcal{F}^{-1}\left\{(k_x^2 + k_y^2 - k_z^2)\bar{P}\right\} \\
&+ p_{3kl} L_k(\epsilon) L_l(\delta) \mathcal{F}^{-1}\left\{\frac{(k_x^2 + k_y^2 - k_z^2)^2}{k^2}\bar{P}\right\} \\
&+ p_{4kl} L_k(\epsilon) L_l(\delta) \mathcal{F}^{-1}\left\{\frac{(k_x^2 + k_y^2 - k_z^2)^3}{k^4}\bar{P}\right\} + S(t),
\end{aligned}
$$

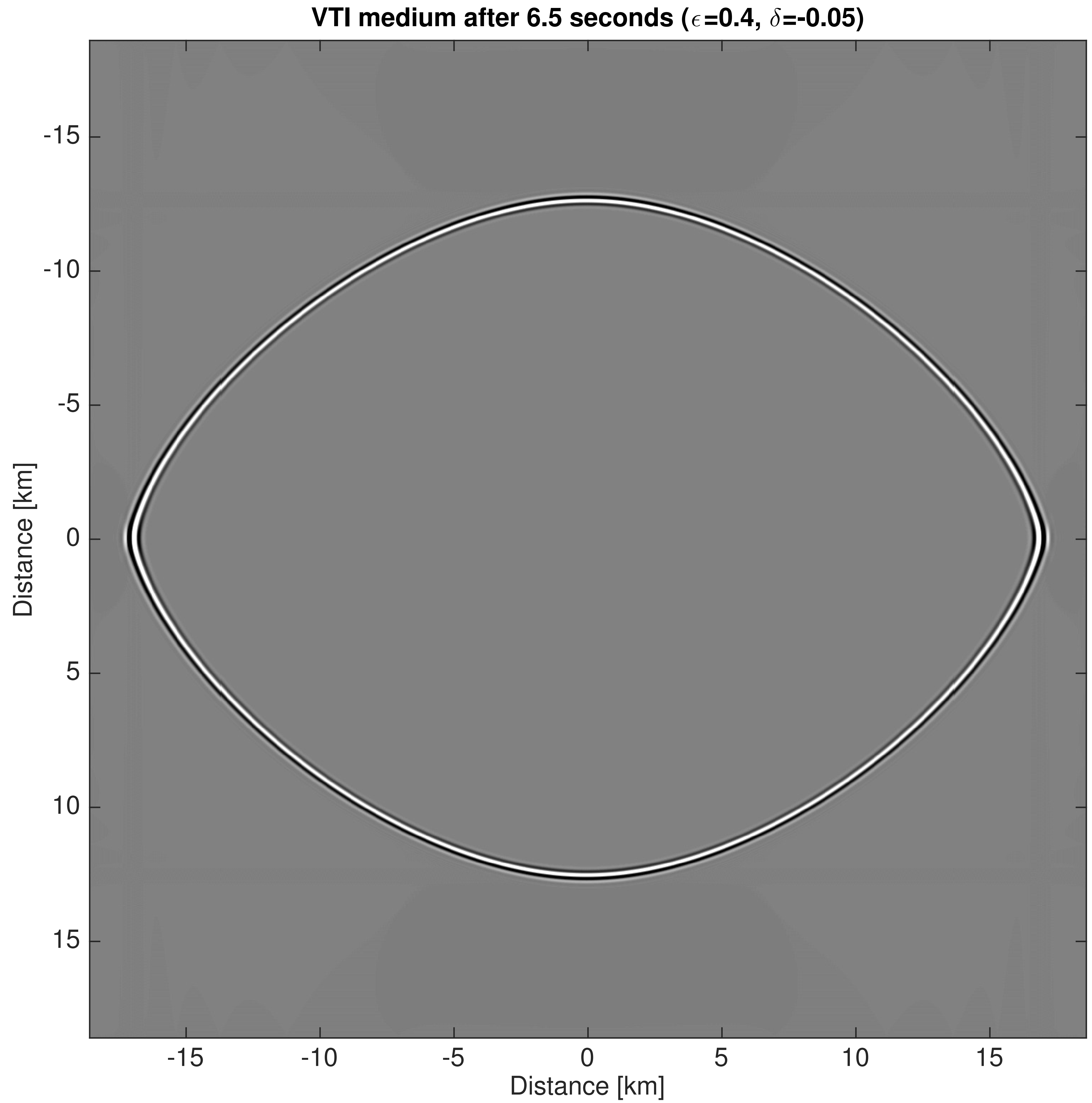
P: wavefield in spatial domain

$\bar{P}$: wavefield in wavenumber domain

$\mathcal{F}^{-1}$: inverse Fourier transform

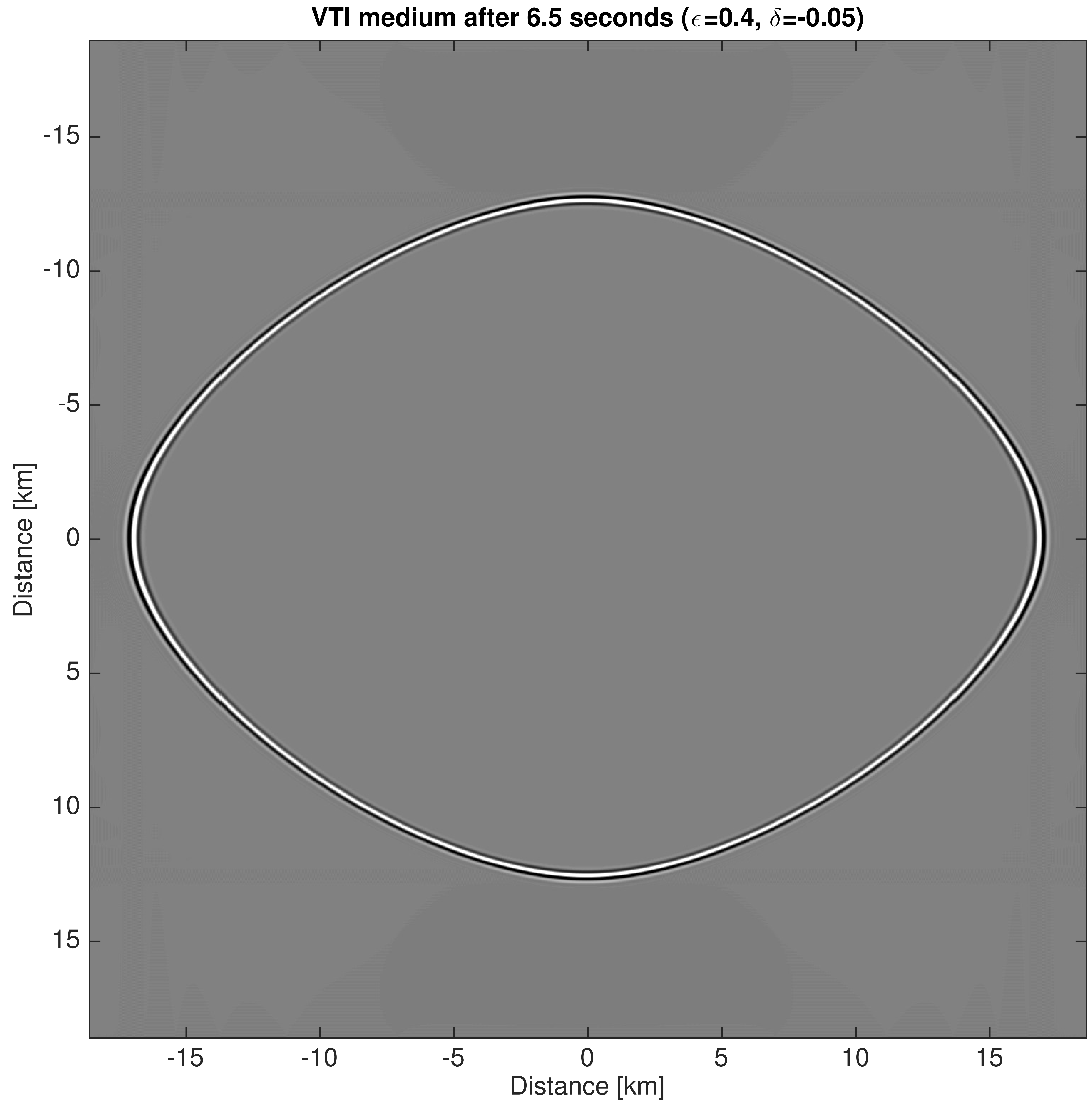
$S(t)$: source function

Forward modeling scheme VTI



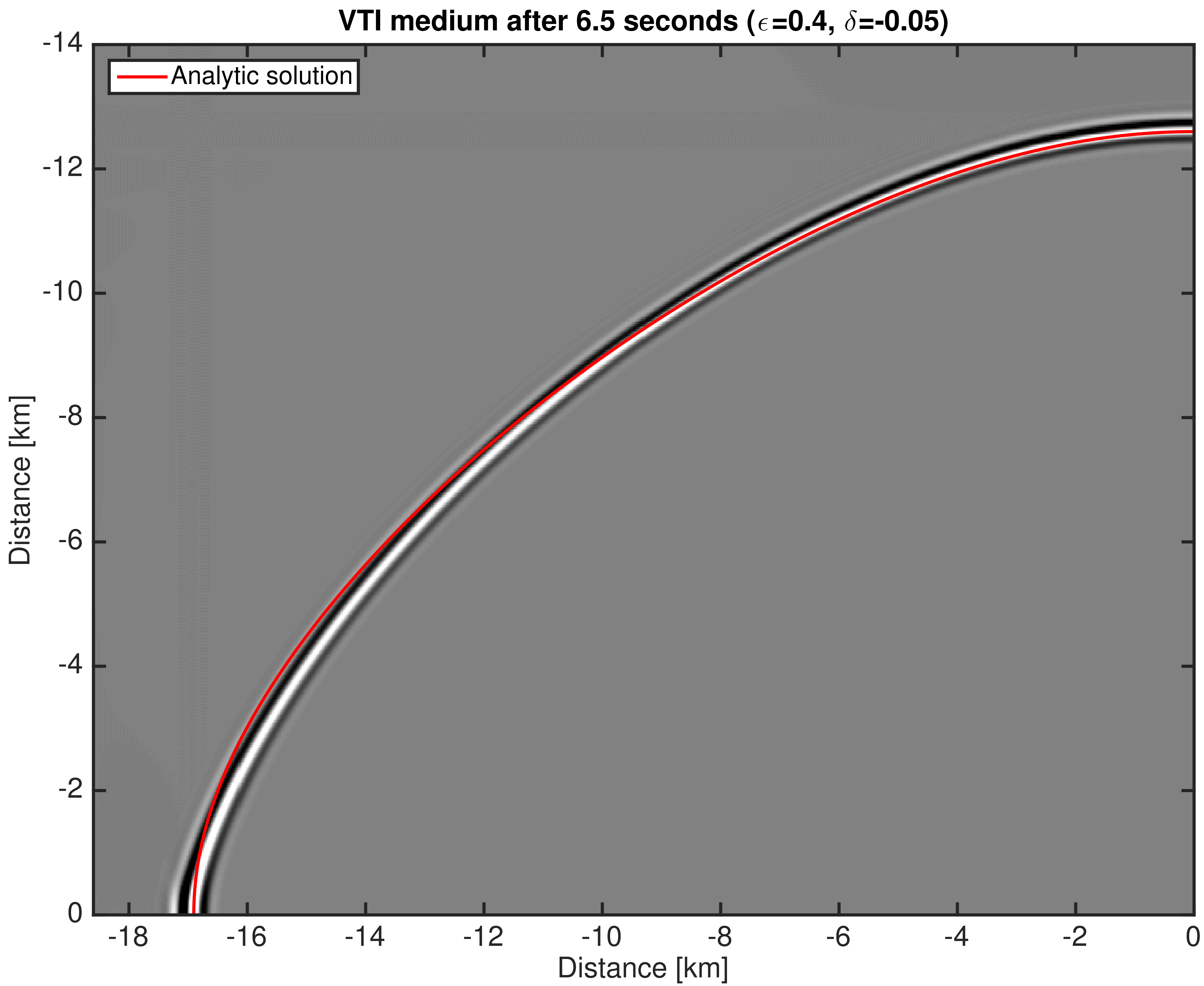
Standard pure P-wave scheme

Forward modeling scheme VTI



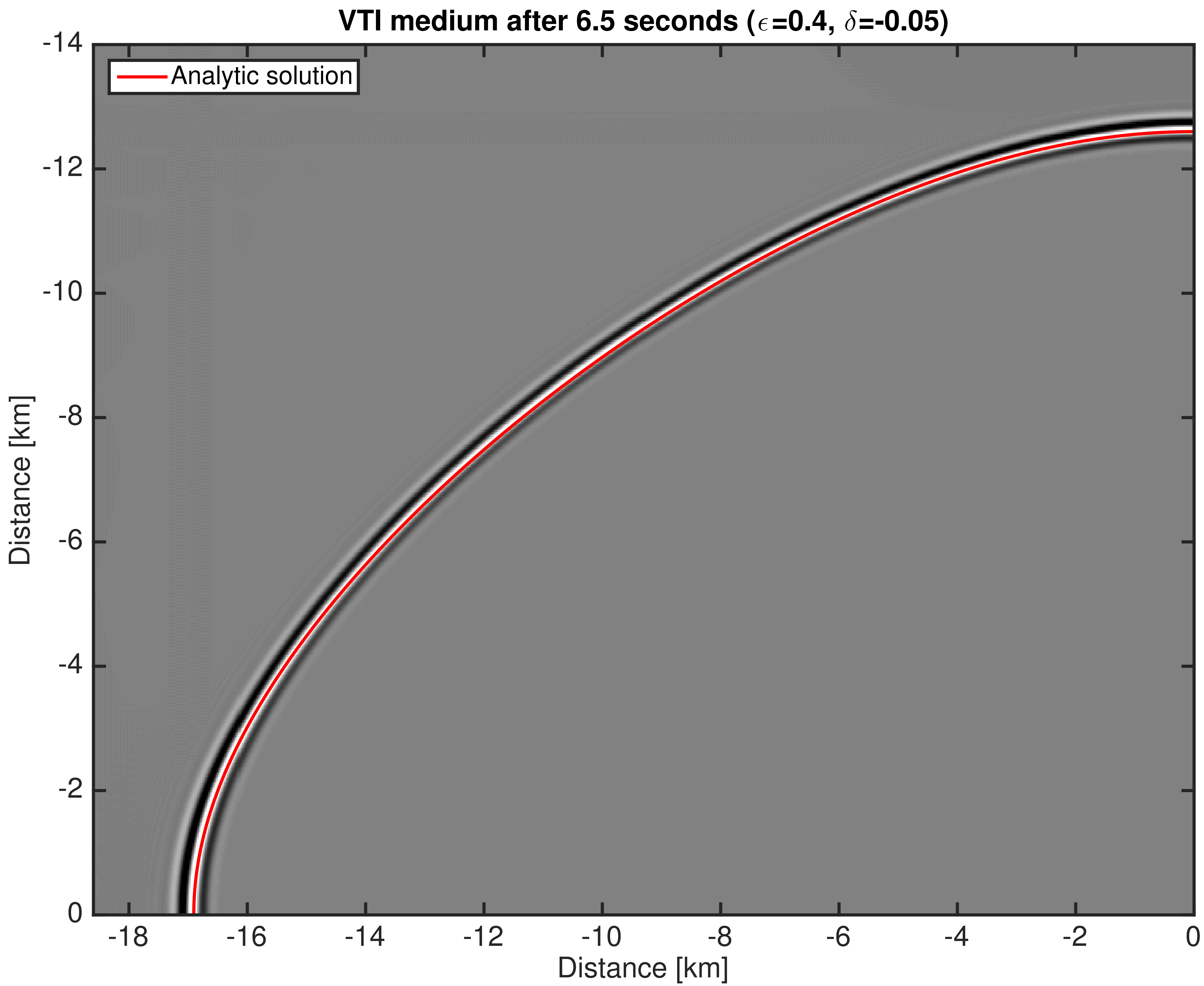
Optimized pure P-wave scheme

Forward modeling scheme VTI



Standard pure P-wave scheme

Forward modeling scheme VTI



Optimized pure P-wave scheme

Forward modeling scheme TTI

TTI medium:

- replace k_{xyz} by rotated wavenumber vectors, e.g.

$$\hat{k}_x = k_x \cos\theta \cos\phi + k_y \cos\theta \sin\phi - k_z sin\theta$$

In the optimized scheme, this would increase the number of terms (FFTs) enormously

- but: special structure of equation allows an efficient modeling scheme

Forward modeling scheme TTI

Recursive computations of terms:

$$u_1 = \left\{ \mathcal{F}^{-1} k^2 \right\} \bar{P}$$

$$u_2 = \Big\{ c_1 \mathcal{F}^{-1} k_x^2 + c_2 \mathcal{F}^{-1} k_y^2 + c_3 \mathcal{F}^{-1} k_z^2 + c_4 \mathcal{F}^{-1} k_x k_y + c_5 \mathcal{F}^{-1} k_x k_z + c_6 \mathcal{F}^{-1} k_y k_z \Big\} \bar{P}$$

$$c_1 = 1 - 2 \sin^2 \theta \cos^2 \phi \text{ etc.}$$

$$u_3 = \Big\{ c_1 \mathcal{F}^{-1} \frac{k_x^2}{k^2} + c_2 \mathcal{F}^{-1} \frac{k_y^2}{k^2} + c_3 \mathcal{F}^{-1} \frac{k_z^2}{k^2} + c_4 \mathcal{F}^{-1} \frac{k_x k_y}{k^2} + c_5 \mathcal{F}^{-1} \frac{k_x k_z}{k^2} + c_6 \mathcal{F}^{-1} \frac{k_y k_z}{k^2} \Big\} \mathcal{F} u_2$$

$$u_4 = ...$$

Followed by a weighted summing:

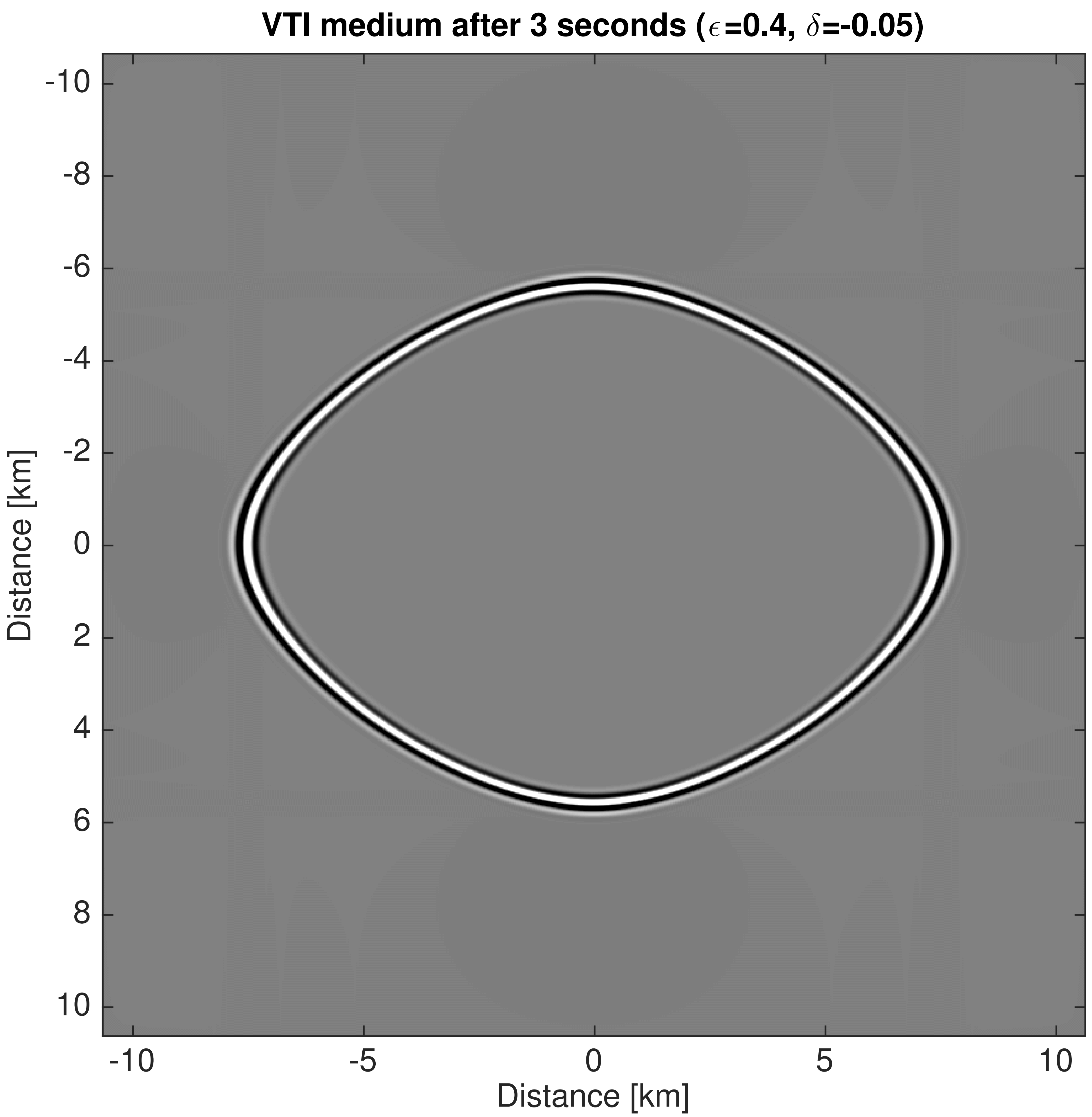
$$\frac{1}{v_{pz}^2} \frac{\partial^2 P}{\partial t^2} = p_{1kl} L_k(\epsilon) L_l(\delta) u_1 + p_{2kl} L_k(\epsilon) L_l(\delta) u_2 + p_{3kl} L_k(\epsilon) L_l(\delta) u_3 + p_{4kl} L_k(\epsilon) L_l(\delta) u_4 + S(t)$$

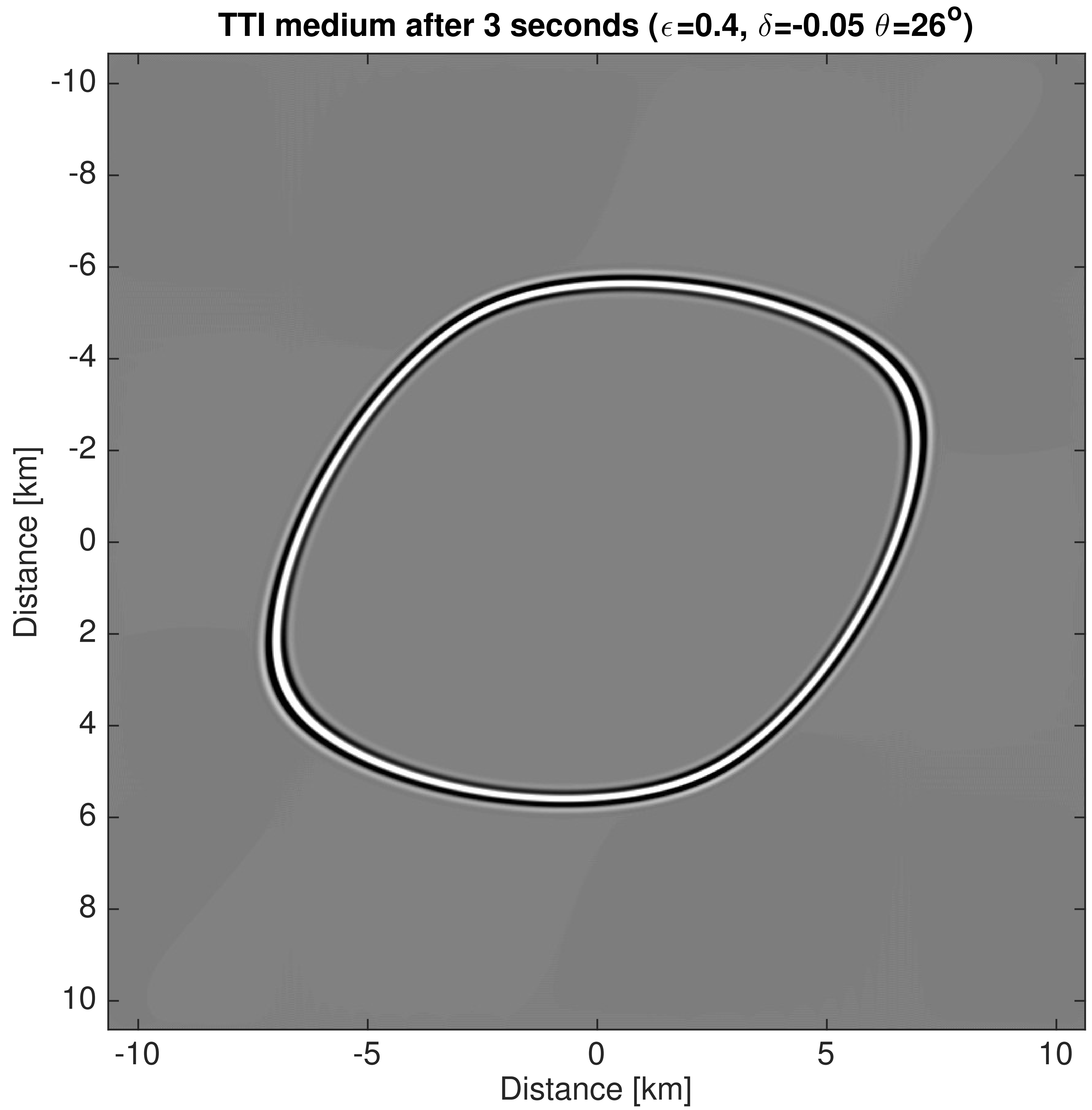
Forward modeling scheme TTI

Overview of computational cost

Method	VTI (2D)	VTI (3D)	TTI (2D)	TTI (3D)
1st order Taylor (standard pure P-wave eq.)	4	4	9	22
1st order Taylor + geometric series (M=1)	4	4	11	35
1st order Taylor + geometric series (M=2)	5	5	13	52
Optimized pure P-wave equation	5	5	13	22

Forward modeling scheme TTI





Forward modeling scheme TTI

3D modeling example:

- 500 cube
- 1 s modeling time

  $v_p = 2\ \mathrm{km/s}$

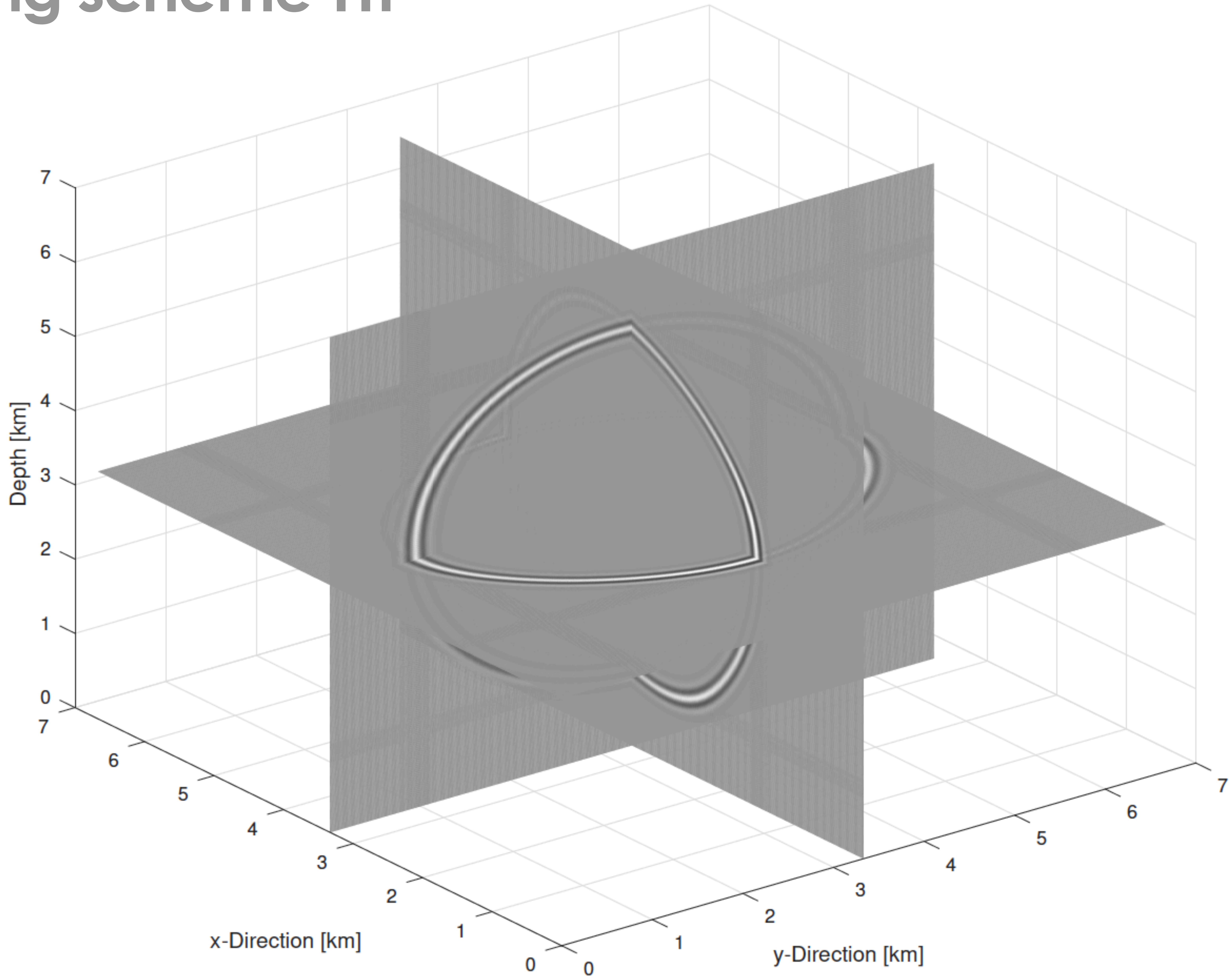
  $\epsilon = 0.4$

  $\delta = -0.1$

  $\theta = 25^o$

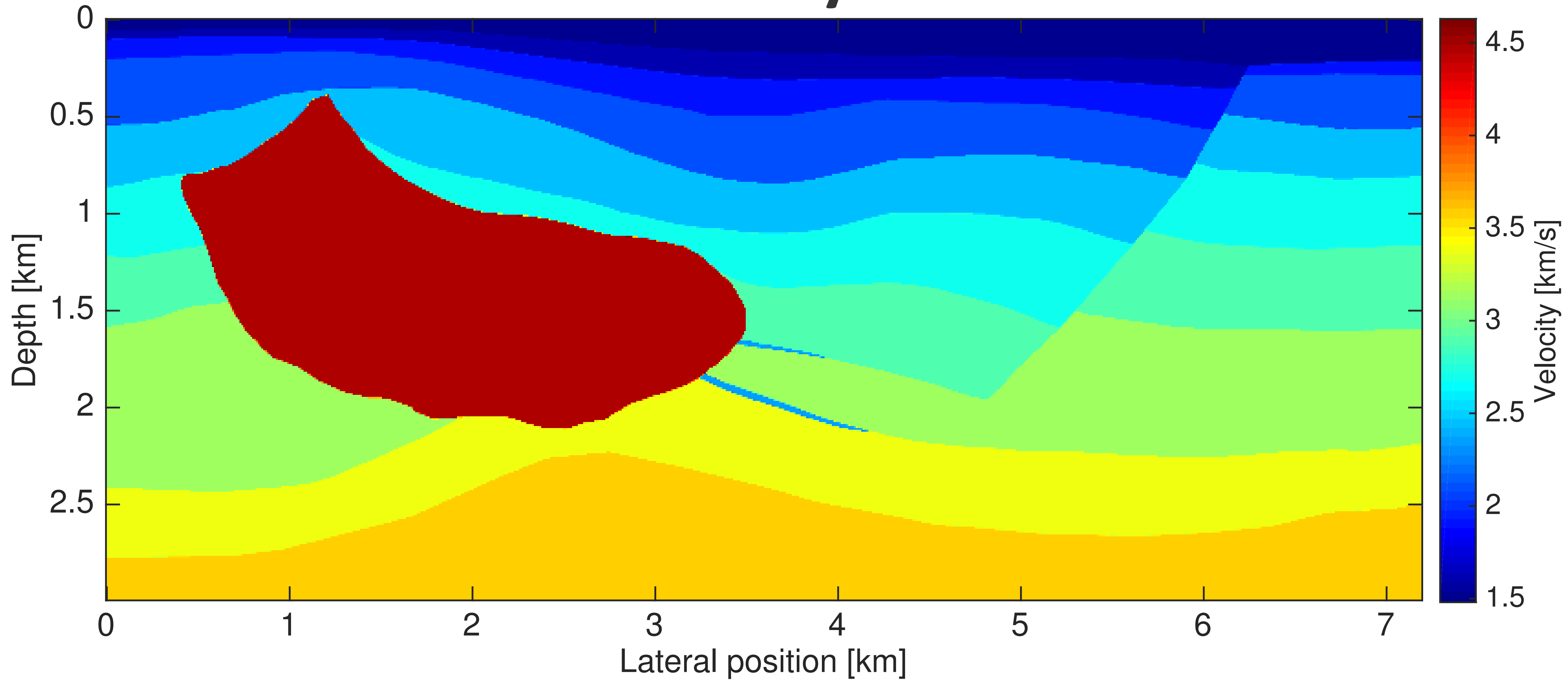
  $\phi = 36^o$

- Modeling time: 48 hours

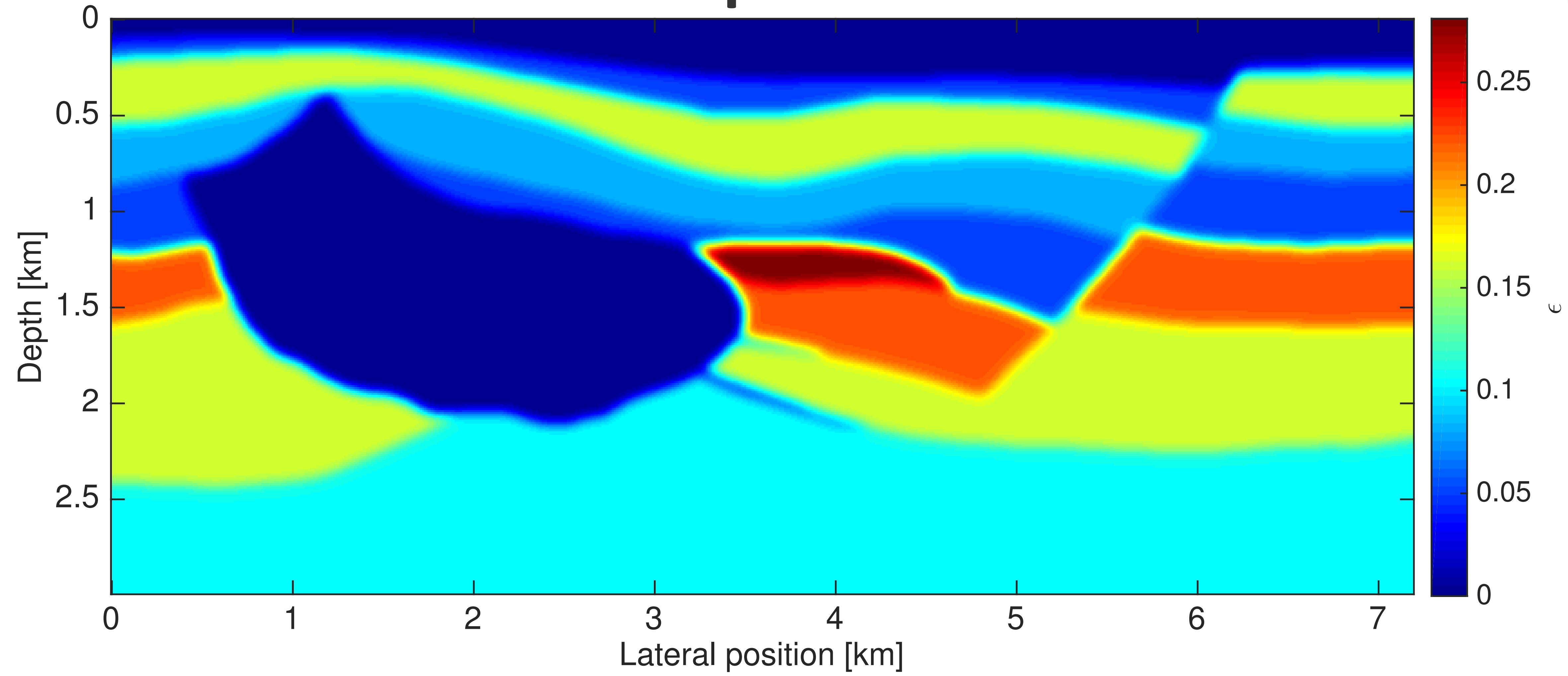


Modeling example: Hess VTI

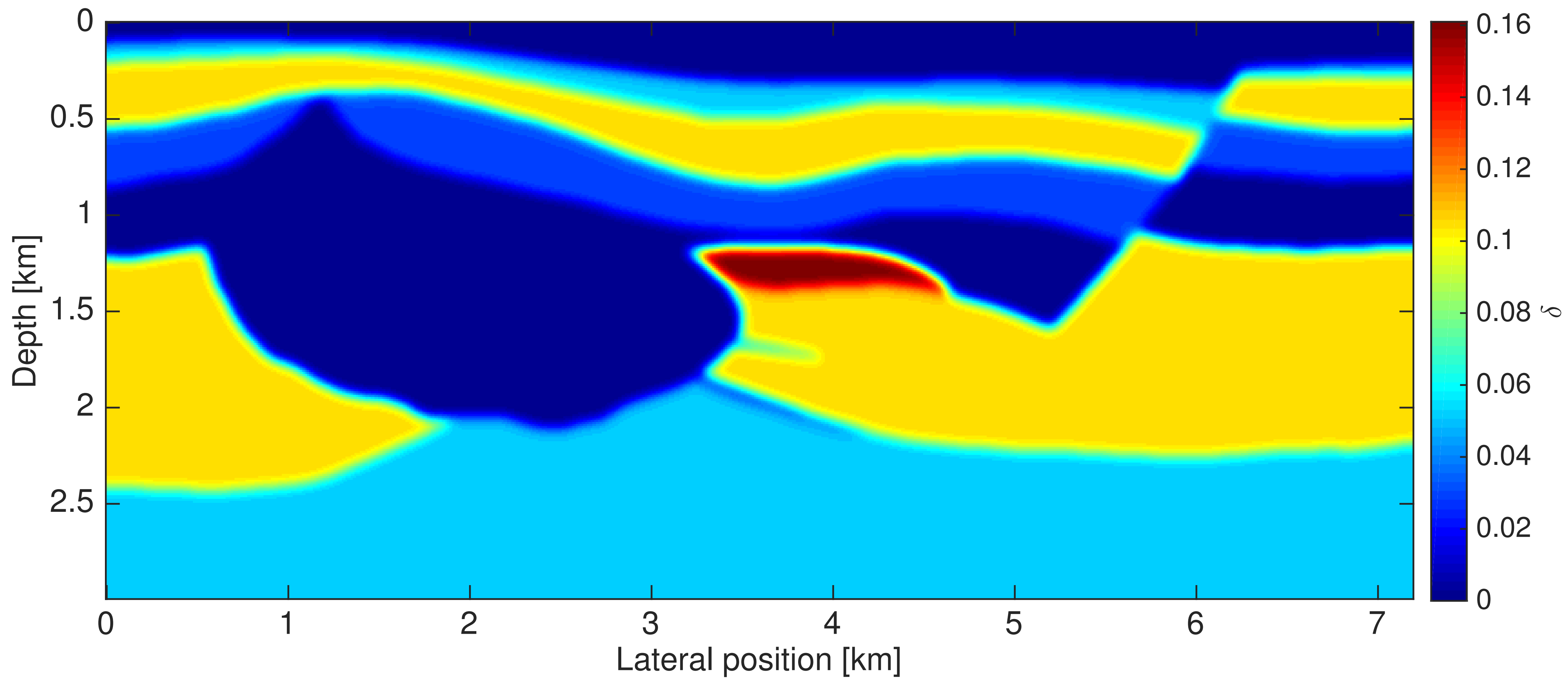
Velocity



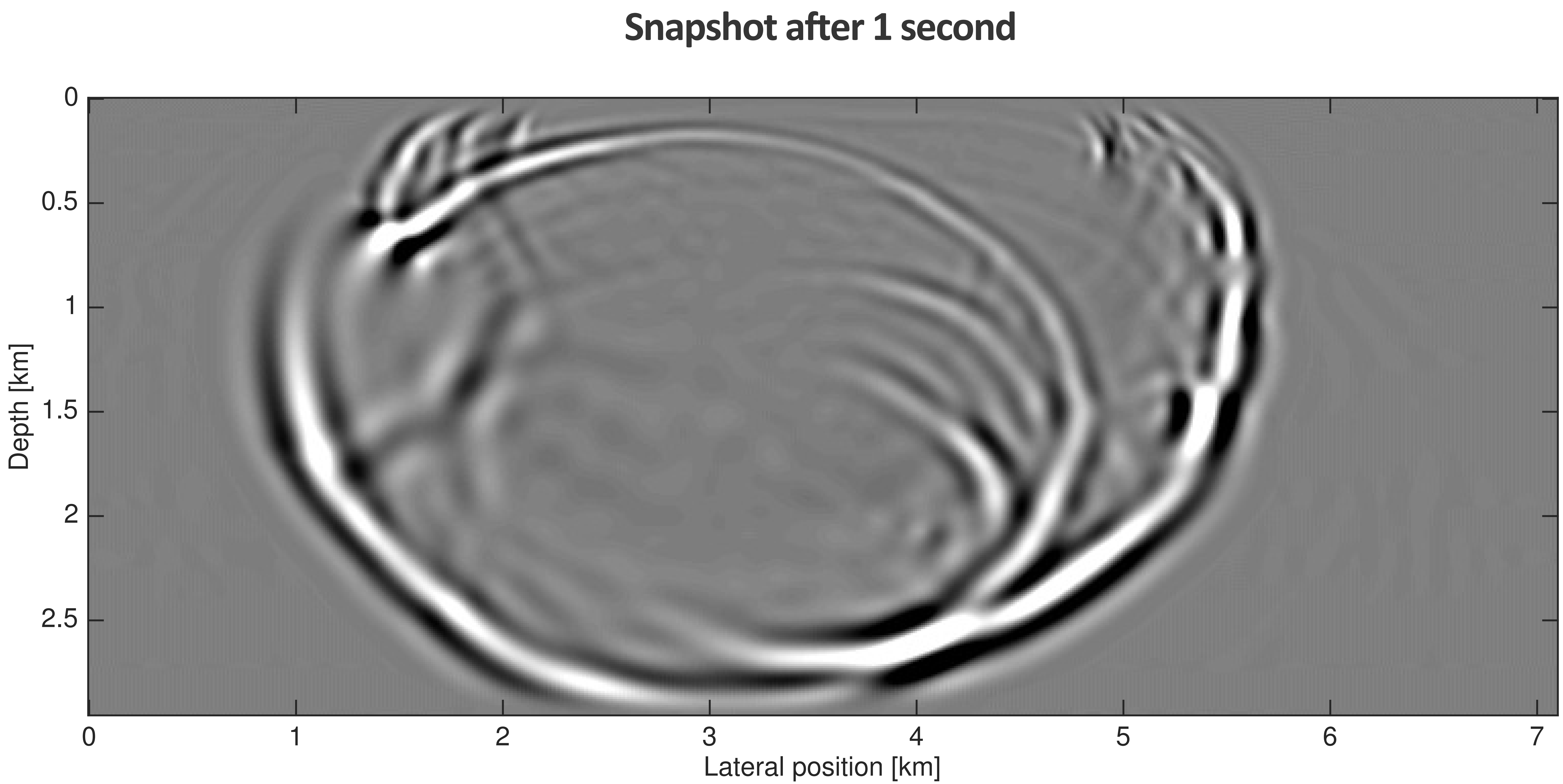
Epsilon



Delta

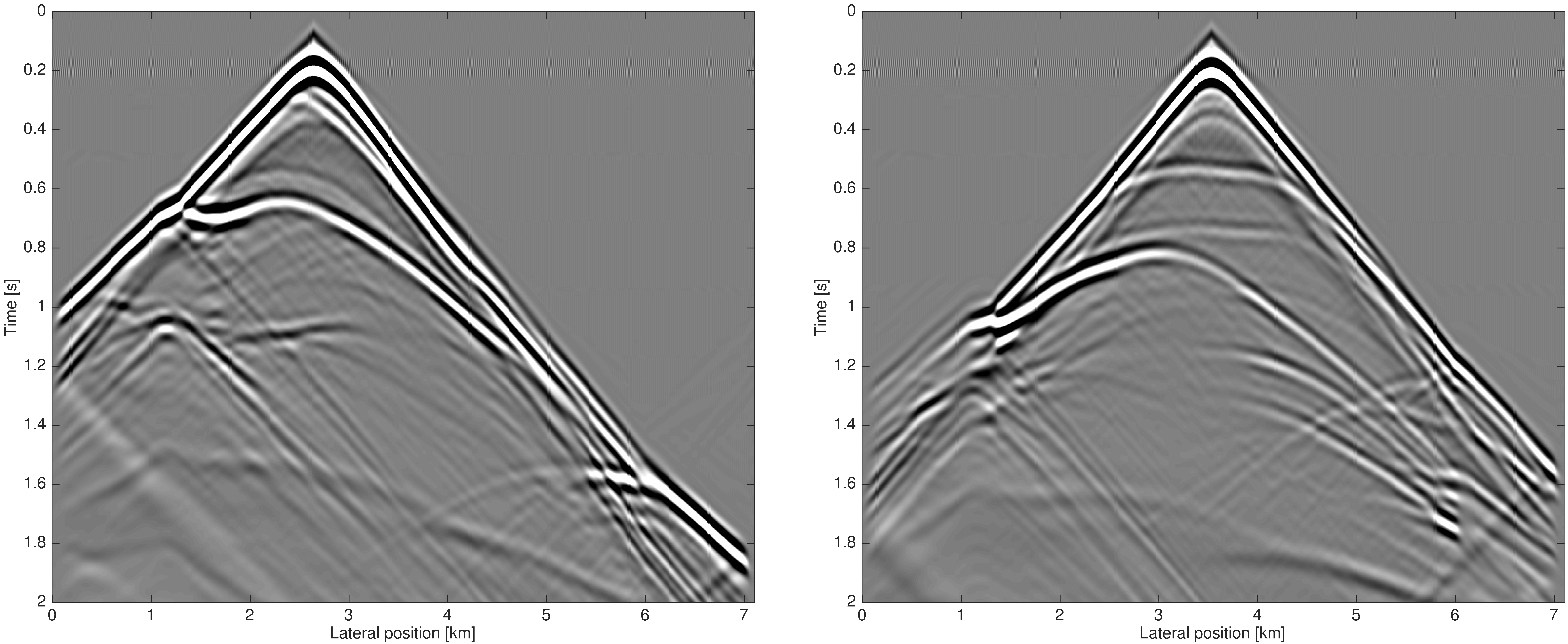


Modeling example



Modeling example

Shot records



Conclusions

Optimized pure P-wave equation:

- optimized to have minimal phase velocity error
- optimal over defined range of phase angles and Thomson parameters
- up to an order of magnitude more accurate than other pure P-wave equations

Forward modeling scheme:

- Similar computational cost for 2/3D VTI
- Same computational cost as standard pure P-wave equation for 3D TTI
- reference code (slow but high accuracy, no dispersion) for Devito

Acknowledgements

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