Solving WRI's data-augmented wave equation in 3D

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SINBAD Consortium Meeting Spring 2015, June 5

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Problem of interest

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$

$H(\mathbf{m}) \in \mathbb{C}^{N \times N}$ discrete PDE

$\mathbf{m} \in \mathbb{R}^{N}$ medium parameters

$P \in \mathbb{R}^{m \times N}$ selects field at receivers

$\mathbf{u} \in \mathbb{C}^{N}$ field

$\mathbf{d} \in \mathbb{C}^{m}$ observed data

$\mathbf{q} \in \mathbb{C}^{N}$ source

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[T. van Leeuwen & F.J. Herrmann, 2013]

WRI

To minimize: $$\min_{\mathbf{m}} \frac{1}{2}\|P\bar{\mathbf{u}} - \mathbf{d}\|_2^2 + \frac{\lambda^2}{2}\|H(\mathbf{m})\bar{\mathbf{u}} - \mathbf{q}\|_2^2$$

at every outer iteration:

- compute $$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$
- evaluate $\bar{\phi}(\mathbf{m}, \bar{\mathbf{u}}, \lambda)$ & $\nabla_{\mathbf{m}}\bar{\phi}(\mathbf{m}, \bar{\mathbf{u}}, \lambda)$
- update $\mathbf{m}$

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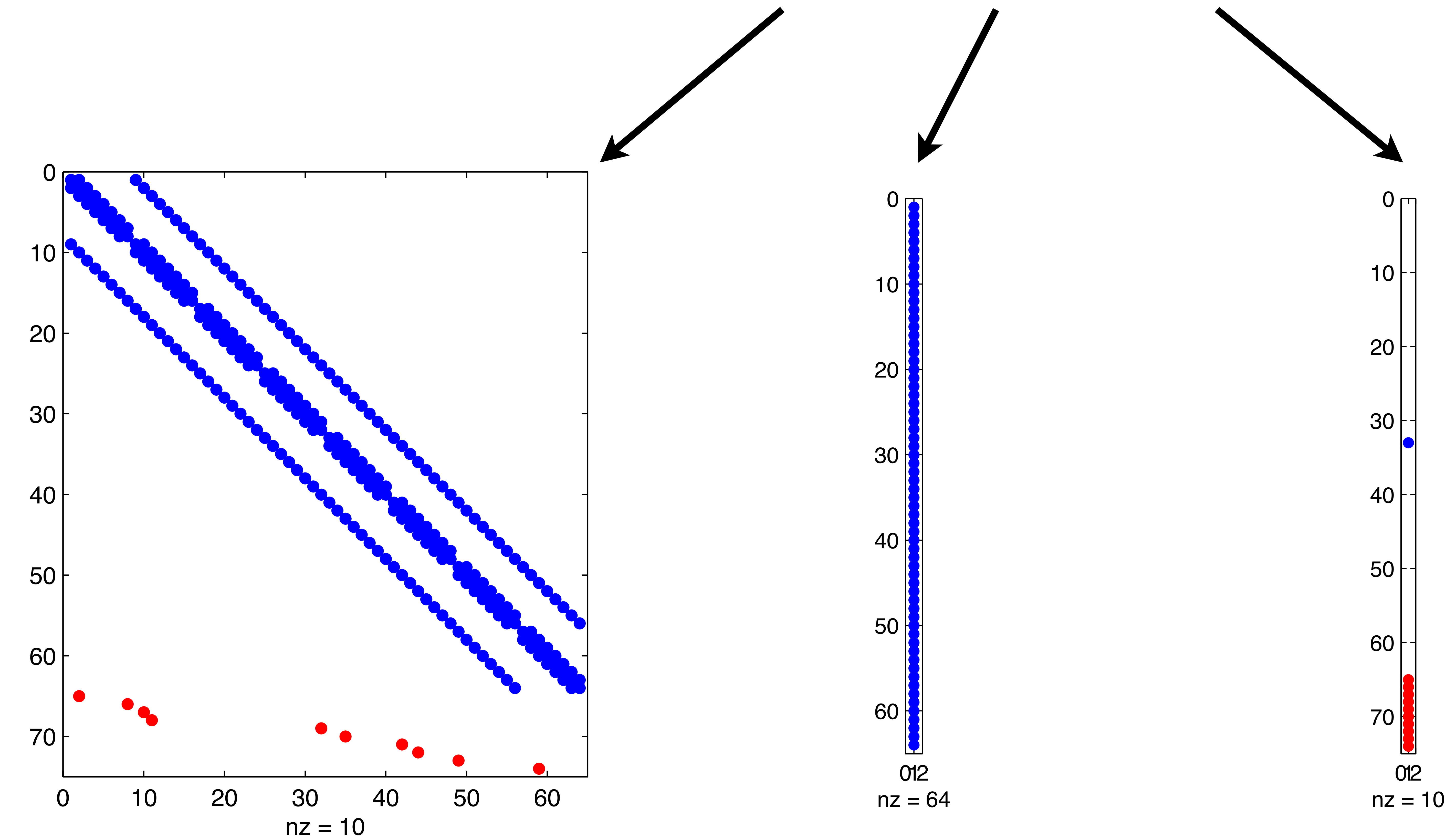
Properties of the problem

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$

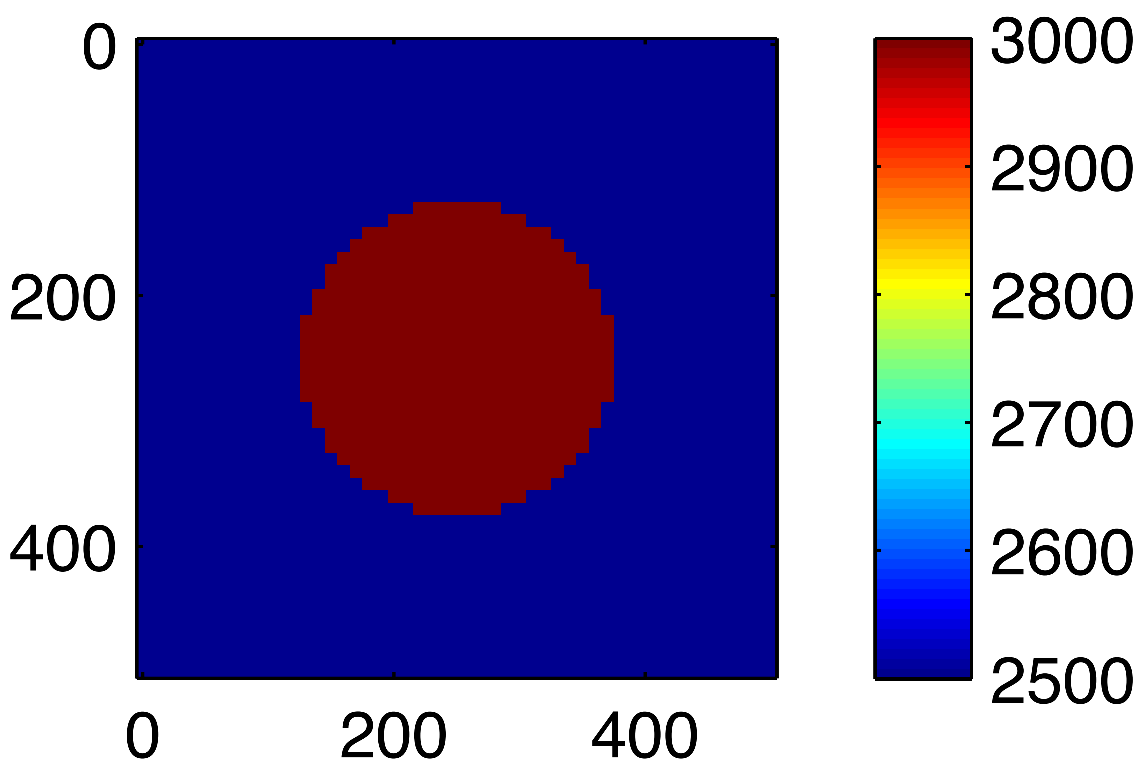
- H is indefinite, non-Hermitian
- inconsistent
- full column rank

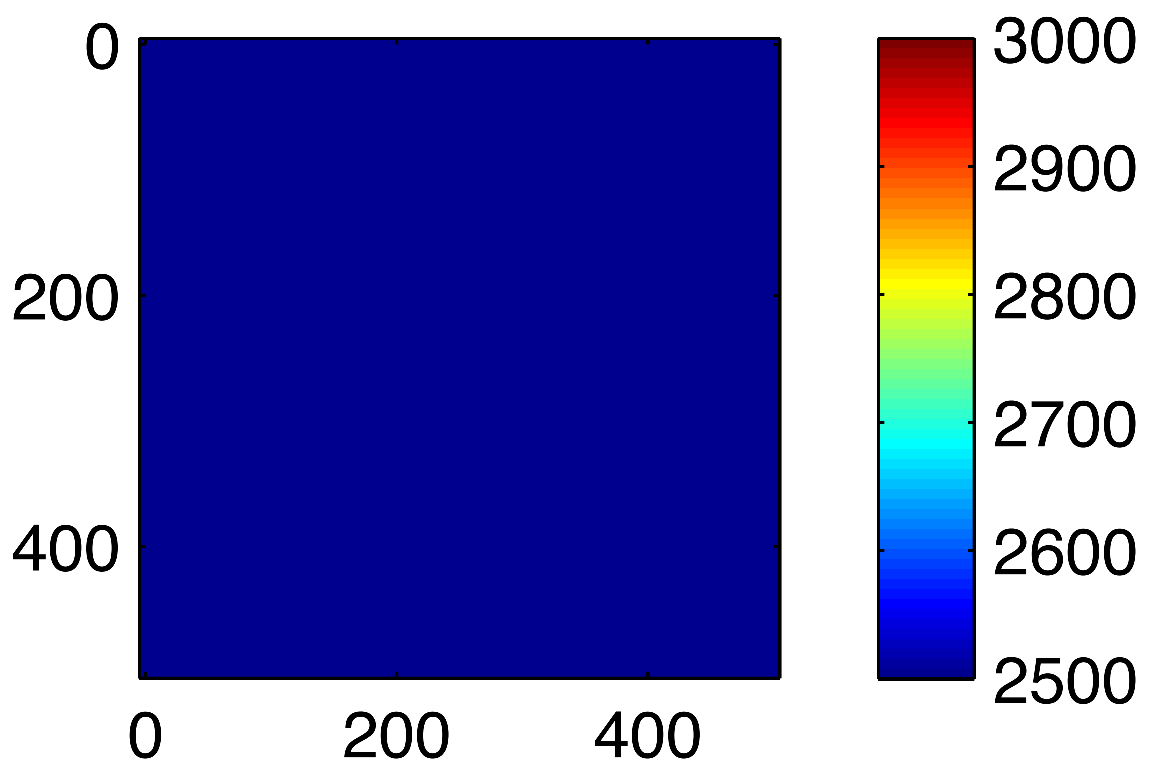
Properties of the problem

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$



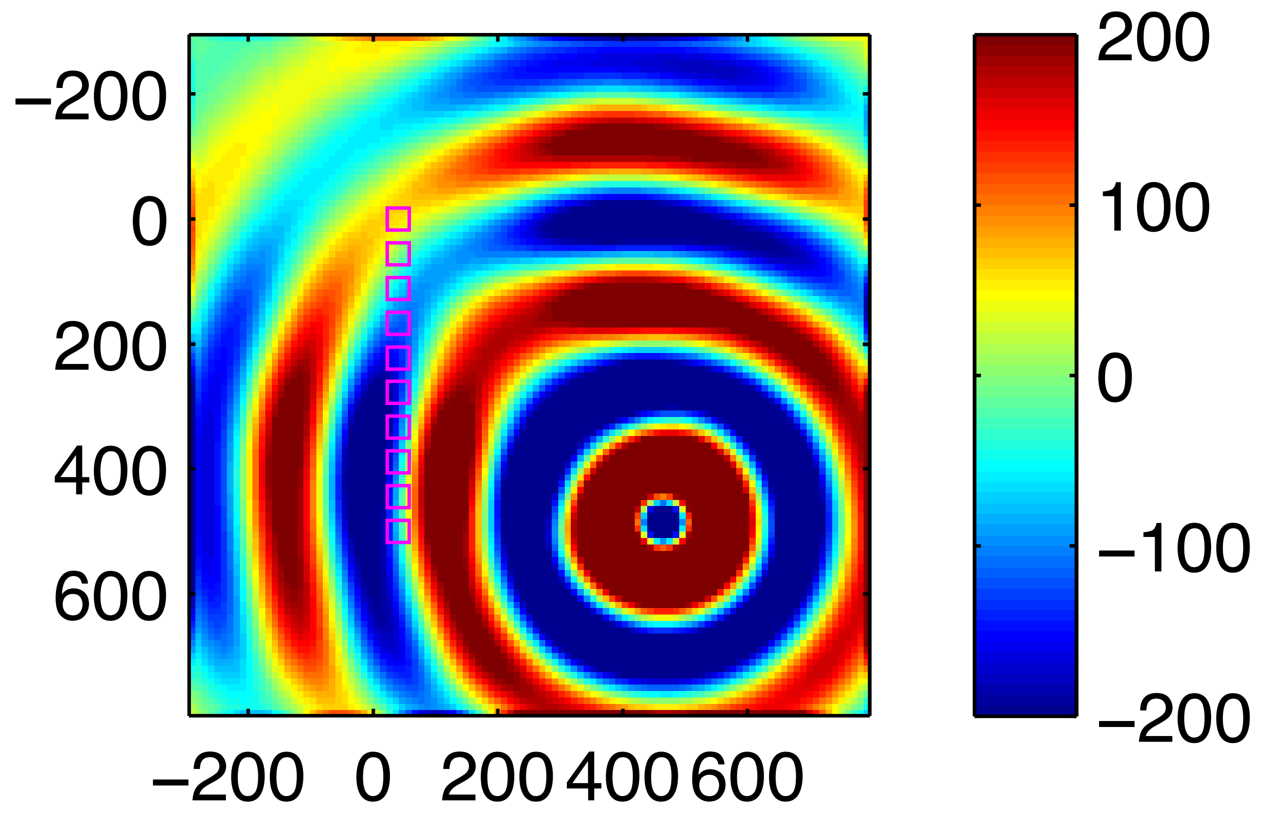
2D example

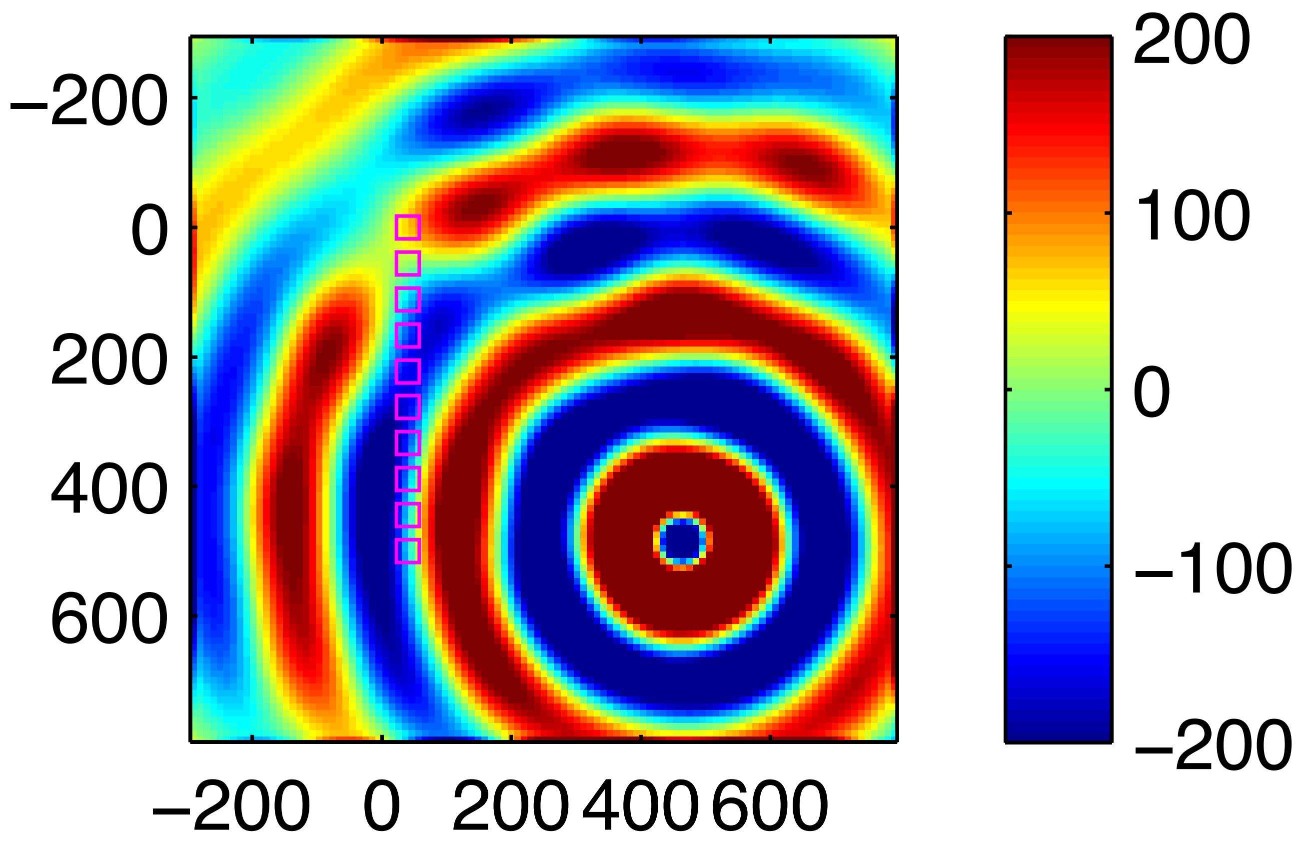


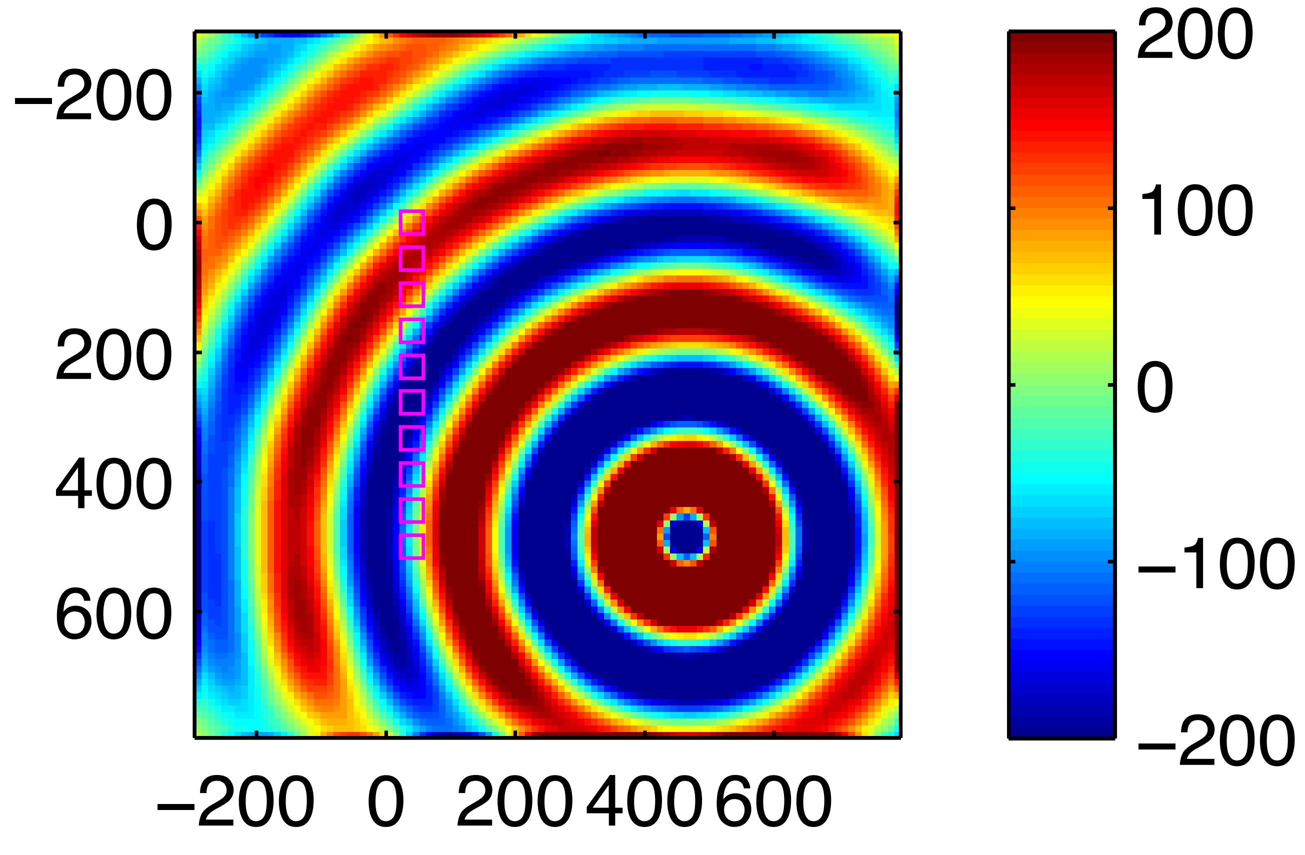


$$\mathbf{u} = H(\mathbf{m}_*)^{-1}\mathbf{q}$$

$$\mathbf{u} = H(\mathbf{m}_0)^{-1}\mathbf{q}$$







$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}_0) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$

Solution of the sub-problem

Main challenge: solve $$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ P \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix} \right\|_2$$

- 2D: direct factorization **[L. M. Delves & I. Barrodale, 1979 ; T. A. Davis, 2011]**

In 3D we want:

- iteratively & matrix-free
- no QR or LU factorizations
- at cost cost of a few PDE solves

Proposed algorithm

LS-problem in normal-equation form:

$$\left(\lambda^2 H(\mathbf{m})^* H(\mathbf{m}) + P^* P\right)\bar{\mathbf{u}} = \lambda^2 H(\mathbf{m})\mathbf{q} + P^*\mathbf{d}$$

Split-preconditioning by λH w/o computations

$$(I + H_\lambda^{-*} P^* P H_\lambda^{-1})\mathbf{y} = \lambda \mathbf{q} + (H_\lambda^*)^{-1} P^* \mathbf{d}, \quad \text{with} \quad H_\lambda \bar{\mathbf{u}} = \mathbf{y}$$

- $m+1$ distinct eigenvalues (identity + low-rank)
- even for inexact Helmholtz

Proposed algorithm

Exploit identity + low-rank structure:

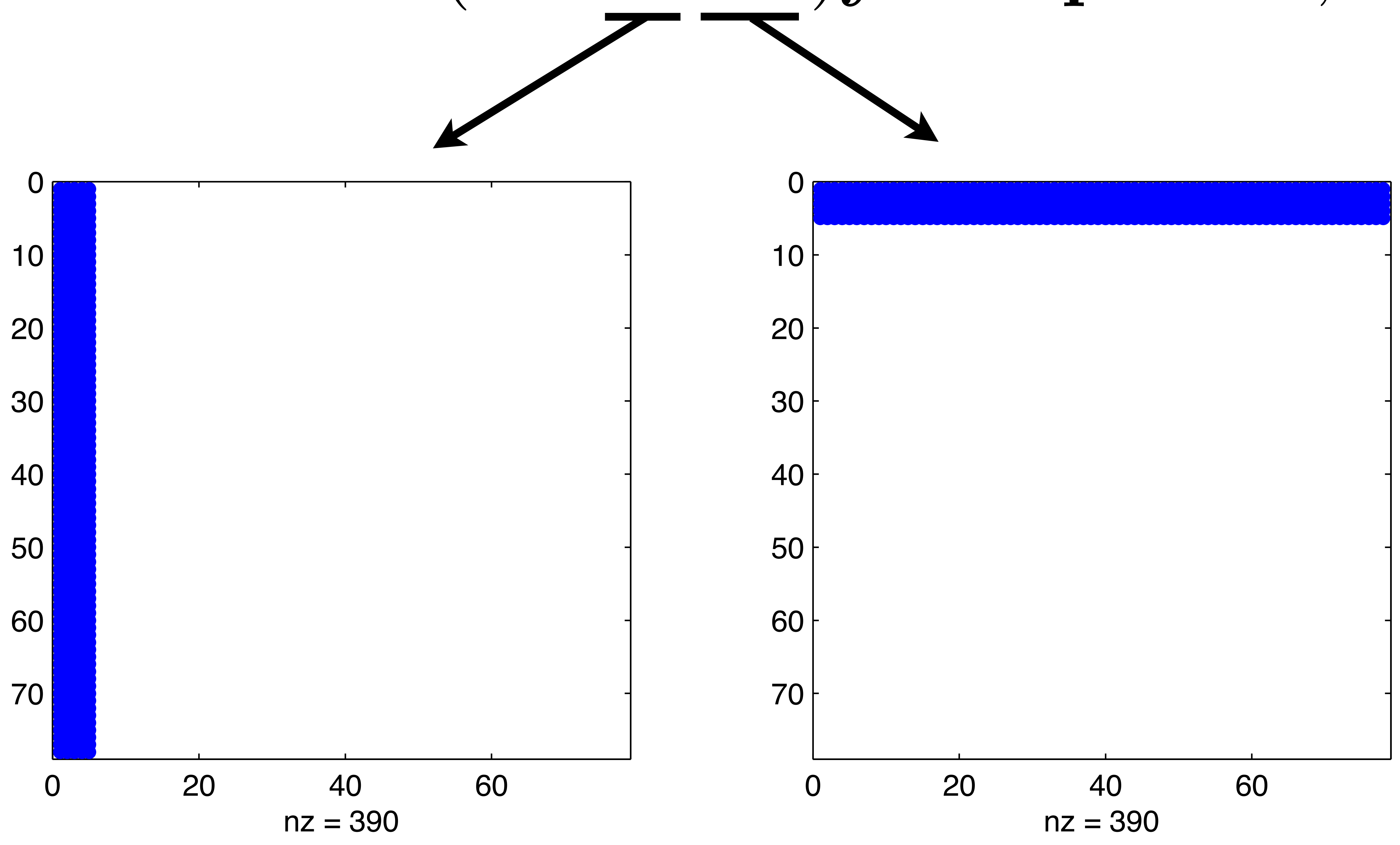
$$(I + H_\lambda^{-*} P^* P H_\lambda^{-1})\mathbf{y} = \lambda \mathbf{q} + (H_\lambda^*)^{-1} P^* \mathbf{d}, \quad \text{with} \quad H_\lambda \bar{\mathbf{u}} = \mathbf{y}$$

by solving $H^{-*} P^* = W$

- n_{rec} Helmholtz problems (inexactly)
- low-rank factorization
- $W \in \mathbb{C}^{N \times m}$ dense but extremely skinny

Proposed algorithm

$$(I + WW^*)\mathbf{y} = \lambda\mathbf{q} + W\mathbf{d}, \quad \text{with} \quad H_\lambda \bar{\mathbf{u}} = \mathbf{y}$$



inverses transformed into identity + low-rank factorization

Proposed algorithm

Leverage low-rank factorization:

$$(I + WW^*)\mathbf{y} = \lambda\mathbf{q} + W\mathbf{d}, \quad \text{with} \quad H_\lambda\bar{\mathbf{u}} = \mathbf{y}$$

and invert system matrix as

$$\mathbf{y} = (I - W(I + W^*W)^{-1}W^*)(\lambda\mathbf{q} + W\mathbf{d}), \quad \text{with} \quad H_\lambda\bar{\mathbf{u}} = \mathbf{y}$$

so we only need to invert $(I + W^*W) \in \mathbb{C}^{m\times m}$

(this is alway small enough to do explicitly)

Proposed algorithm

for angular frequency ω **do**
  // solve m Helmholtz problems
  $H_{\lambda}^{*} W = P^{*}$
  $M = (I + W^{*}W)^{-1}$
  for right hand side i **do**
    $\mathbf{y}_i = \left(I - WMW^{*}\right)\left(\lambda \mathbf{q}_i + W\mathbf{d}_i\right)$
    // solve for $\bar{\mathbf{u}}_i$
    $H_{\lambda}\bar{\mathbf{u}}_i = \mathbf{y}_i$
  end for
end for

Proposed algorithm

Matrix-free algorithm

- no direct solves
- related mildly overdetermined systems **[L. M. Delves & I. Barrodale, 1979]**

Computational cost:

- 1 PDE per receiver
- 1 PDE per source

Memory requirements:

- 1 vector per receiver (W)
- system matrix (H)
- storage for solving systems with H

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Proposed algorithm

Inexact solutions to the linear systems:

for angular frequency ω **do**
  // solve m Helmholtz problems inexactly
  → $H_\lambda^* \hat{W} = P^* + R_W$
  $\hat{M} = (I + \hat{W}^* \hat{W})^{-1}$
  for right hand side $\mathbf{b}_i$ **do**
    $\hat{\mathbf{y}}_i = \left(I - \hat{W}\hat{M}\hat{W}^*\right)\left(\lambda \mathbf{q}_i + \hat{W}\mathbf{d}_i\right)$
    // solve for $\bar{\mathbf{u}}_i$ inexactly
    → $H_\lambda \hat{\mathbf{u}}_i = \hat{\mathbf{y}}_i + \mathbf{r_u}$
  end for
end for

Proposed algorithm

error propagation (1 right-hand-side, 1 receiver case):

$$H_\lambda^* \hat{\mathbf{w}} = \mathbf{p}^* + \mathbf{r}_\mathbf{w}$$

$$(I + \hat{\mathbf{w}}\hat{\mathbf{w}}^*)\hat{\mathbf{y}} = \lambda \mathbf{q} + \hat{\mathbf{w}} d$$

$$H_\lambda \hat{\mathbf{u}} = \hat{\mathbf{y}} + \mathbf{r}_\mathbf{u}$$

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Proposed algorithm

error propagation (1 right-hand-side, 1 receiver case):

$$H_\lambda^* \hat{\mathbf{w}} = \mathbf{p}^* + \mathbf{r}_\mathbf{w}$$

$$(I + \hat{\mathbf{w}}\hat{\mathbf{w}}^*)\hat{\mathbf{y}} = \lambda \mathbf{q} + \hat{\mathbf{w}} d$$

$\longrightarrow$ solve as: $\hat{\mathbf{y}} = (I - \hat{m}\hat{\mathbf{w}}\hat{\mathbf{w}}^*)(\lambda \mathbf{q} + \hat{\mathbf{w}} d)$

with $\hat{m} = \dfrac{1}{1 + \hat{\mathbf{w}}^*\hat{\mathbf{w}}}$

$$H_\lambda \hat{\mathbf{u}} = \hat{\mathbf{y}} + \mathbf{r}_\mathbf{u}$$

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Proposed algorithm

- several bounds derived
- Multi-stage error propagation makes deriving (useful) bounds in terms of observables very challenging.
- Very pessimistic bound may be all what is possible.
- The aim is to obtain (asymptotically) correct dependence on condition numbers / spectral norms of involved matrices and relative residuals.
- work in progress

Suggested PDE-solver

need to store 1 vector per receiver

-> use PDE-solver with low-memory & setup requirements

Helmholtz:

- CGMN (only 4 vectors) / CARP-CG **[A. Bjorck & T. Elfving, 1979; D. Gordon & R. Gordon, 2010; T. van Leeuwen & F.J. Herrmann, 2014]**
- shifted-Laplacian w/ multi-grid **[Y.A. Erlangga, 2008; H. Calandra et al., 2013]**
- combination of the above **[R. Lago & F.J. Herrmann, 2015]**

Simultaneous receivers

Simultaneous sources reduce the number of sources to be modeled.

Can we use similar ideas with the proposed algorithm?

Memory and computational cost now depends on sources + receivers.

Simultaneous receivers

What is the number of receivers is too large, storage wise?

Can we approximate the least-squares problem using randomization & subsampling (simultaneous receivers)?

Use ideas from algorithms such as

- **[V Rokhlin & M Tygert, 2008]**
- Blendenpik **[H. Avron et. al., 2010]**
- LSRN **[X. Meng, M. A. Saunders, M. W. Mahoney, 2014]**

Simultaneous receivers

Blendenpik:

- Randomize (mix the rows) and subsample a very overdetermined system.
- Use R from QR of the approximated and well conditioned problem as a preconditioner for LSQR to solve the original problem.
- Define randomize & subsample matrix as: $V = SFD$,

$$V \in \mathbb{C}^{l \times m}, \quad l < m$$

$$D \in \mathbb{R}^{m \times m} \quad \text{random } [+1 \text{ , -1}] \text{ on the diagonal}$$

$$F \in \mathbb{C}^{m \times m} \quad \text{DFT matrix}$$

$$S \in \mathbb{R}^{l \times m} \quad \text{subsampling matrix, restriction of the identity}$$

Simultaneous receivers

Initial attempt in this work:

apply randomization and subsampling to the receiver block only for a one-step approximation:

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ VP \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ V\mathbf{d} \end{pmatrix} \right\|_2$$

$$V \in \mathbb{C}^{l \times m}, \quad l < m$$

What should V be ? ongoing research, use $V = SFD$

to illustrate the principle

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Simultaneous receivers

Initial attempt in this work:

apply randomization and subsampling to the receiver block only for a one-step approximation:

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ VP \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ V\mathbf{d} \end{pmatrix} \right\|_2$$

$$V \in \mathbb{C}^{l \times m}, \quad l < m$$

reduces

- # of PDE solves
- # vectors to be stored

Simultaneous receivers

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}} \left\| \begin{pmatrix} \lambda H(\mathbf{m}) \\ VP \end{pmatrix} \mathbf{u} - \begin{pmatrix} \lambda \mathbf{q} \\ V\mathbf{d} \end{pmatrix} \right\|_2$$

Approximates system matrix and right hand side as:

$$\left(\lambda^2 H(\mathbf{m})^* H(\mathbf{m}) + P^* V^* V P\right)\bar{\mathbf{u}} = \lambda^2 H(\mathbf{m})\mathbf{q} + P^* V^* V\mathbf{d}.$$

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Simultaneous receivers

Example:

investigate error in the fields and in the resulting gradient, introduced by subsampling and randomization.

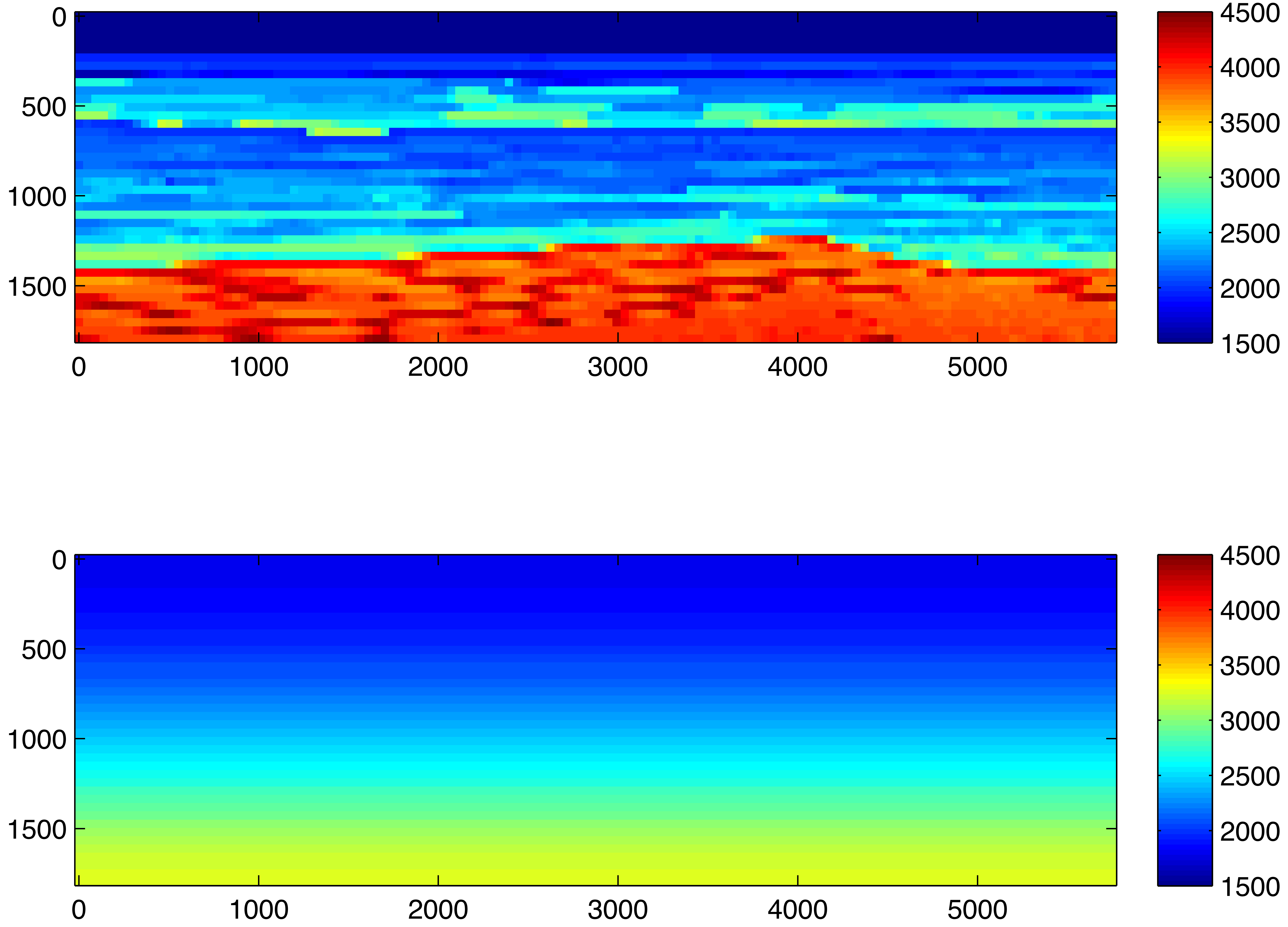
for now, assume:

- solve linear systems exactly
- fixed number of sources (30)
- 1 frequency (4Hz)

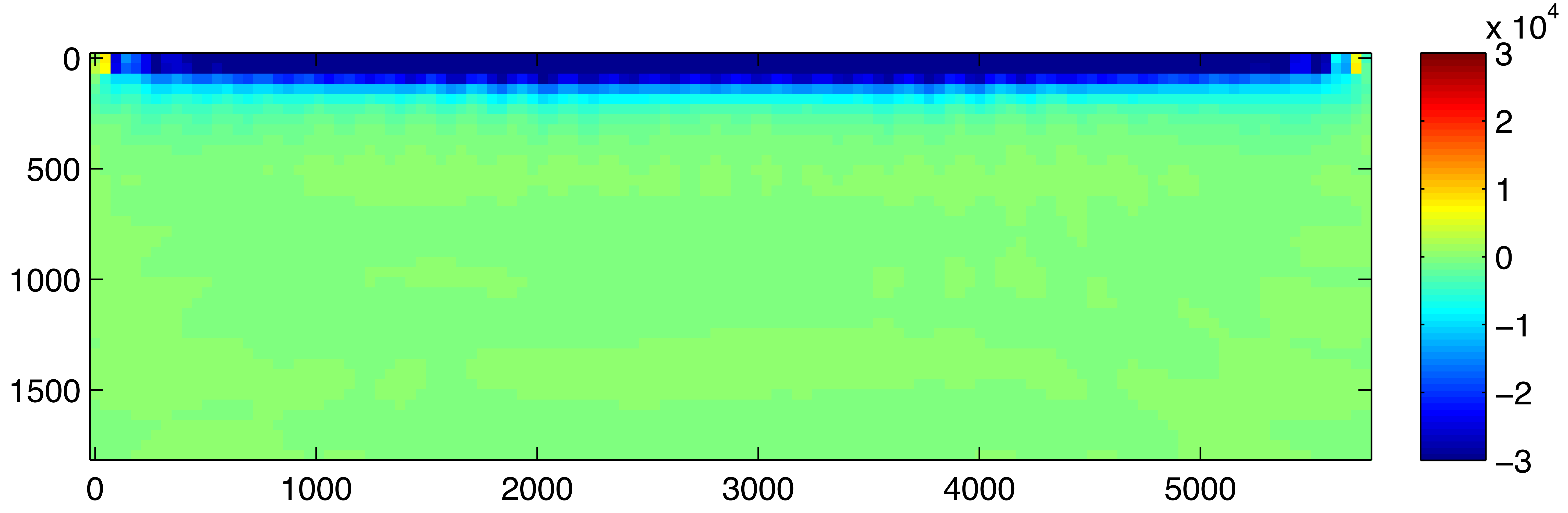
Simultaneous receivers

Example:

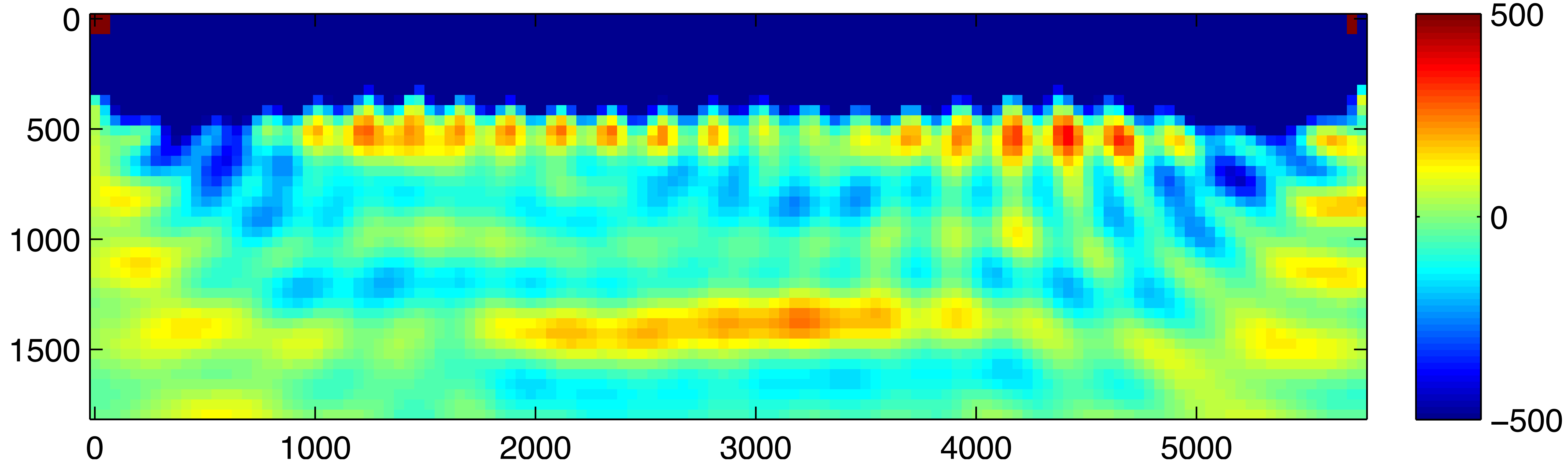
- 100 receivers in total (~55 m interval)
- only work with randomized subsets of varying size (previous slides)
- 30 sources (~200 m interval)
- 1 frequency (4Hz)



Simultaneous receivers



True gradient,
same image twice, different color scale



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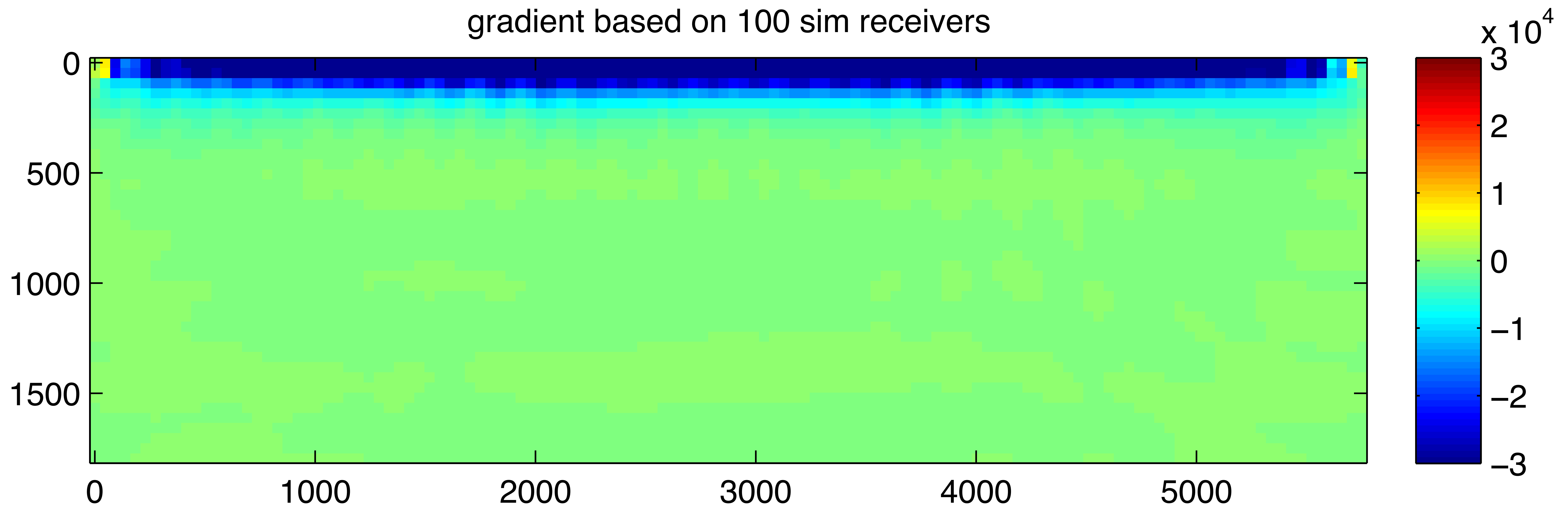

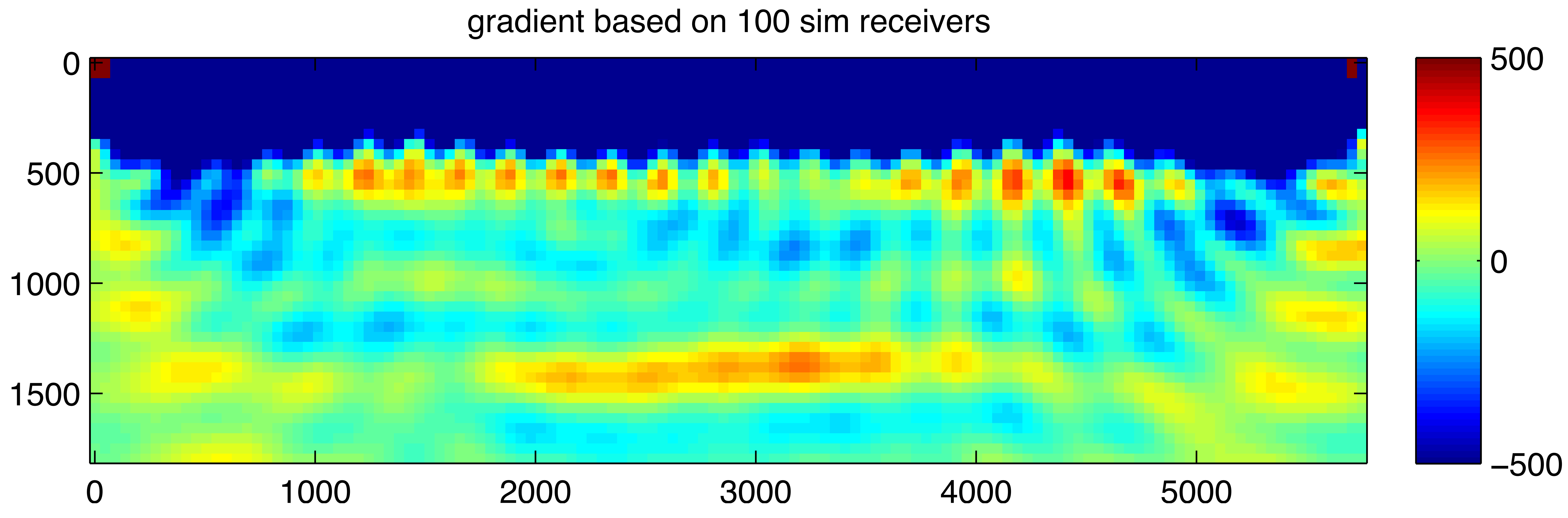

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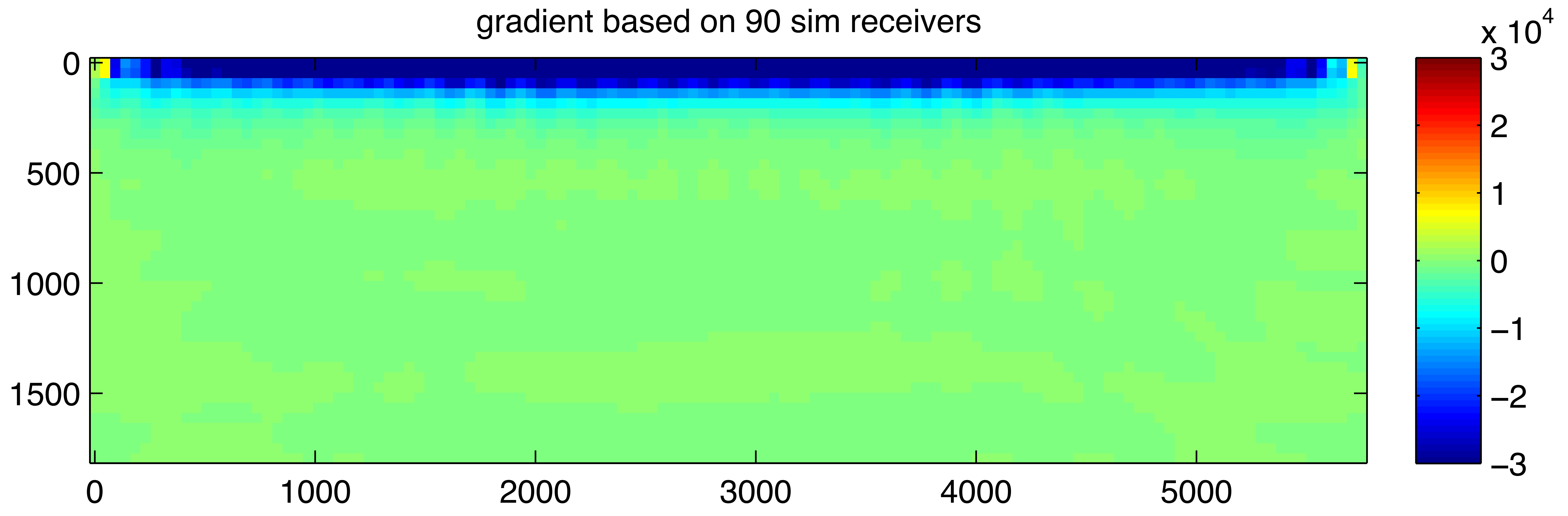
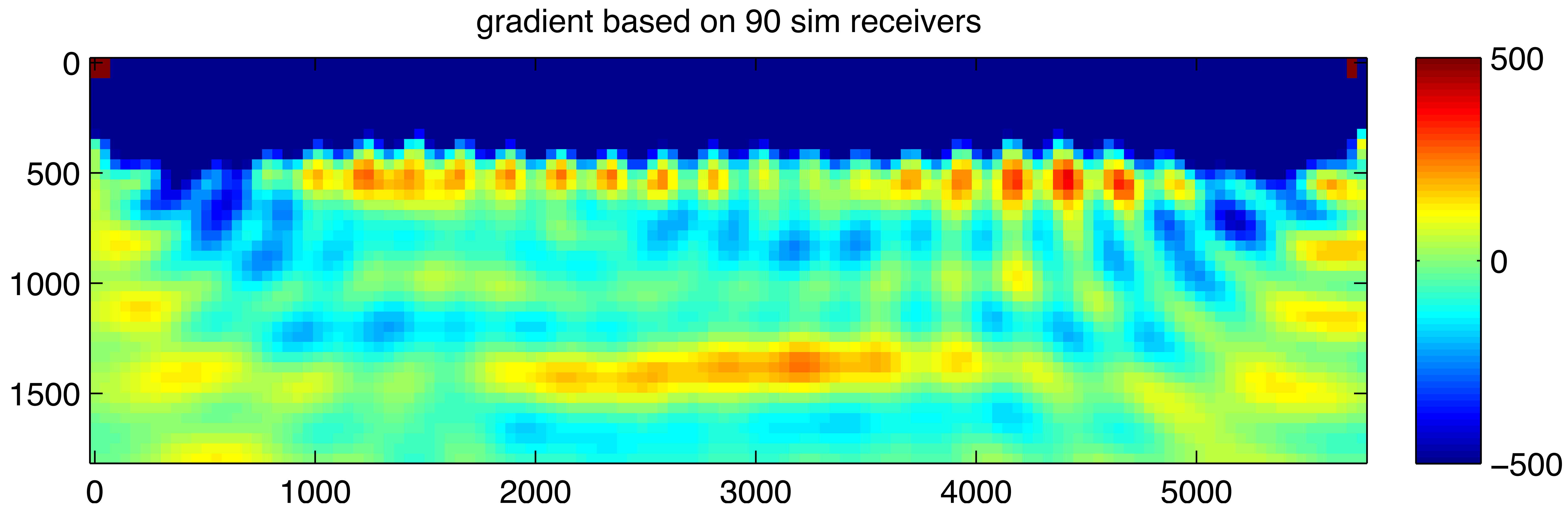
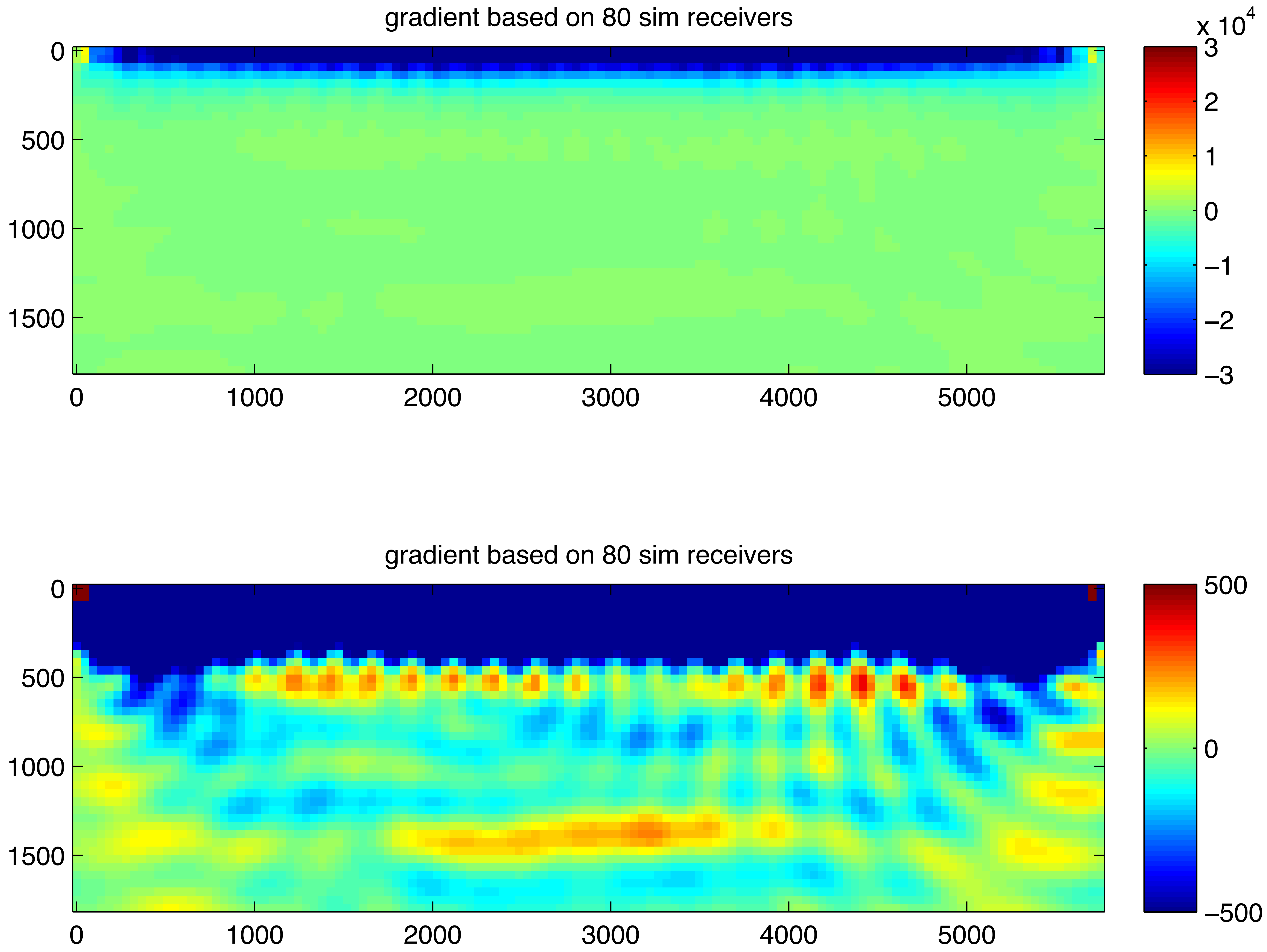
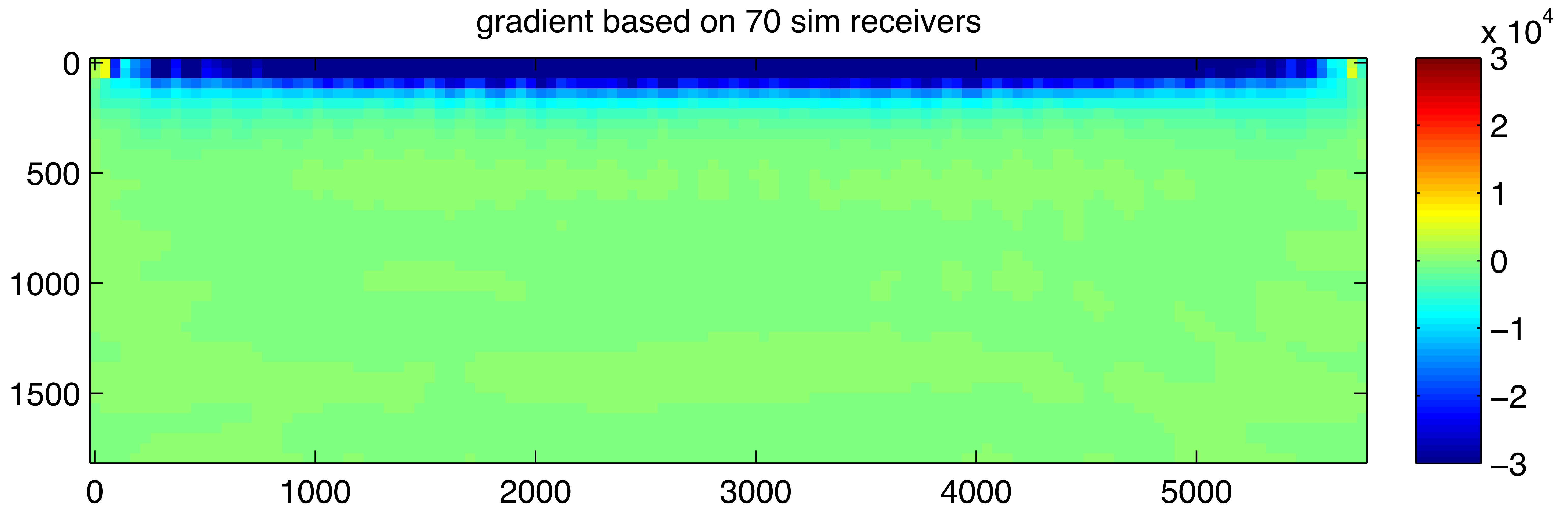


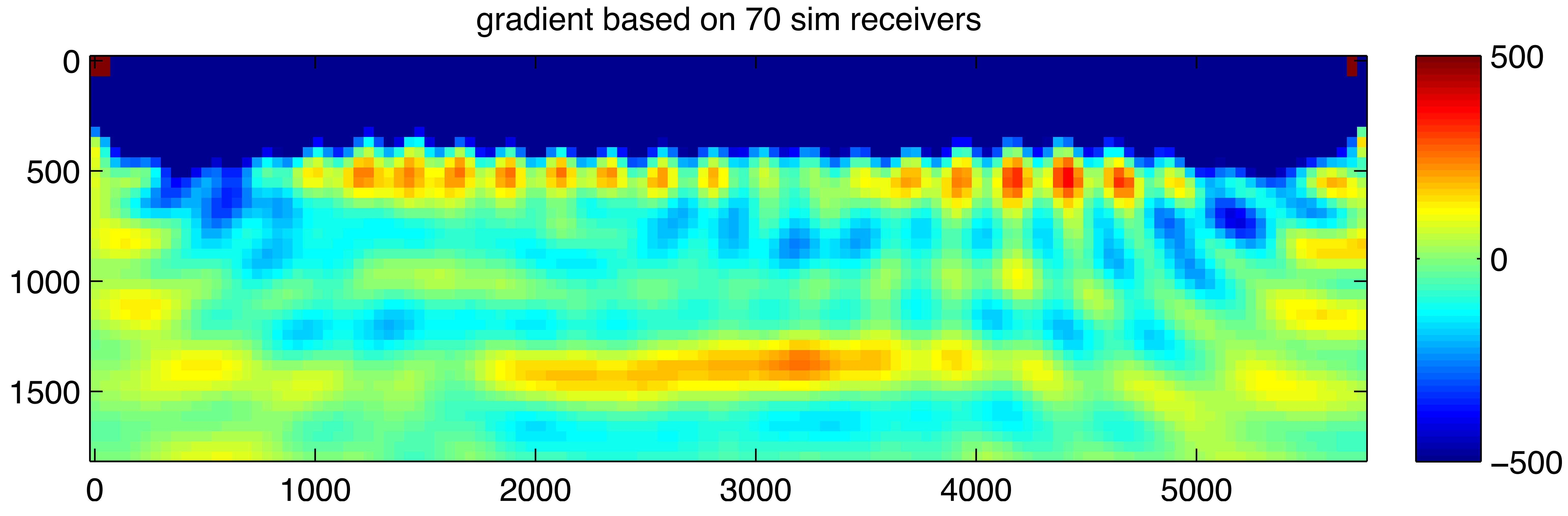


Simultaneous receivers



Simultaneous receivers





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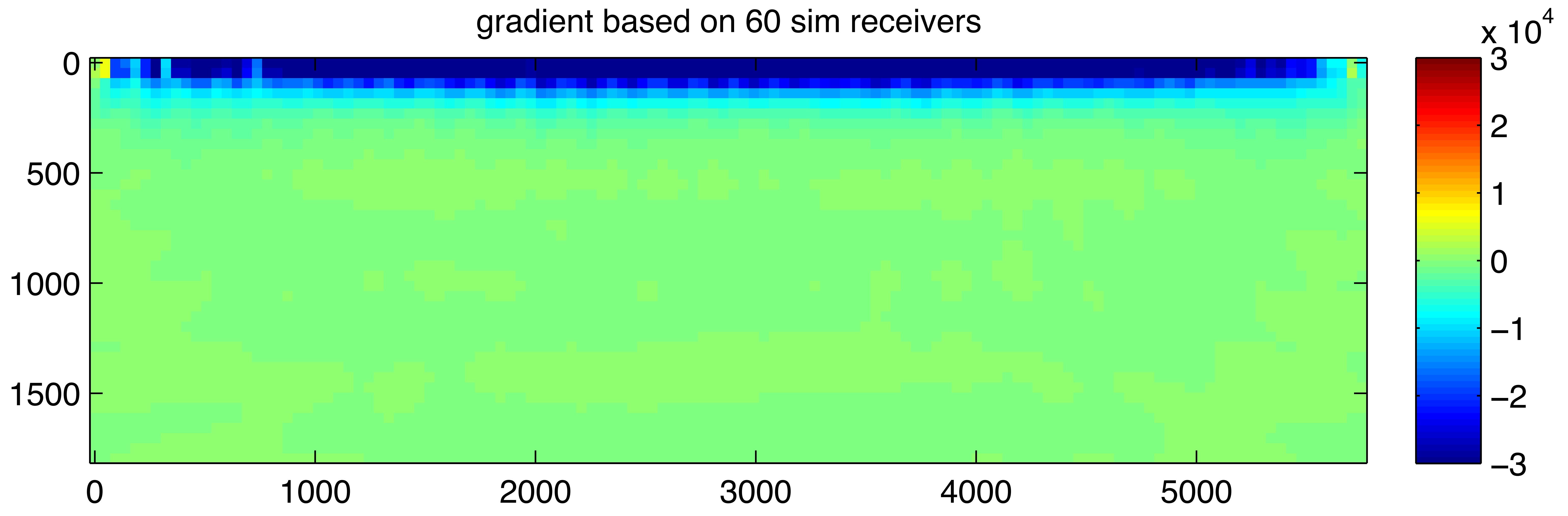

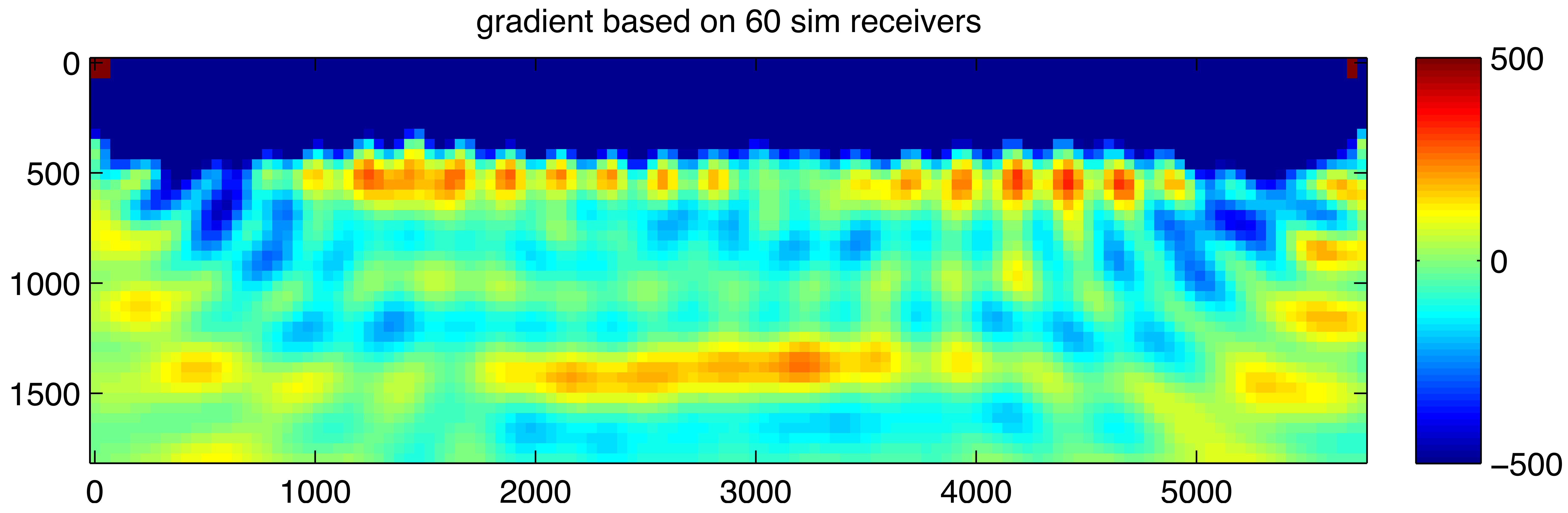

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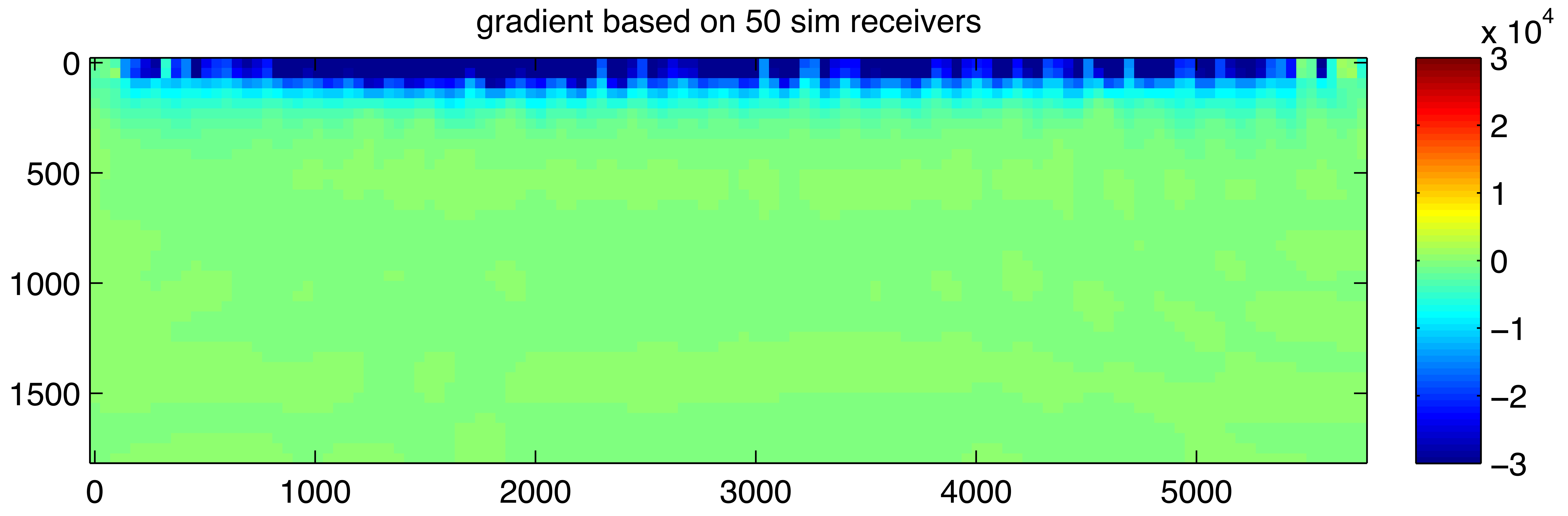

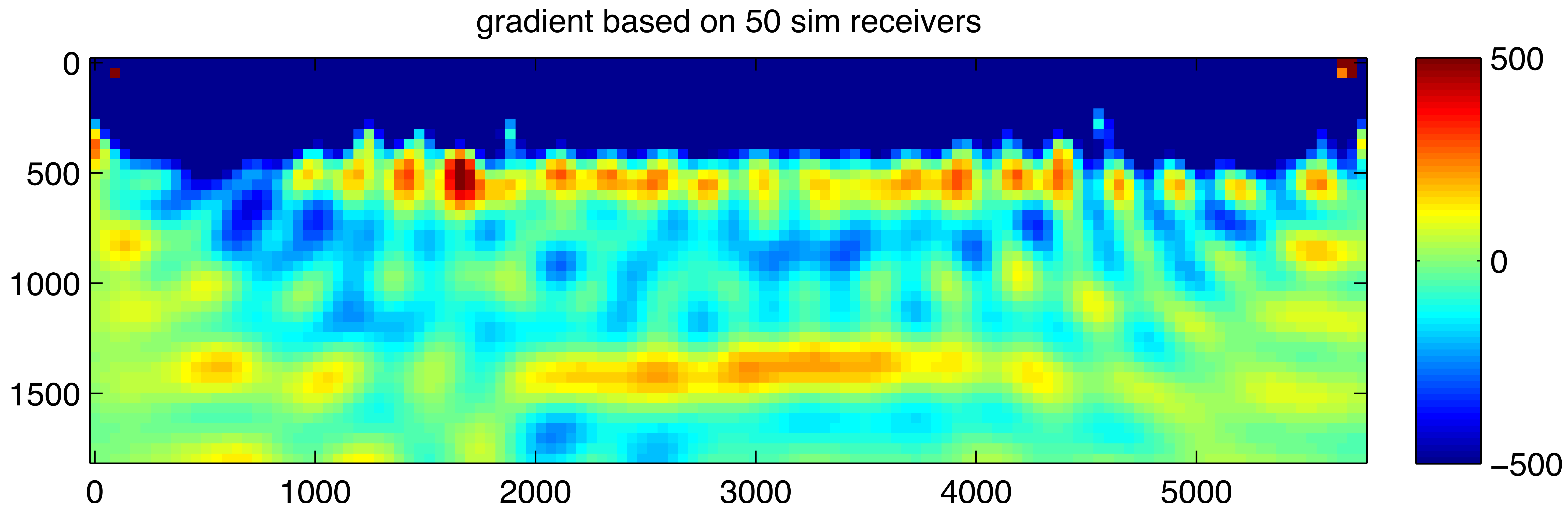

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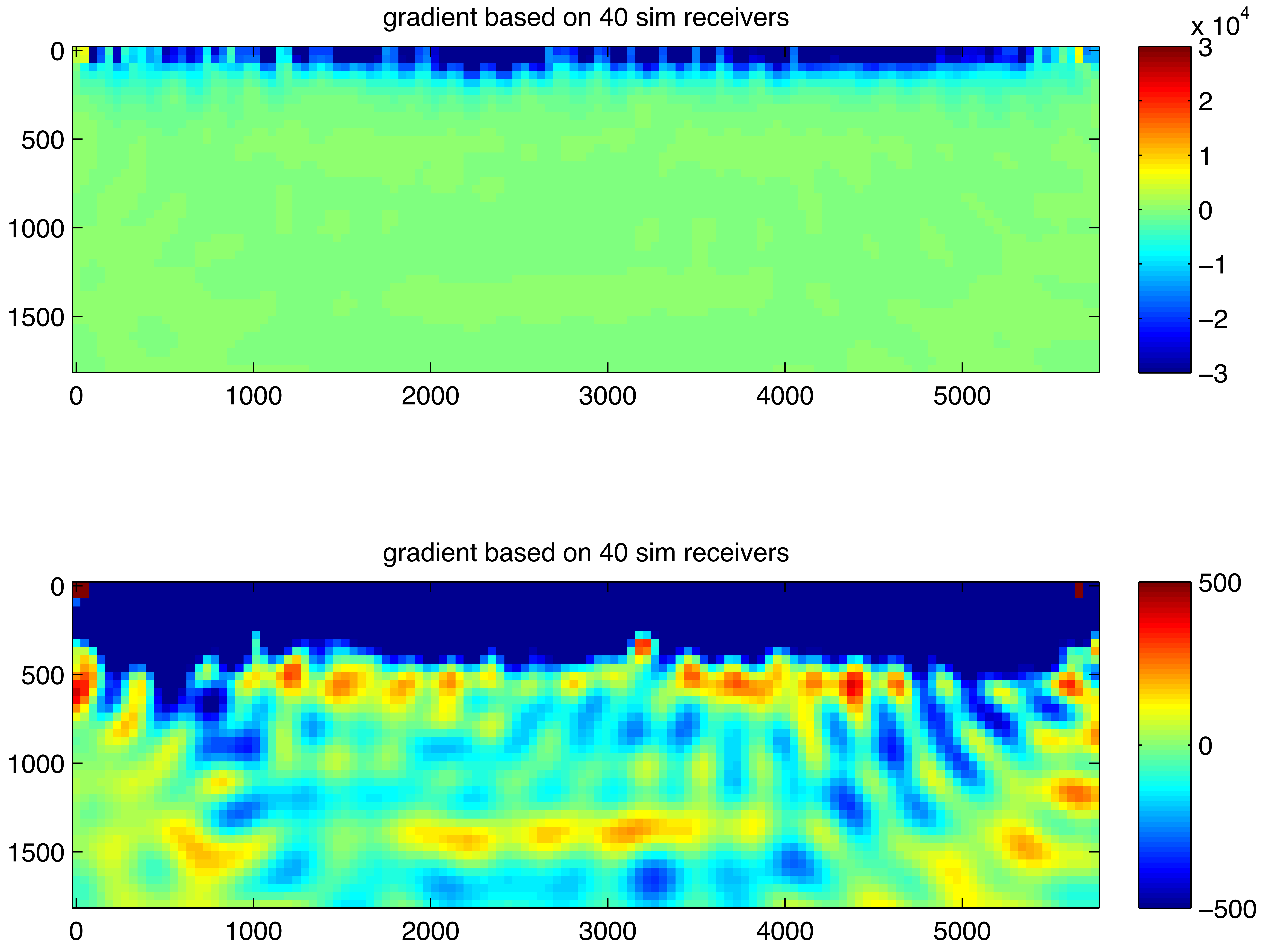

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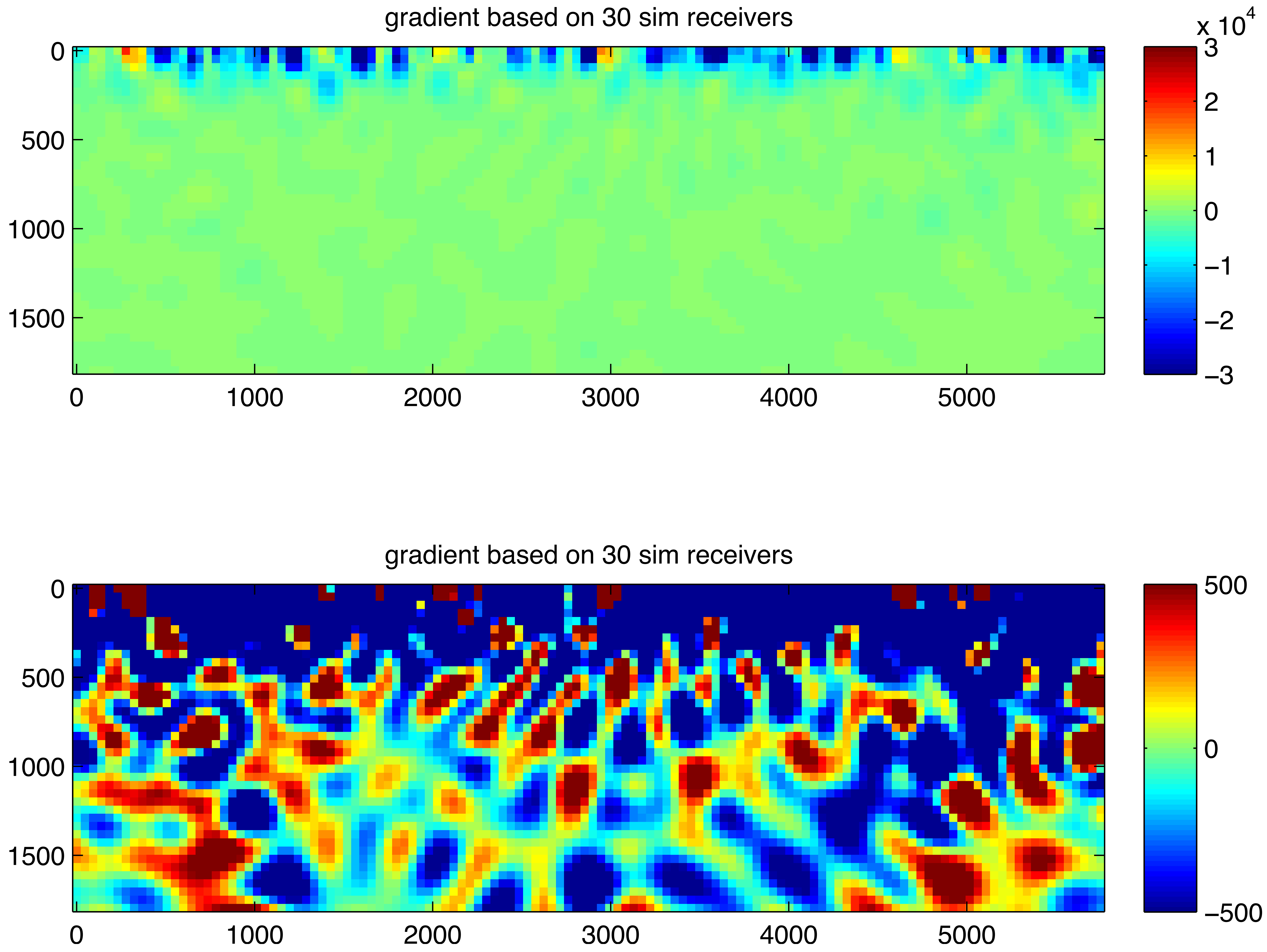

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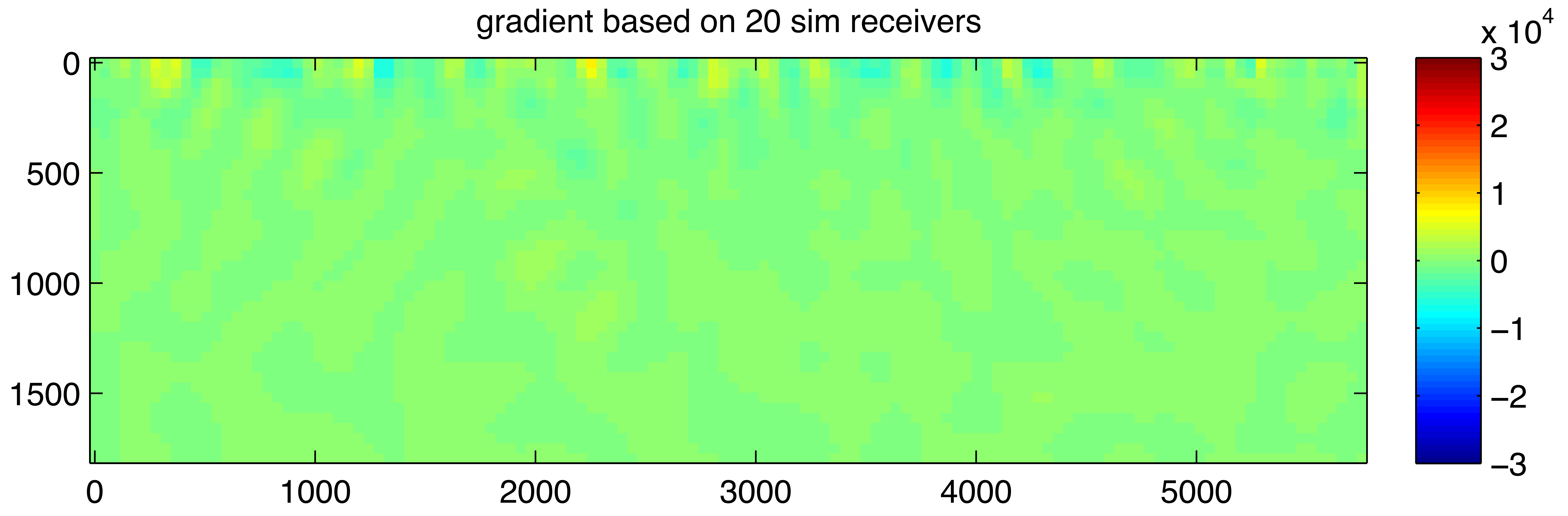
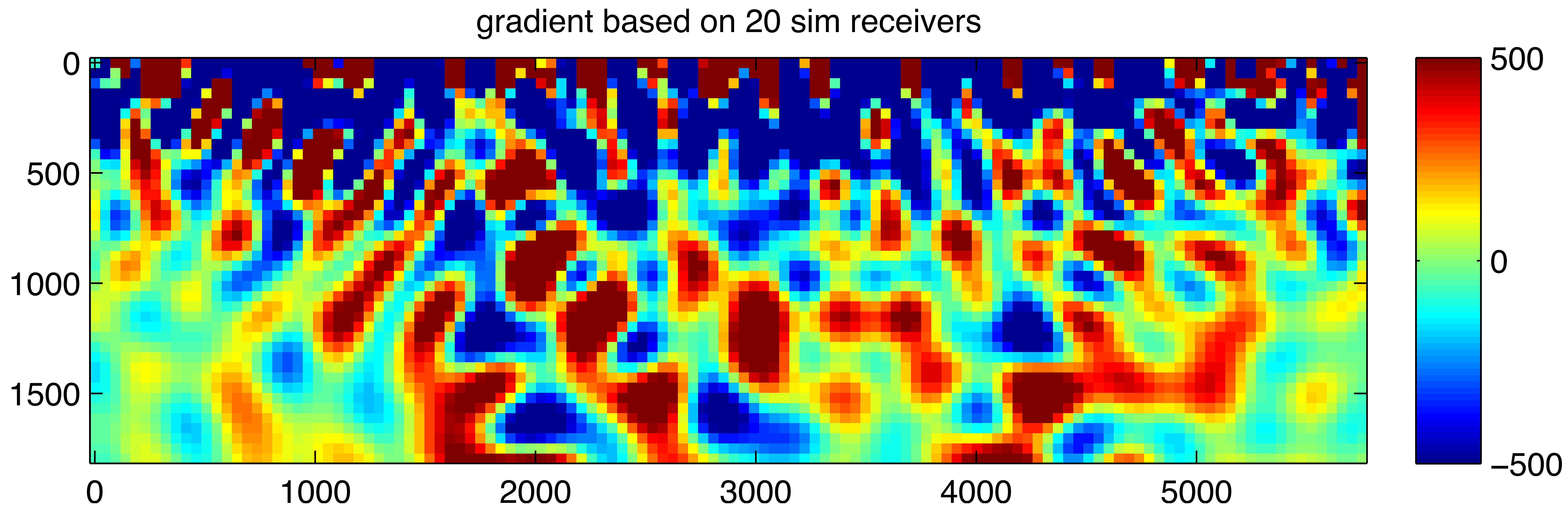
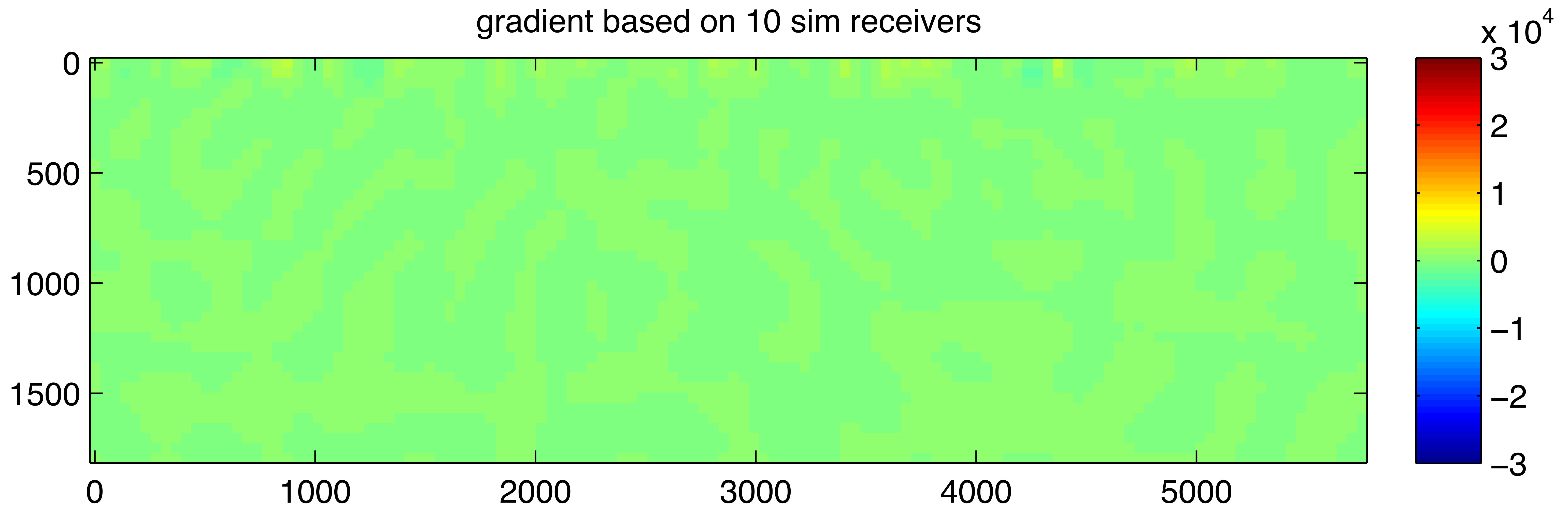
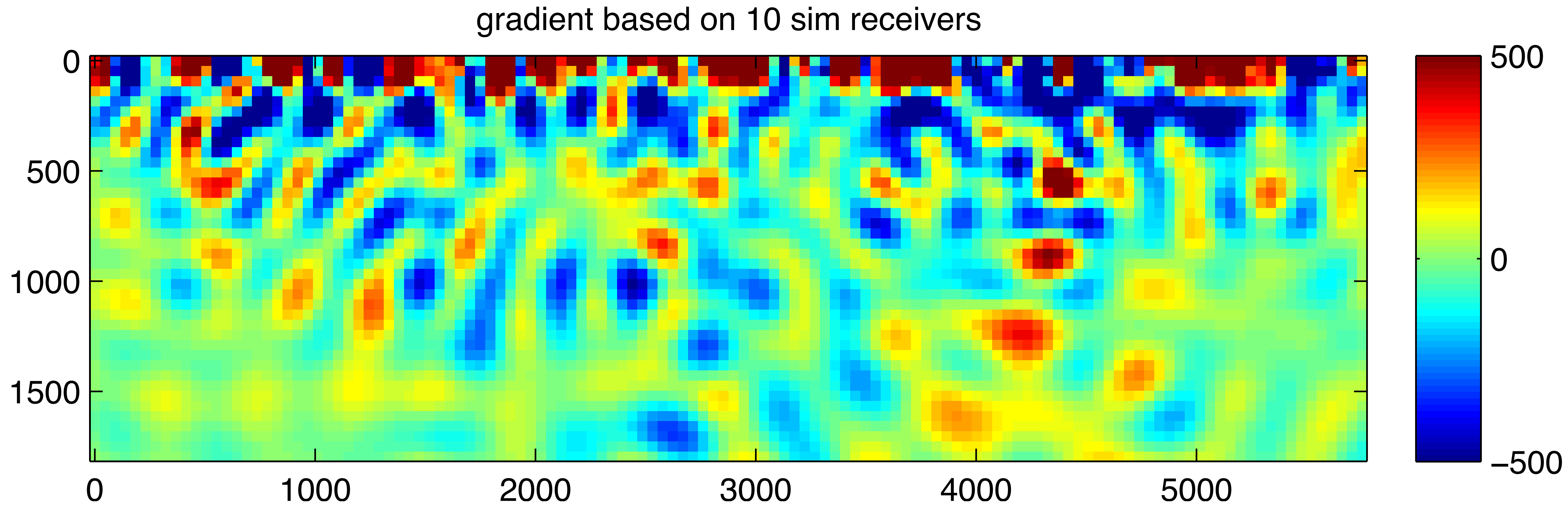


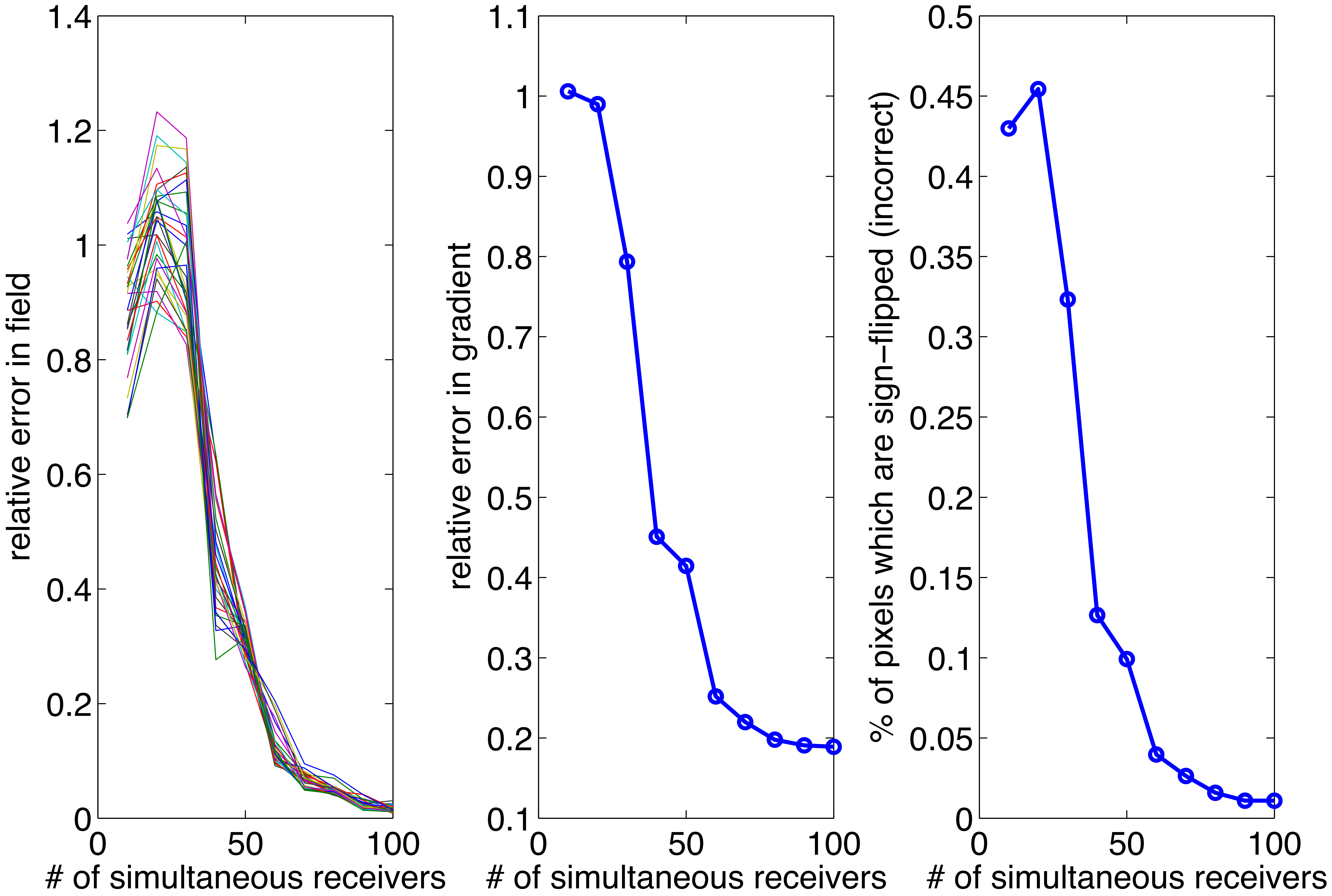


Simultaneous receivers



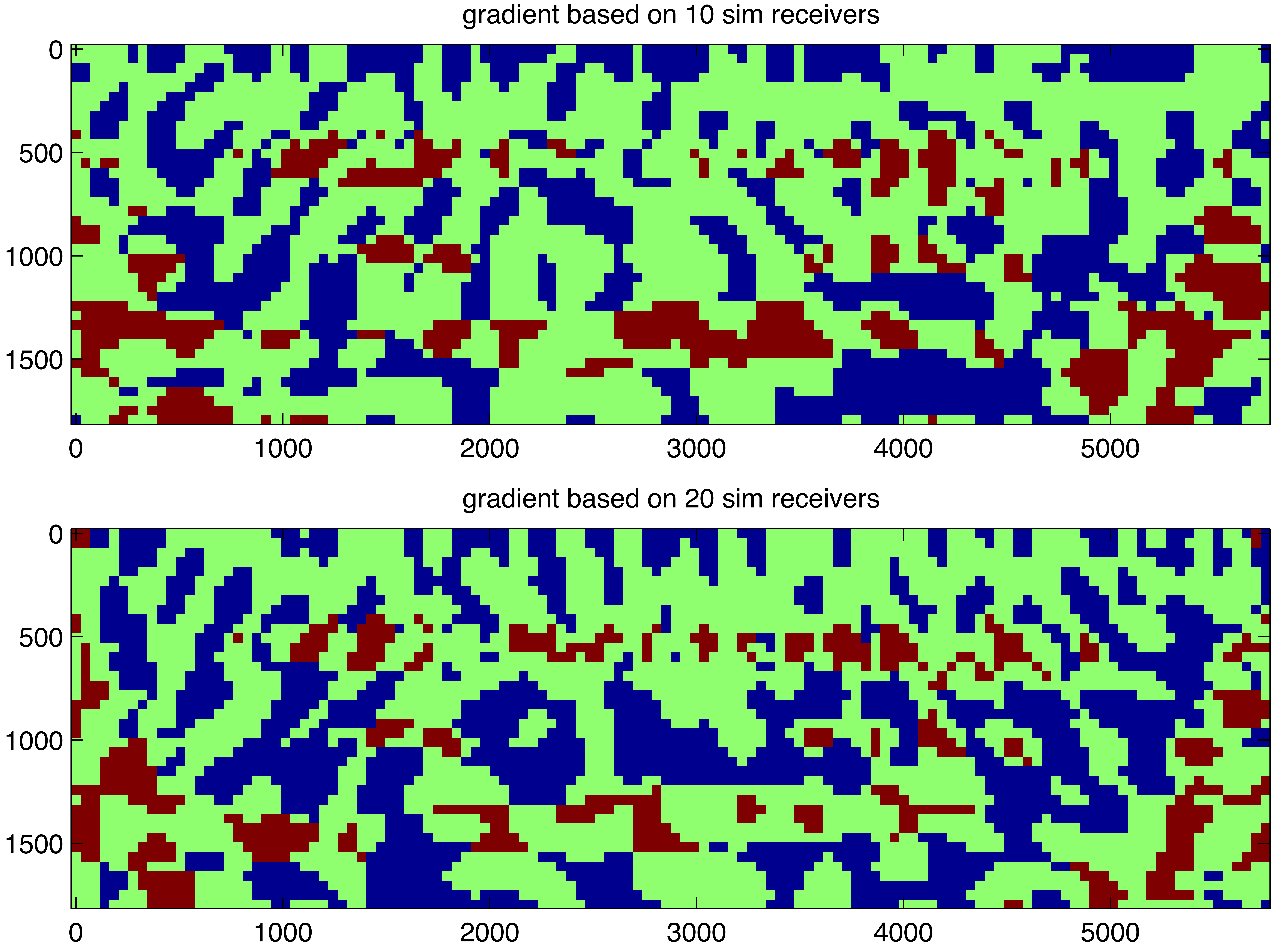


Simultaneous receivers



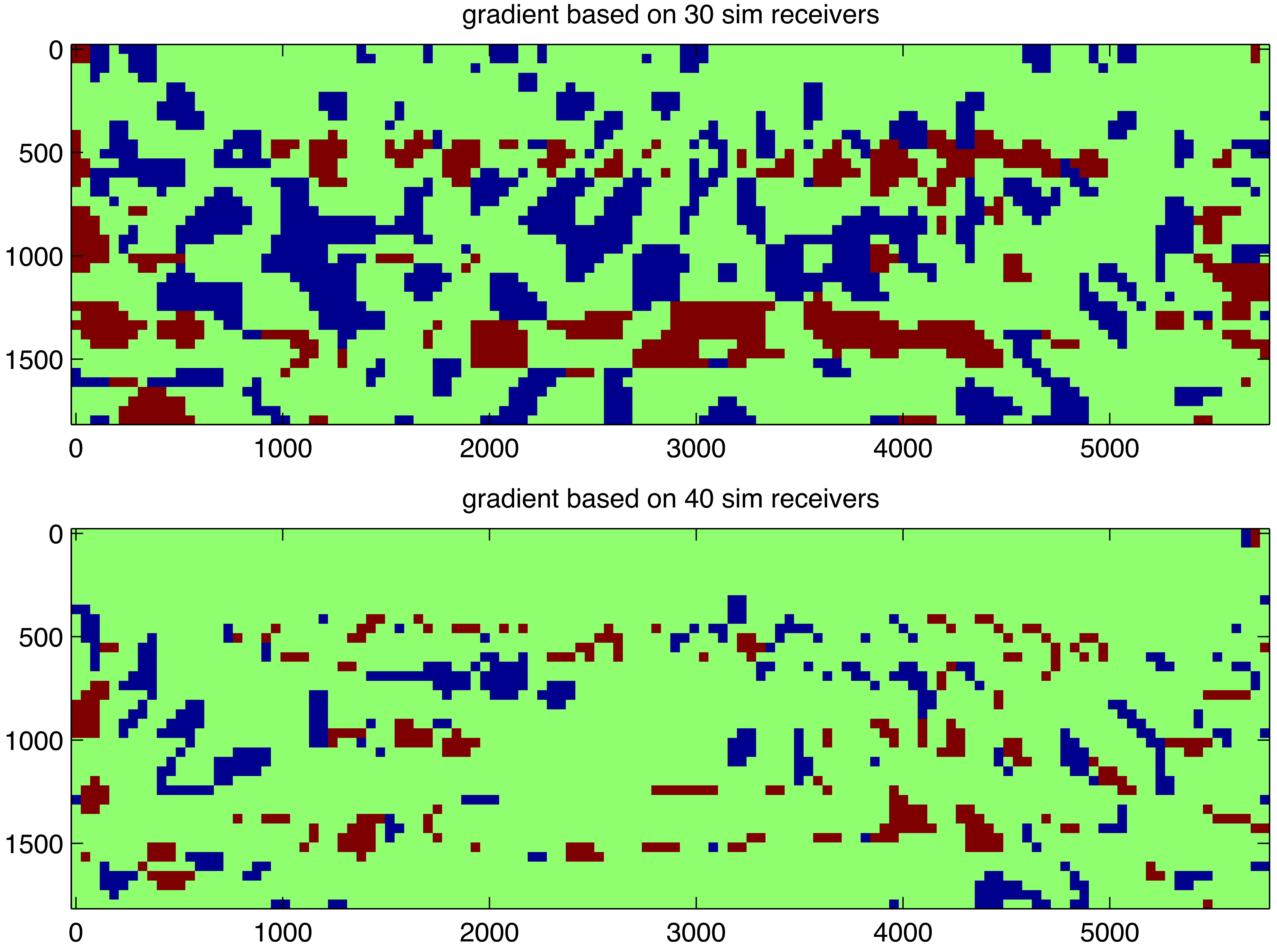
Simultaneous receivers

$$\text{sign}(\mathbf{g}_*) - \text{sign}(\hat{\mathbf{g}})$$



Simultaneous receivers

$\text{sign}(\mathbf{g}_*) - \text{sign}(\hat{\mathbf{g}})$



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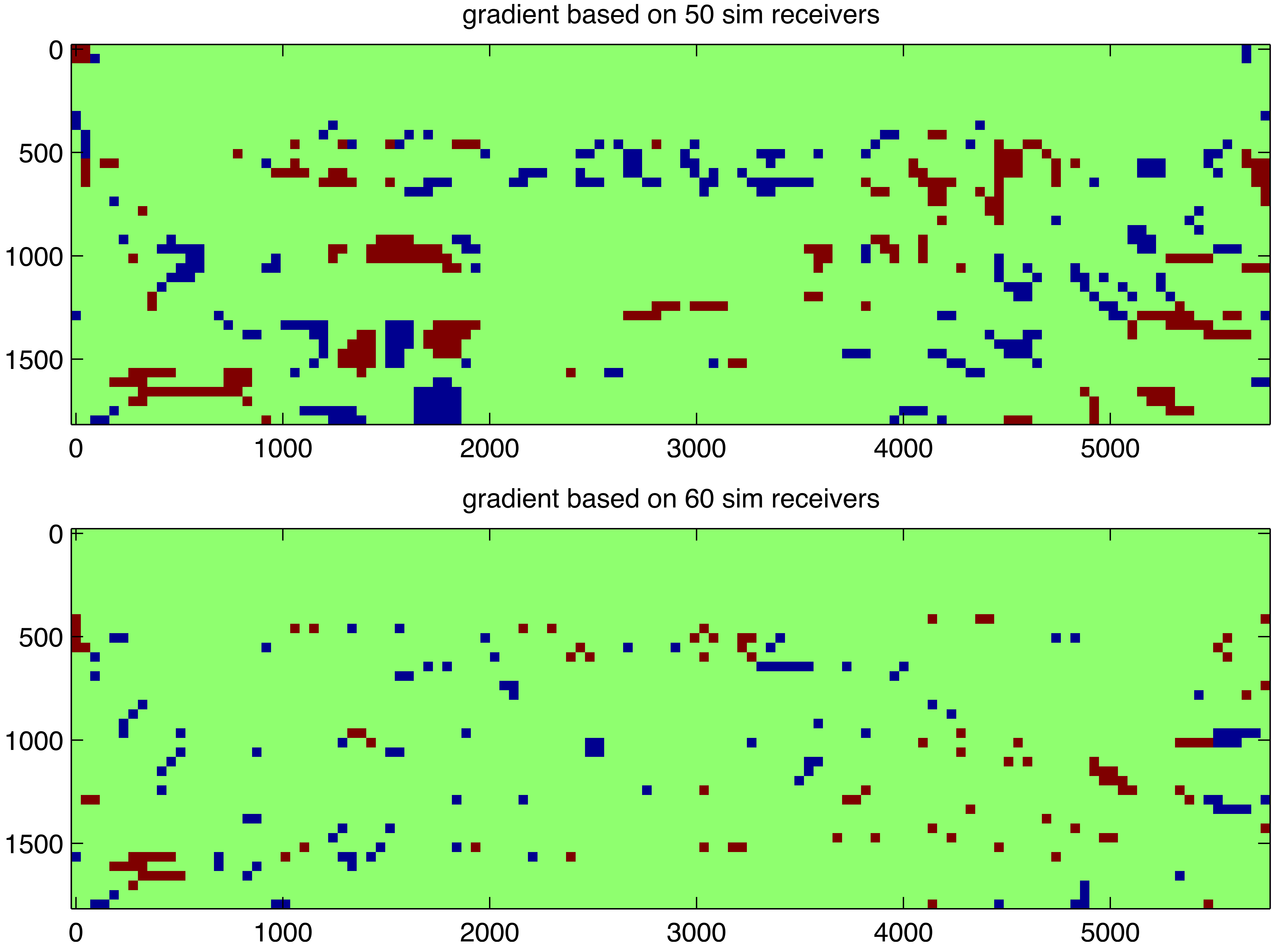

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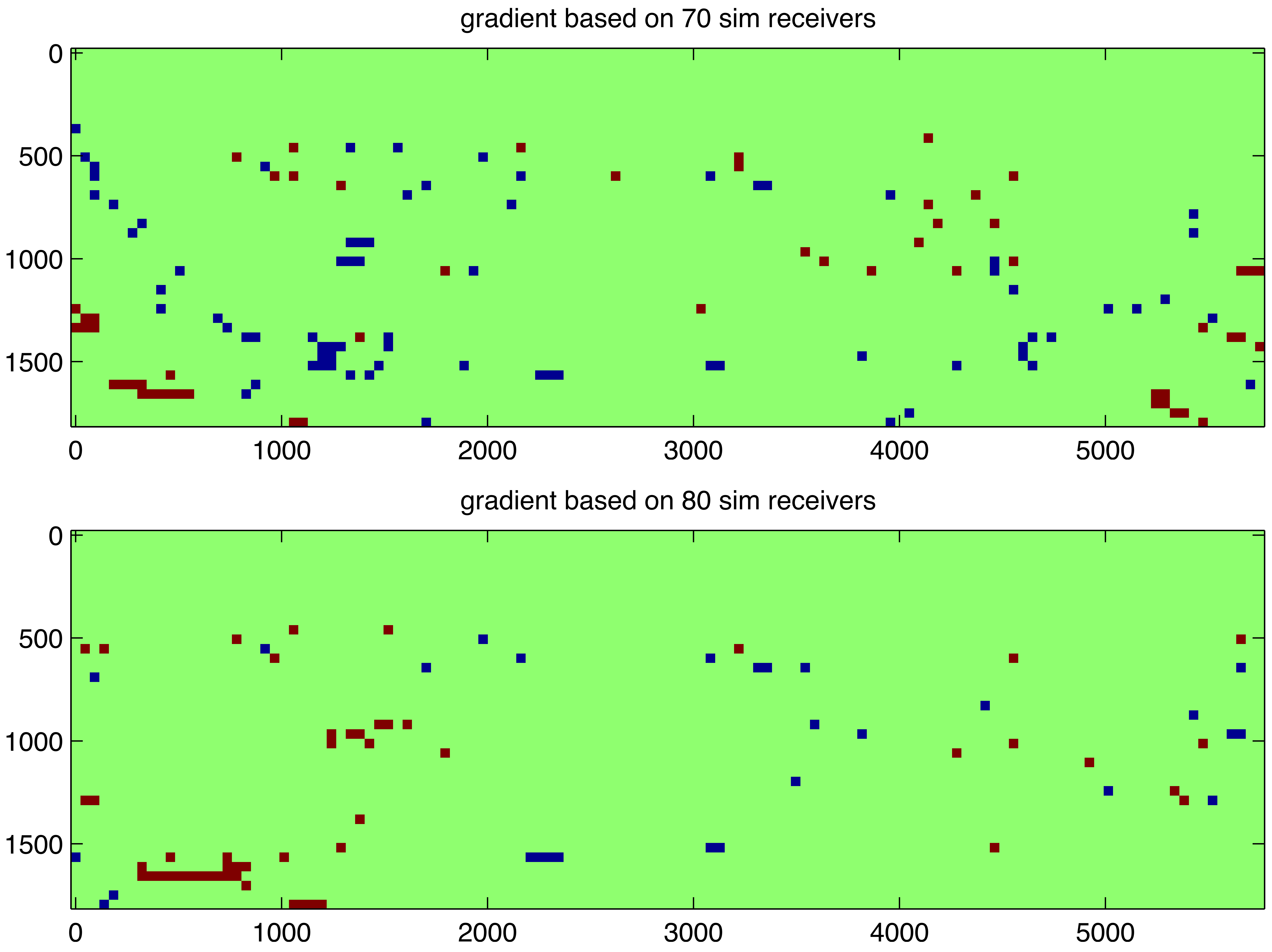

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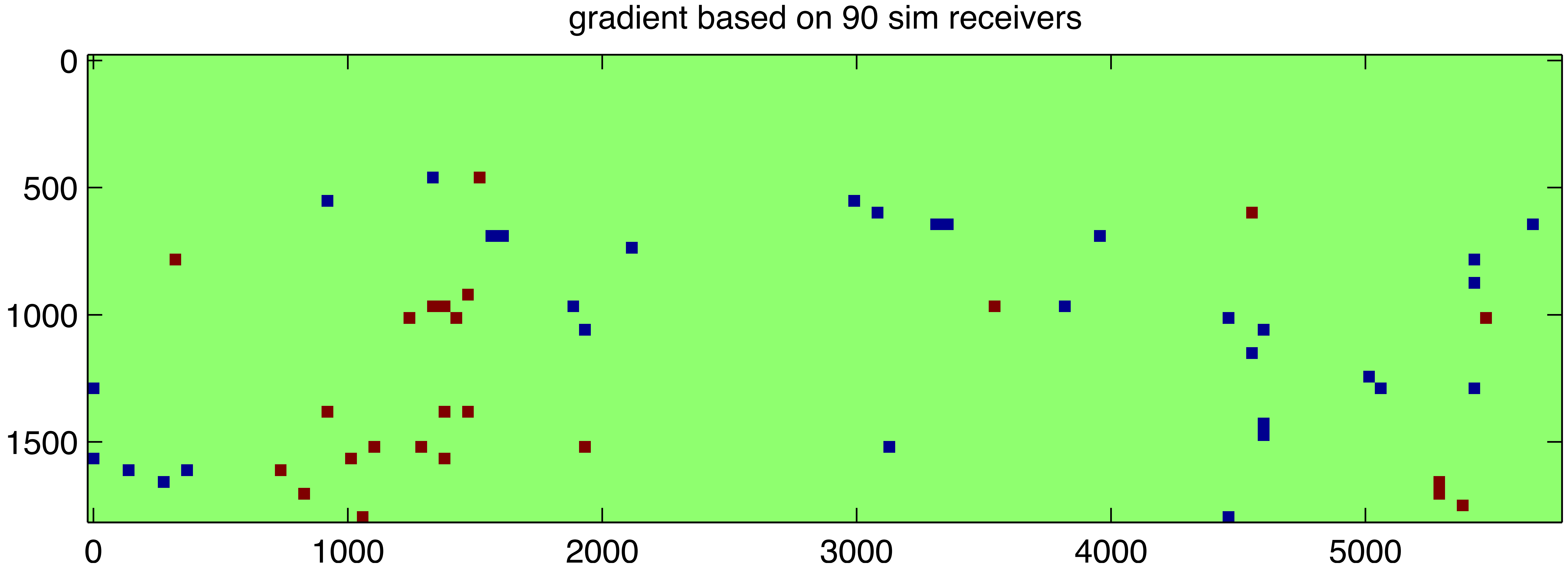
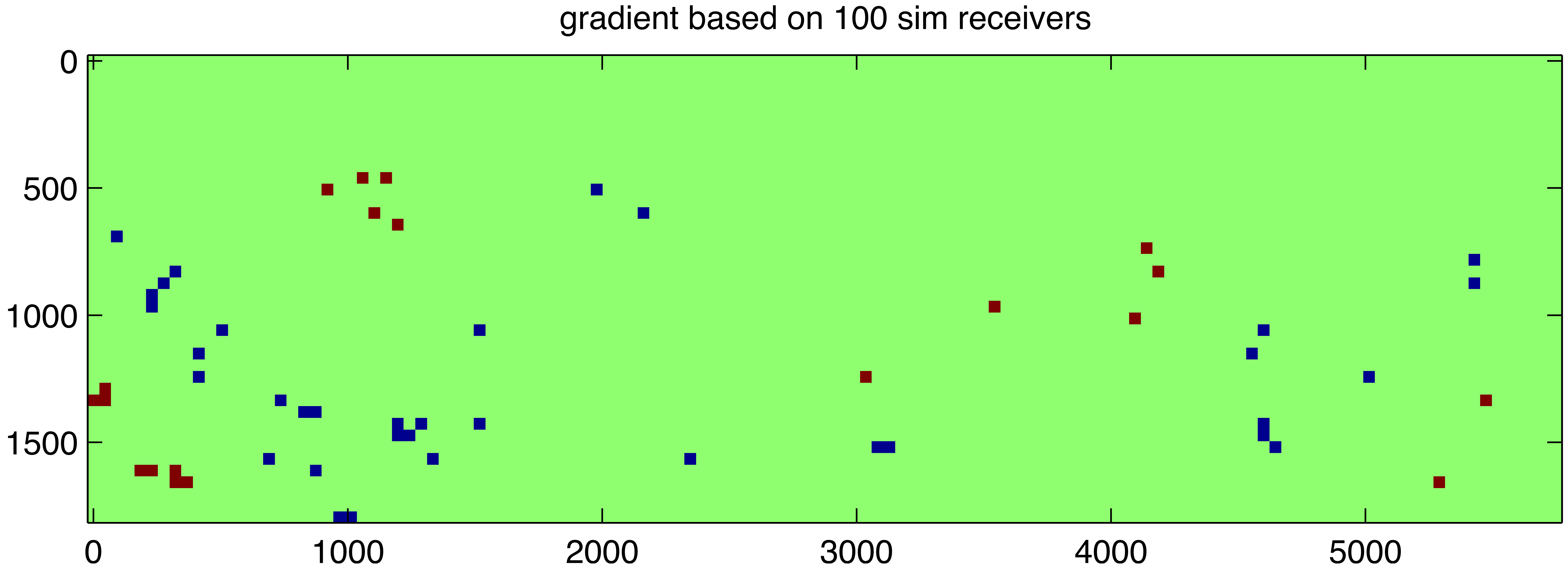
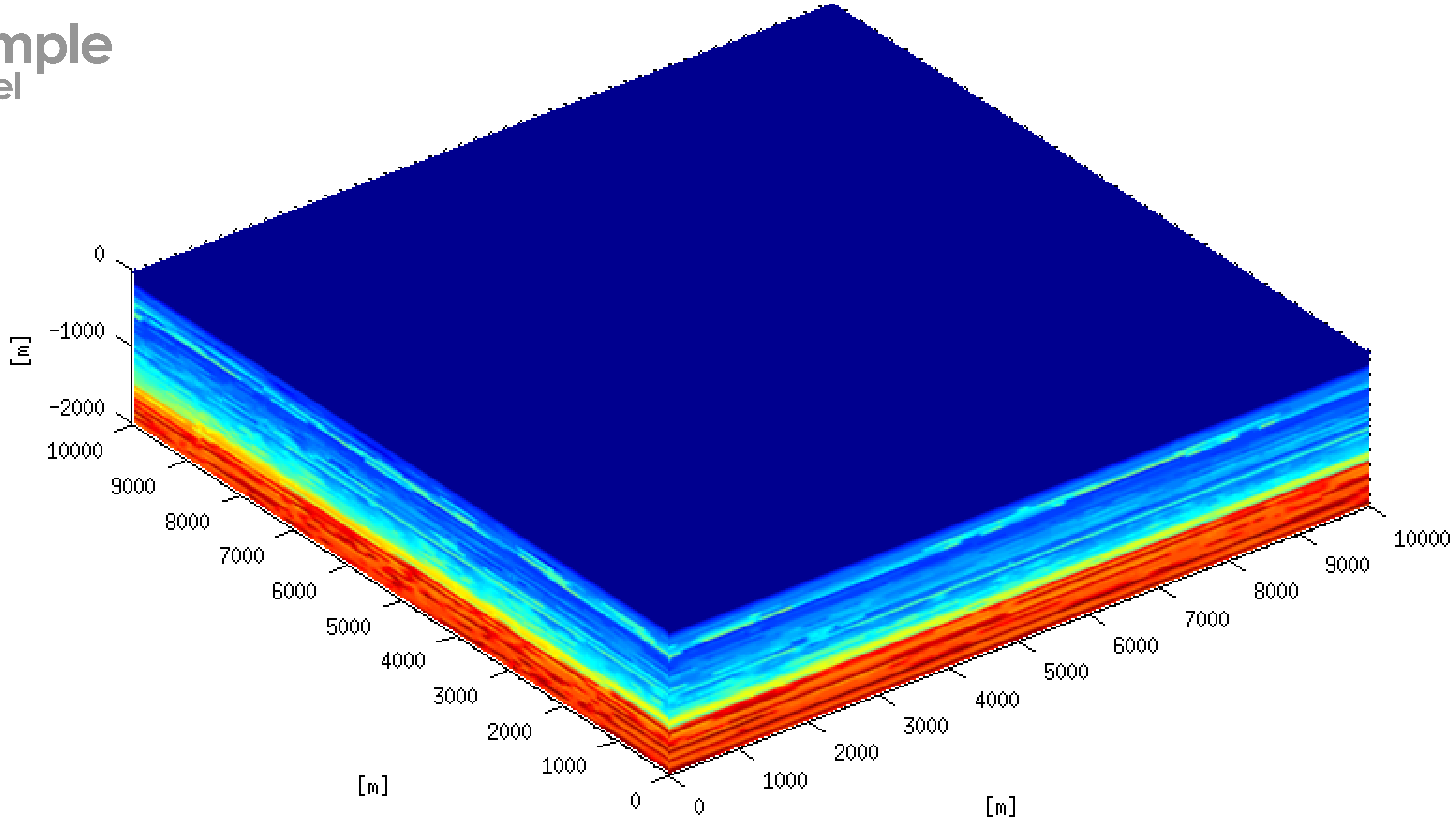


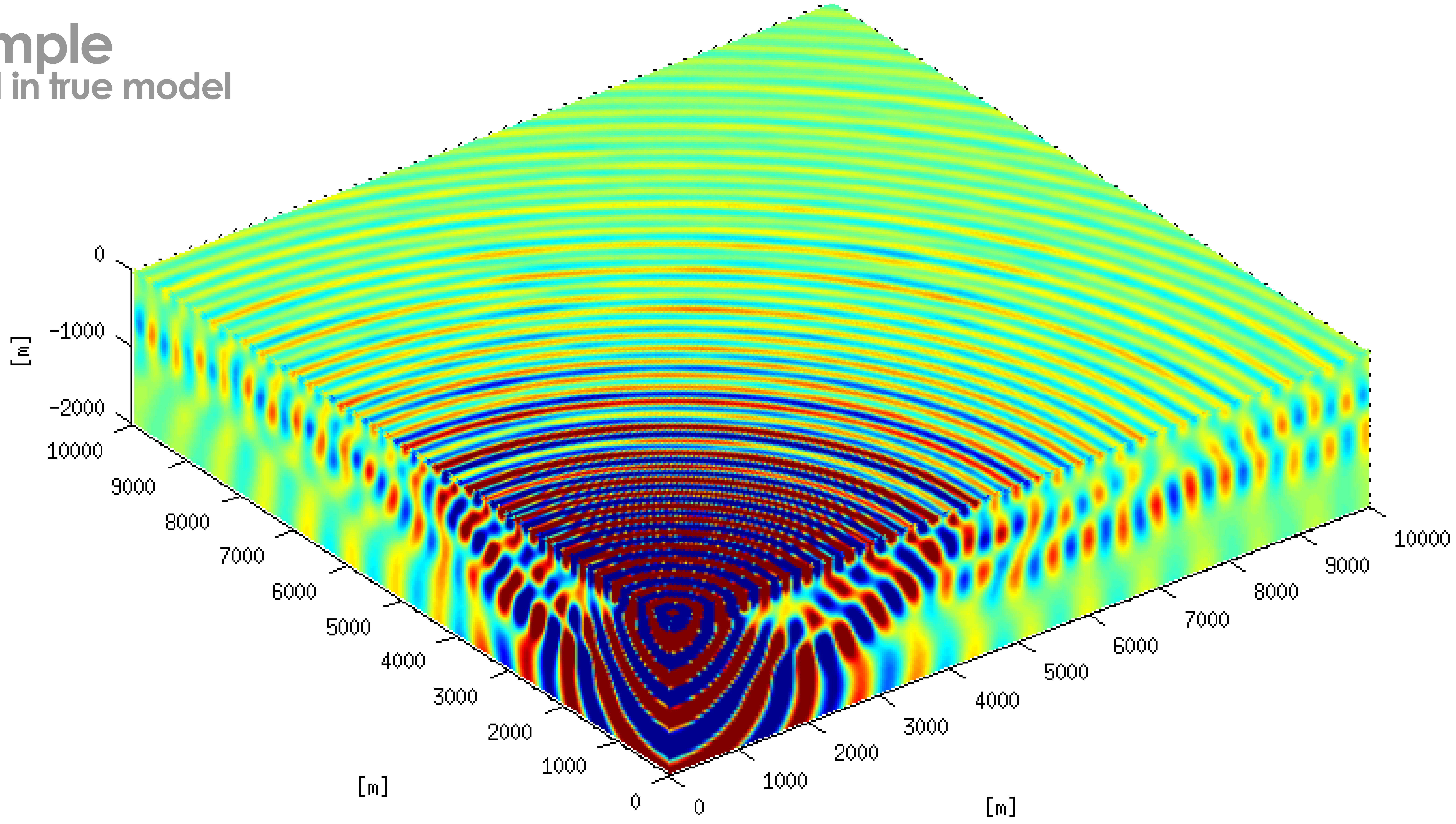


3D Example
– true model

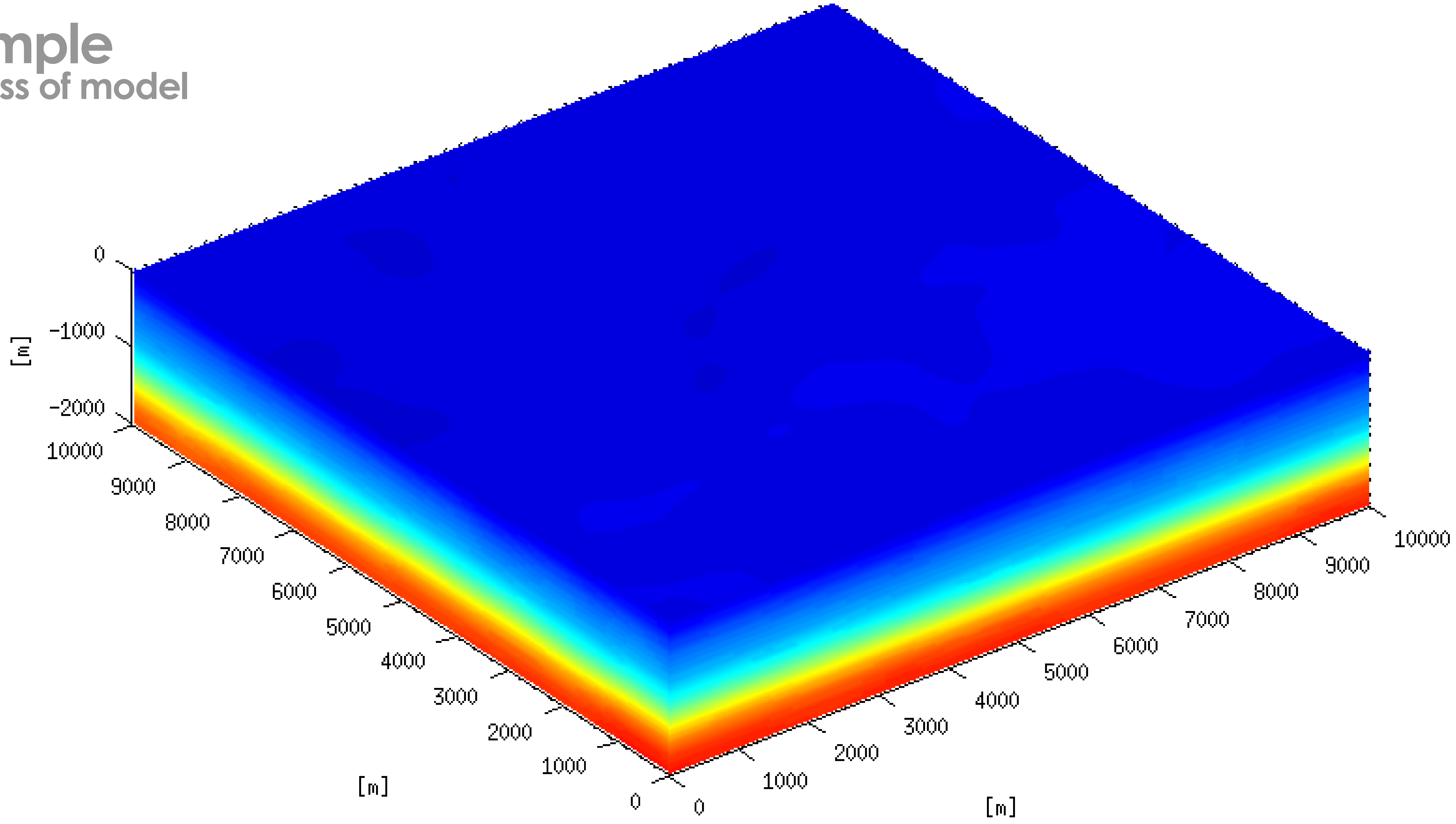


10 x 10 x 2 km, 5 Hz, 27-point discretization, ~1e7 grid points, source at [0,0,0]

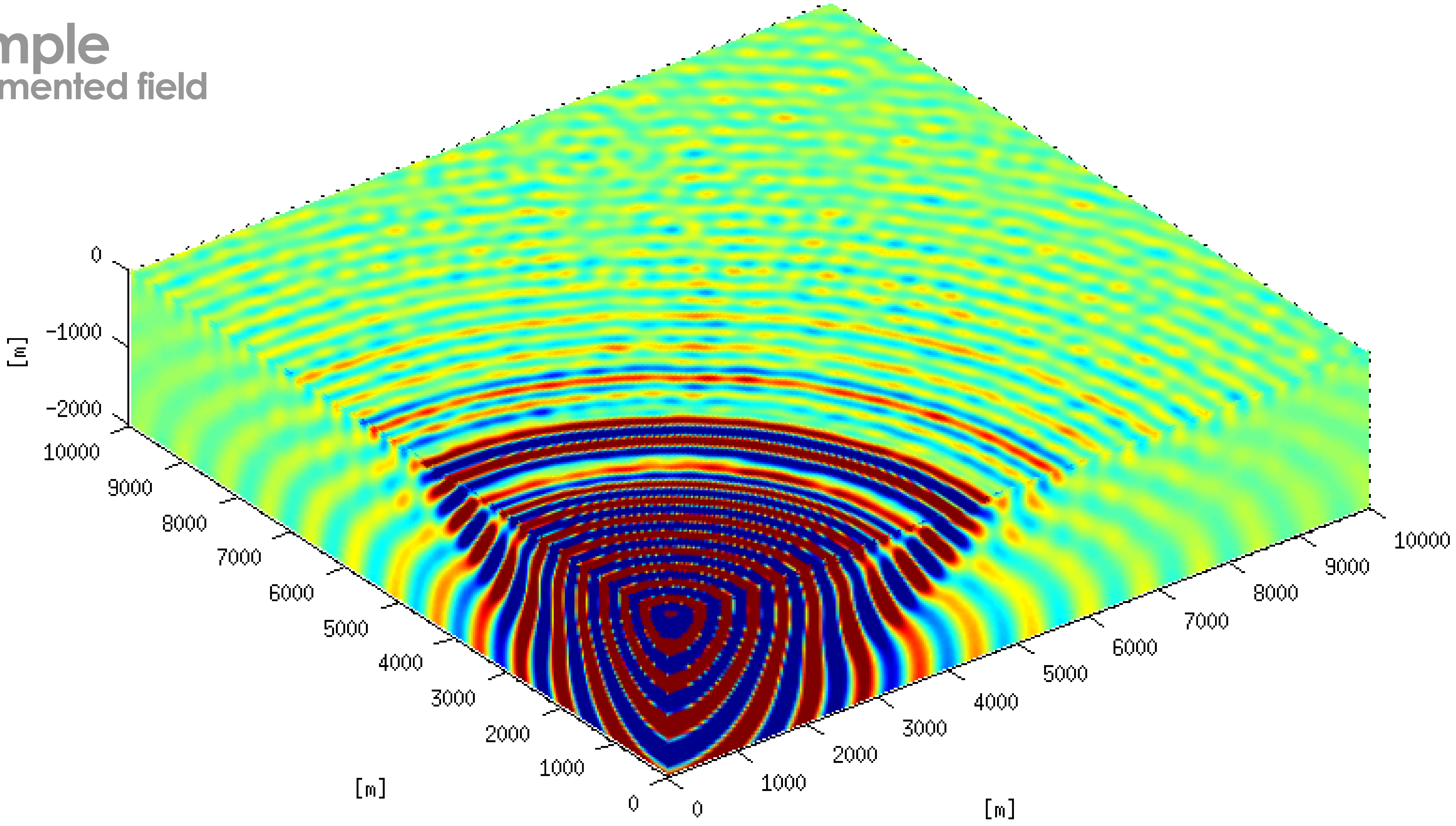
3D Example
– wavefield in true model



3D Example
– initial guess of model



3D Example
– data-augmented field



Conclusions

- Enables WRI in 3D.
- Accepts any Helmholtz solver for the sub-problems.
- Compute 1 Helmholtz problem per source and 1 per receiver.
- Store 1 vector per receiver.
- Can use simultaneous receivers to reduce computational cost and memory use.
- Proposed algorithm might be used for other large-scale mildly overdetermined problems w/ many variables & few constraints.

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Future work

- Answer some open questions on how to combine simultaneous receiver and simultaneous source principles.
- Adapt optimization algorithms to be able to work with simultaneous sources & receivers. (partially done)
- Error bound derivations are work in progress.

Acknowledgements

Tristan van Leeuwen, Art Petrenko & Rafael Lago for the CGMN & CARP-CG implementation

SINBAD

NSERC CRSNG

This work was in part financially supported by the Natural Sciences and Engineering Research Council of Canada via the Collaborative Research and Development Grant DNOISEII (375142--08). This research was carried out as part of the SINBAD II project which is supported by the following organizations: BG Group, BGP, CGG, Chevron, ConocoPhillips, DownUnder GeoSolutions, Hess, Petrobras, PGS, Schlumberger, Sub Salt Solutions and Woodside

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