Uncertainty quantification for Wavefield-Reconstruction Inversion using a positive-definite approximated Hessian

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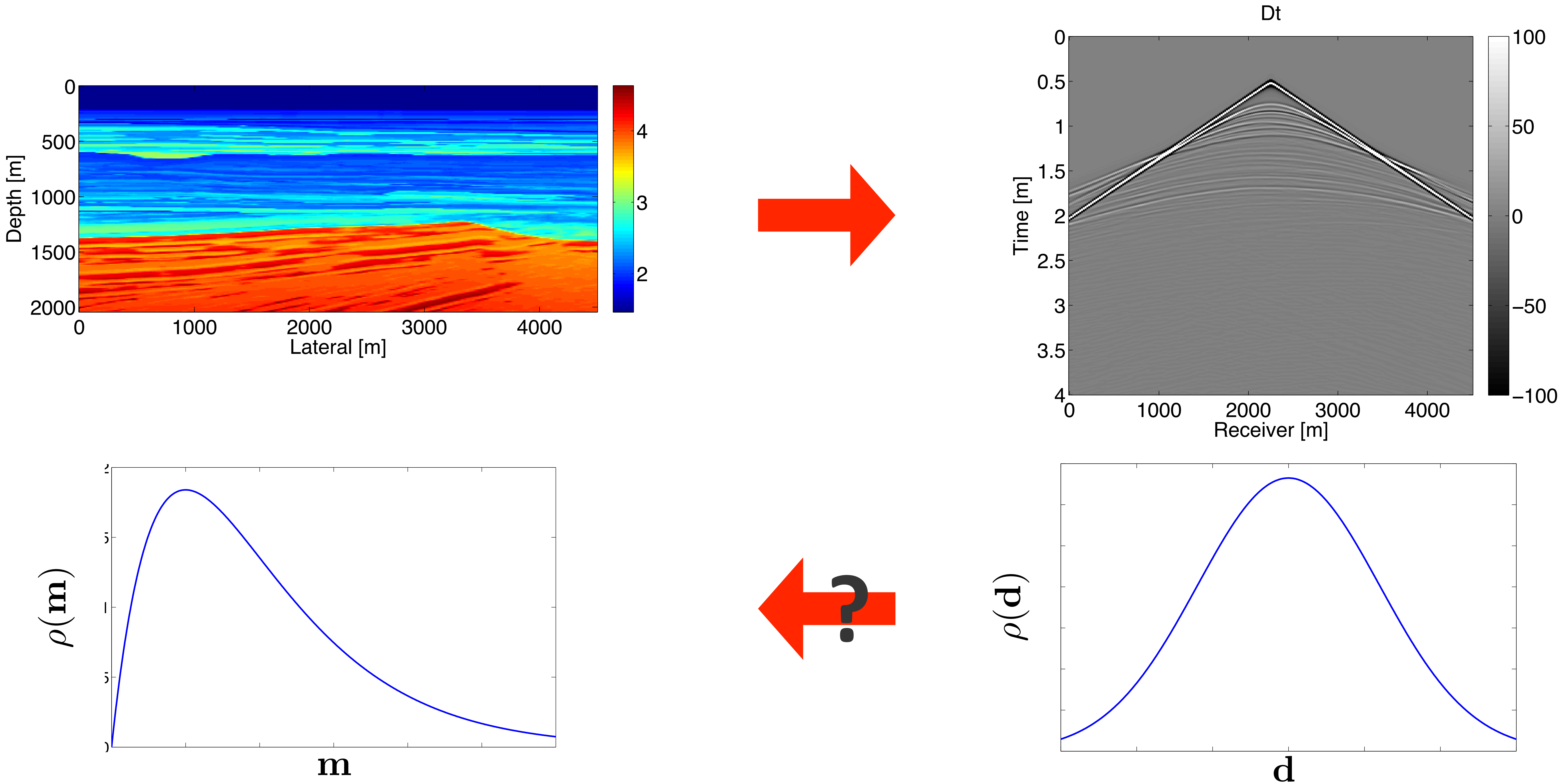
SLIM
University of British Columbia

Motivation



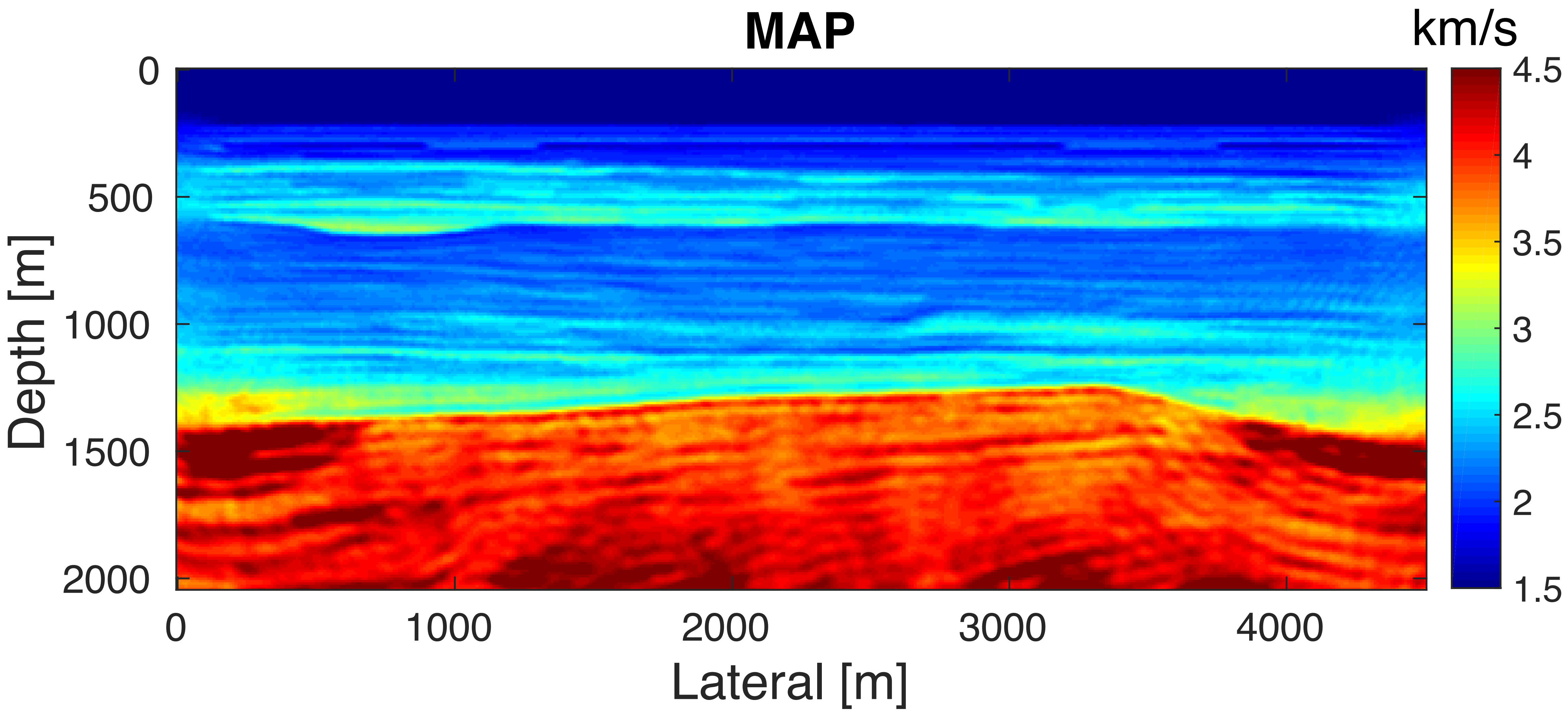
Distribution of model

Distribution of data

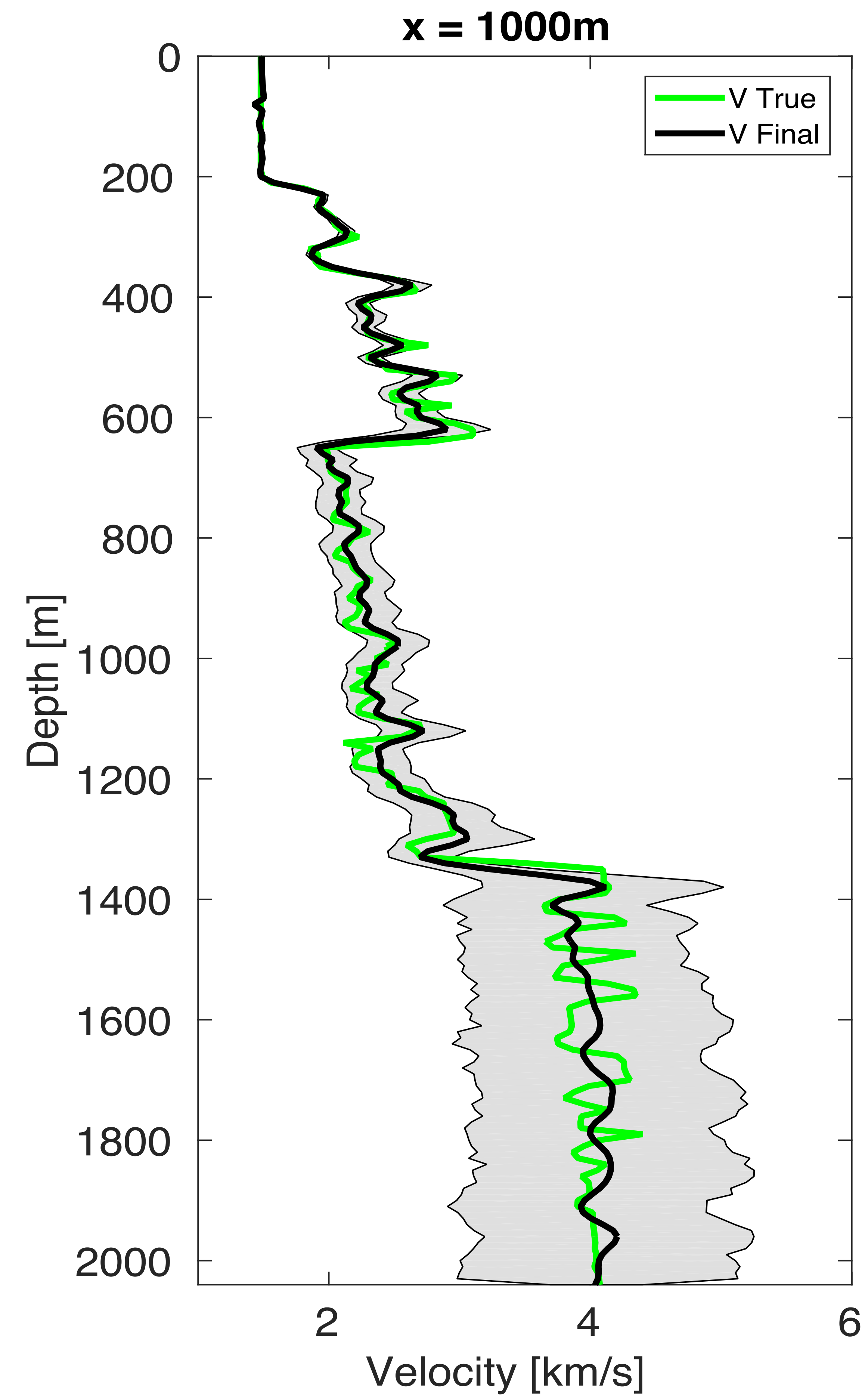
Motivation

Uncertainty of veolocity, density and elastic parameters → Uncertainty of porosity, saturation and permeability → Uncertainty of oil/gas volume

Motivation

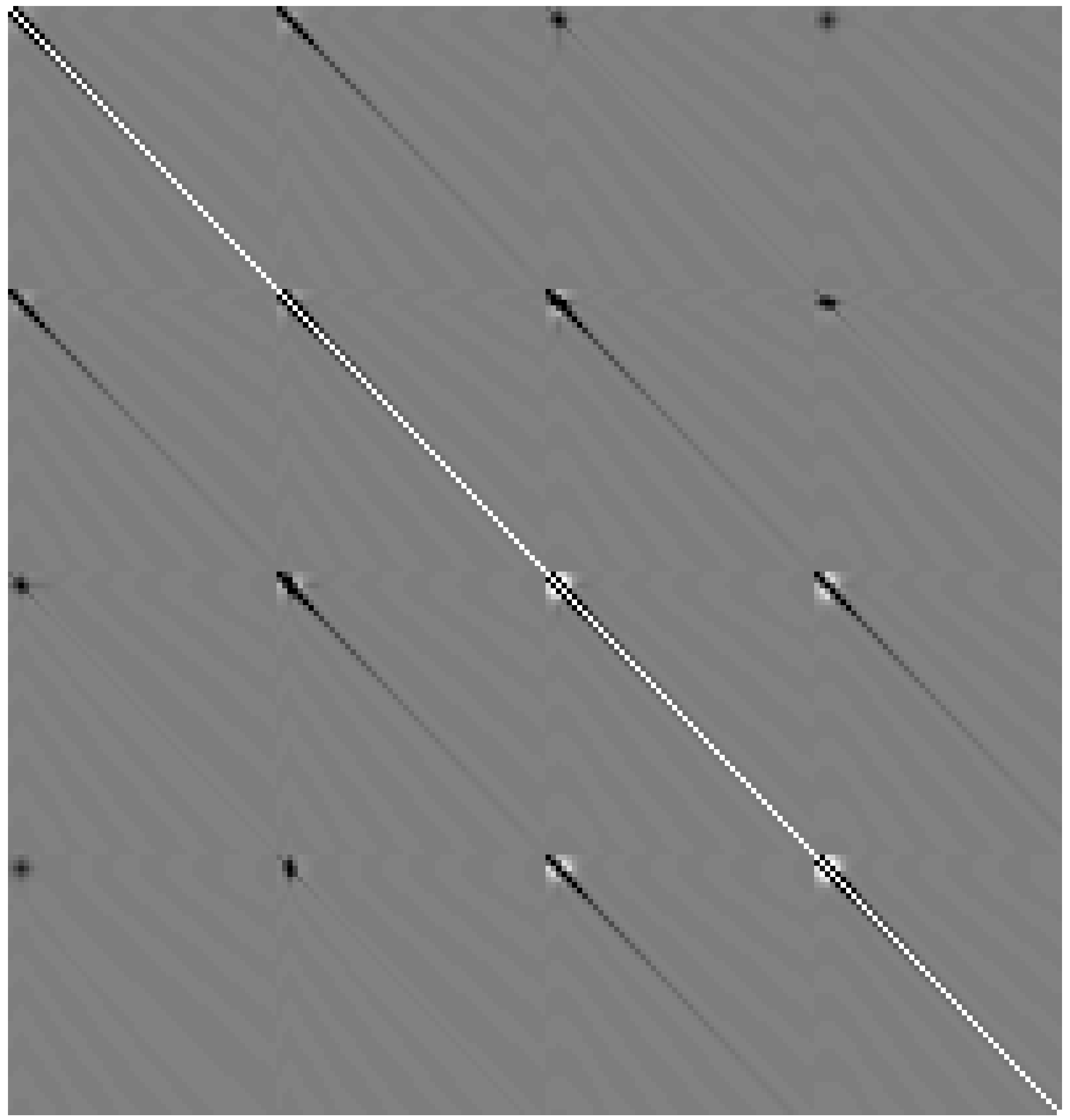


Inversion result



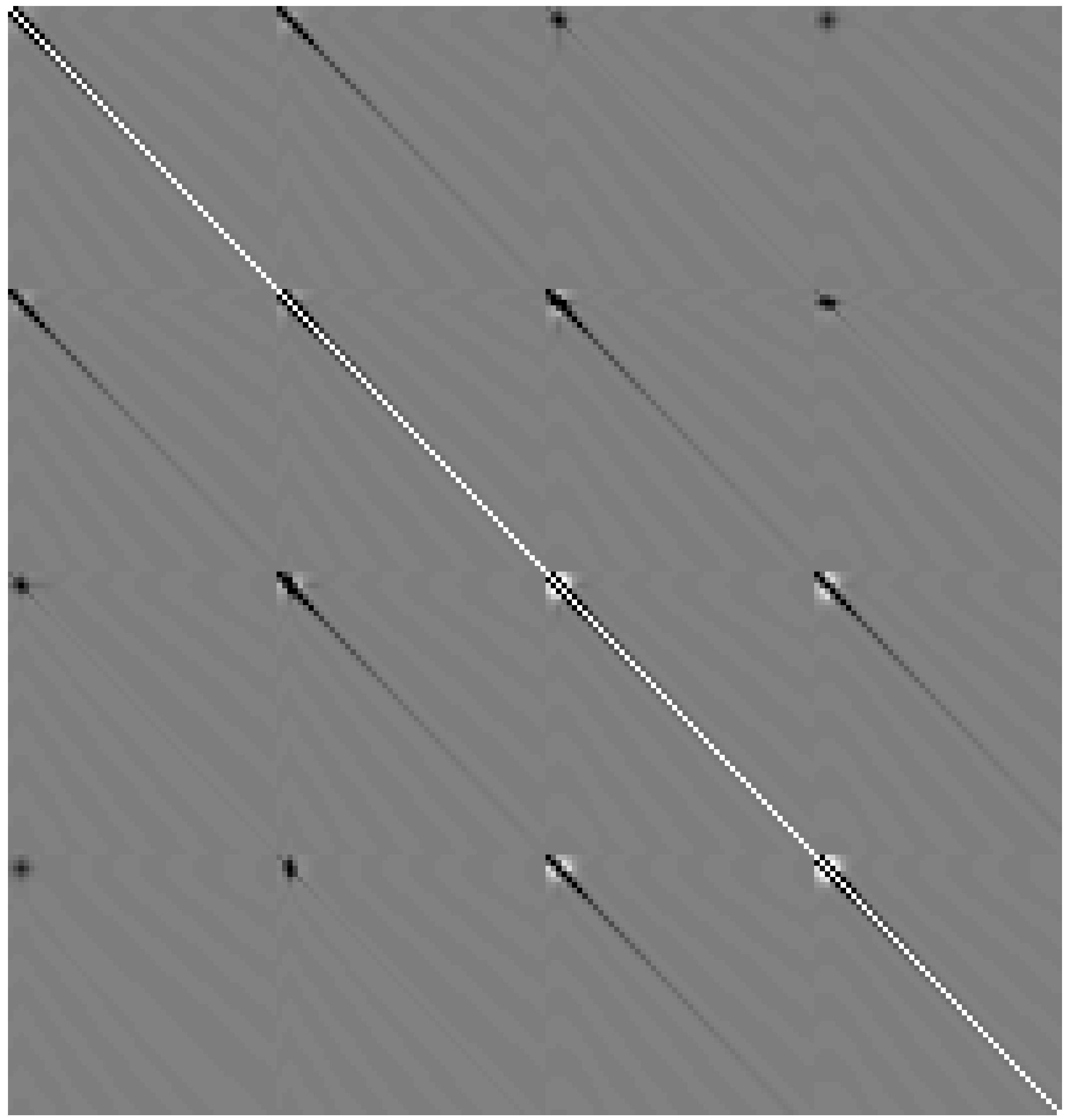
Confidence Interval

Motivation



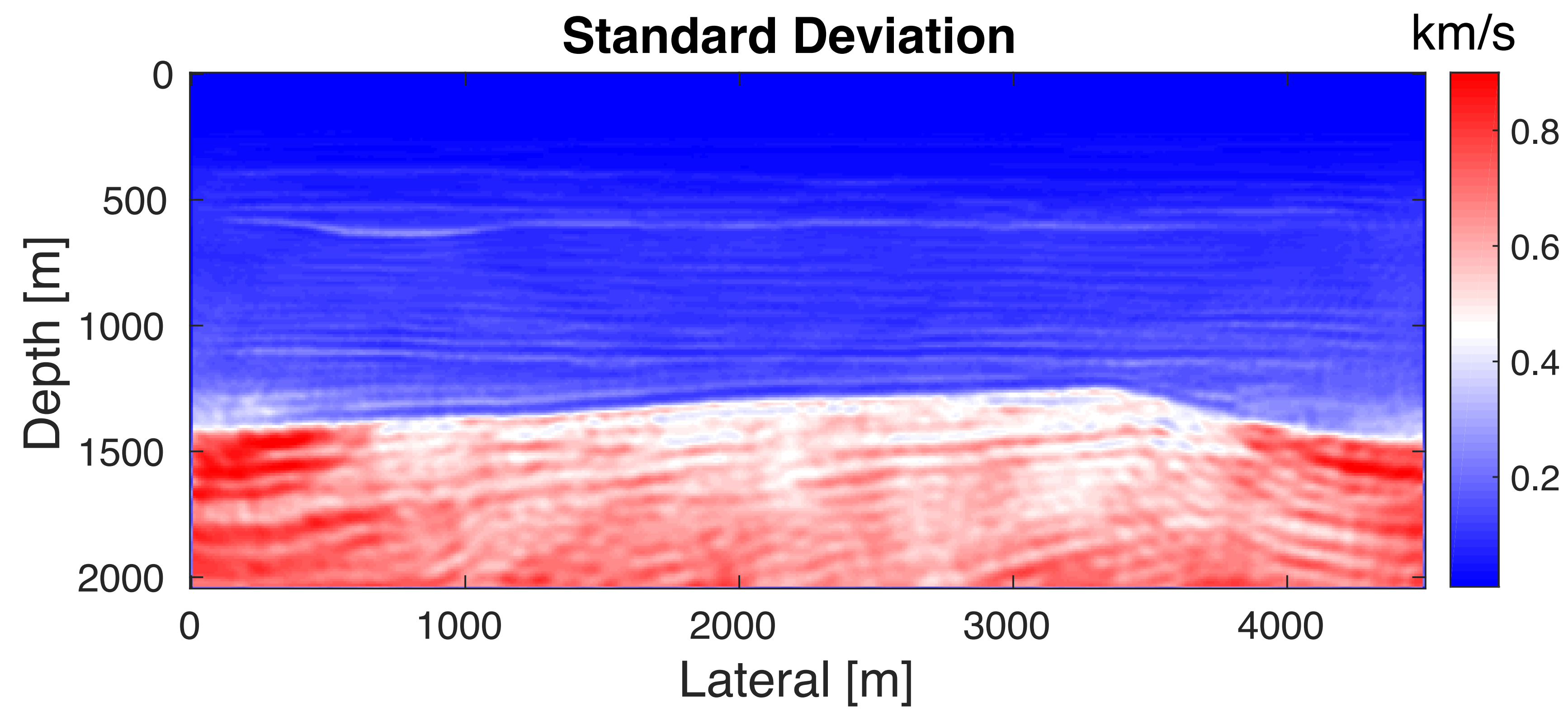
Hessian

Motivation



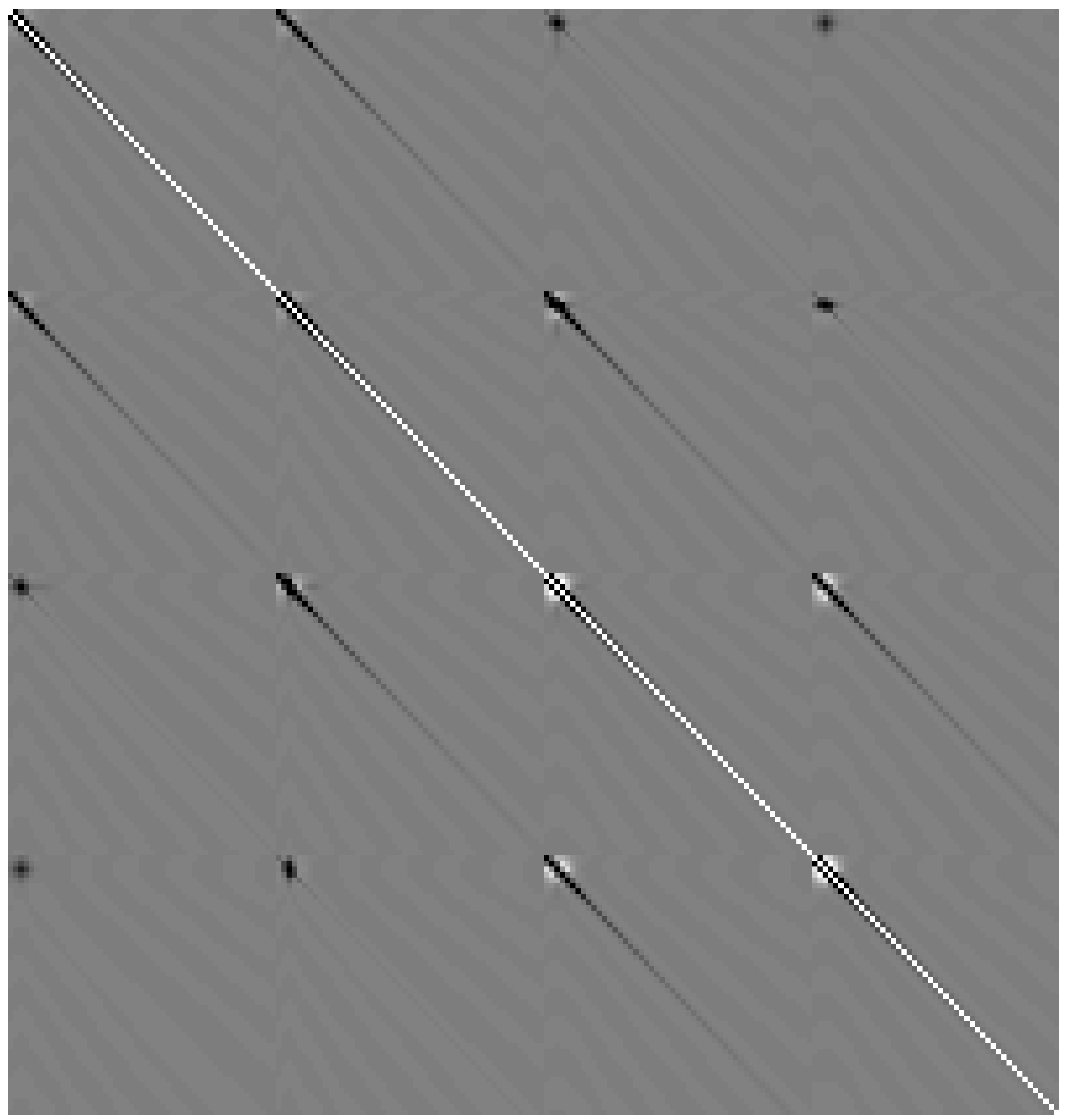
Hessian

statistical inversion



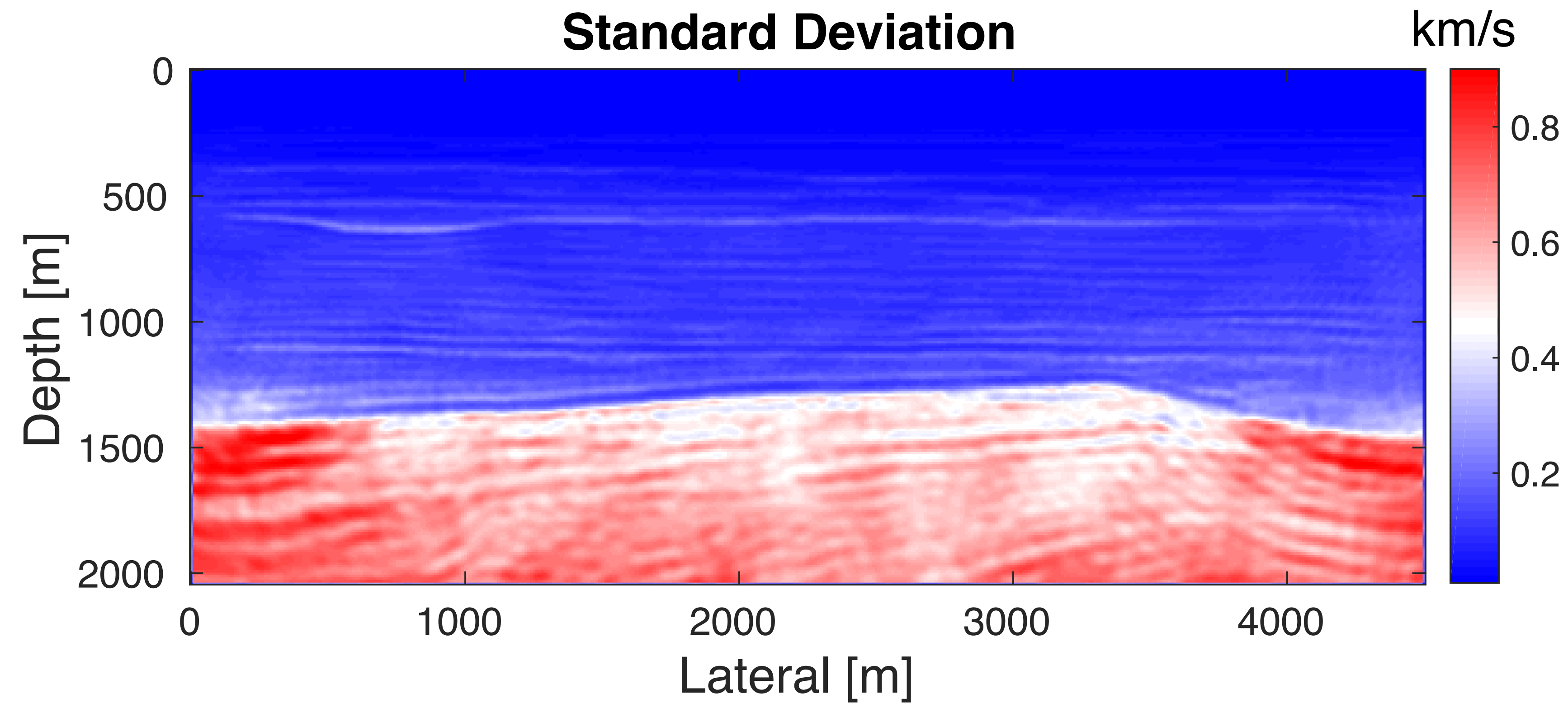
standard deviation

Motivation



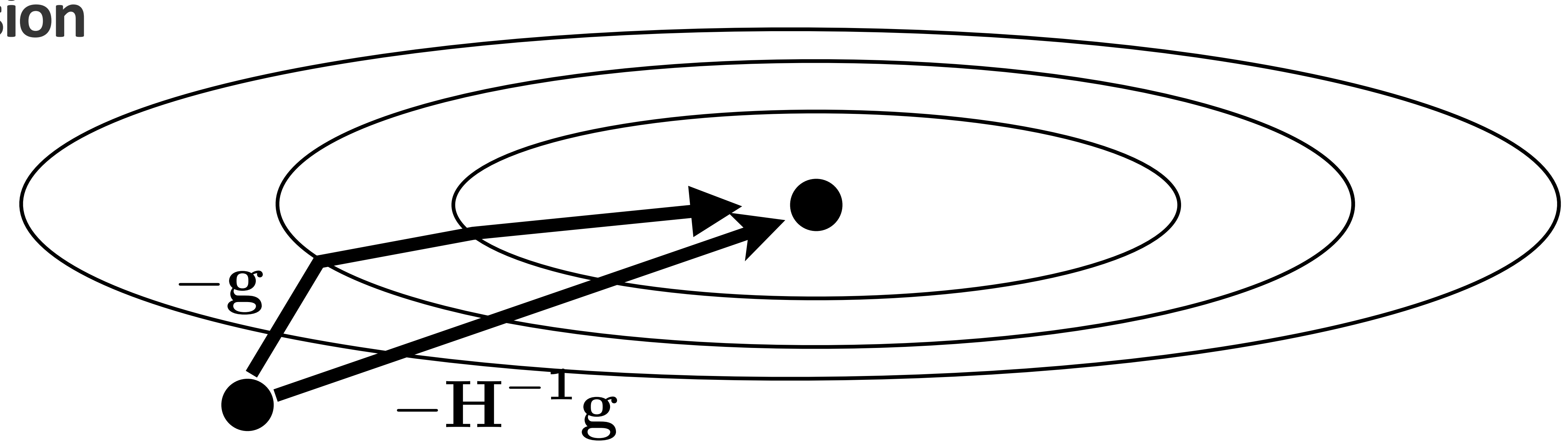
Hessian

statistical
inversion



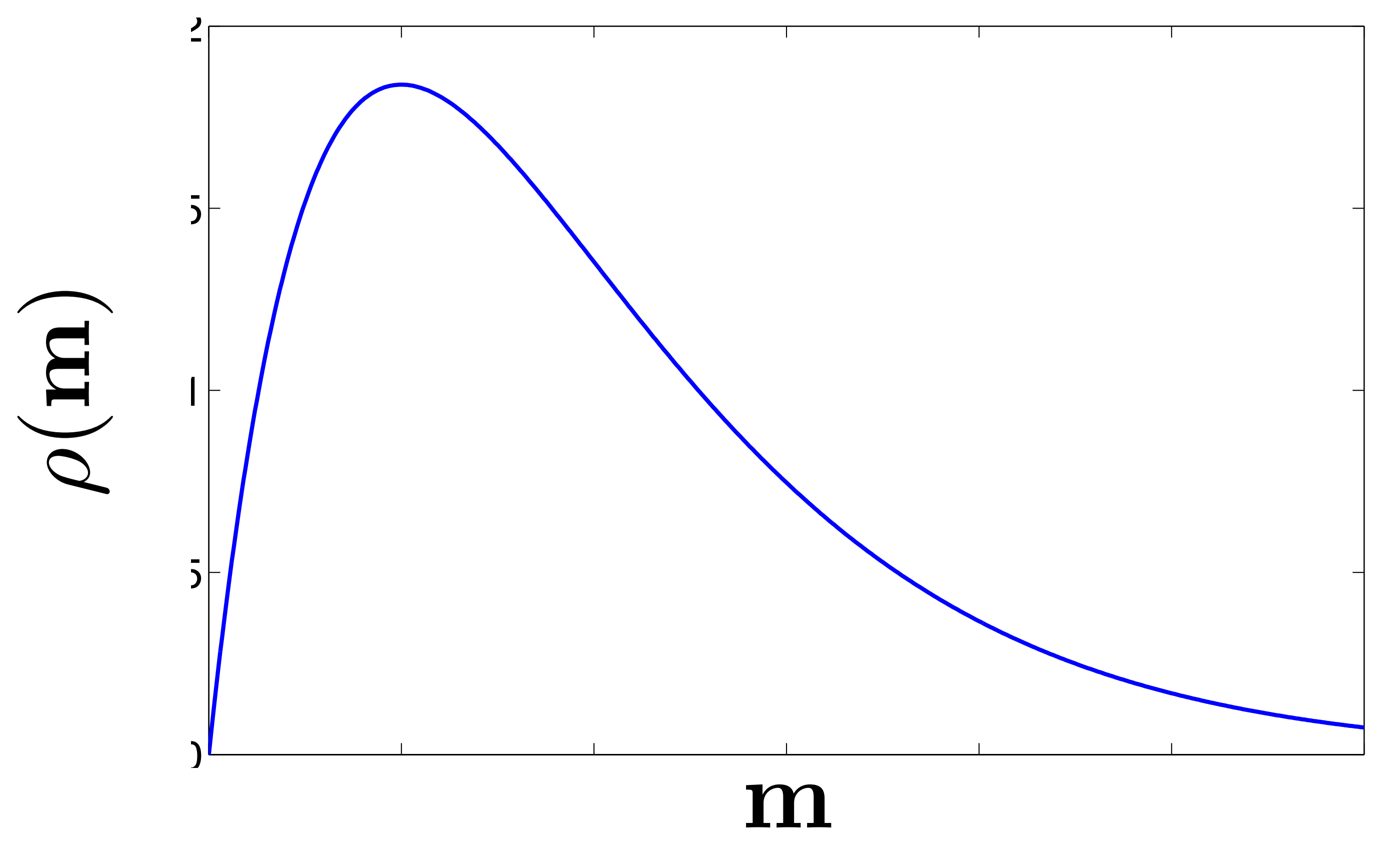
standard deviation

deterministic
inversion

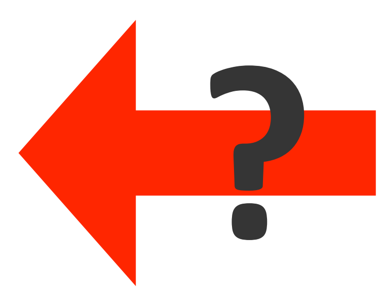


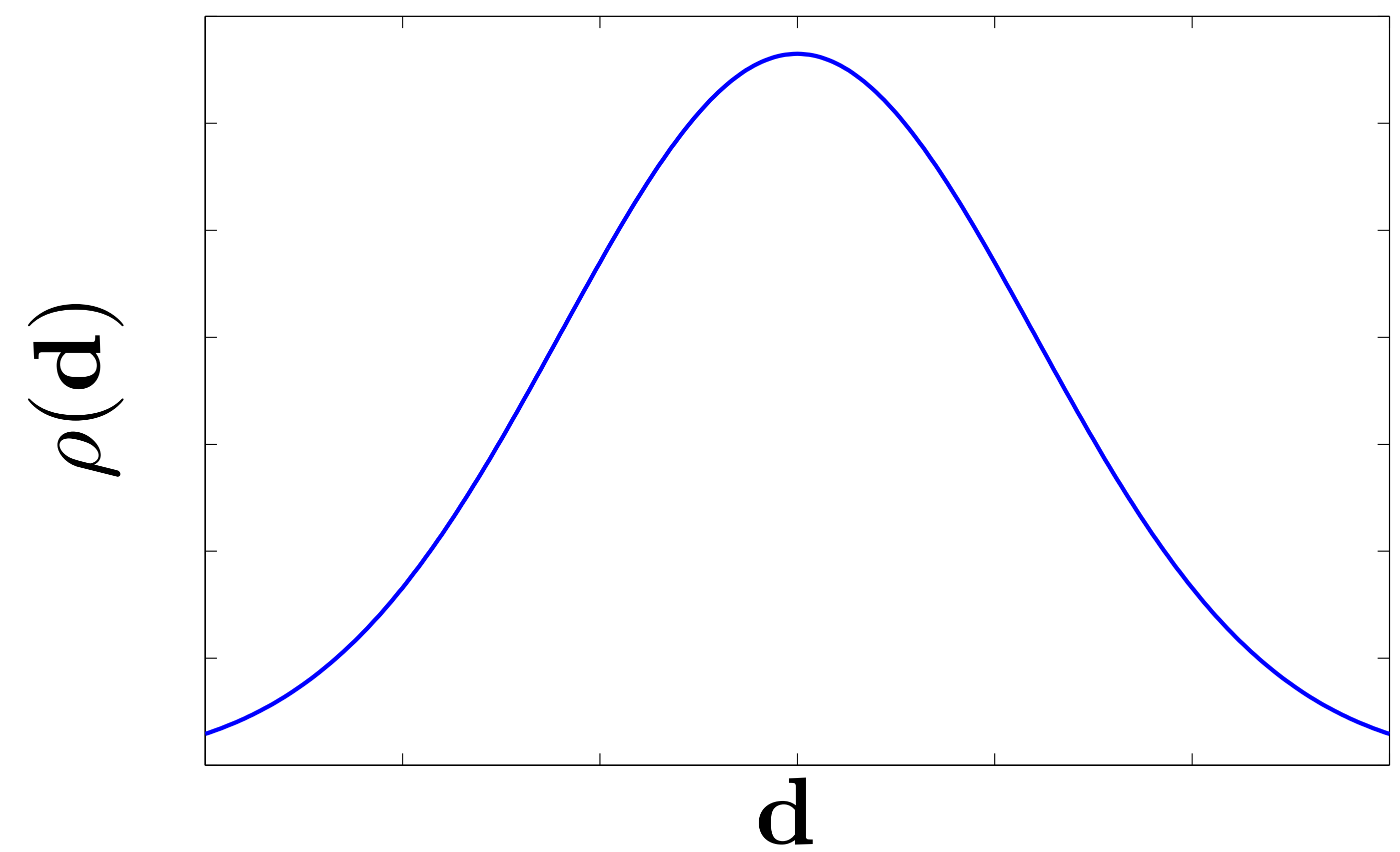
Newton type method

Bayesian theory



Distribution of model





Distribution of data

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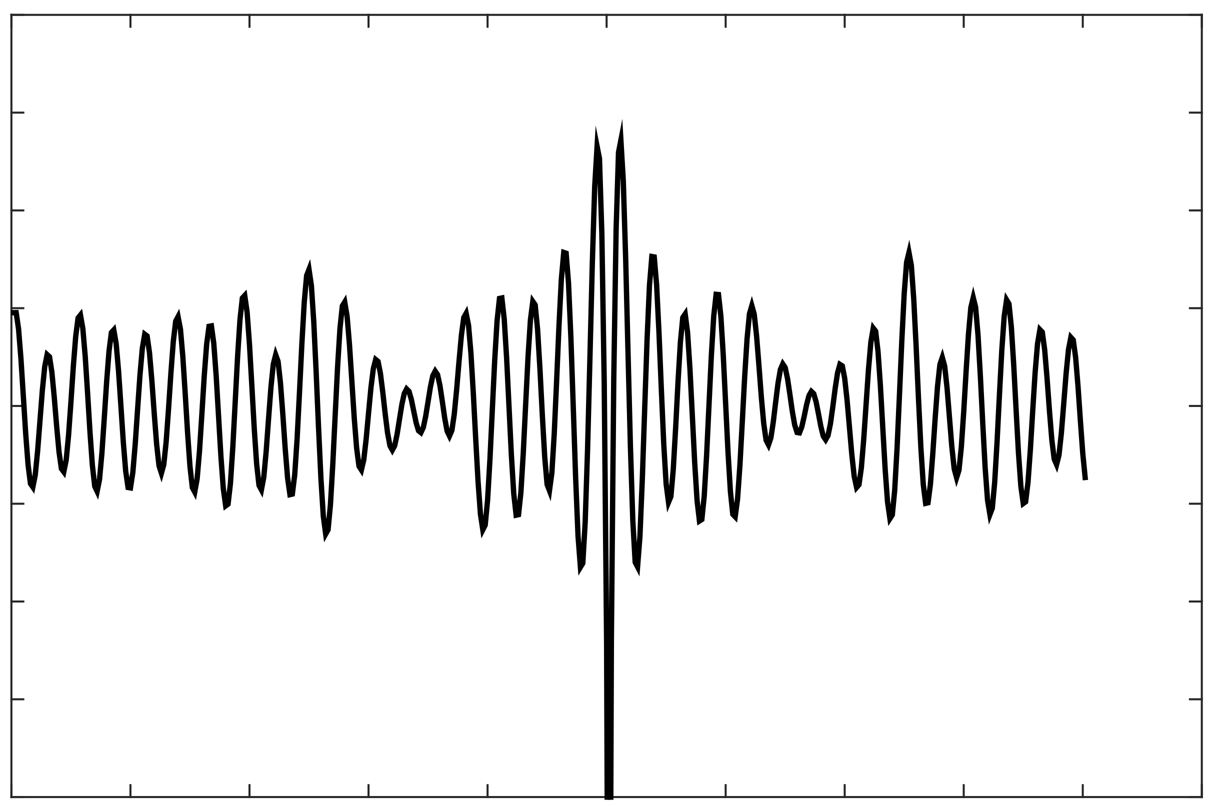
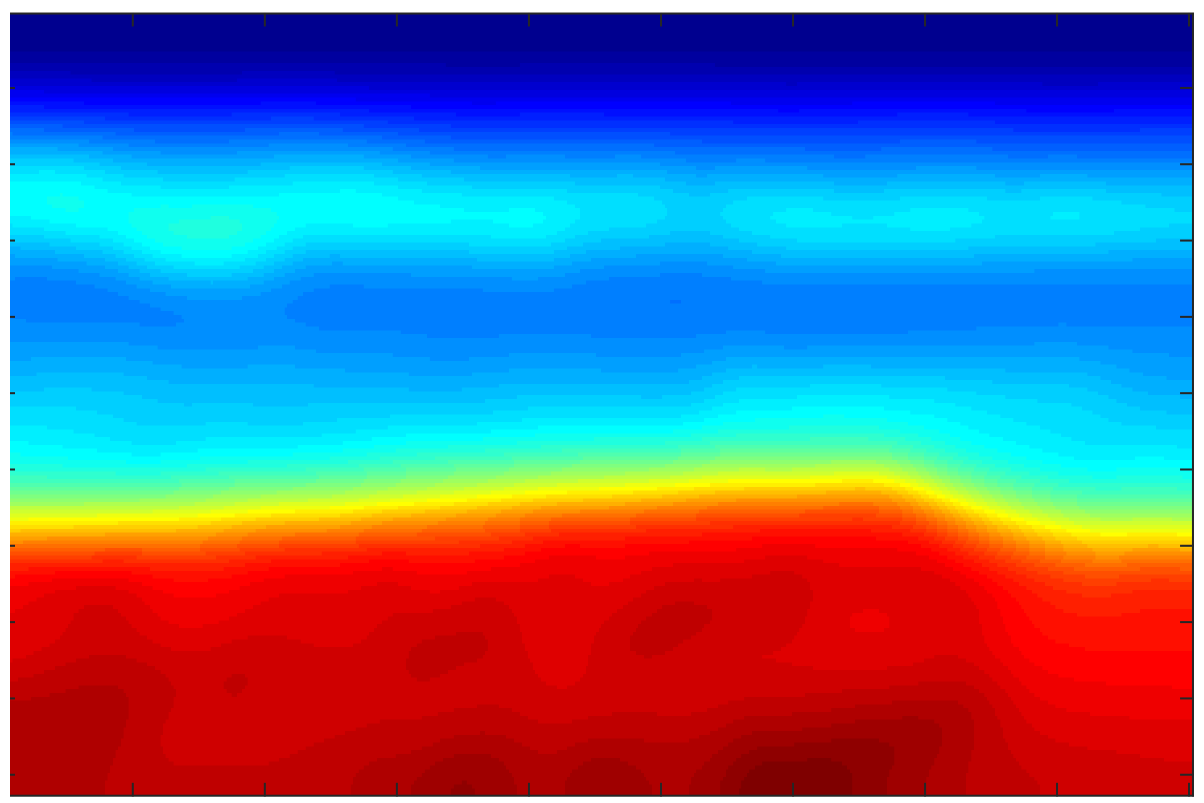
Bayesian theory

Bayesian inference:

$$\rho_{\text{post}}(\mathbf{m}) \propto \rho_{\text{like}}(\mathbf{m}|\mathbf{d}_{obs})\rho_{\text{prior}}(\mathbf{m})$$



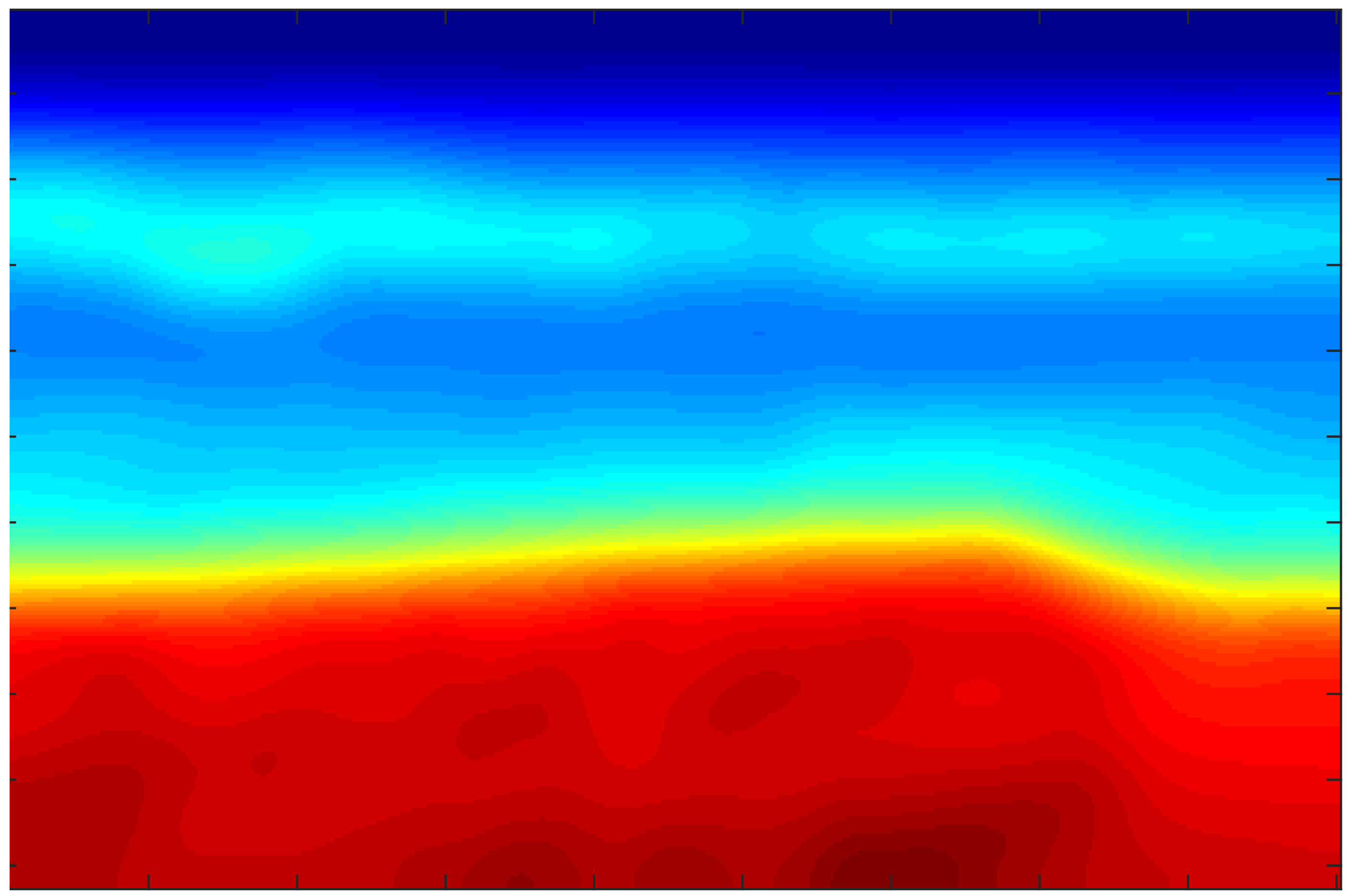


Bayesian theory

Prior information

Bayesian theory

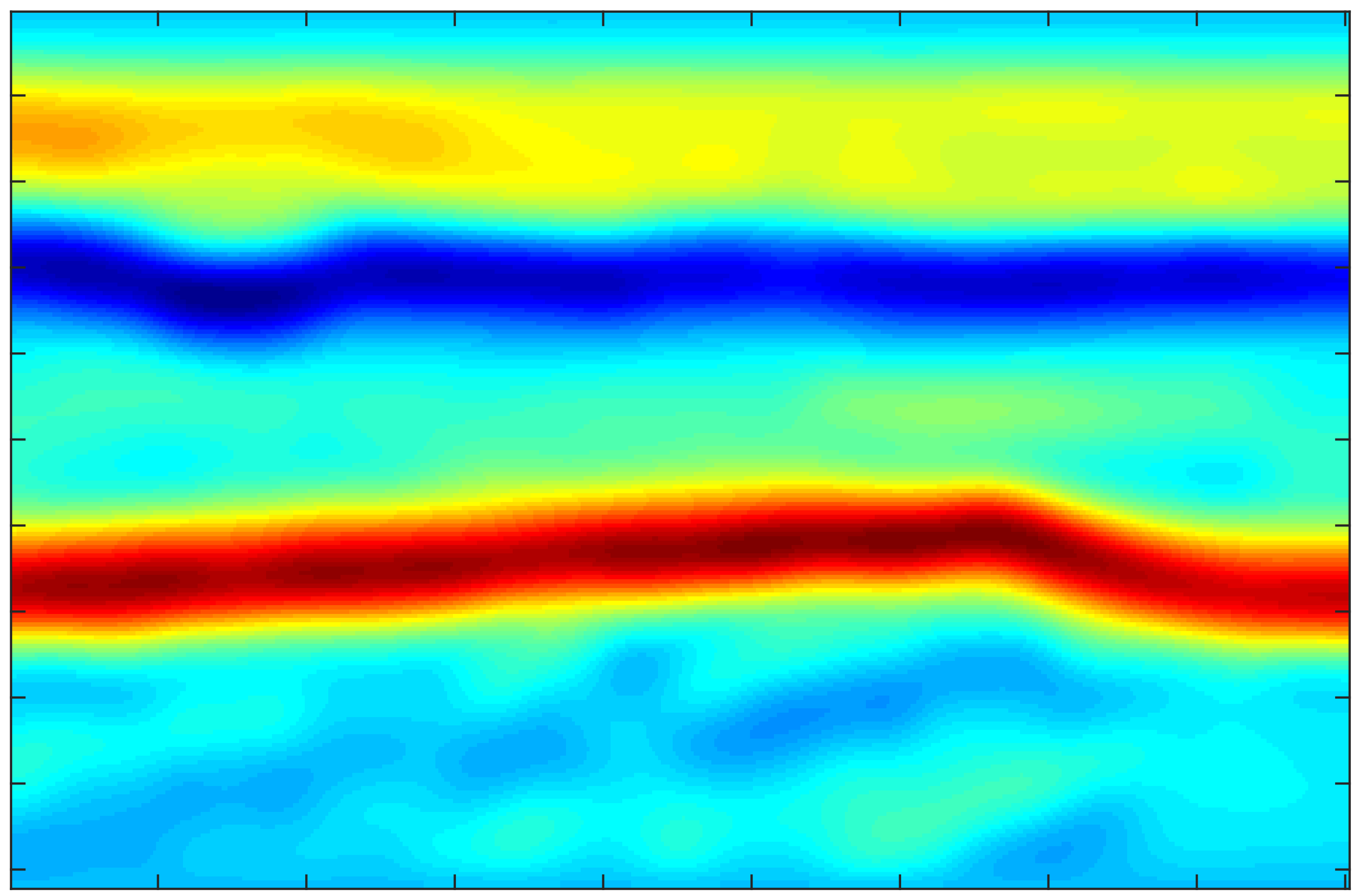
Prior information



$$\rho_{\mathrm{prior}}(\mathbf{m}) \propto \exp\left(-\|\mathbf{m} - \mathbf{m}_{\mathrm{prior}}\|^2_{\mathbf{\Sigma}^{-1}_{\mathrm{prior}}}\right)$$

Bayesian theory

Prior information

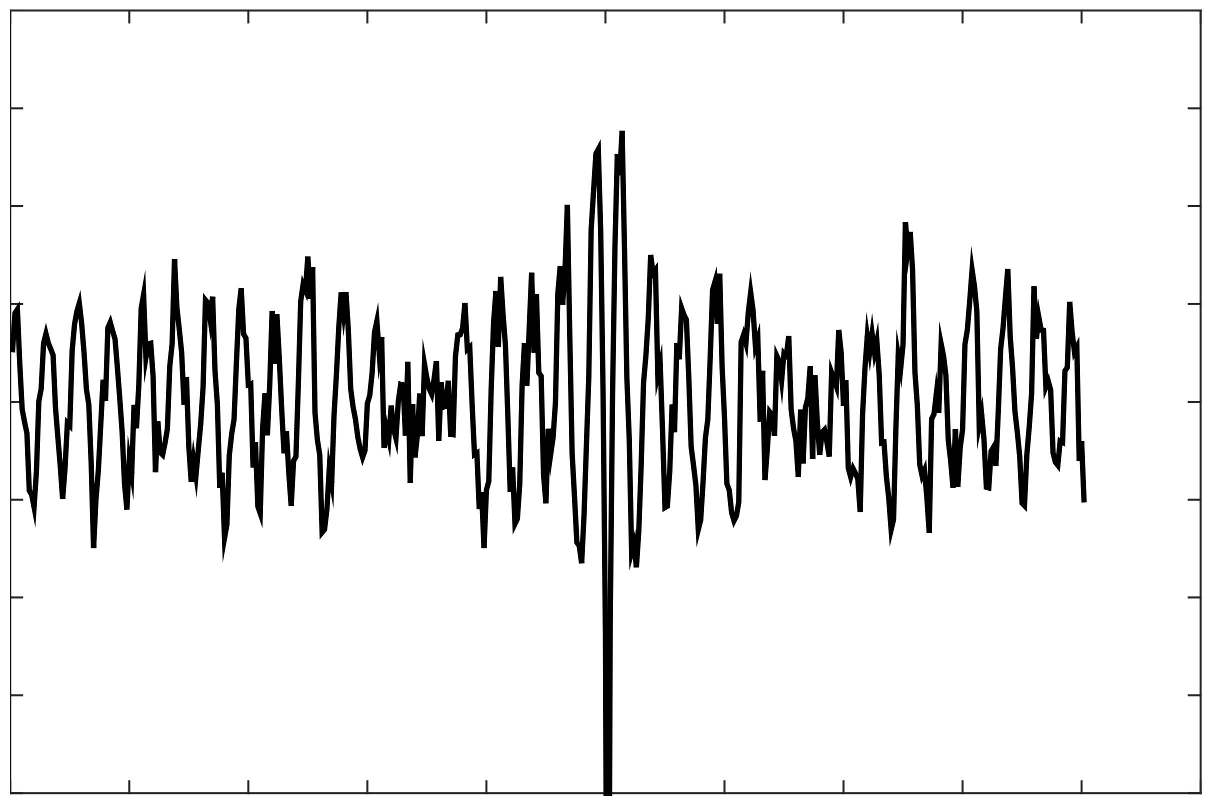


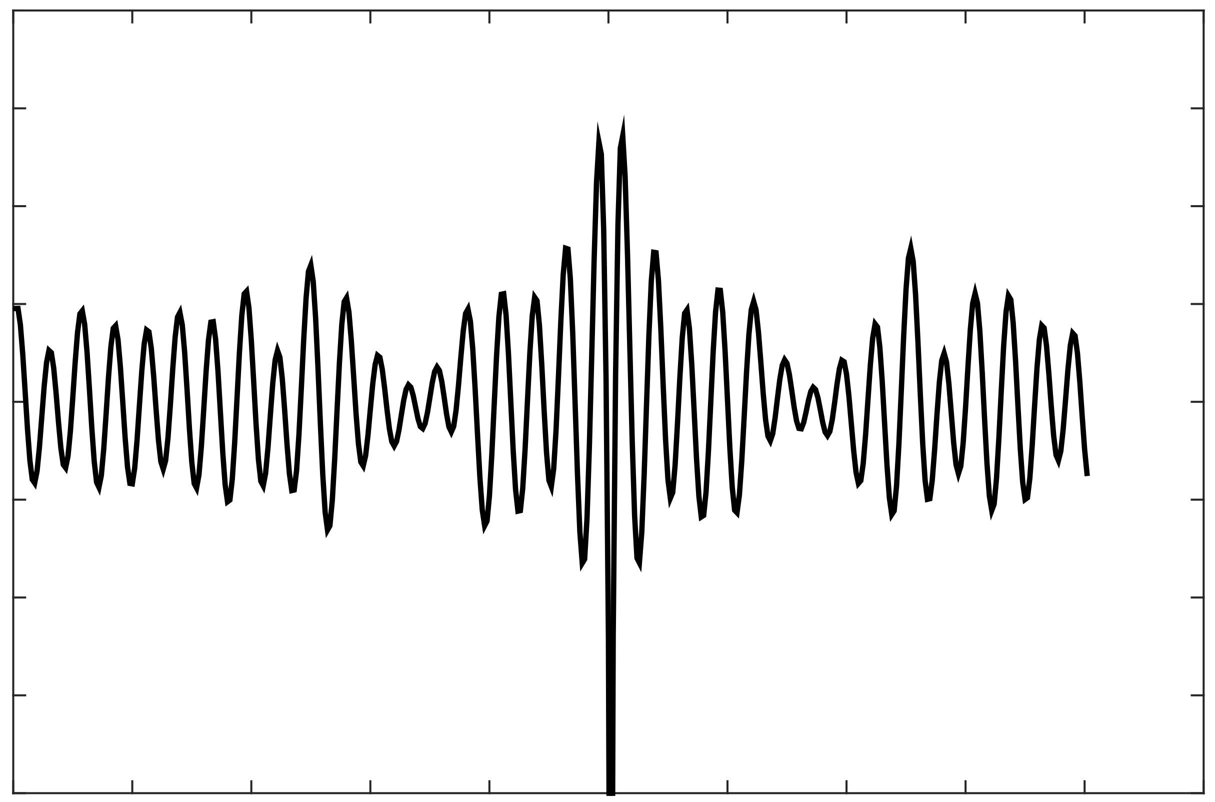
$$\rho_{\text{prior}}(\mathbf{m}) \propto \exp\left(-\|\mathbf{m}-\mathbf{m}_{\text{prior}}\|^2_{\mathbf{\Sigma}^{-1}_{\text{prior}}}\right)$$

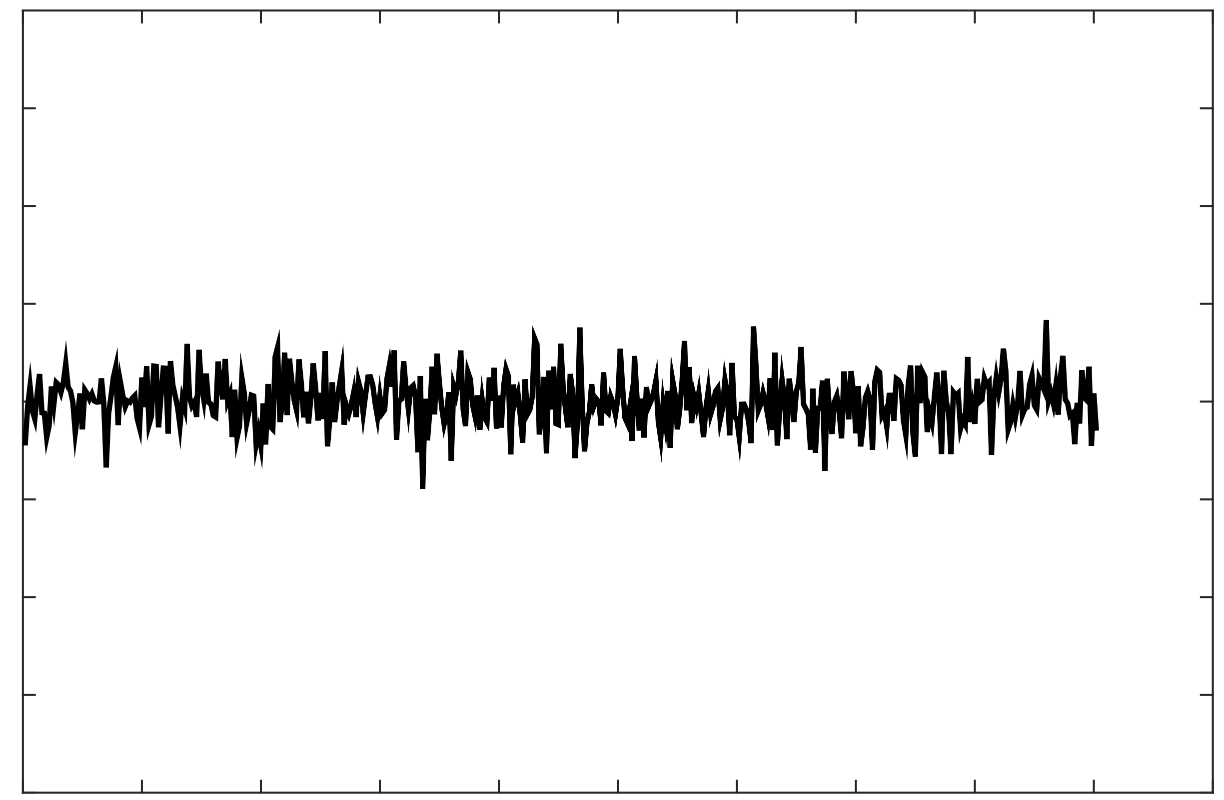
$$\rho_{\text{prior}}(\mathbf{m}) \propto \exp\left(-\|\nabla\mathbf{m}\|^2_{\mathbf{\Sigma}^{-1}_{\text{prior}}}\right)$$

Bayesian theory

Gaussian distribution assumption:



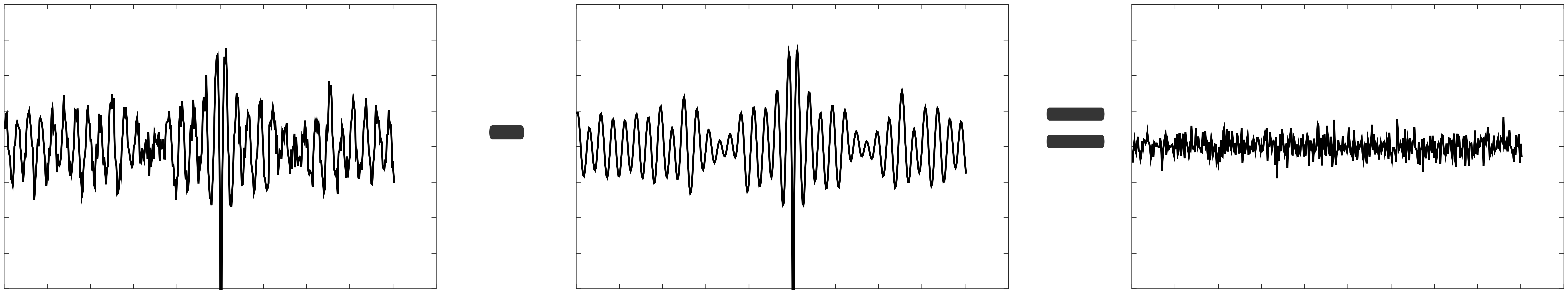




$$\mathbf{d}_{\text{obs}} = f(\mathbf{m}) + \text{noise}$$

Bayesian theory

Gaussian distribution assumption:



$\mathbf{d}_{\text{obs}}$ $f(\mathbf{m})$ noise

$$\rho_{\text{like}}(\mathbf{m}|\mathbf{d}_{\text{obs}}) \propto \exp\left(-\|f(\mathbf{m})-\mathbf{d}_{\text{obs}}\|^2_{\mathbf{\Sigma}^{-1}_{\text{noise}}}\right)$$

Bayesian theory

Posterior distribution of FWI:

$$\rho_{\text{post}}(\mathbf{m}) \propto \exp\left(-\|\mathbf{P}\mathbf{A}(\mathbf{m})^{-1}\mathbf{q} - \mathbf{d}_{\text{obs}}\|^2_{\boldsymbol{\Sigma}^{-1}_{\text{noise}}} - \|\mathbf{m} - \mathbf{m}_{\text{prior}}\|^2_{\boldsymbol{\Sigma}^{-1}_{\text{prior}}}\right)$$

Bayesian theory

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Strong nonlinearity
Many local minima

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Bayesian theory

Posterior distribution of FWI:

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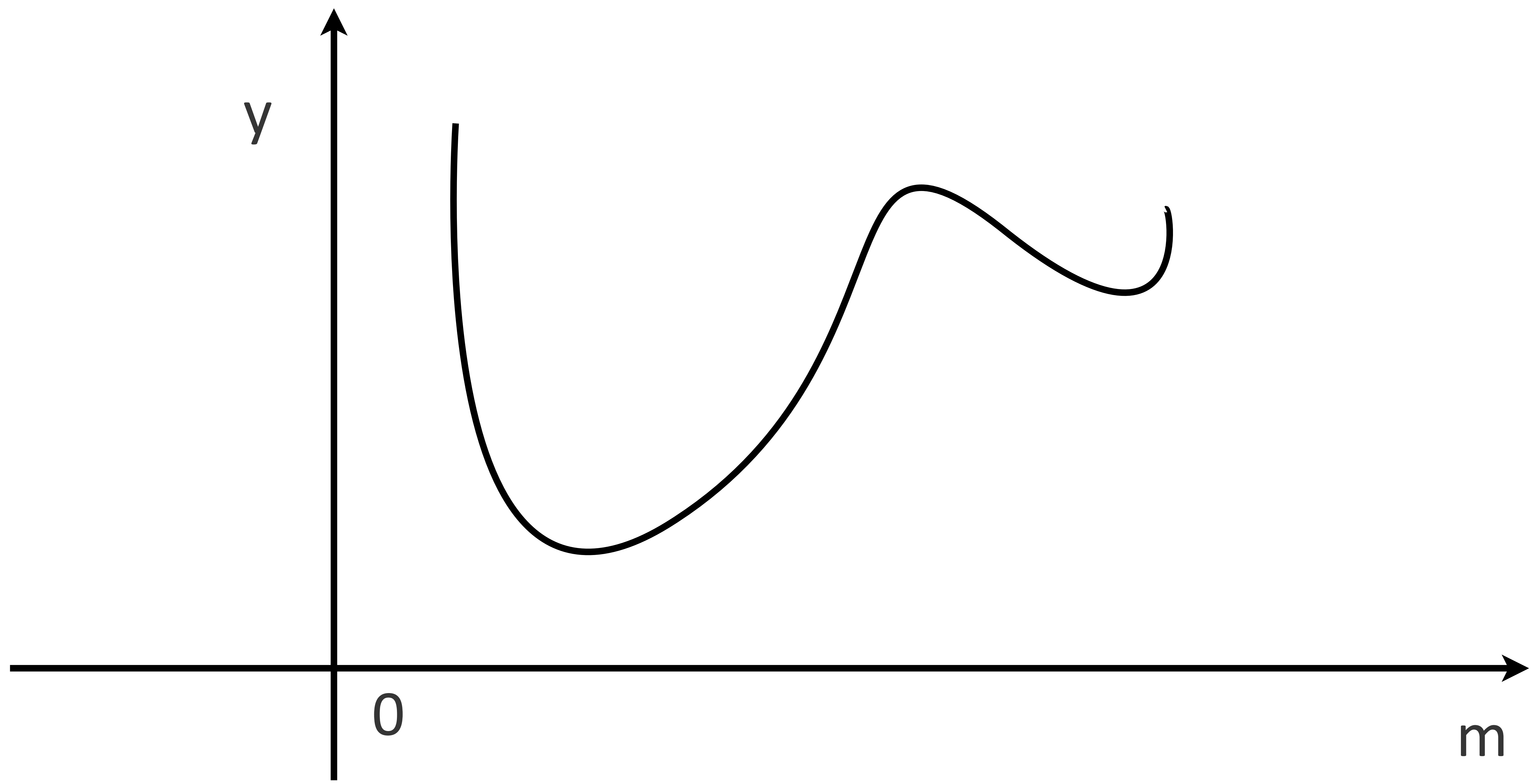
Strong nonlinearity

Many local minima



Bayesian theory

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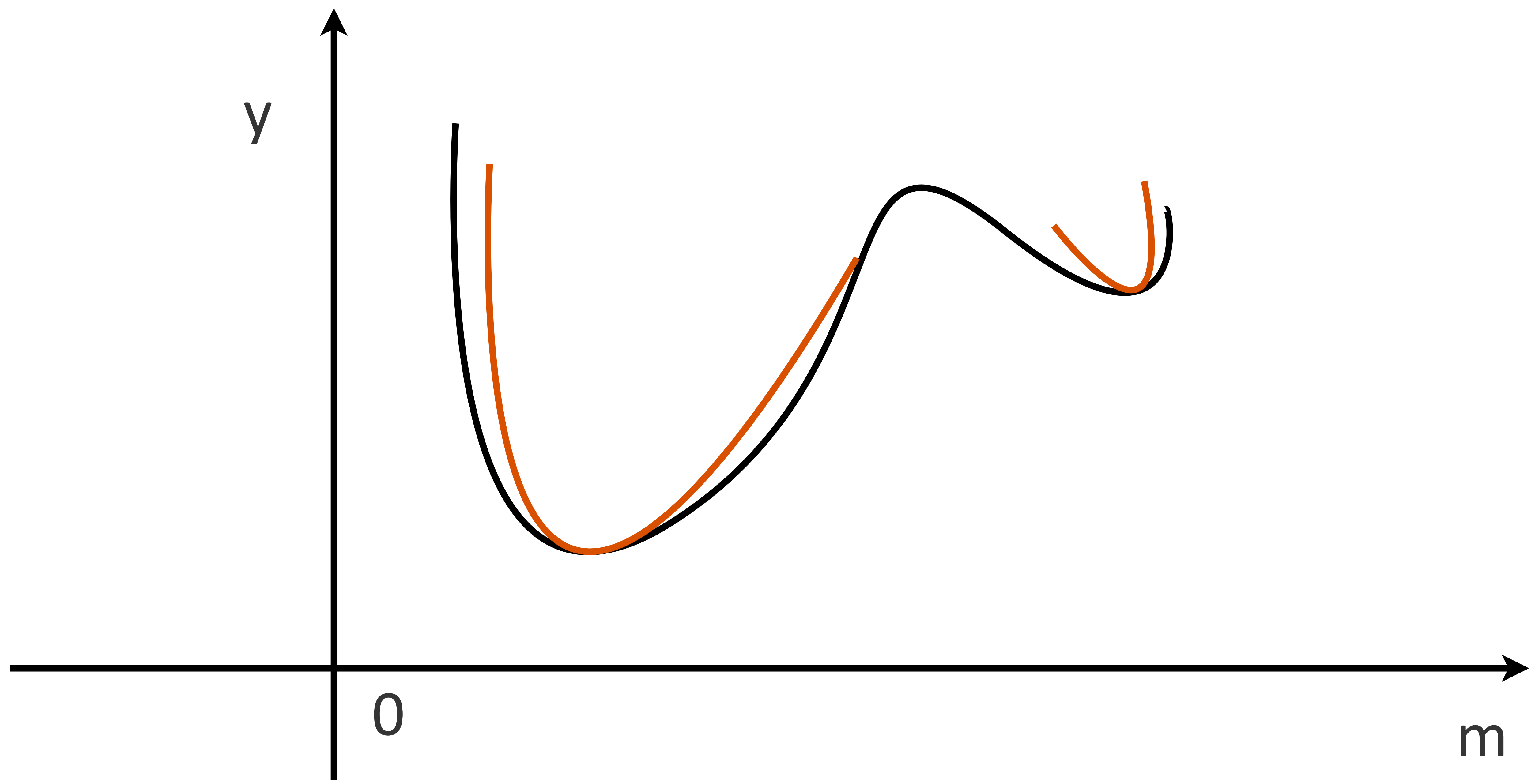

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Bayesian and WRI

Posterior distribution of WRI:

$$\rho_{\text{post}}(\mathbf{m}) \propto$$

$$\exp\left(\underbrace{-\|\mathbf{P}\mathbf{u}(\mathbf{m}) - \mathbf{d}_{\text{obs}}\|^2_{\boldsymbol{\Sigma}^{-1}_{\text{noise}}} - \lambda^2\|\mathbf{A}(\mathbf{m})\mathbf{u}(\mathbf{m}) - \mathbf{q}\|^2_{\boldsymbol{\Sigma}^{-1}_{\text{pde}}}}_{\text{Likelihood}} \underbrace{- \|\mathbf{m} - \mathbf{m}_{\text{prior}}\|^2_{\boldsymbol{\Sigma}^{-1}_{\text{prior}}}}_{\text{Prior}}\right)$$

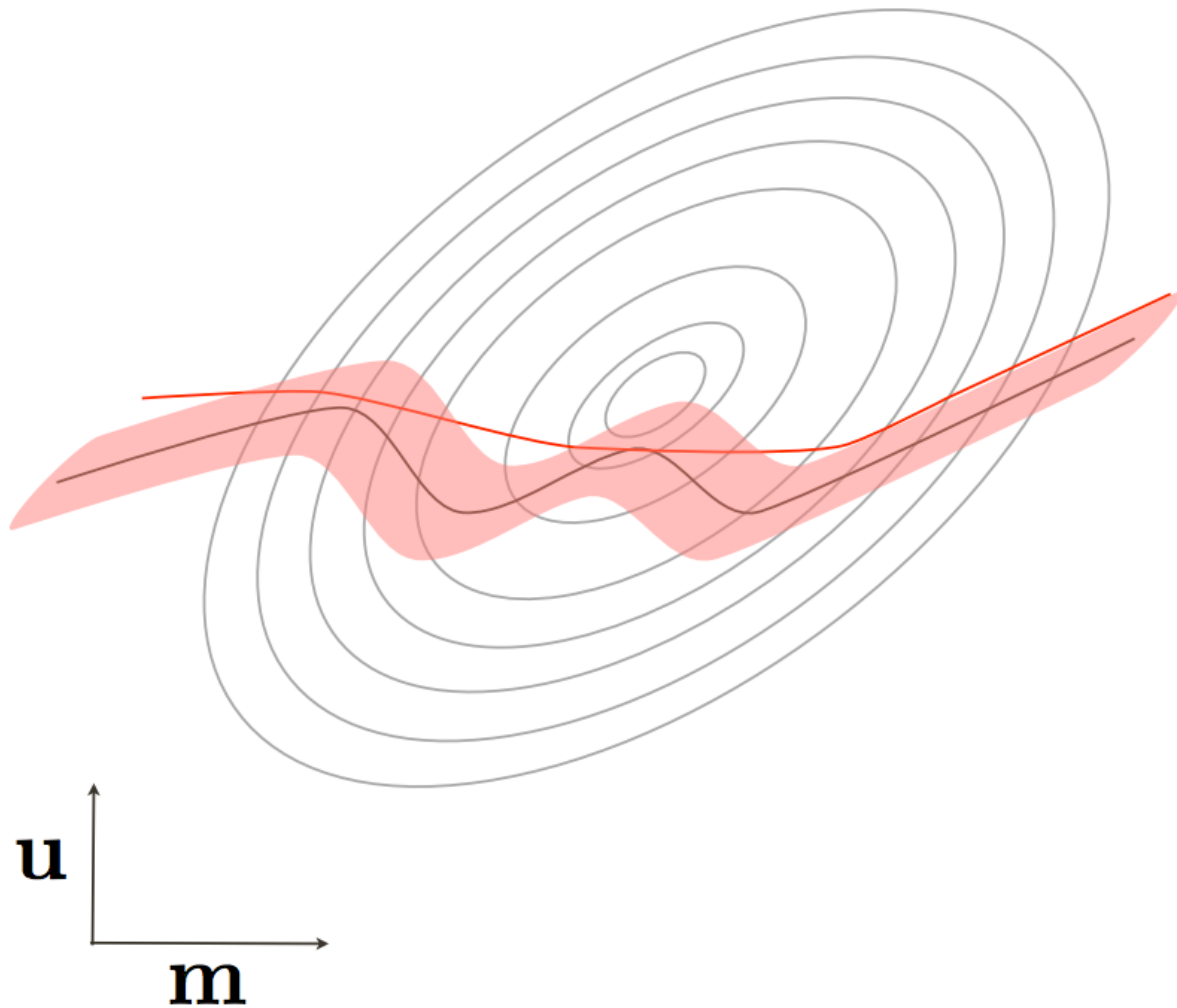
where,

$$\begin{pmatrix} \lambda\boldsymbol{\Sigma}^{-1/2}_{\text{pde}}\mathbf{A} \\ \boldsymbol{\Sigma}^{-1/2}_{\text{noise}}\mathbf{P} \end{pmatrix} \mathbf{u} = \begin{pmatrix} \lambda\boldsymbol{\Sigma}^{-1/2}_{\text{pde}}\mathbf{q} \\ \boldsymbol{\Sigma}^{-1/2}_{\text{noise}}\mathbf{d}_{\text{obs}} \end{pmatrix}$$

WRI vs FWI

[van Leeuwen, T and Herrmann, F J , 2013]

Larger # of degrees of freedom



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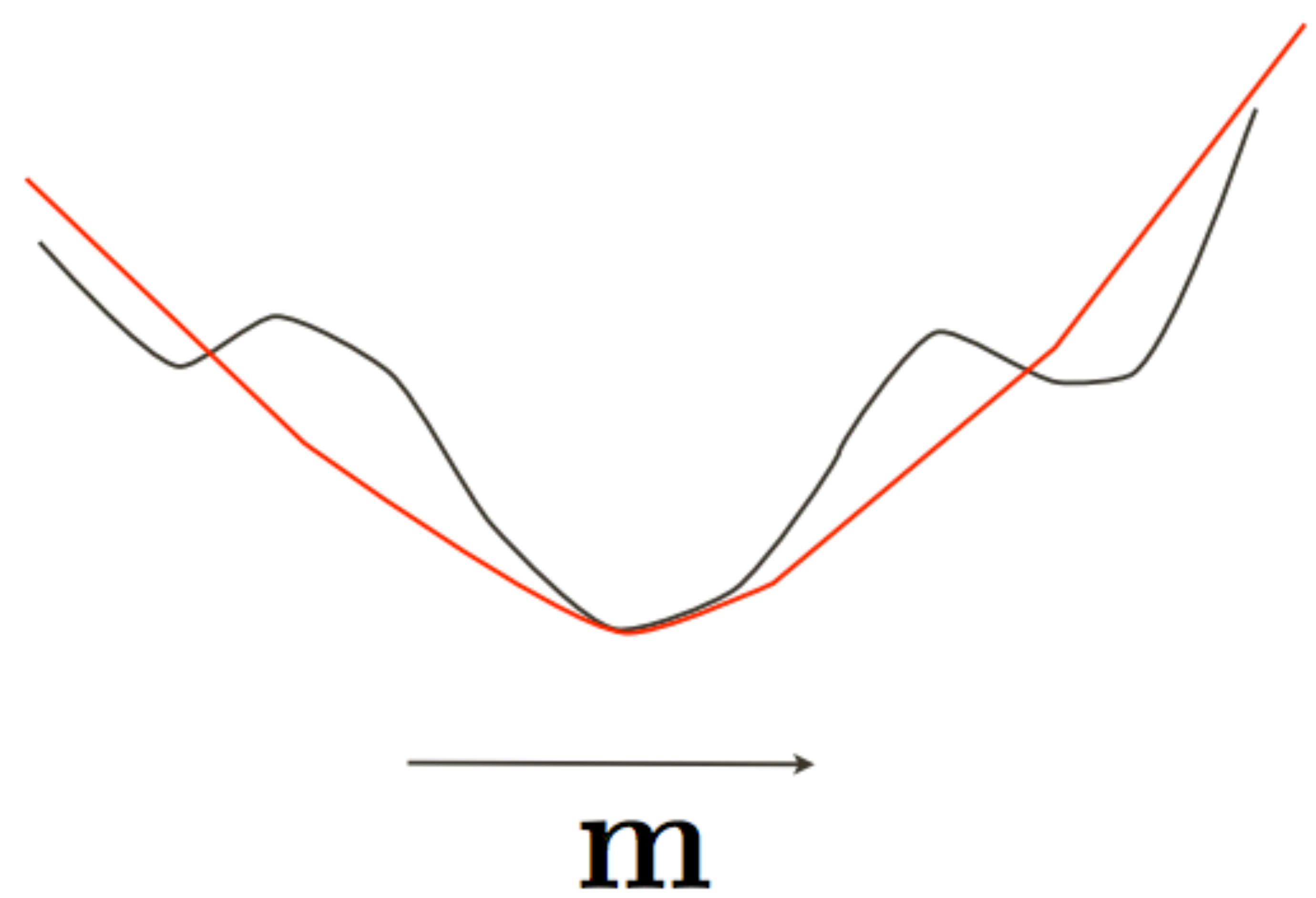

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Bayesian and WRI

Mean and covariance of the model:

$$\mathbb{E}(\mathbf{m}) = \int \mathbf{m}\rho_{\text{post}}(\mathbf{m})d\mathbf{m}$$

$$\text{Cov}(\mathbf{m}) = \int (\mathbb{E}(\mathbf{m}) - \mathbf{m})^2 \rho_{\text{post}}(\mathbf{m})d\mathbf{m}$$

Quantify the uncertainty

Goal : Quantify the uncertainty based on the posterior distribution $\rho_{\mathrm{post}}(\mathbf{m})$

Solution:

- Integrate the posterior distribution

Quantify the uncertainty

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Quantify the uncertainty

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Solution:

- Integrate the posterior distribution
- McMC method to sample the posterior distribution

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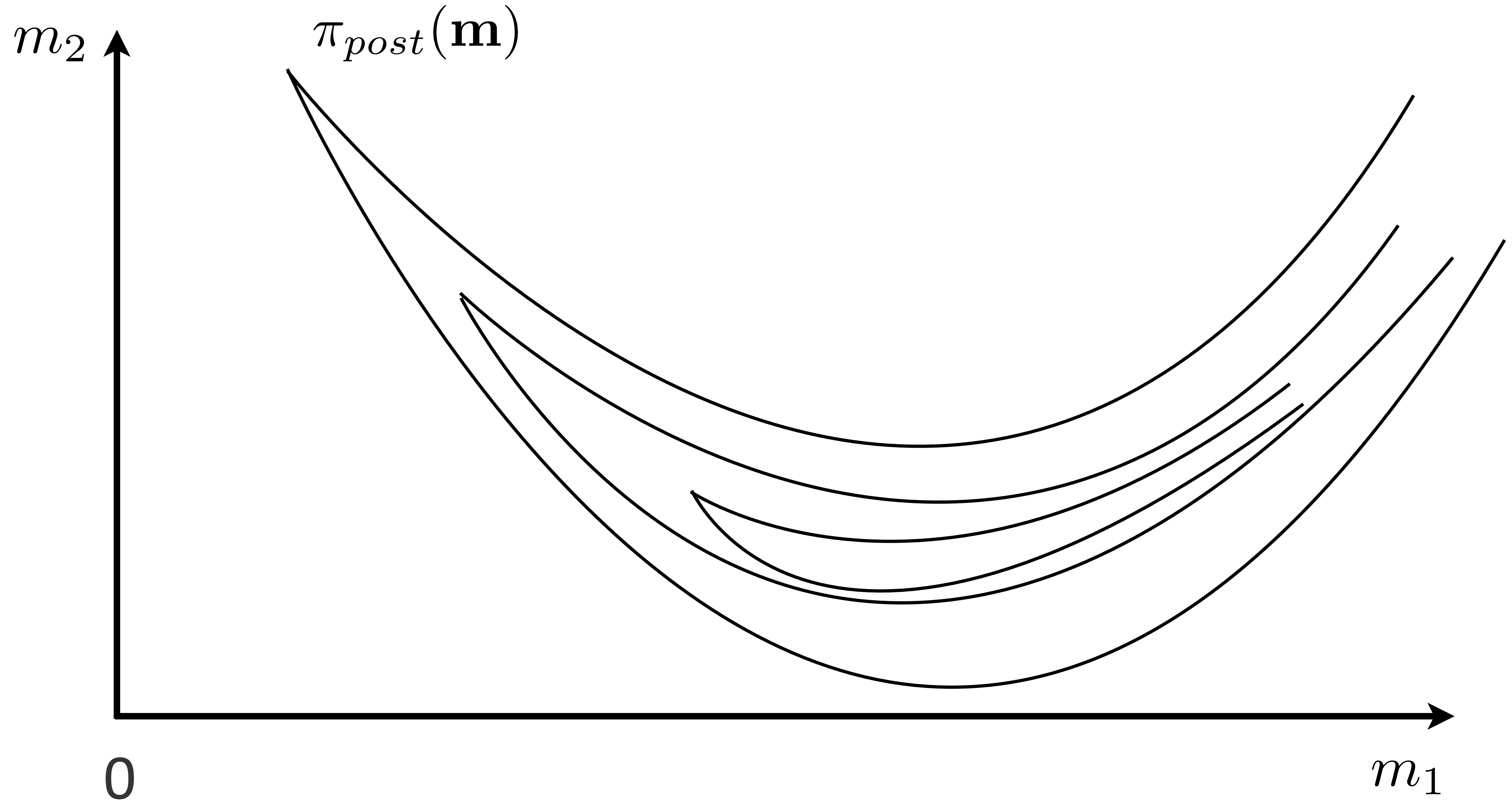
McMC method

Metropolis-Hasting Method:



At sample $\mathbf{m}_k$

Draw sample $\mathbf{y}$ from the proposal distribution $\tilde{\pi}_k(\mathbf{m})$

if $\min(1, \dfrac{\pi_{post}(\mathbf{y})\tilde{\pi}_y(\mathbf{m}_k)}{\pi_{post}(\mathbf{m}_k)\tilde{\pi}_k(\mathbf{y})}) > \alpha$

    set $\mathbf{m}_{k+1} = \mathbf{y}$

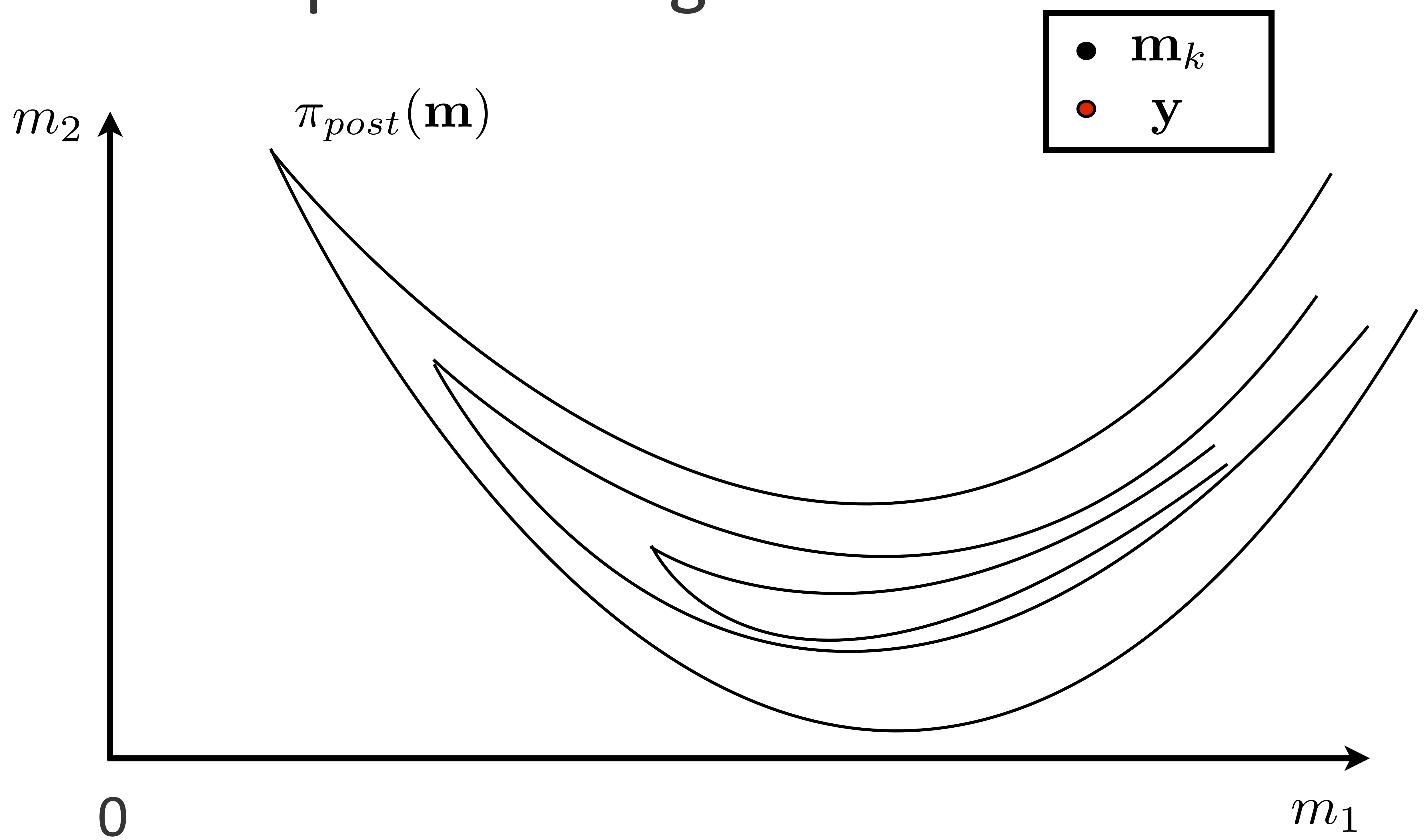
else

    regenerate $\mathbf{y}$

end

McMC method

Metropolis-Hasting Method:



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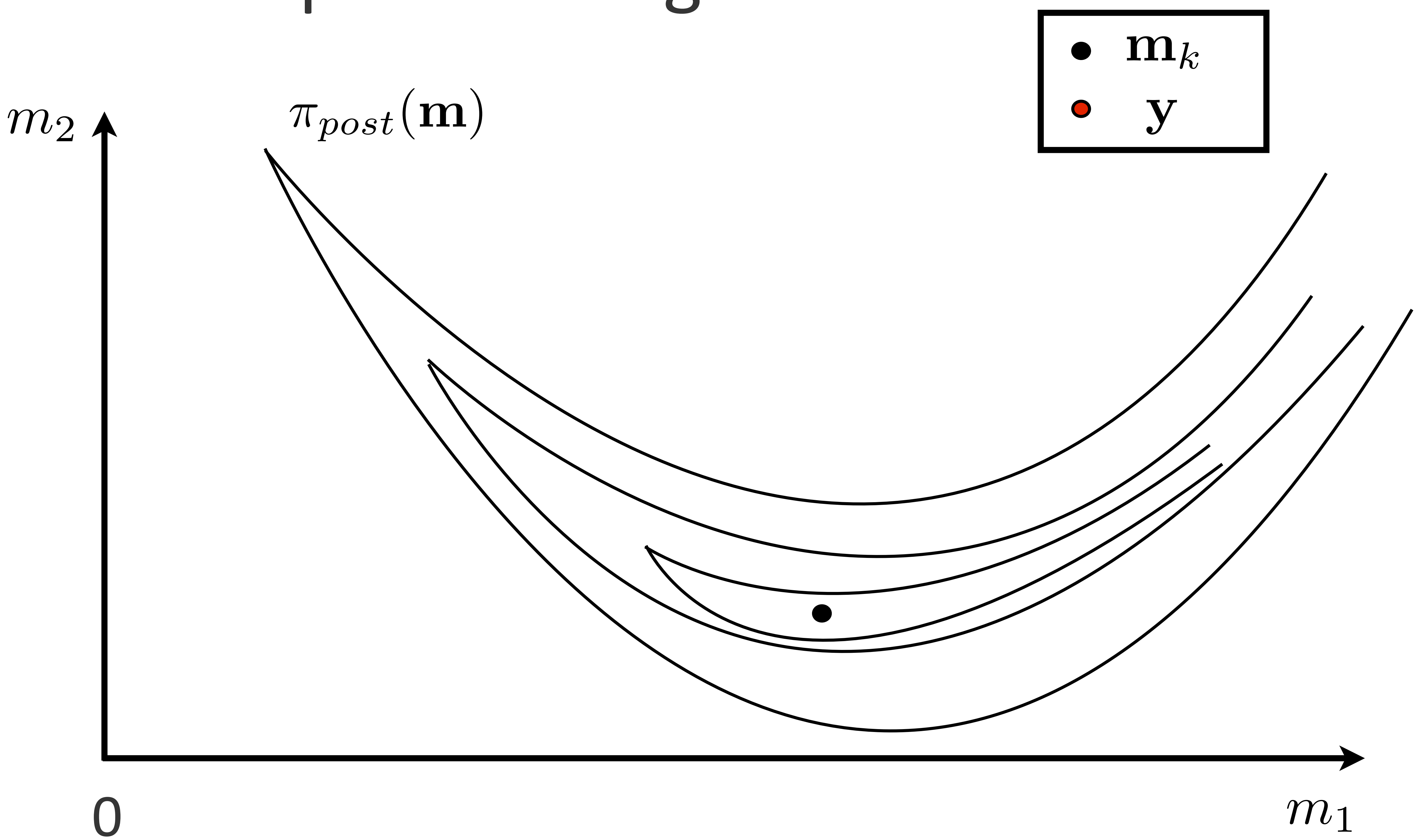

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[D. Higdon, *et al*, 2010]

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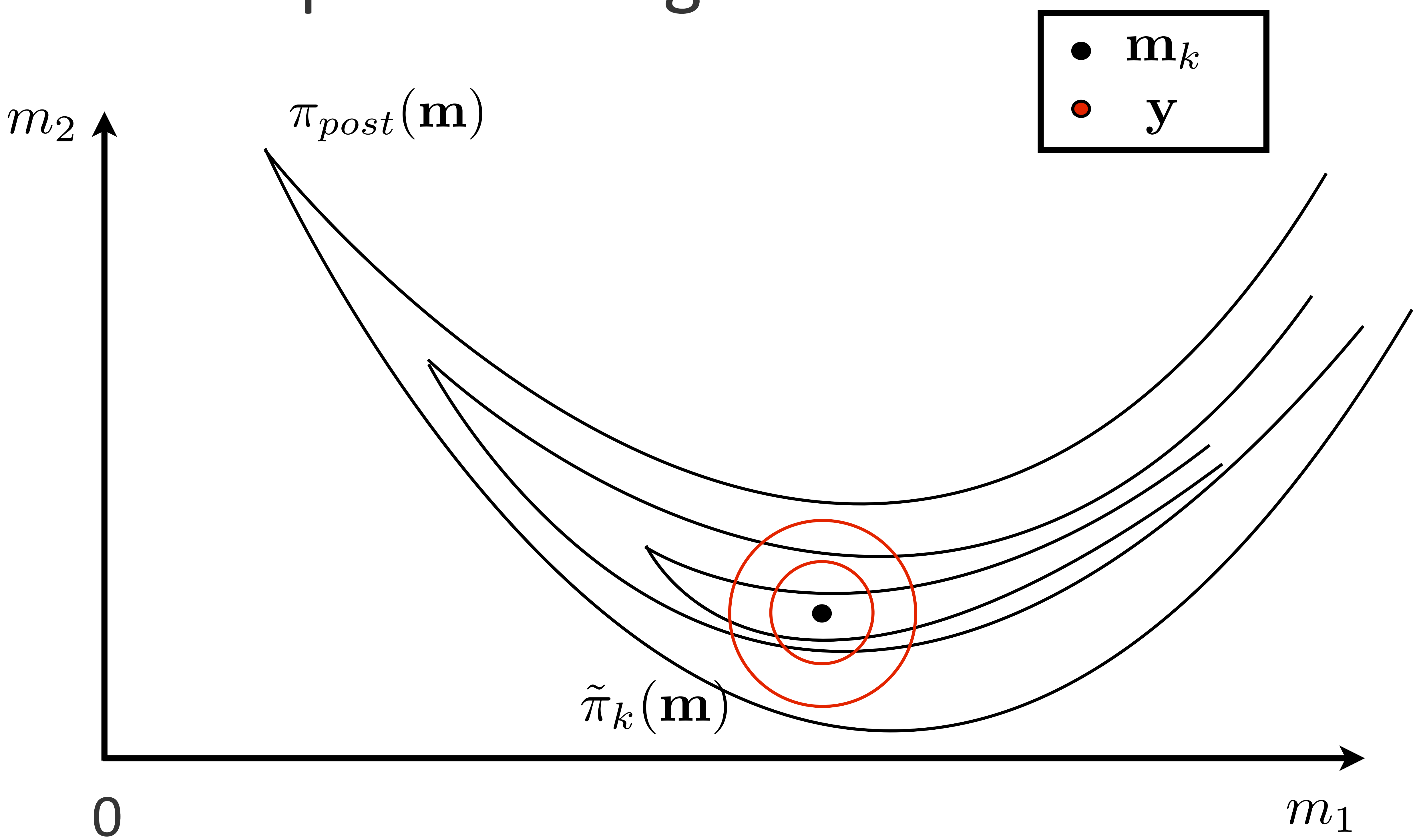

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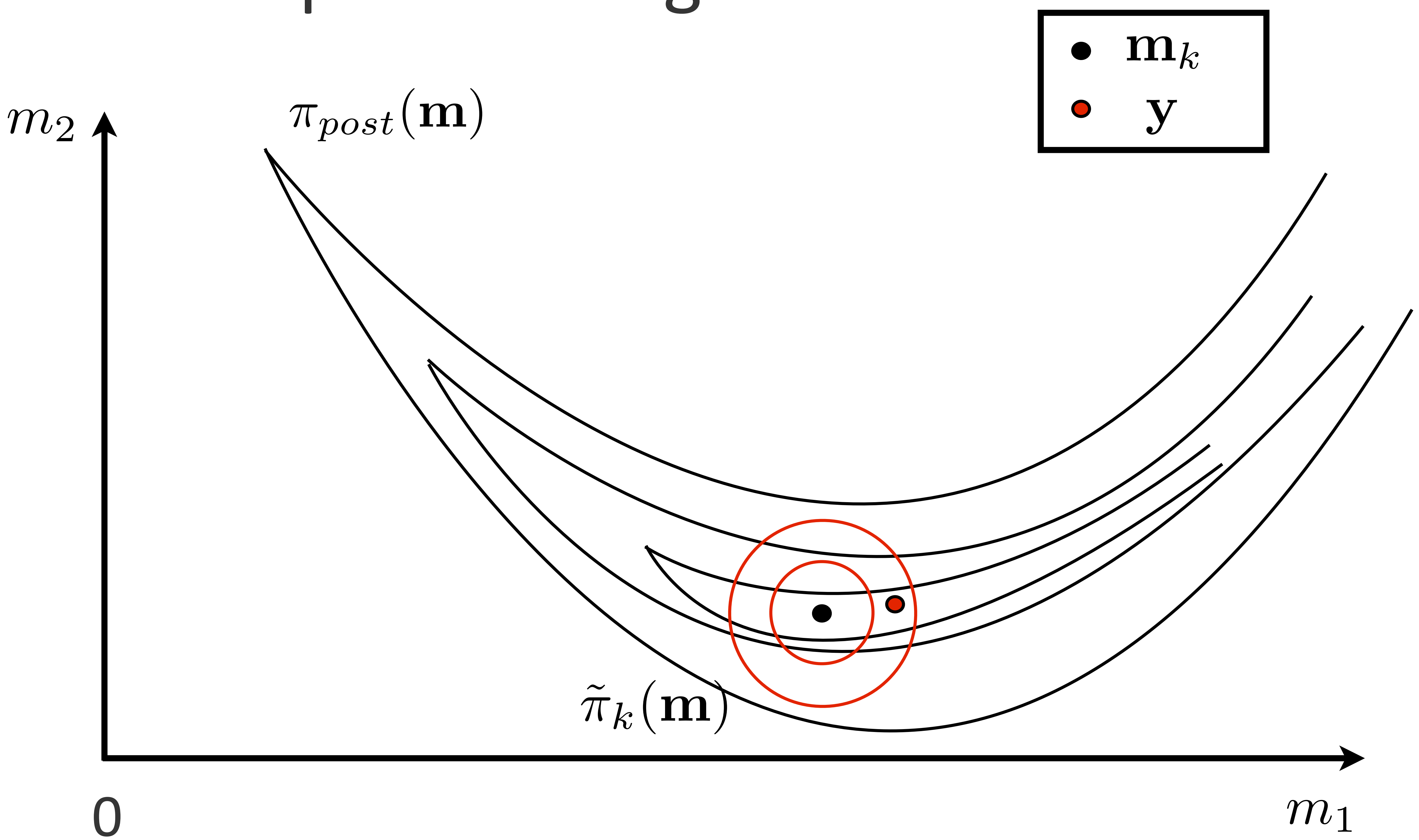

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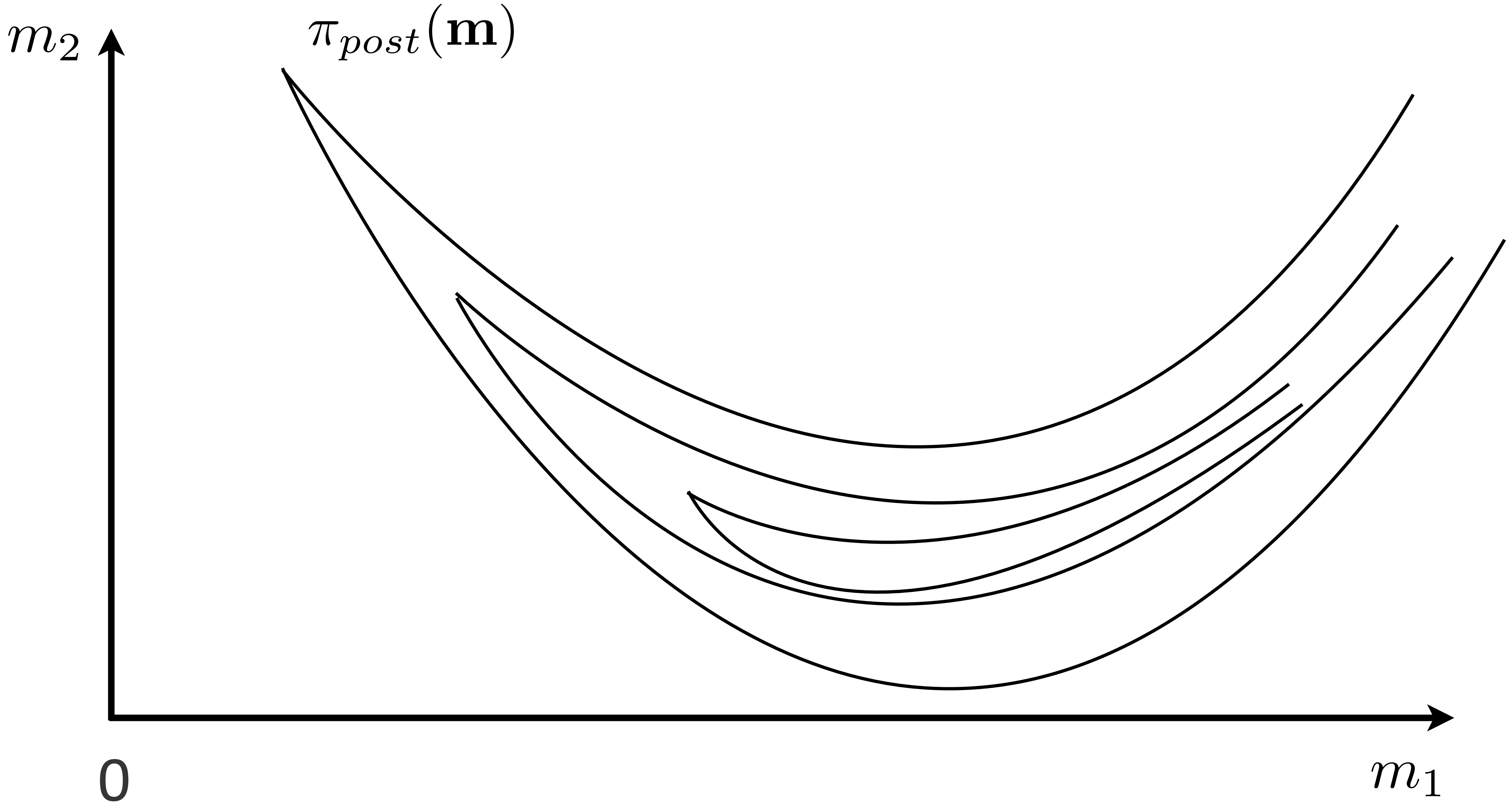
McMC method

Random Walk: $\tilde{\pi}_k(\mathbf{m}) \backsim \mathcal{N}(\mathbf{m}_k, \alpha\mathbf{I})$



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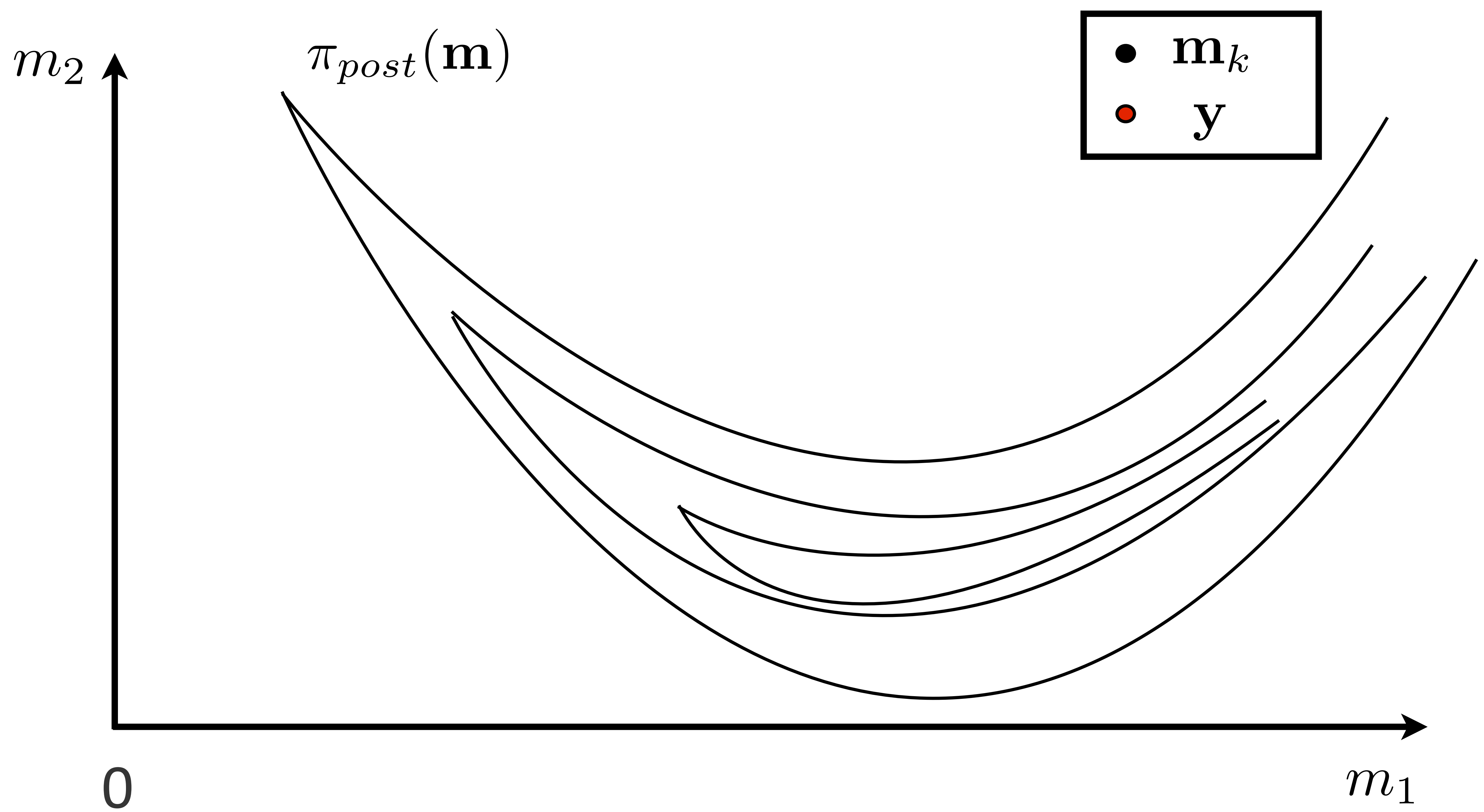

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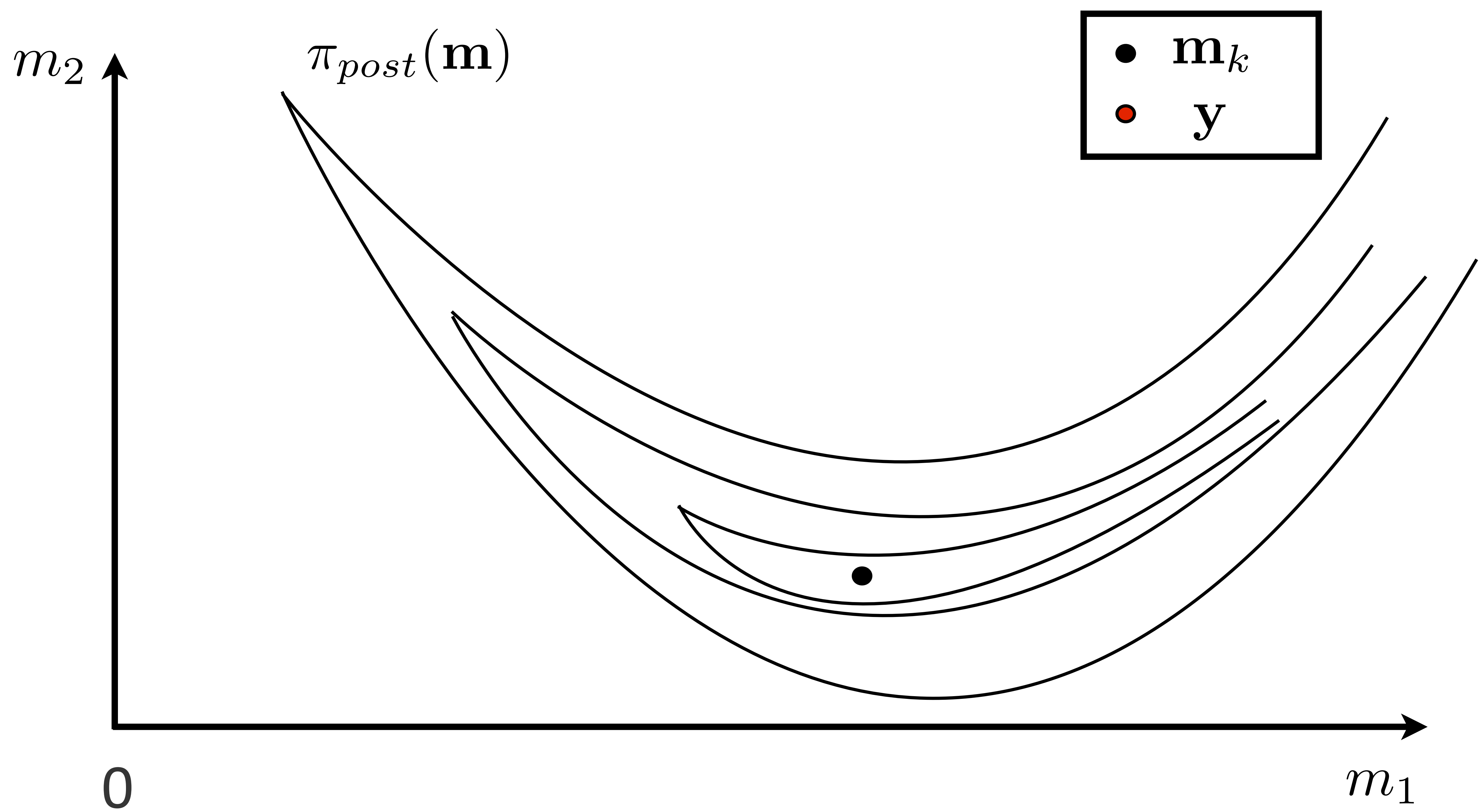

[D. Higdon, *et al*, 2010]

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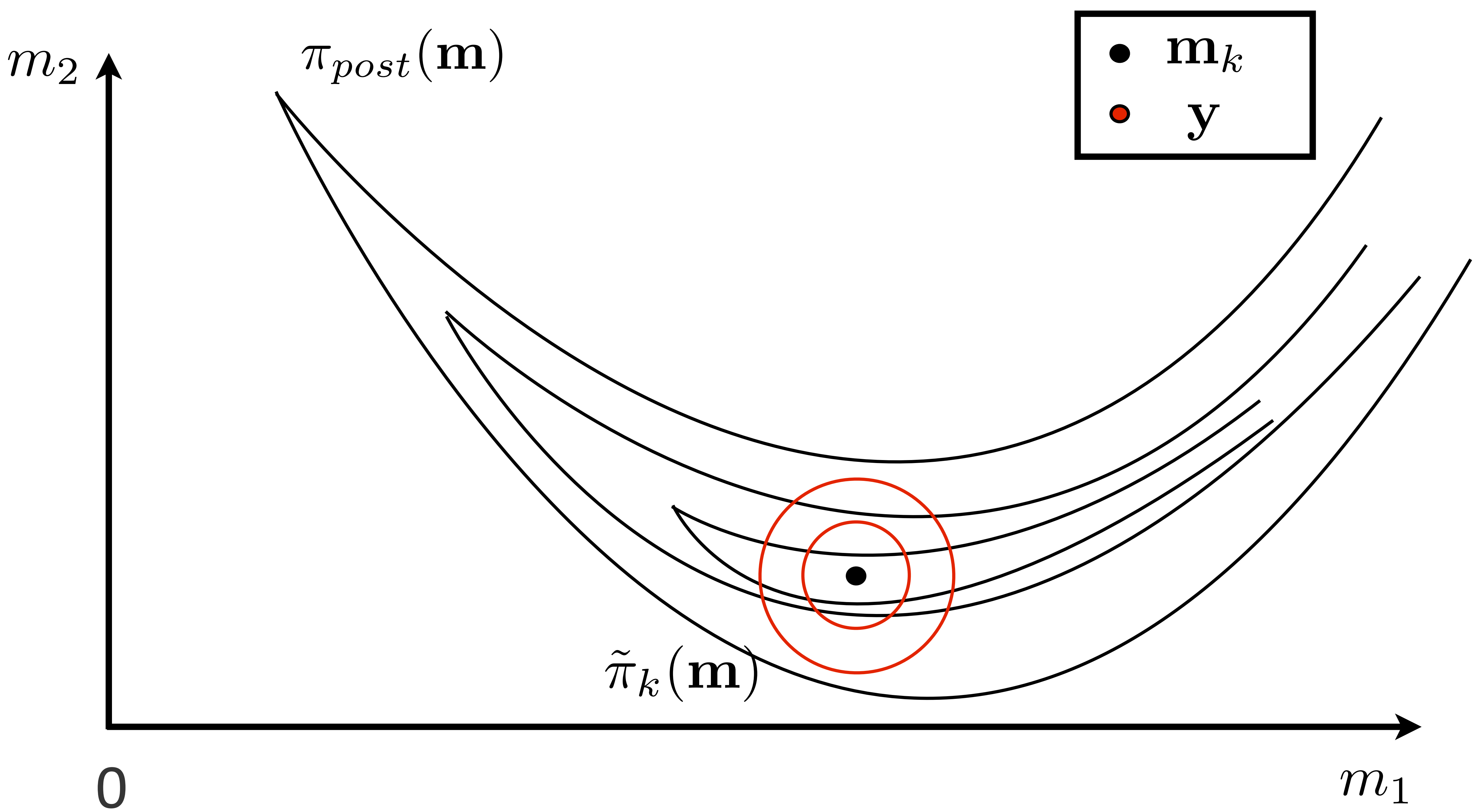

Wednesday, 28 October, 15

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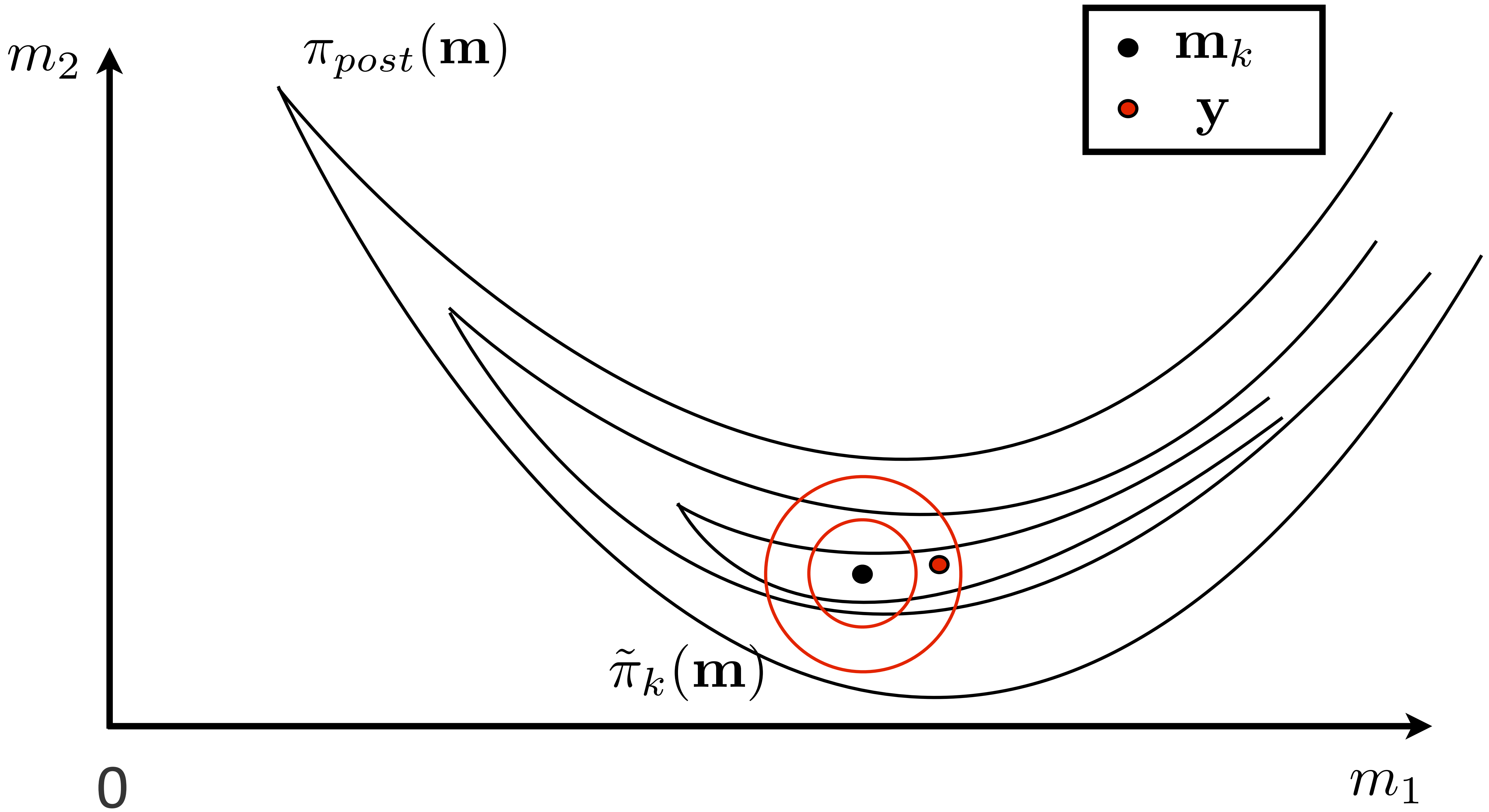

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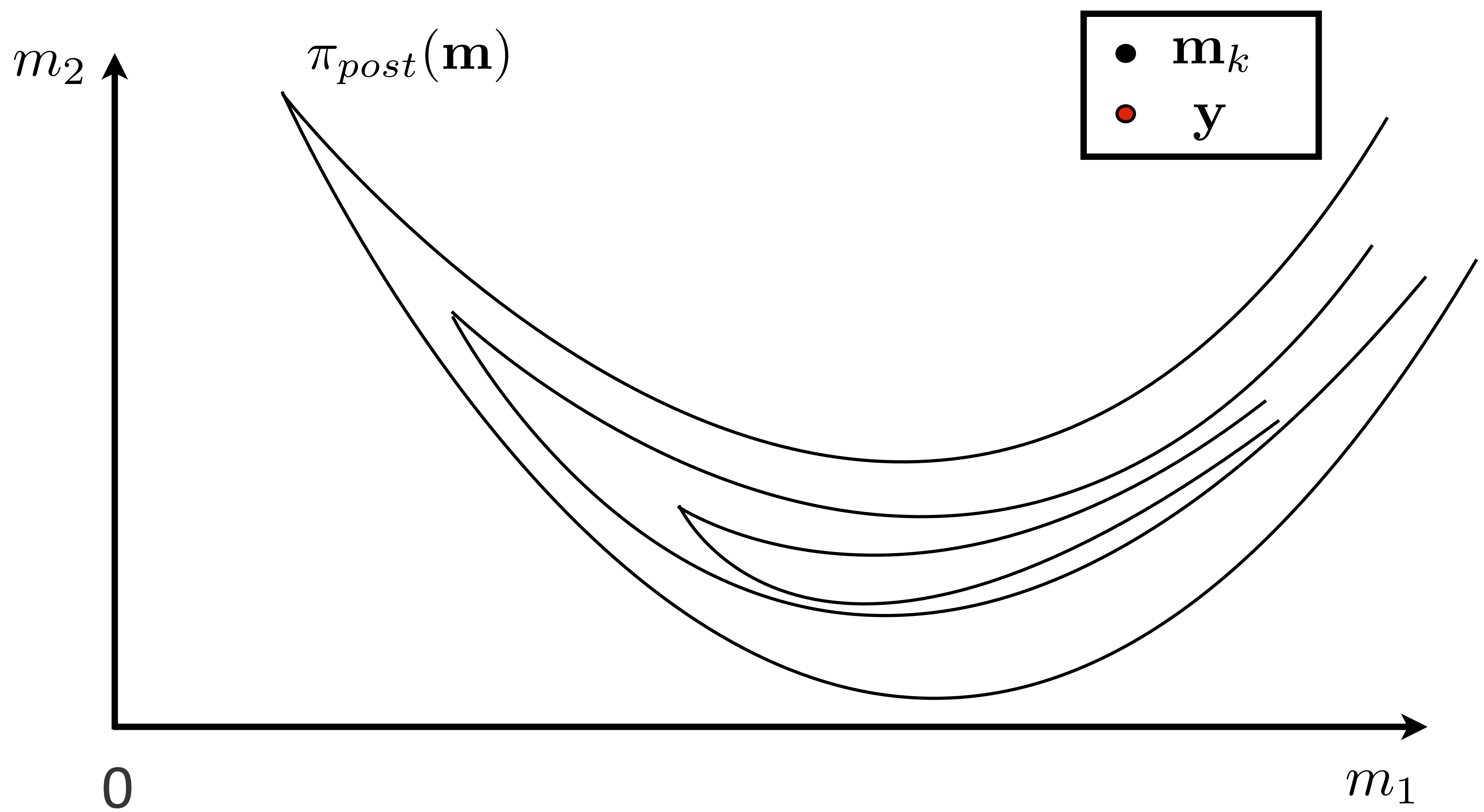
McMC method

Langevin McMC method: $\tilde{\pi}_k(\mathbf{m}) \backsim \mathcal{N}(\mathbf{m}_k - \mathbf{L}\mathbf{g}_k, \mathbf{L})$



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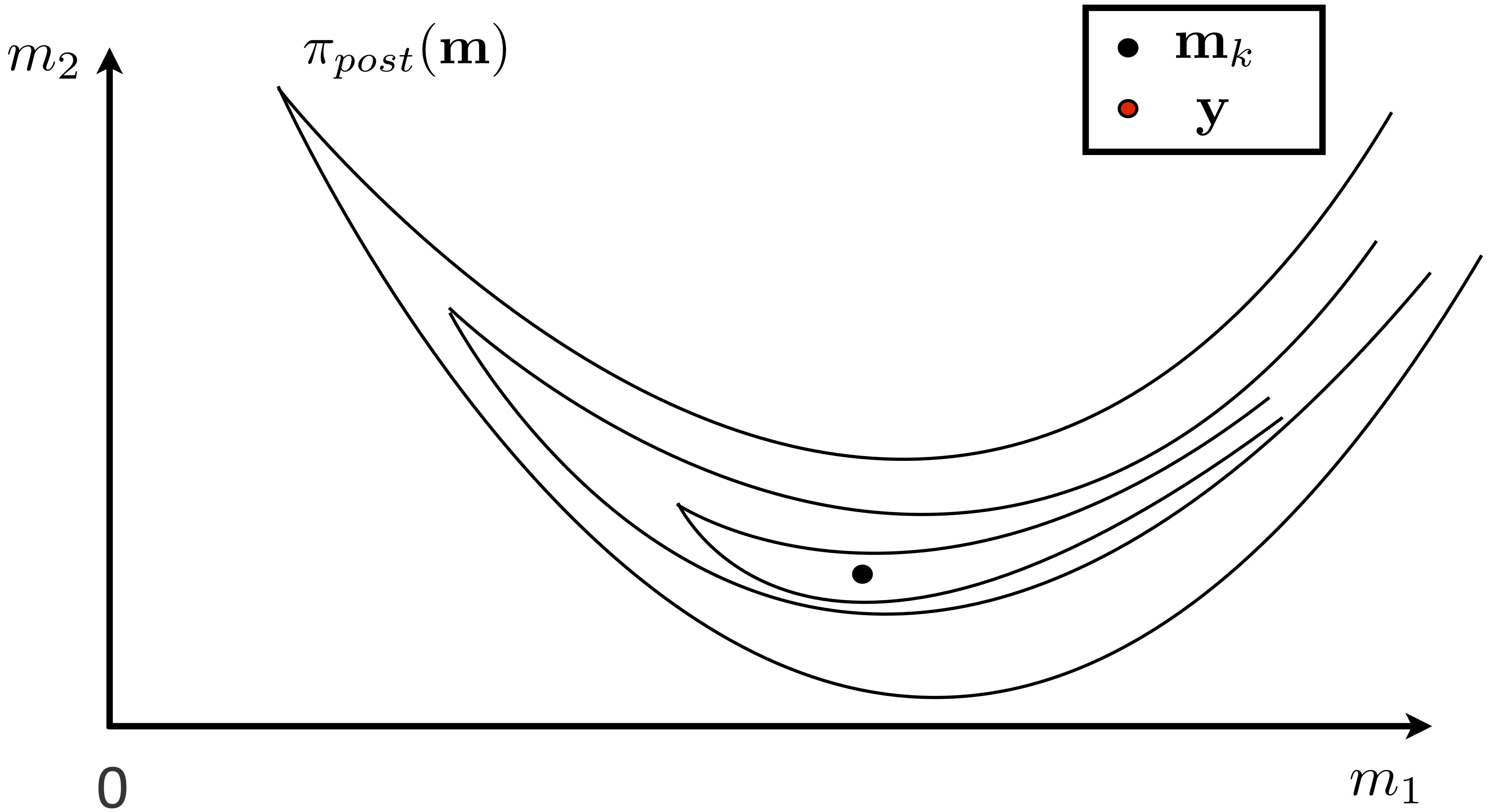

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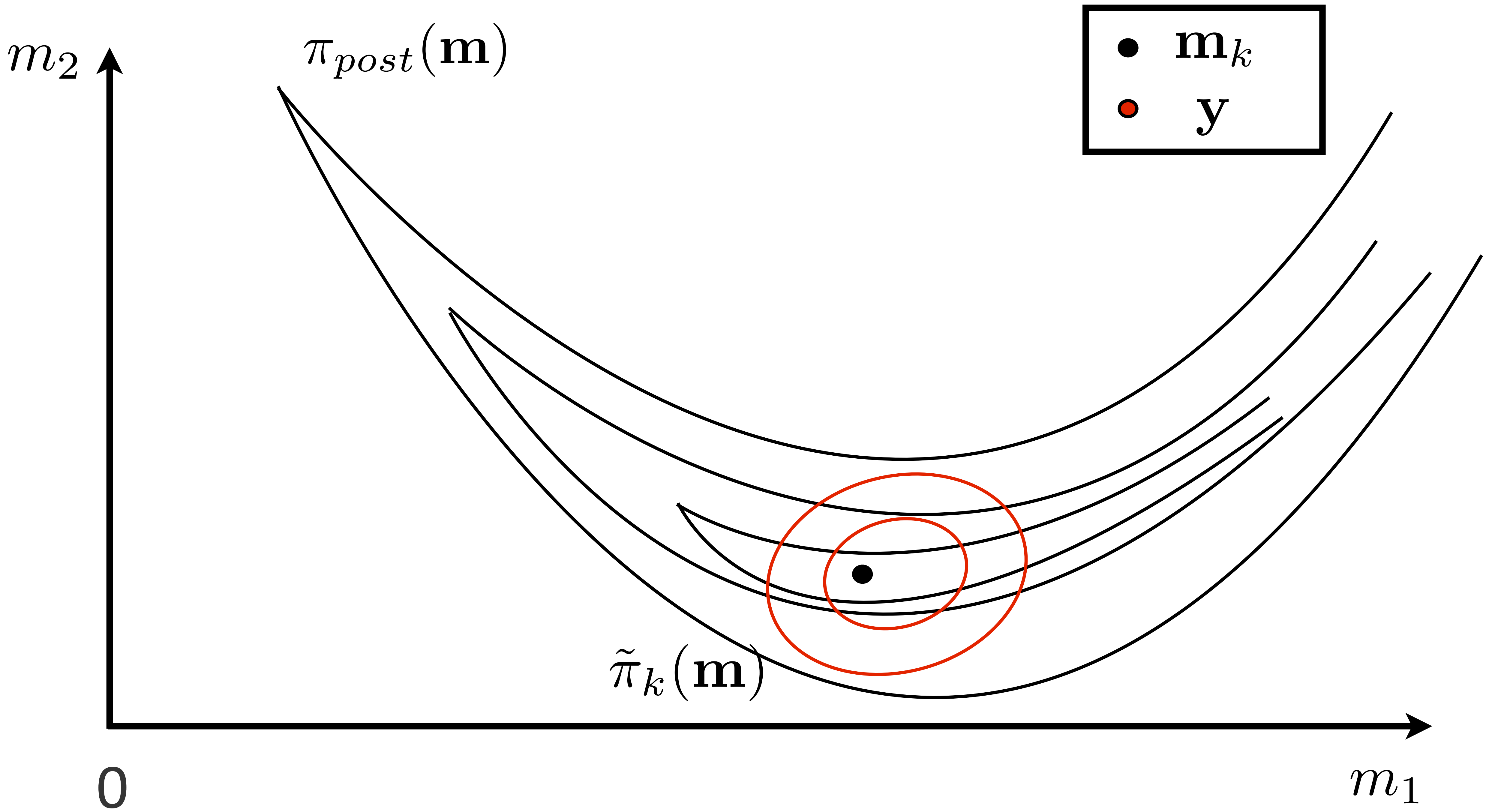

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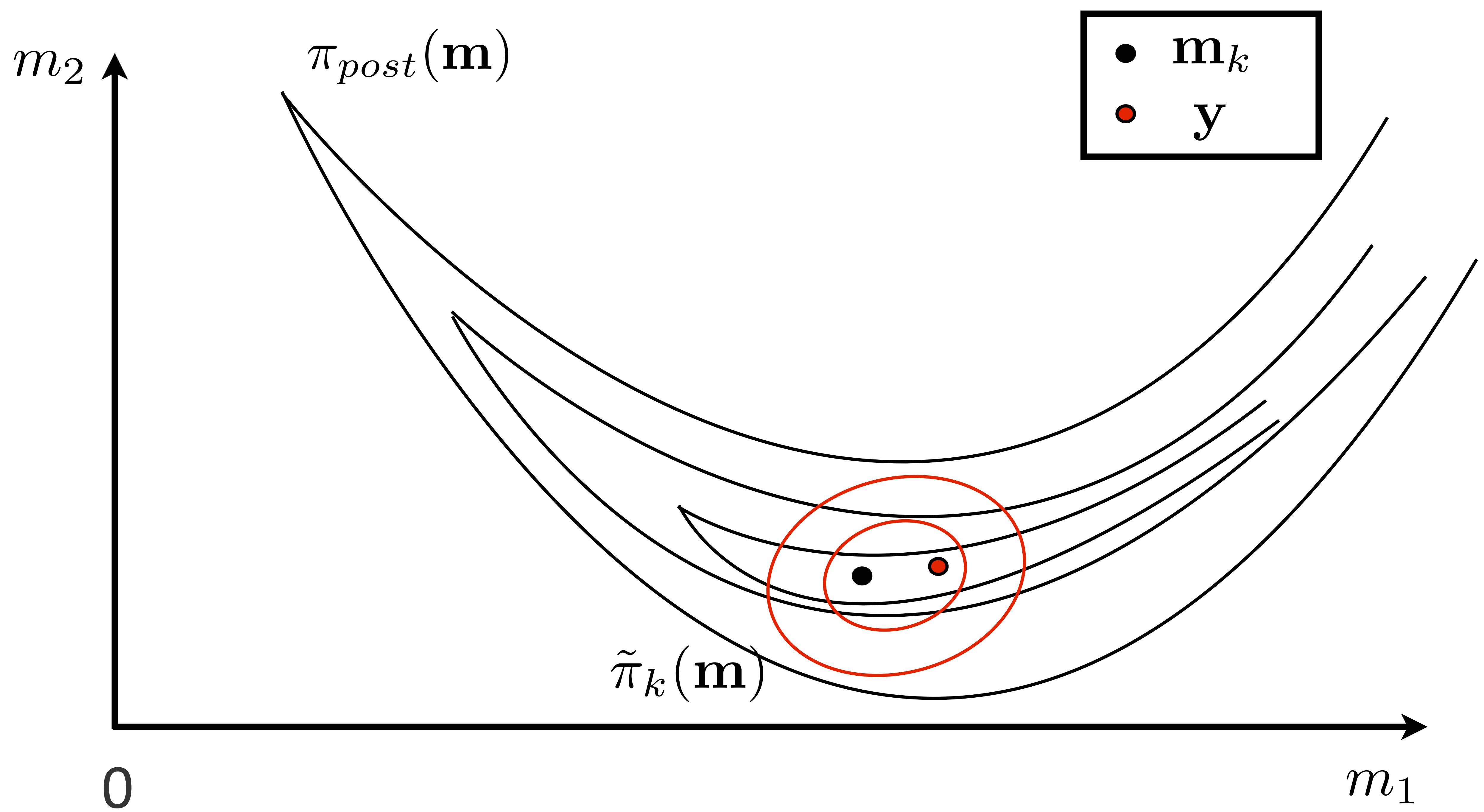

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McMC method

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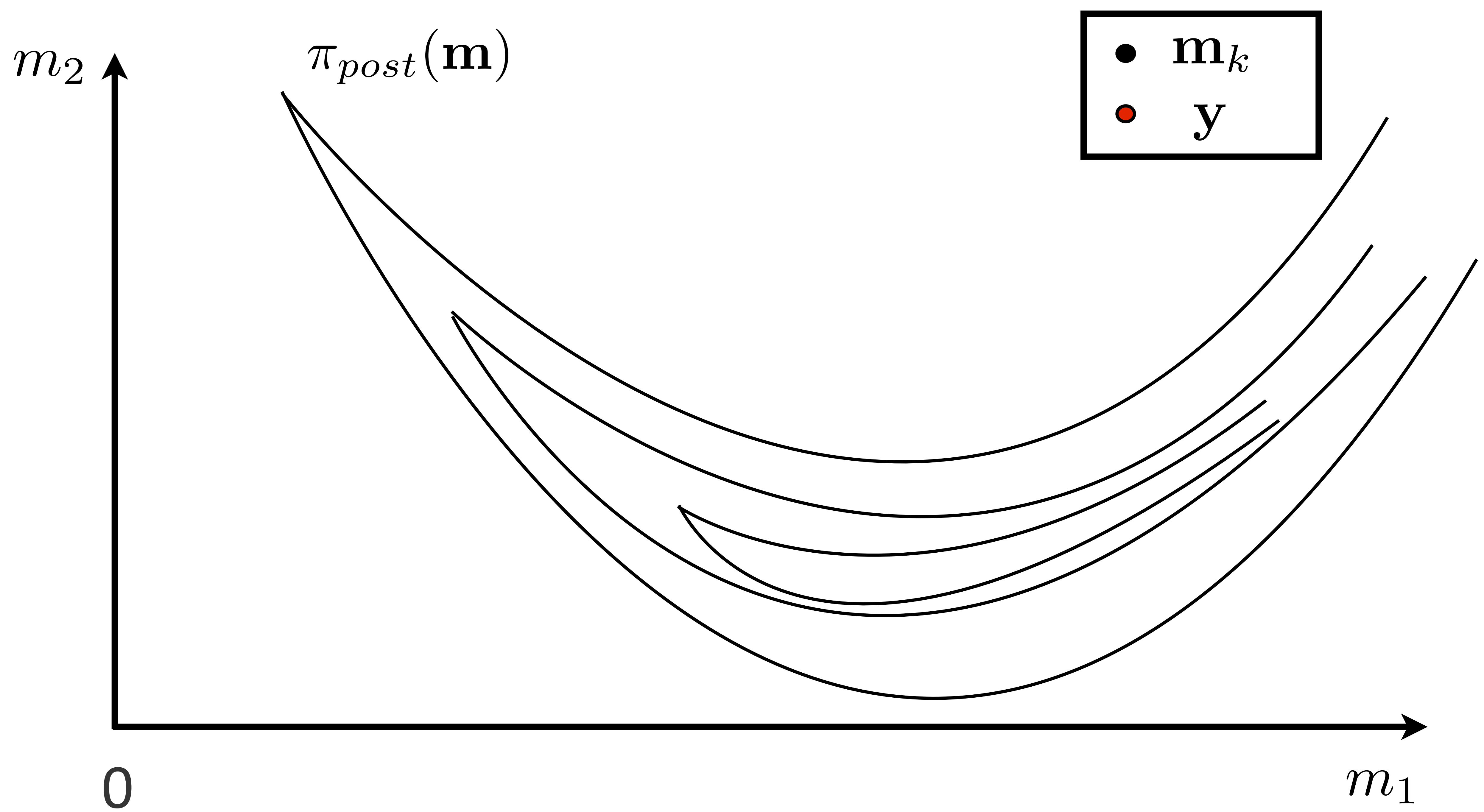

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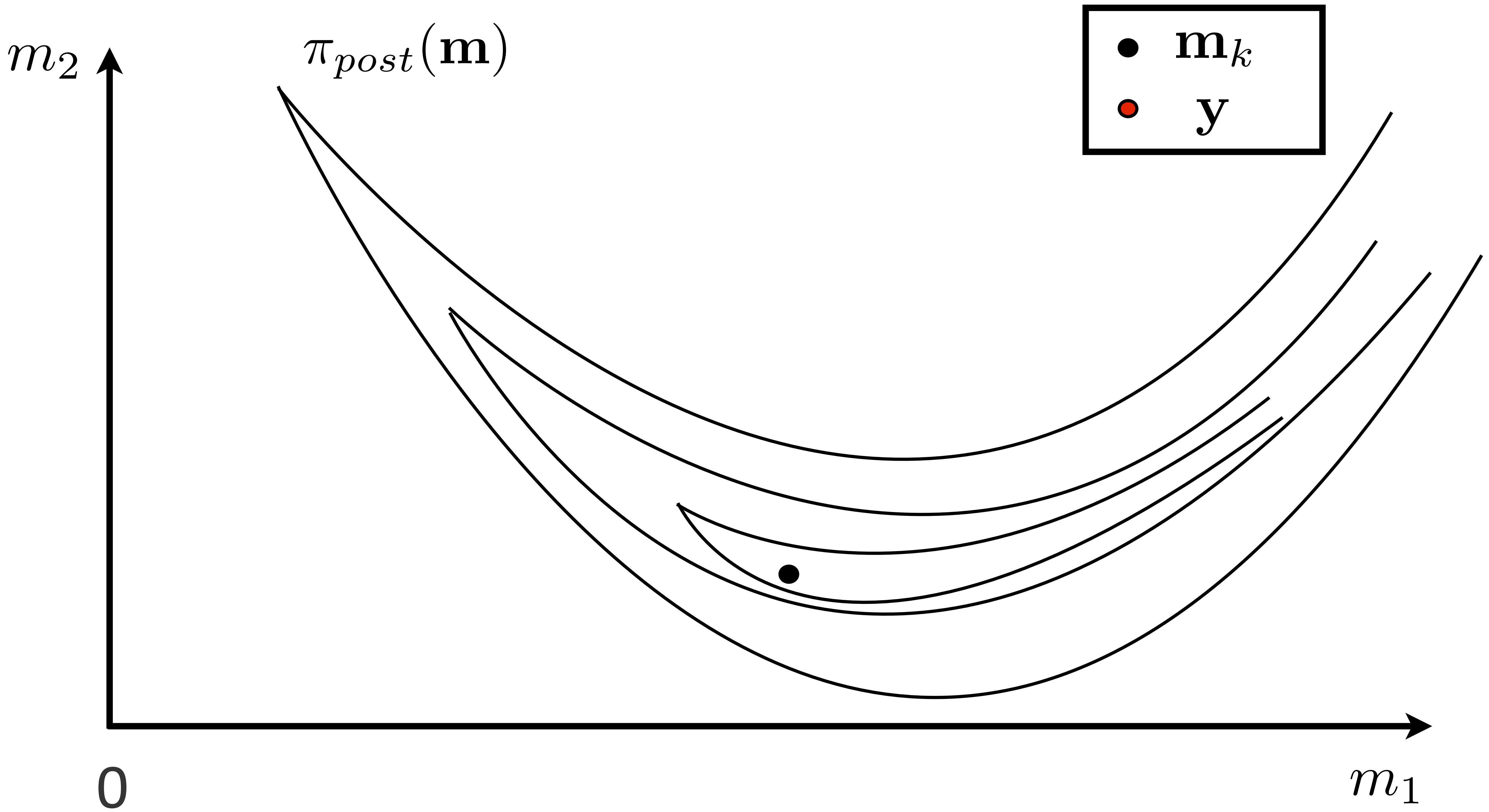

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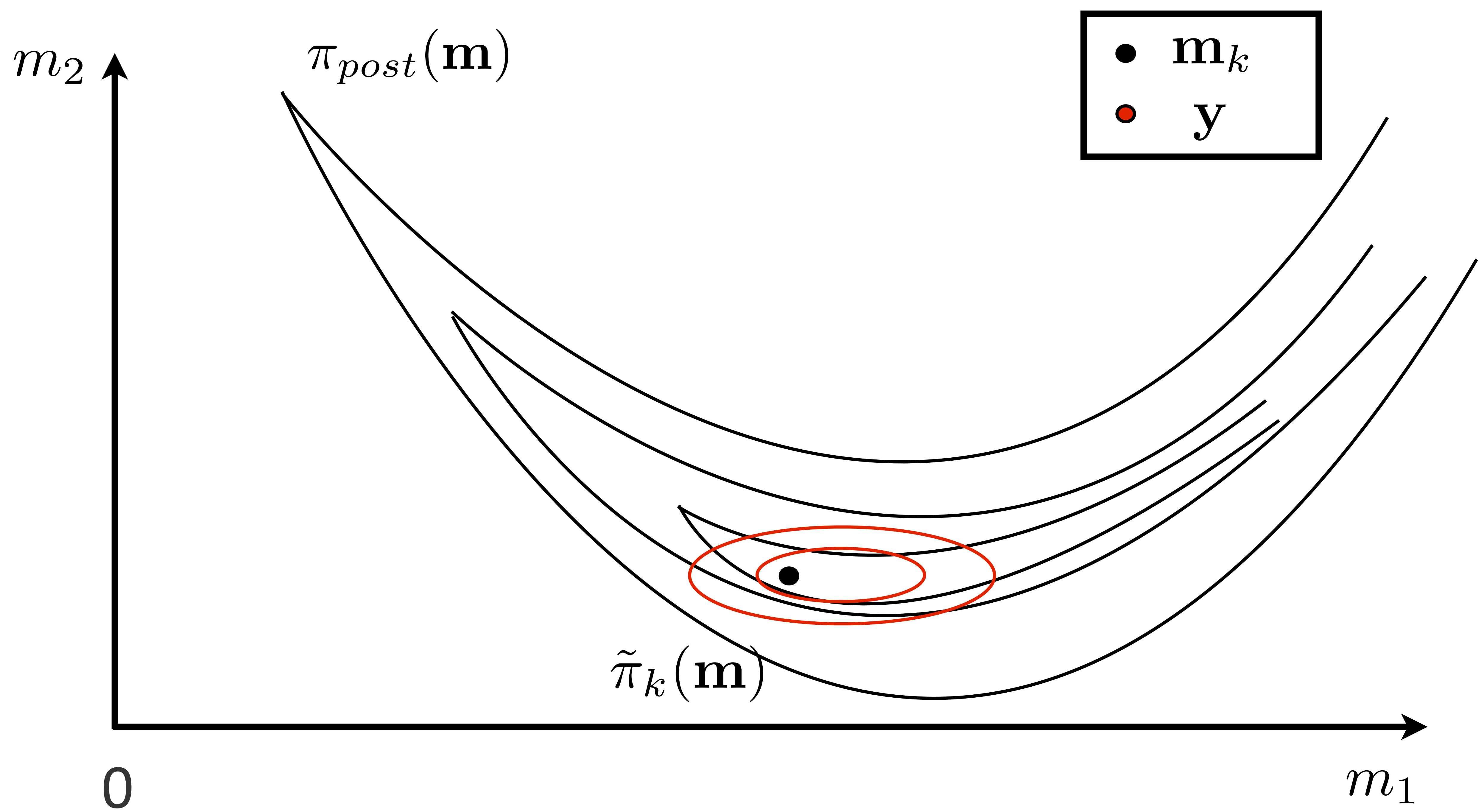

[James Martin *et al*, 2012]

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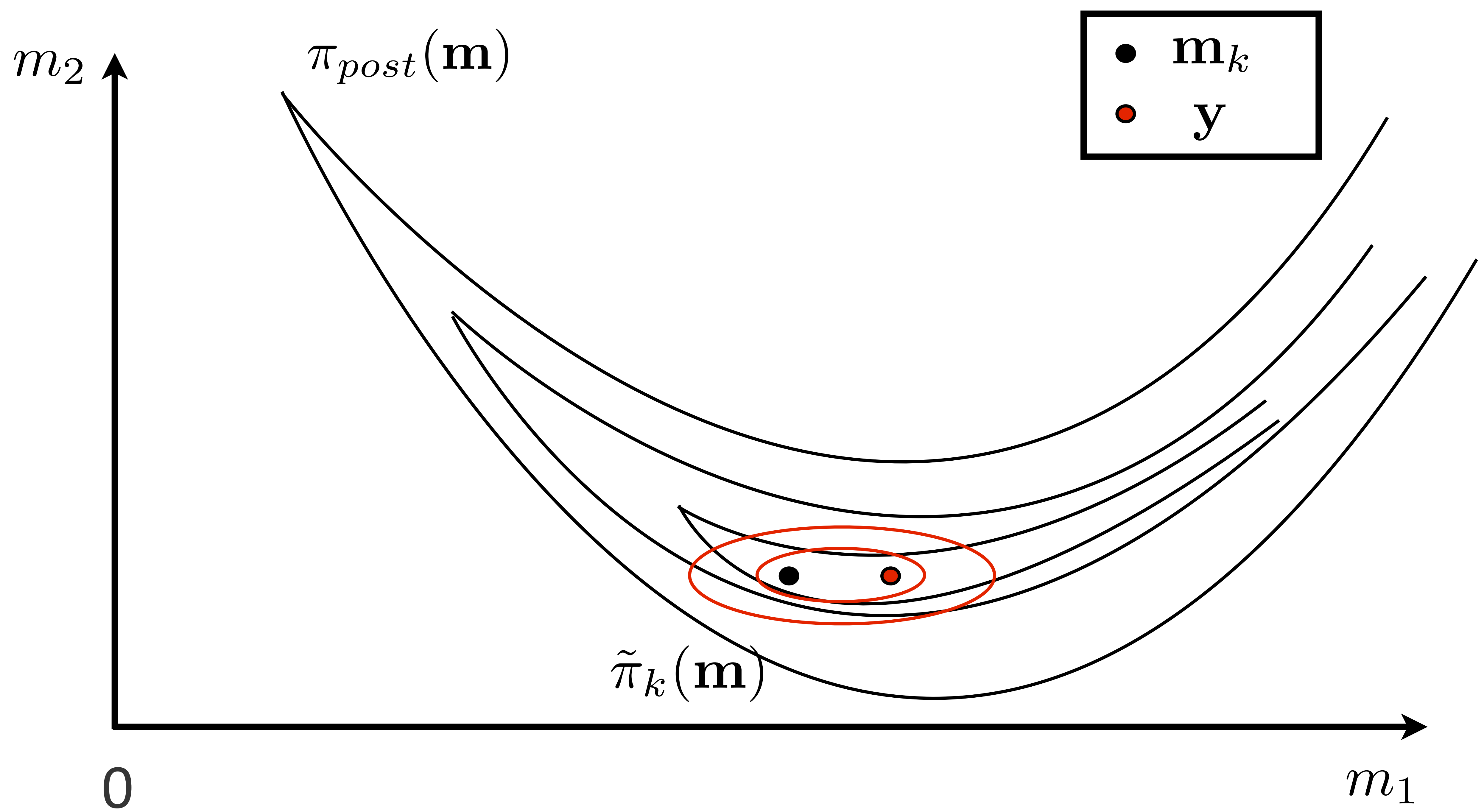

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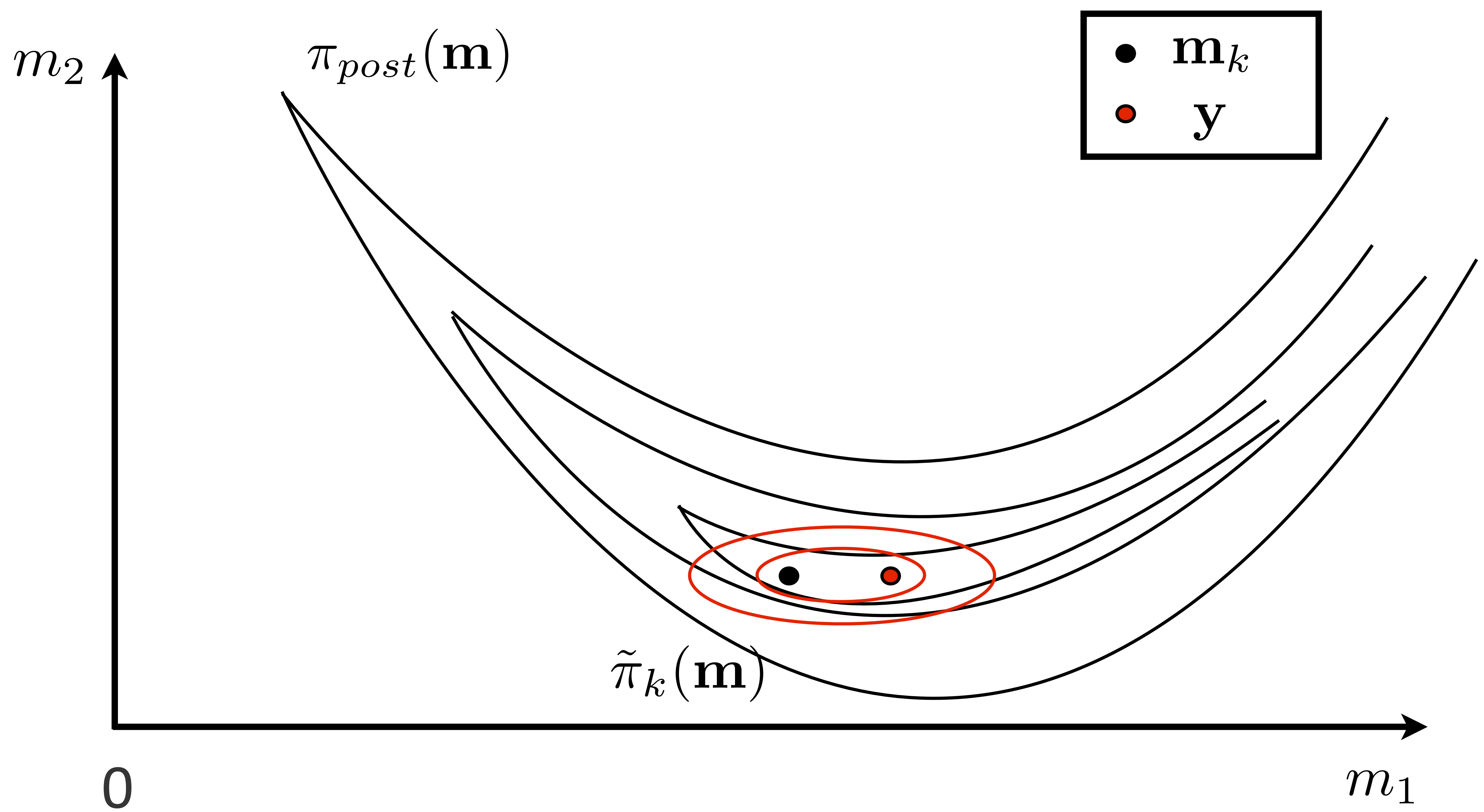

Computational cost:

1) Low rank approximation of the Hessian.

2) Number of PDE solvers ~ Number of samples.

Quantify the uncertainty

Goal : Quantify the uncertainty based on the posterior distribution $\rho_{\mathrm{post}}(\mathbf{m})$

Solution:

- Integrate the posterior distribution

- McMC method to sample the posterior distribution
  - Advantage: the true uncertainty can be quantified
  - Disadvantage: huge computational cost

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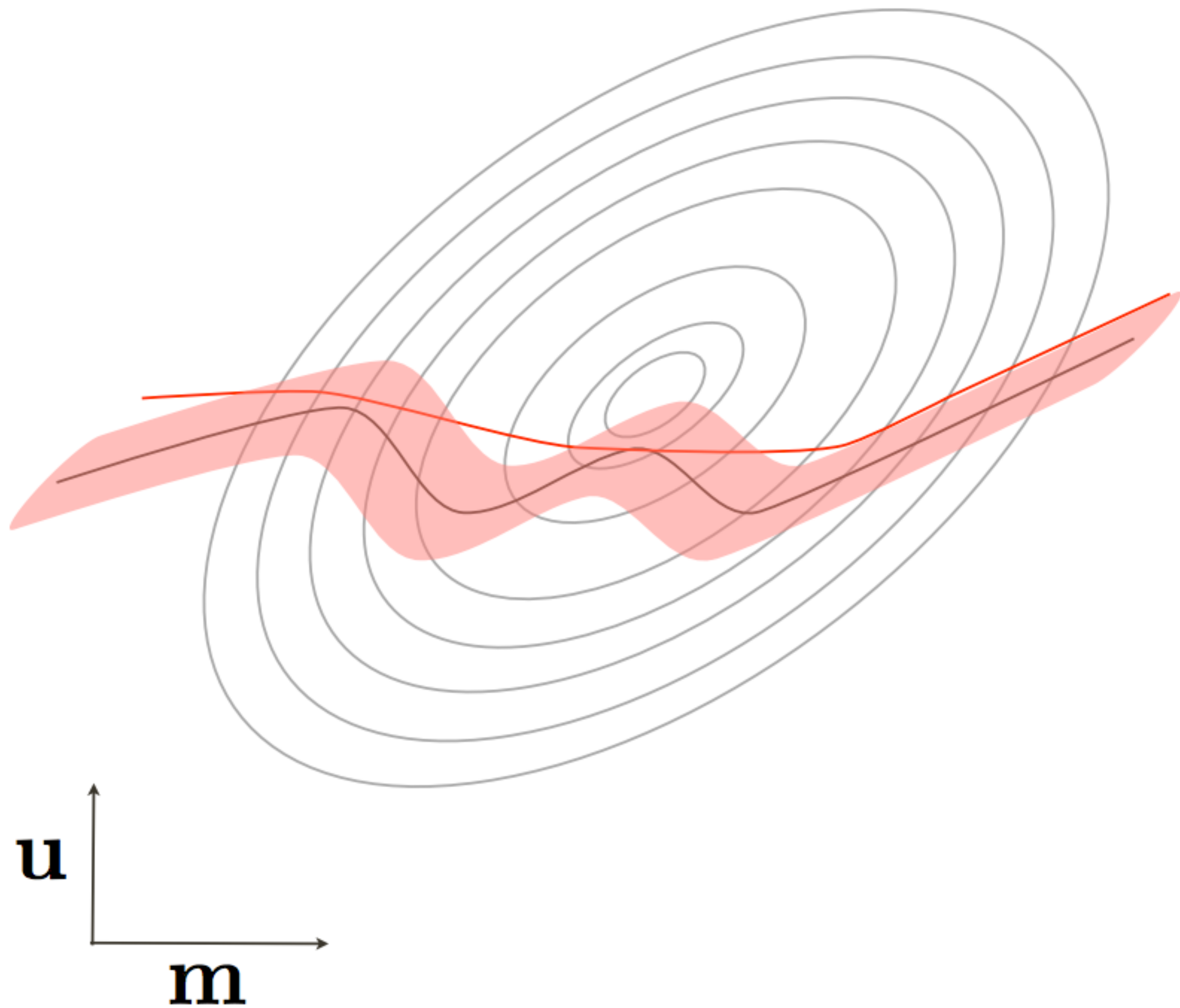
Solution:

- Integrate the posterior distribution
- McMC method to sample the posterior distribution
  - Advantage: the true uncertainty can be quantified
  - Disadvantage: huge computational cost
- Use an approximate distribution to quantify the uncertainty

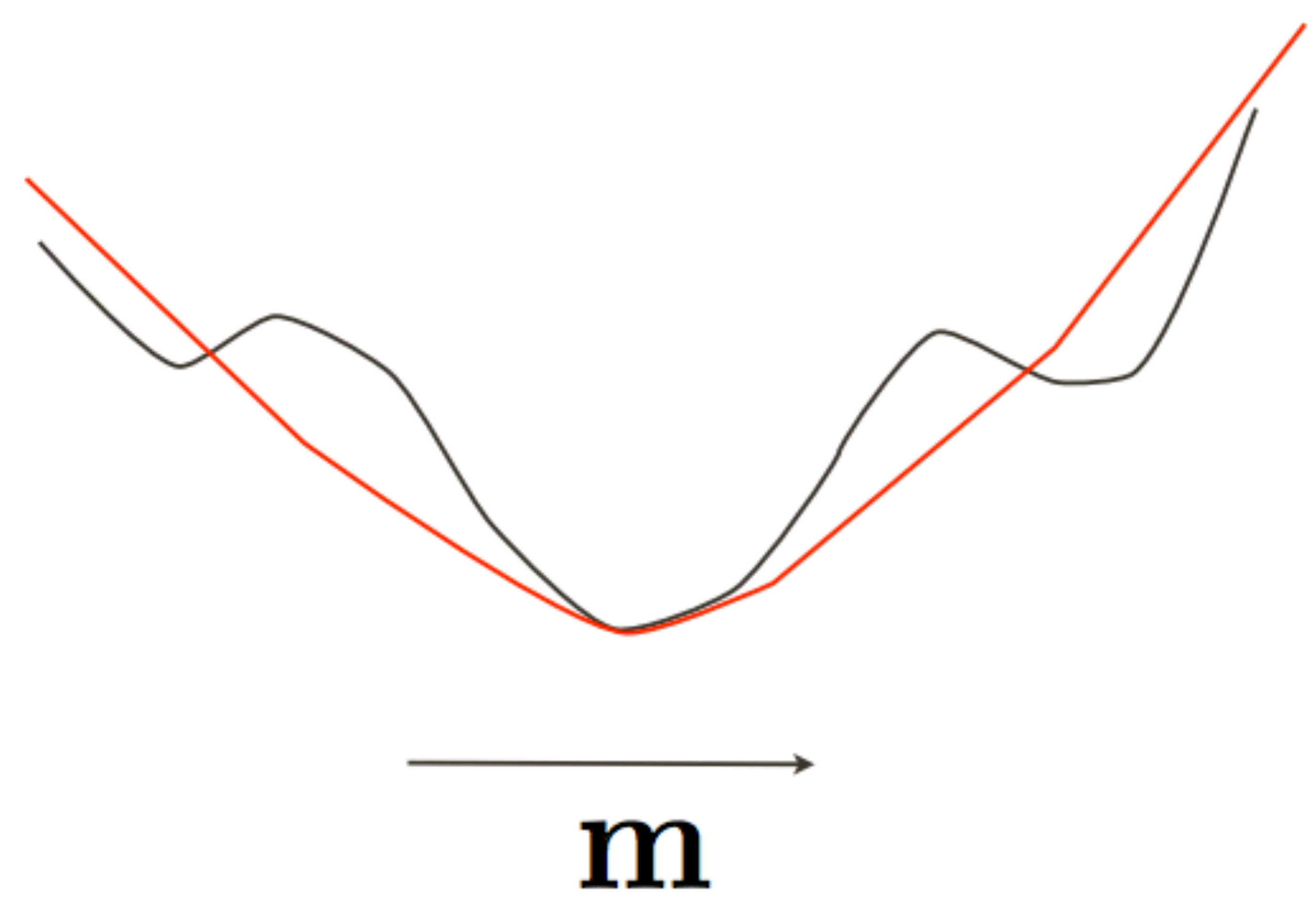
WRI vs FWI

[van Leeuwen, T and Herrmann, F J , 2013]

Larger # of degrees of freedom

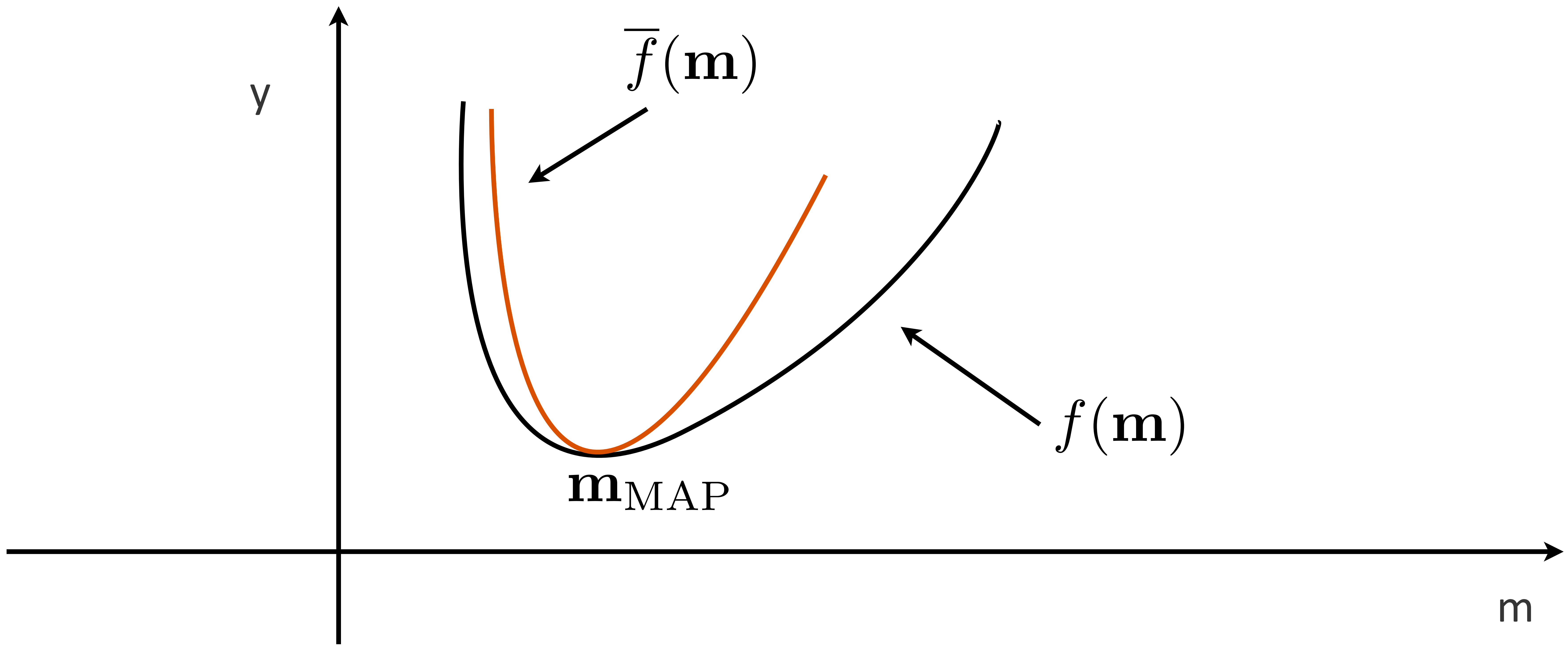


"more convex"



Quadratic approximation

$$f(\mathbf{m}) \approx f(\mathbf{m}_{\mathrm{MAP}}) + \mathbf{g}^T(\mathbf{m} - \mathbf{m}_{\mathrm{MAP}}) + \frac{1}{2}(\mathbf{m} - \mathbf{m}_{\mathrm{MAP}})^T \mathbf{H}(\mathbf{m} - \mathbf{m}_{\mathrm{MAP}}) := \overline{f}(\mathbf{m})$$



Hessian of WRI

Misfit function of WRI:

$$f(\mathbf{m}, \mathbf{u}(\mathbf{m})) = \|\mathbf{Pu} - \mathbf{d}\|^{\mathbf{2}} + \lambda^{\mathbf{2}}\|\mathbf{A}(\mathbf{m})\mathbf{u} - \mathbf{q}\|^{\mathbf{2}}$$

Hessian of WRI:

$$\mathbf{H} = \frac{\partial^{\mathbf{2}}\mathbf{f}}{\partial\mathbf{m}^{\mathbf{2}}} - \frac{\partial^{\mathbf{2}}\mathbf{f}}{\partial\mathbf{m}\partial\mathbf{u}}\frac{\partial^{\mathbf{2}}\mathbf{f}}{\partial\mathbf{u}^{\mathbf{2}}}^{-\mathbf{1}}\frac{\partial^{\mathbf{2}}\mathbf{f}}{\partial\mathbf{u}\partial\mathbf{m}}$$

Hessian of WRI

$$\mathbf{H} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} - \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2}^{-1} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}}$$

Dense and not positive-definite

where,

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2} = \mathbf{P^T P} + \lambda^2 \mathbf{A^T A}$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} = \lambda^2 \left(\frac{\partial \mathbf{A u}}{\partial \mathbf{m}}\right)^{\mathbf{T}} \left(\frac{\partial \mathbf{A u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}} = \lambda^2 \left(\frac{\partial \mathbf{A^T (A u - q)}}{\partial \mathbf{m}}\right) + \lambda^2 \mathbf{A^T} \left(\frac{\partial \mathbf{A u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} = \lambda^2 \left(\frac{\partial \mathbf{A^T (A u - q)}}{\partial \mathbf{m}}\right)^{\mathbf{T}} + \lambda^2 \left(\frac{\partial \mathbf{A u}}{\partial \mathbf{m}}\right)^{\mathbf{T}} \mathbf{A}$$

Hessian of WRI

$$\mathbf{H} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} - \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2}^{-1} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}}$$

Dense and not positive-definite

where,

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2} = \mathbf{P}^{\mathbf{T}}\mathbf{P} + \lambda^{\mathbf{2}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} = \lambda^{\mathbf{2}}\left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)^{\mathbf{T}}\left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}} = \lambda^{\mathbf{2}}\left(\frac{\partial \mathbf{A}^{\mathbf{T}}(\mathbf{A}\mathbf{u} - \mathbf{q})}{\partial \mathbf{m}}\right) + \lambda^{\mathbf{2}}\mathbf{A}^{\mathbf{T}}\left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} = \lambda^{\mathbf{2}}\left(\frac{\partial \mathbf{A}^{\mathbf{T}}(\mathbf{A}\mathbf{u} - \mathbf{q})}{\partial \mathbf{m}}\right)^{\mathbf{T}} + \lambda^{\mathbf{2}}\left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)^{\mathbf{T}}\mathbf{A}$$

$$\boxed{\begin{array}{l} \lambda \to \infty \text{ or } \mathbf{m} \to \mathbf{m}_t \\ \mathbf{A}\mathbf{u} - \mathbf{q} \to 0 \end{array}}$$

Hessian of WRI

$$\mathbf{H} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} - \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2}^{-1} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}}$$

Dense and not positive-definite

where,

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2} = \mathbf{P}^{\mathbf{T}}\mathbf{P} + \lambda^2 \mathbf{A}^{\mathbf{T}}\mathbf{A}$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} = \lambda^2 \left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)^{\mathbf{T}} \left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}} = \xcancel{\lambda^2 \left(\frac{\partial \mathbf{A}^{\mathbf{T}}(\mathbf{A}\mathbf{u} - \mathbf{q})}{\partial \mathbf{m}}\right)} + \lambda^2 \mathbf{A}^{\mathbf{T}} \left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} = \xcancel{\lambda^2 \left(\frac{\partial \mathbf{A}^{\mathbf{T}}(\mathbf{A}\mathbf{u} - \mathbf{q})}{\partial \mathbf{m}}\right)^{\mathbf{T}}} + \lambda^2 \left(\frac{\partial \mathbf{A}\mathbf{u}}{\partial \mathbf{m}}\right)^{\mathbf{T}} \mathbf{A}$$

$$\boxed{\begin{array}{l} \lambda \to \infty \text{ or } \mathbf{m} \to \mathbf{m}_t \\ \mathbf{A}\mathbf{u} - \mathbf{q} \to 0 \end{array}}$$

Positive-definite approximation of Hessian

Approximated Hessian:

$$
\begin{aligned}
\mathbf{H} &= \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} - \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}\partial \mathbf{u}} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2}^{-1} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}\partial \mathbf{m}} \\
&\approx \lambda^2 \mathbf{G}^T\mathbf{G} - \lambda^4 \mathbf{G}^T \mathbf{A}(\lambda^2 \mathbf{A}^T \mathbf{A} + \mathbf{P}^T\mathbf{P})^{-1}\mathbf{A}^T\mathbf{G} \\
&= \lambda^2 \mathbf{G}^T\left(\frac{1}{\lambda^2}\mathbf{A}^{-T}\mathbf{P}^T \underbrace{(\mathbf{I} + \frac{1}{\lambda^2}\mathbf{P}\mathbf{A}^{-1}\mathbf{A}^{-T}\mathbf{P}^T)^{-1}}_{n_r^2} \underbrace{\mathbf{P}\mathbf{A}^{-1}}_{n_r n_g}\right)\underbrace{\mathbf{G}}_{\text{diagonal}}
\end{aligned}
$$

Positive-definite approximation of Hessian

Approximated Hessian:

$$
\begin{aligned}
\mathbf{H} &= \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m}^2} - \frac{\partial^2 \mathbf{f}}{\partial \mathbf{m} \partial \mathbf{u}} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}^2}^{-1} \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{m}} \\
&\approx \lambda^2 \mathbf{G}^T \mathbf{G} - \lambda^4 \mathbf{G}^T \mathbf{A} (\lambda^2 \mathbf{A}^T \mathbf{A} + \mathbf{P}^T \mathbf{P})^{-1} \mathbf{A}^T \mathbf{G} \\
&= \lambda^2 \mathbf{G}^T \Big(\frac{1}{\lambda^2} \mathbf{A}^{-T} \mathbf{P}^T \underbrace{\Big(\mathbf{I} + \frac{1}{\lambda^2} \mathbf{P} \mathbf{A}^{-1} \mathbf{A}^{-T} \mathbf{P}^T\Big)}_{\mathbf{S}}{}^{-1} \underbrace{\mathbf{P} \mathbf{A}^{-1} \Big) \mathbf{G}}_{\mathbf{W}}
\end{aligned}
$$

Positive-definite approximation of Hessian

$$\mathbf{H} = \sum_{n_f, n_s} \mathbf{W}^{\mathbf{T}} \mathbf{S} \mathbf{W}$$

Positive-definite approximation of Hessian

$$\mathbf{H} = \sum_{n_f, n_s} \mathbf{W}^{\mathbf{T}} \, \mathbf{S} \, \mathbf{W}$$

Computational cost:

$n_f * (n_s + n_r)$

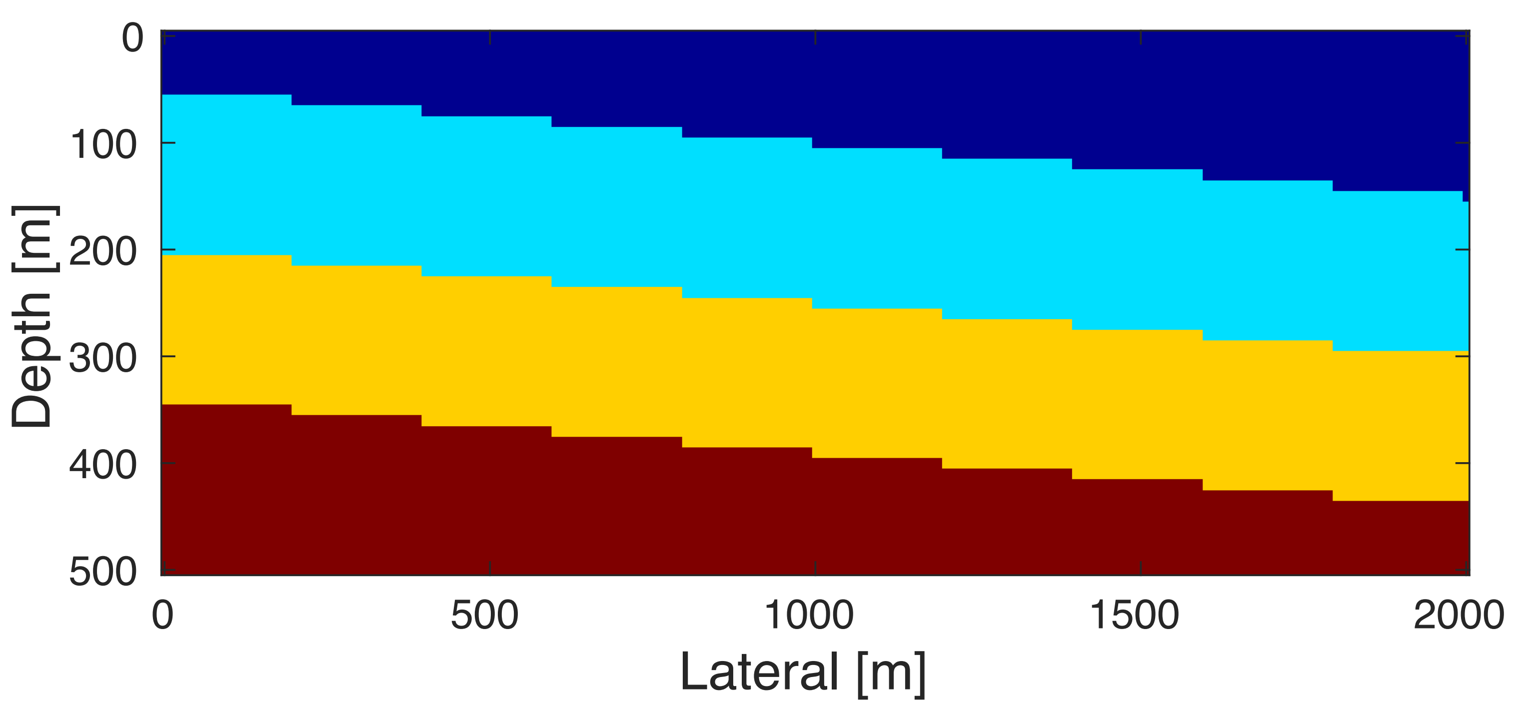
storage cost:

$n_f * n_g * (n_s + n_r)$

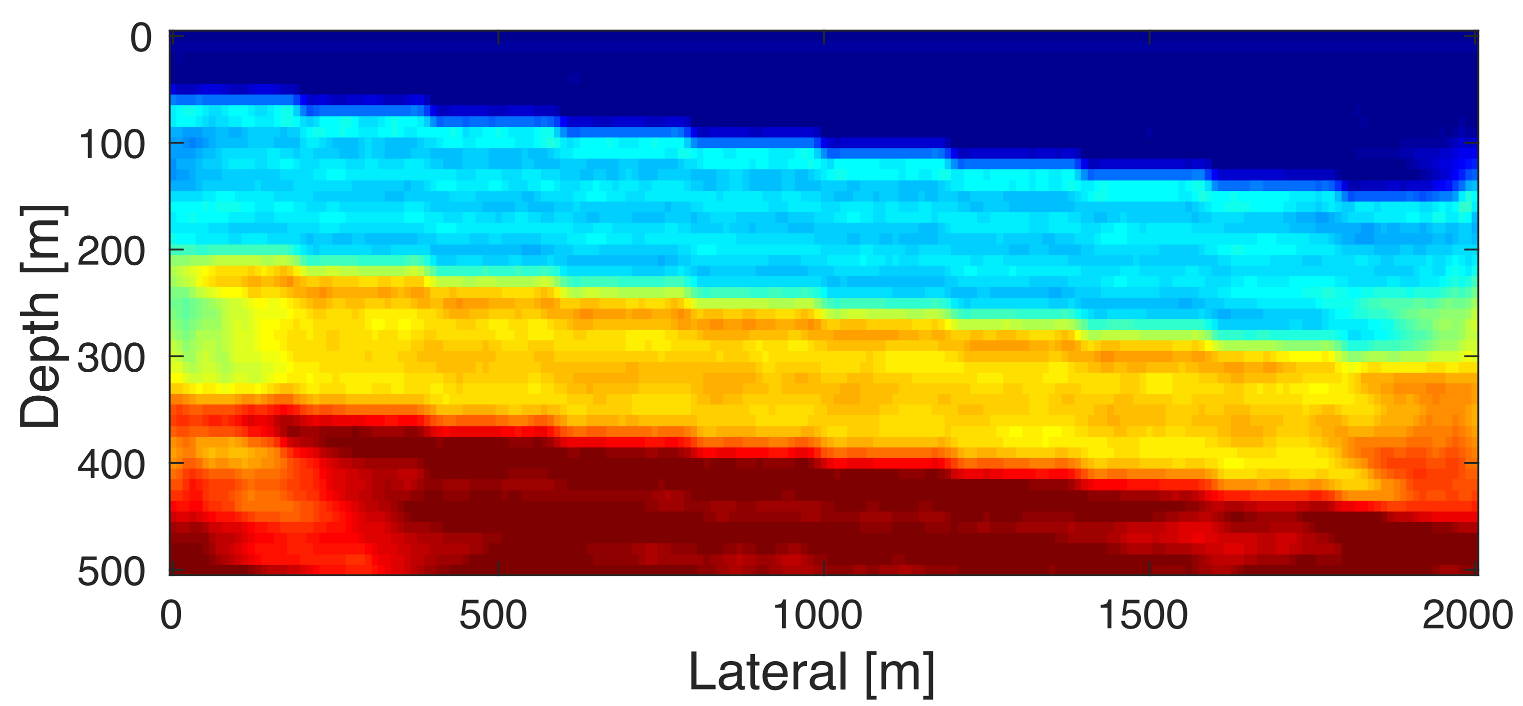
Positive-definite approximation of Hessian

Evaluate the Hessian with three different models:

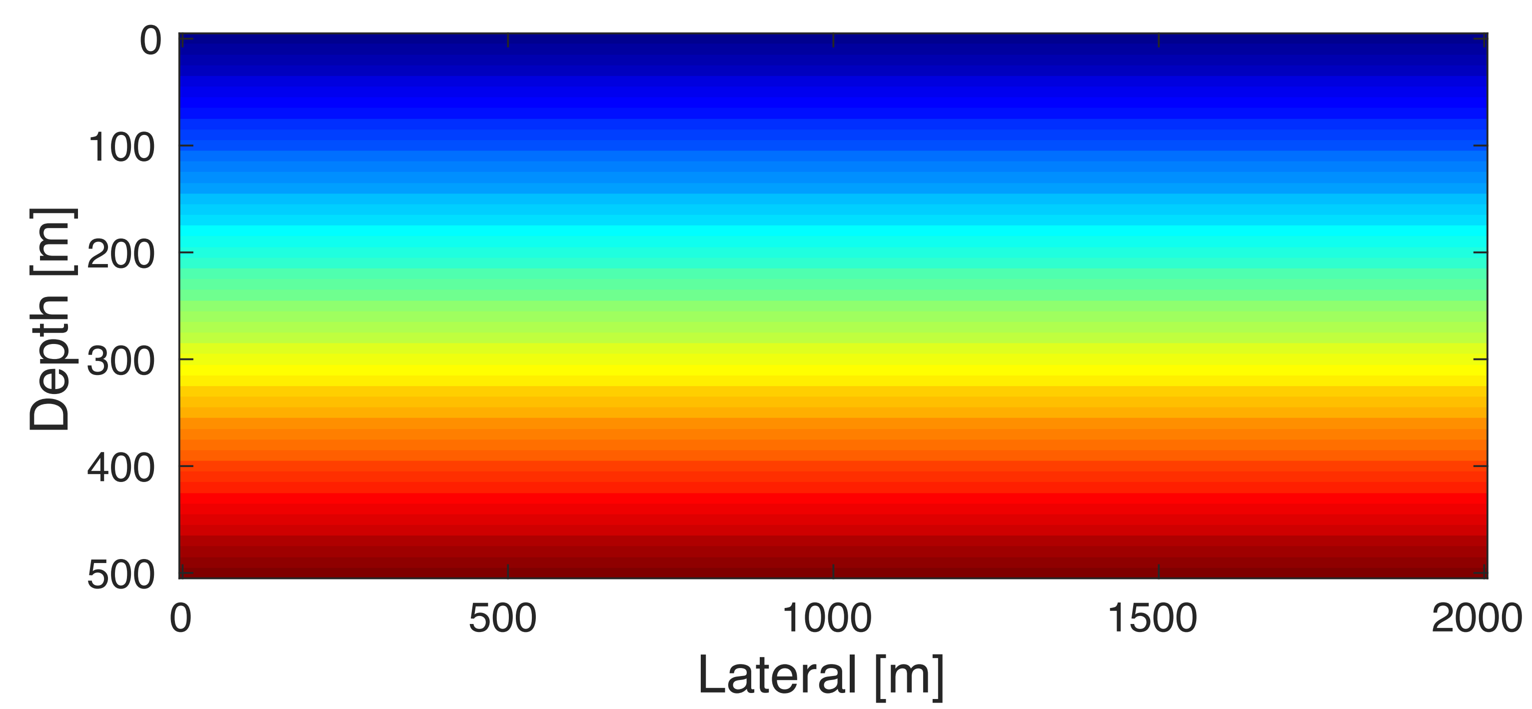
- True model
- Model from inversion
- Initial model



$\mathbf{m}_t$



$\mathbf{m}_n$



$\mathbf{m}_0$

Positive-definite approximation of Hessian

Evaluate the Hessian with three different model:

- True model
- Model from inversion
- Initial model

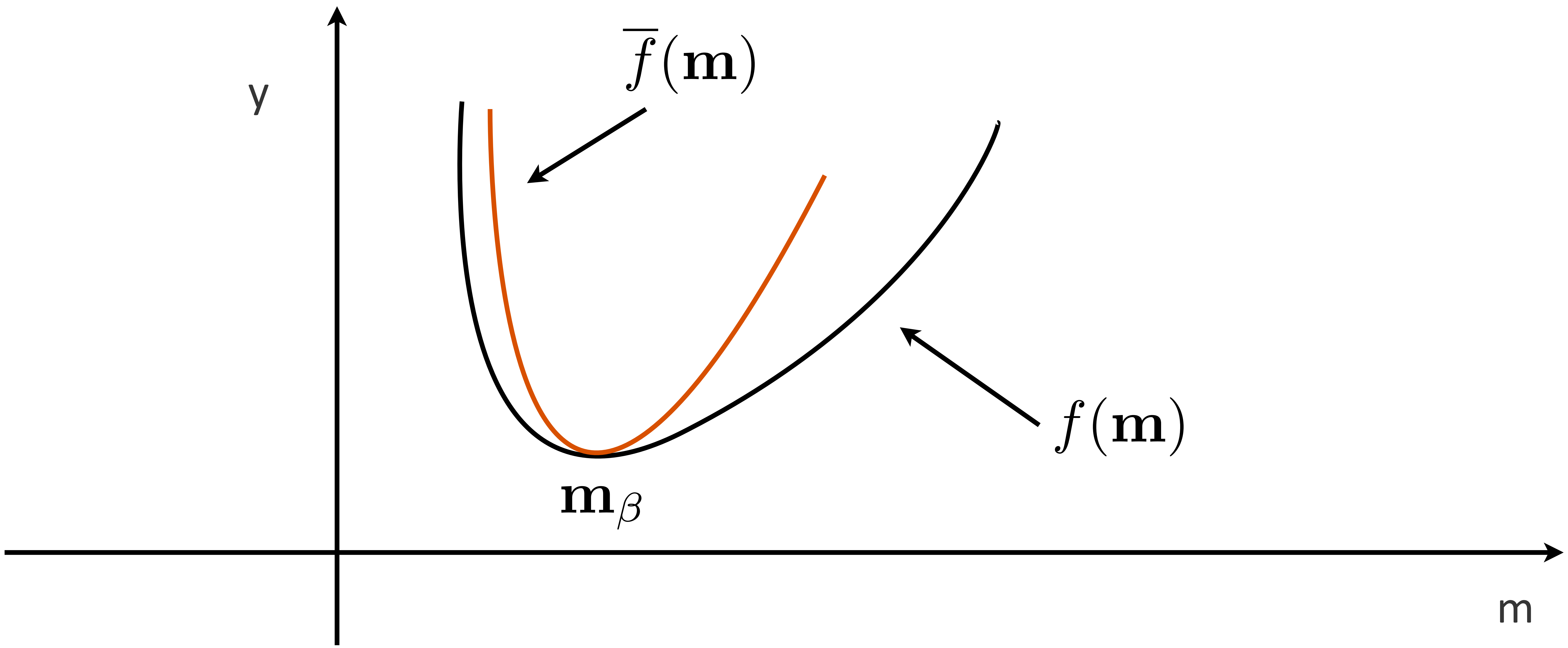
Evaluate the Hessian with three different λ choices:

- $\lambda = 1e0$
- $\lambda = 1e3$
- $\lambda = 1e6$

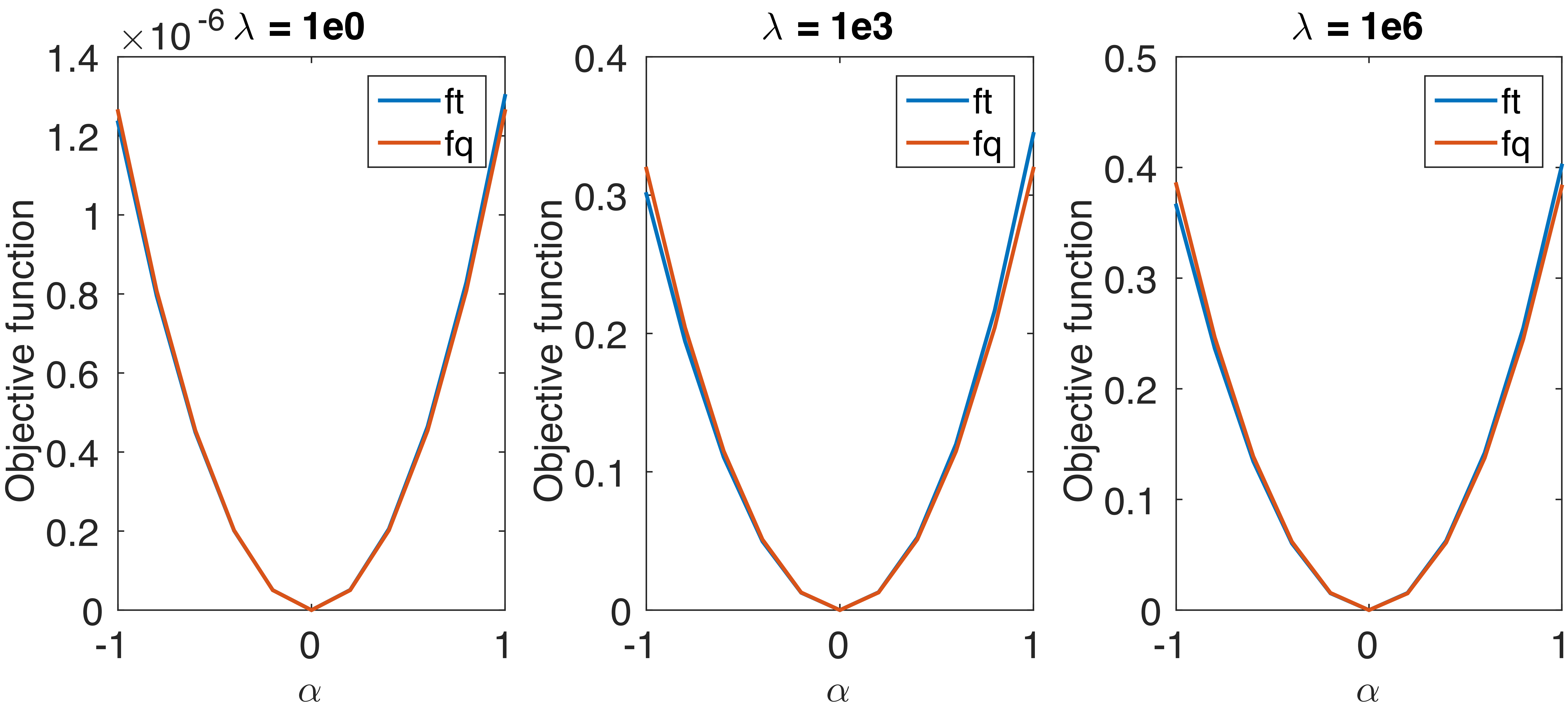
Quadratic approximation

$$f(\mathbf{m}) \approx f(\mathbf{m}_\beta) + \mathbf{g}^T(\mathbf{m} - \mathbf{m}_\beta) + \frac{1}{2}(\mathbf{m} - \mathbf{m}_\beta)^T\mathbf{H}(\mathbf{m} - \mathbf{m}_\beta) := \overline{f}(\mathbf{m})$$

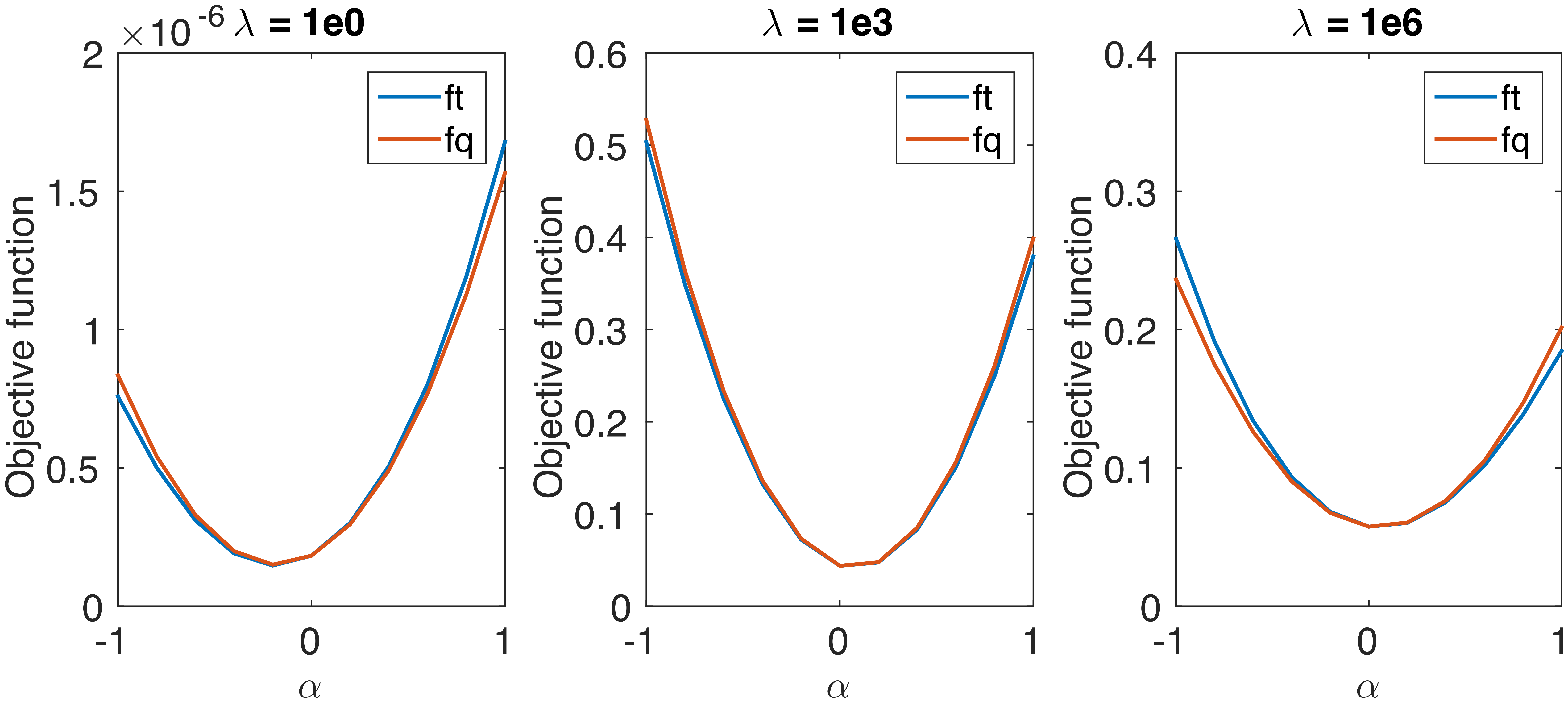
$$\mathbf{m} = \mathbf{m}_\beta + \alpha \mathrm{d}\mathbf{m},\ \beta \in \{t, n, 0\}$$



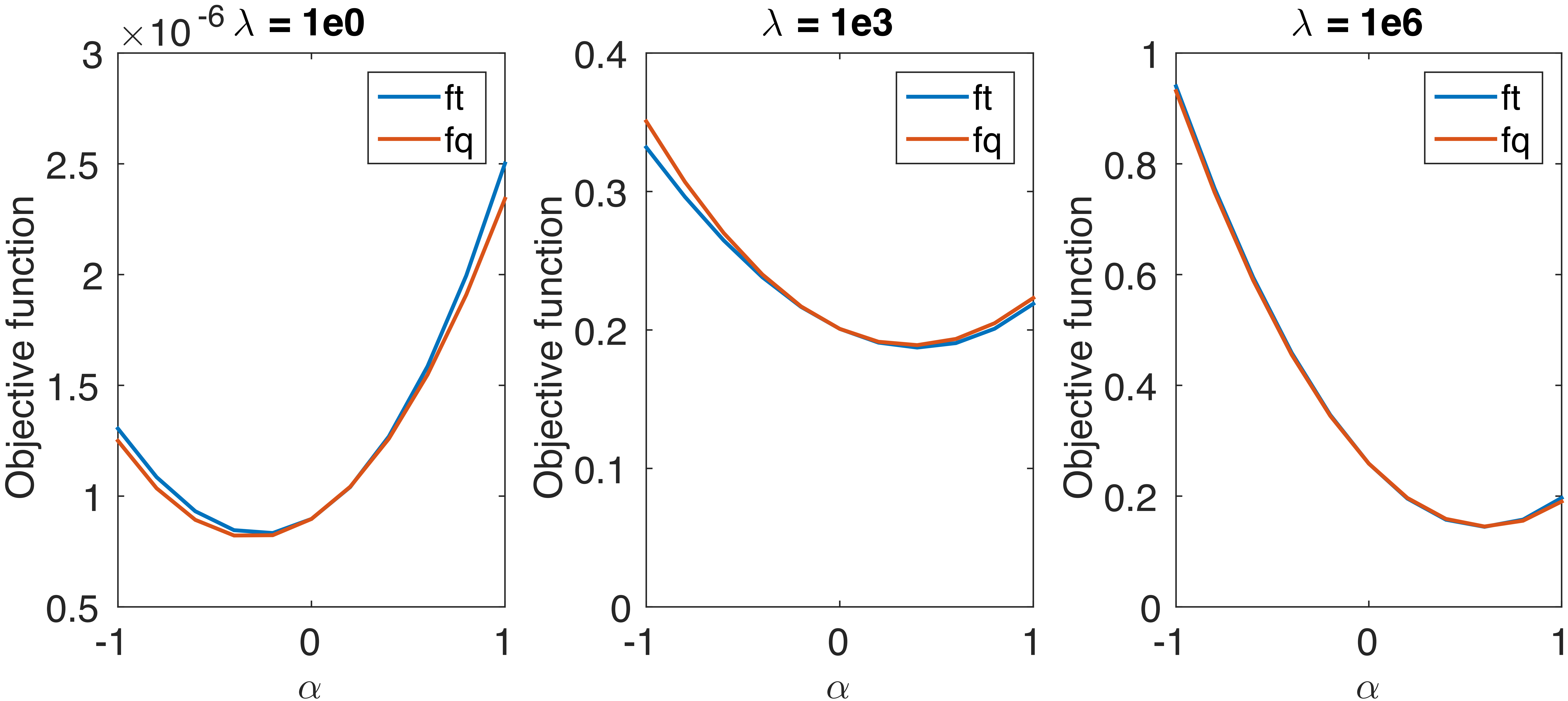
Quadratic approximation with approximated Hessian at m_t



Quadratic approximation with approximated Hessian at m_n



Quadratic approximation with approximated Hessian at m_0



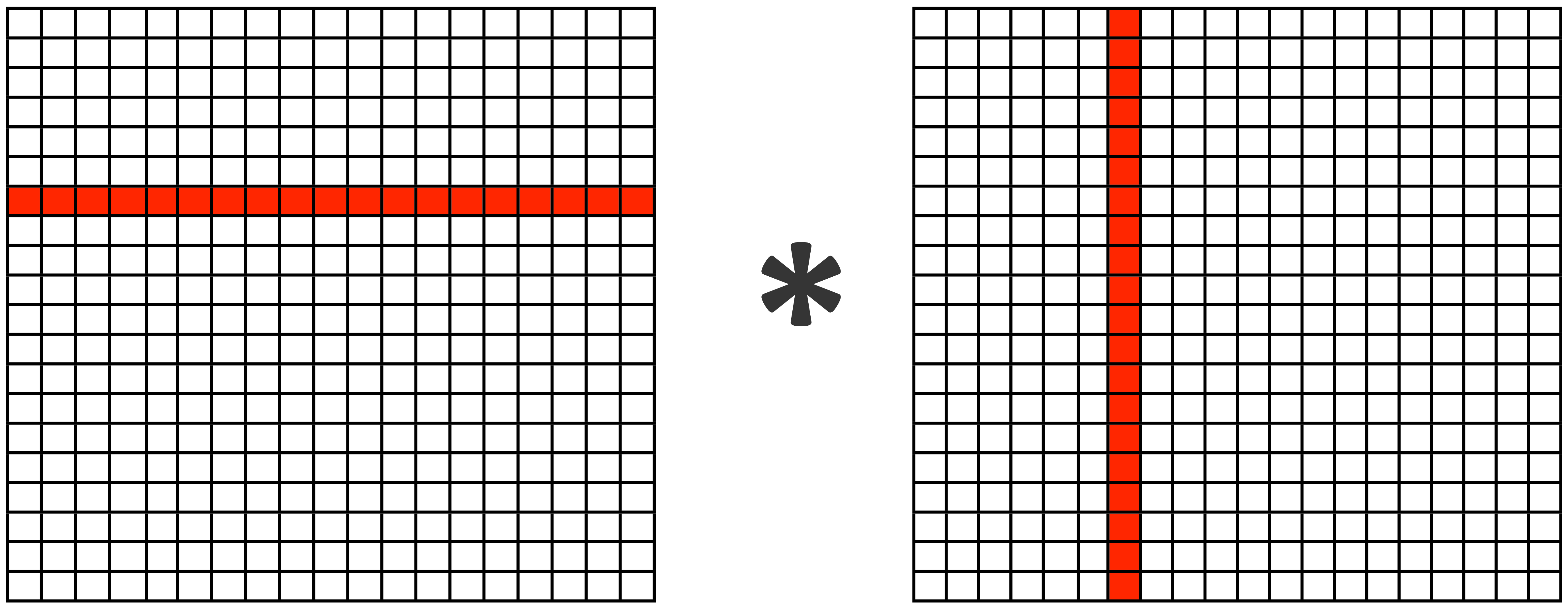
Diagonal part of the Hessian

$$\mathbf{H} = \lambda^2 \mathbf{G}^\mathbf{T}\left(\frac{1}{\lambda^2}\mathbf{A}^{-\mathbf{T}}\mathbf{P}^\mathbf{T}\left(\mathbf{I} + \frac{1}{\lambda^2}\mathbf{P}\mathbf{A}^{-1}\mathbf{A}^{-\mathbf{T}}\mathbf{P}^\mathbf{T}\right)^{-1}\mathbf{P}\mathbf{A}^{-1}\right)\mathbf{G}$$
$$= \mathbf{B}^\mathbf{T}\mathbf{B}$$

Diagonal part of the Hessian:

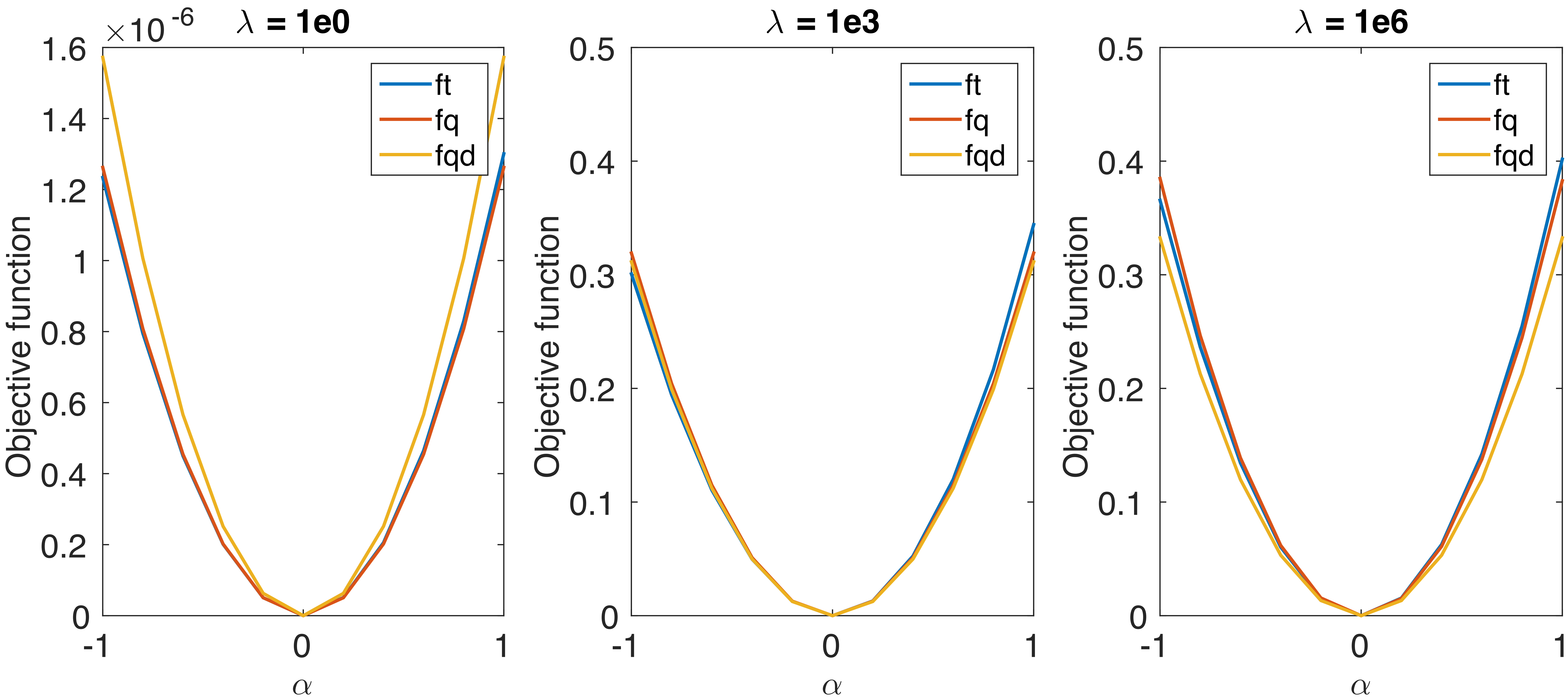
$$h_{i,i} = \mathbf{B}(:,i)^\mathbf{T} * \mathbf{B}(:,i)$$

$h_{i,i}$ =

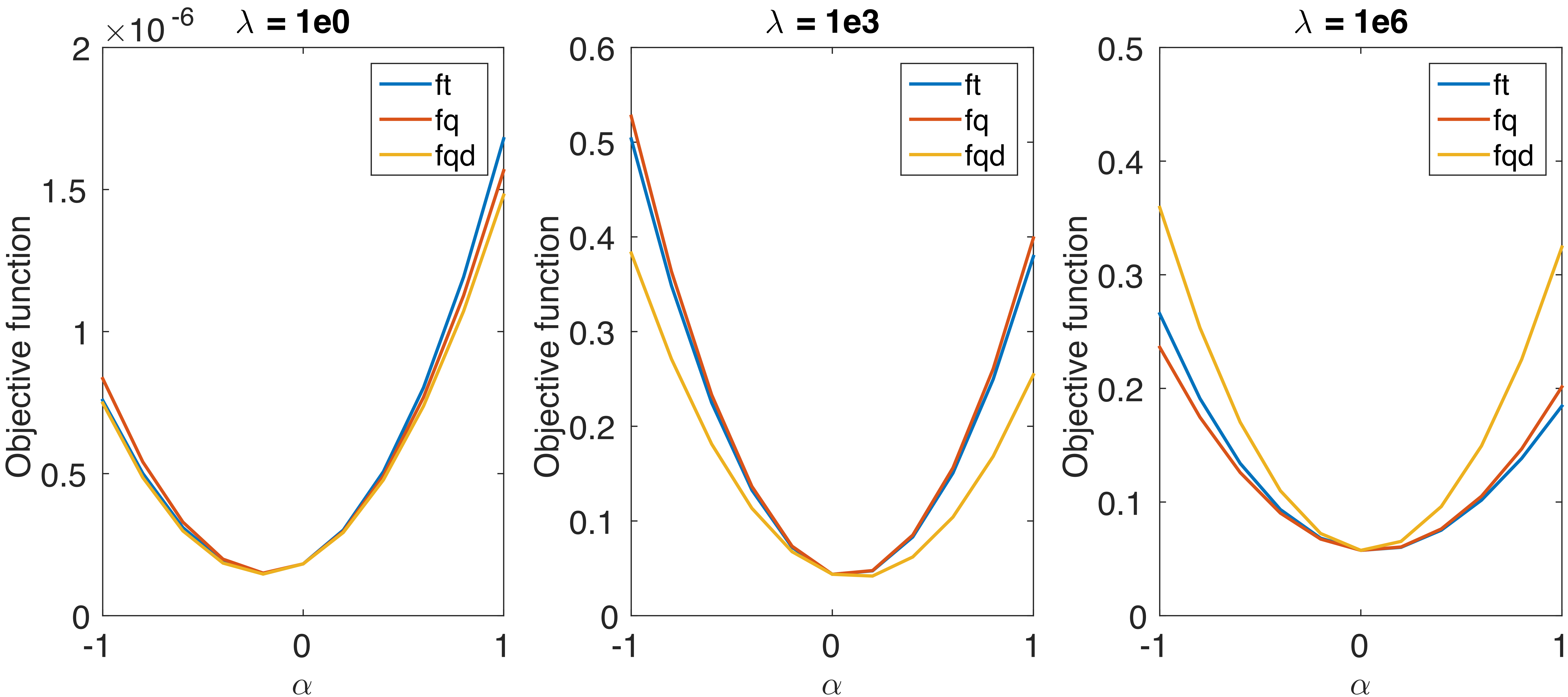


$\mathbf{B}^T$ $\quad$ $\mathbf{B}$

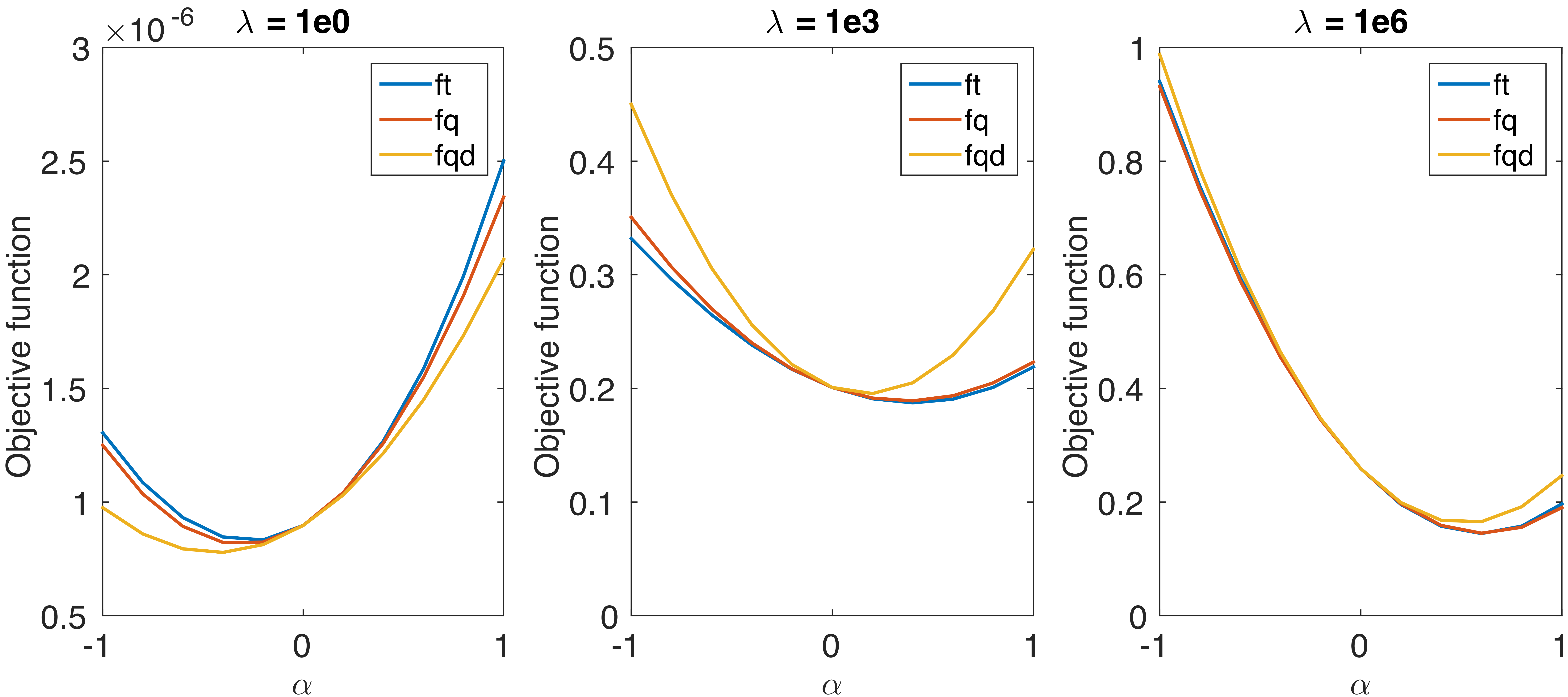
Quadratic approximation with diagonal part of approximated Hessian at m_t



Quadratic approximation with diagonal part of approximated Hessian at m_n



Quadratic approximation with diagonal part of approximated Hessian at m_0



Positive-definite approximation of Hessian

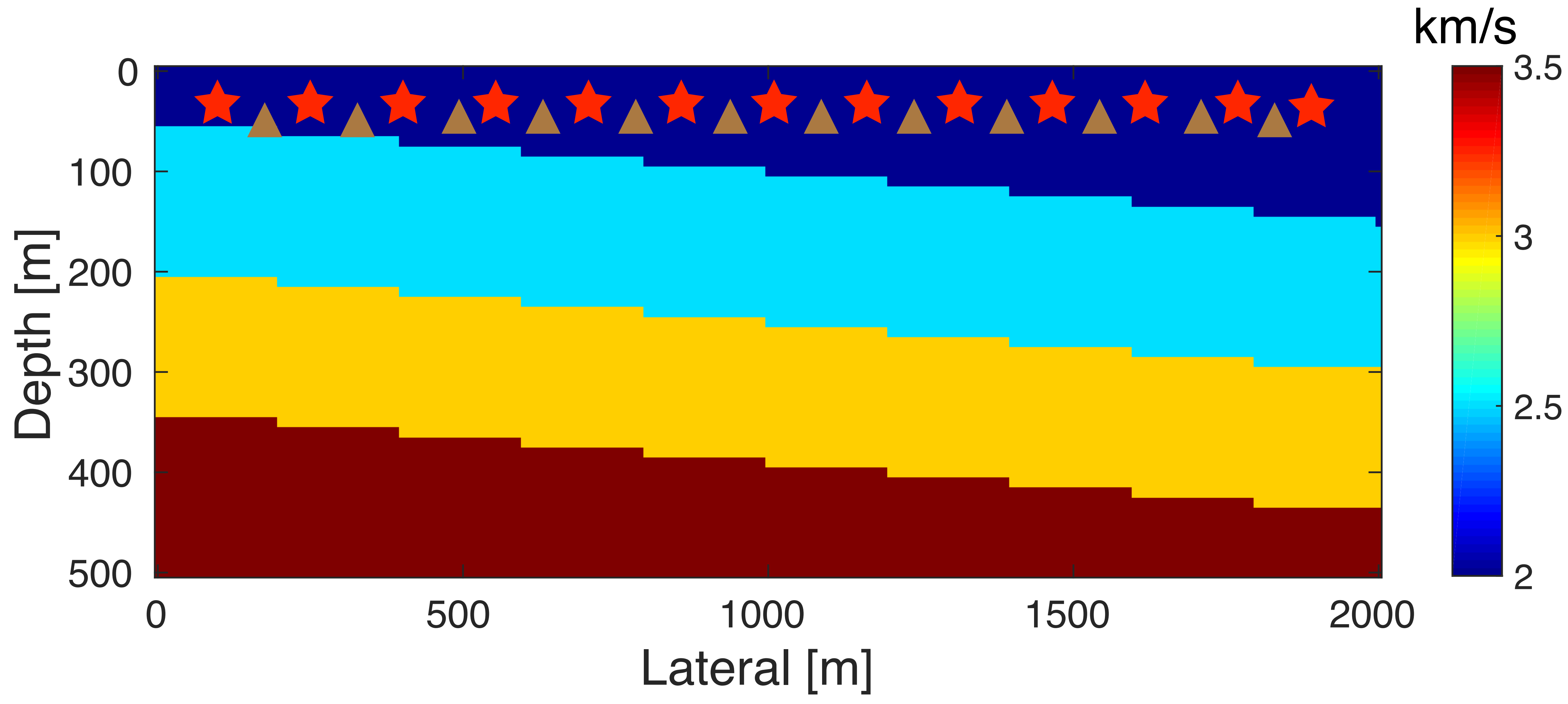
The application of the approximated Hessian and its diagonal part:

- Invert the MAP point with Newton type method (PDE free and matrix free)
- Quantify the uncertainty of the inversion result (standard deviation)

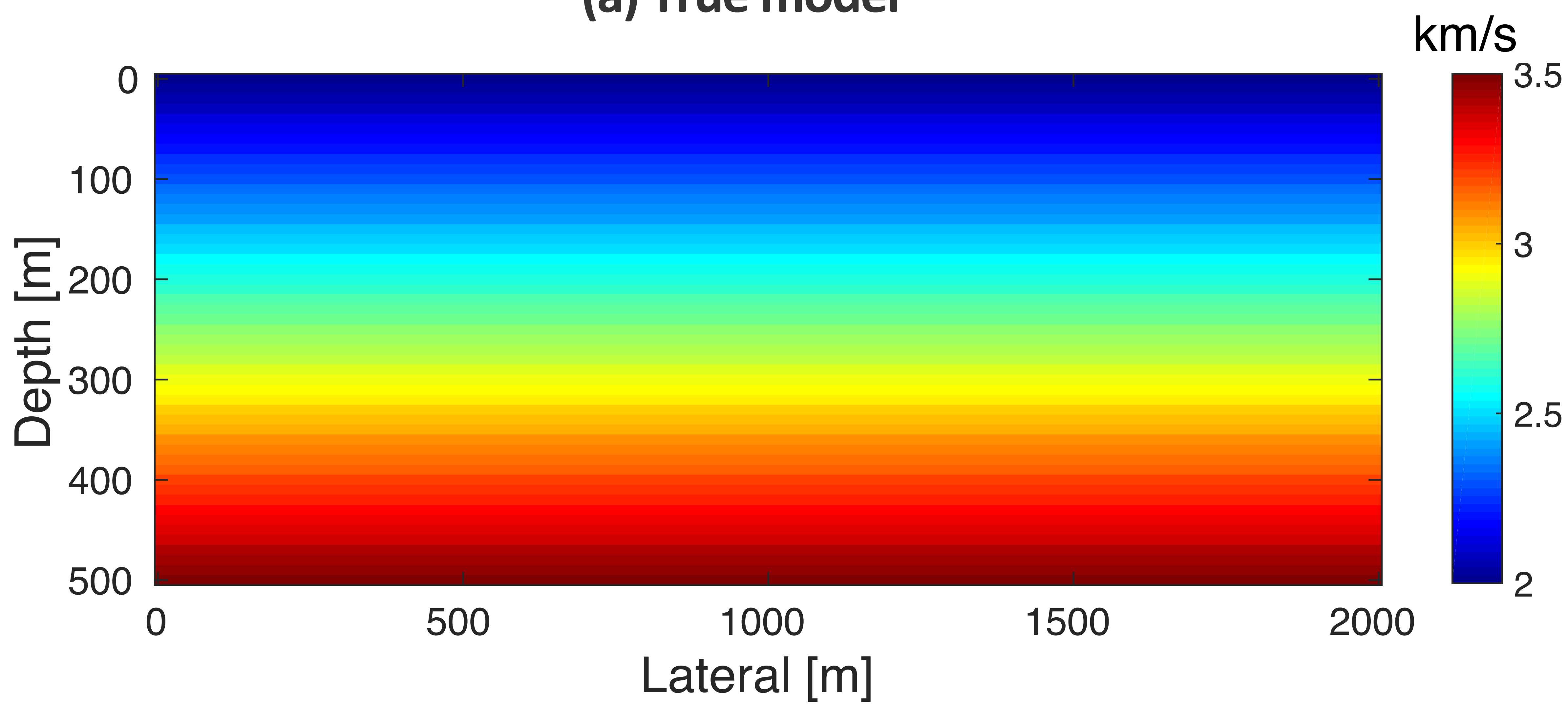
Workflow
– uncertainty quantification

Solve the deterministic WRI problem to obtain the MAP estimate. → Compute the Hessian at the MAP estimate and generate the Gaussian distribution. → Quantify the uncertainty of the model.

Numerical experiment (Full acquisition)



(a) True model

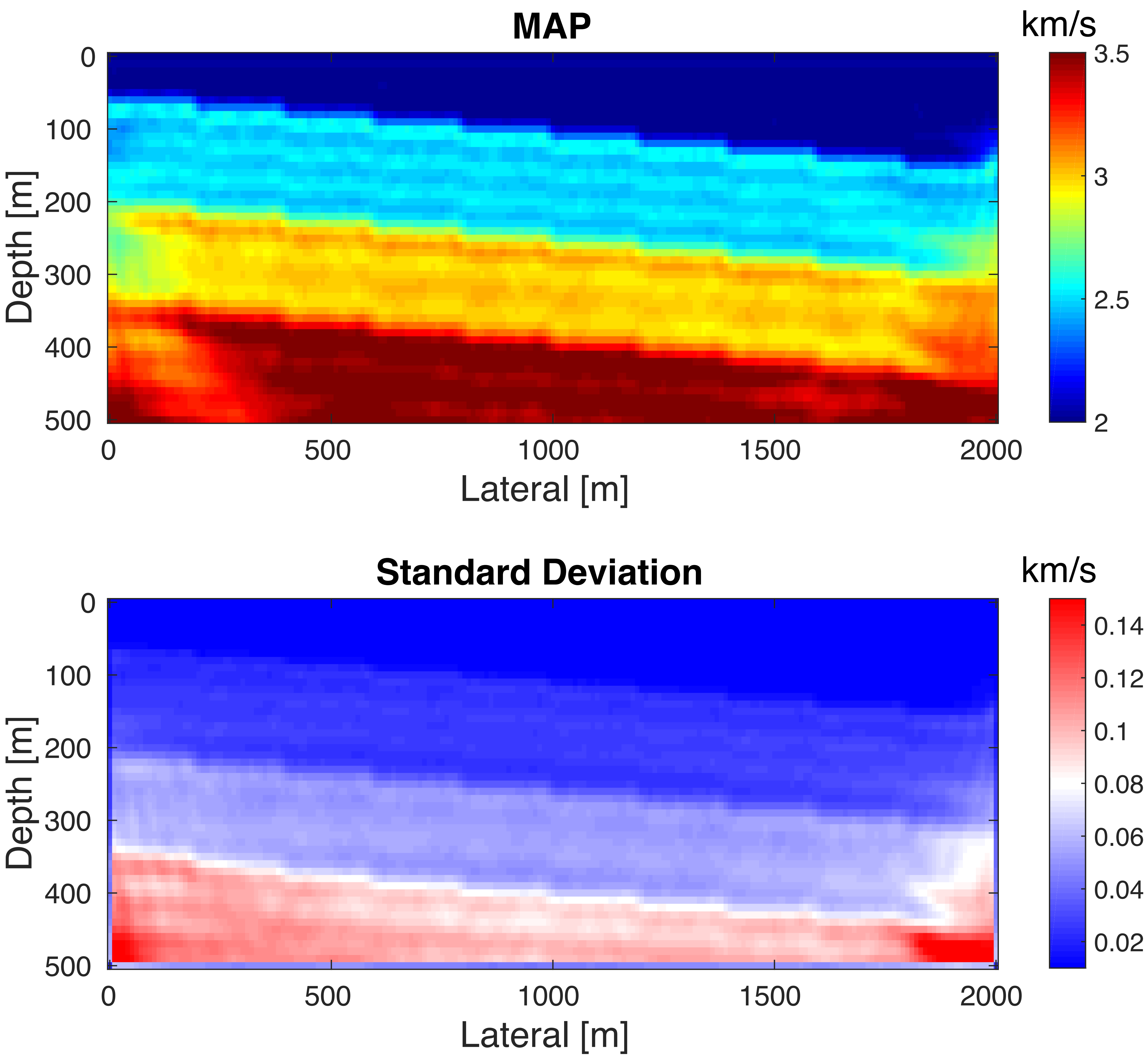


(b) Initial model

Model size: 500m x 2000m
Source spacing: 80m
Receiver spacing: 20m
Fixed spread 2km
Frequency : 10-31 Hz

Standard deviation of data noise: 0.1
Standard deviation of pde: 0.1
Standard deviation of model: 1
lambda: 1

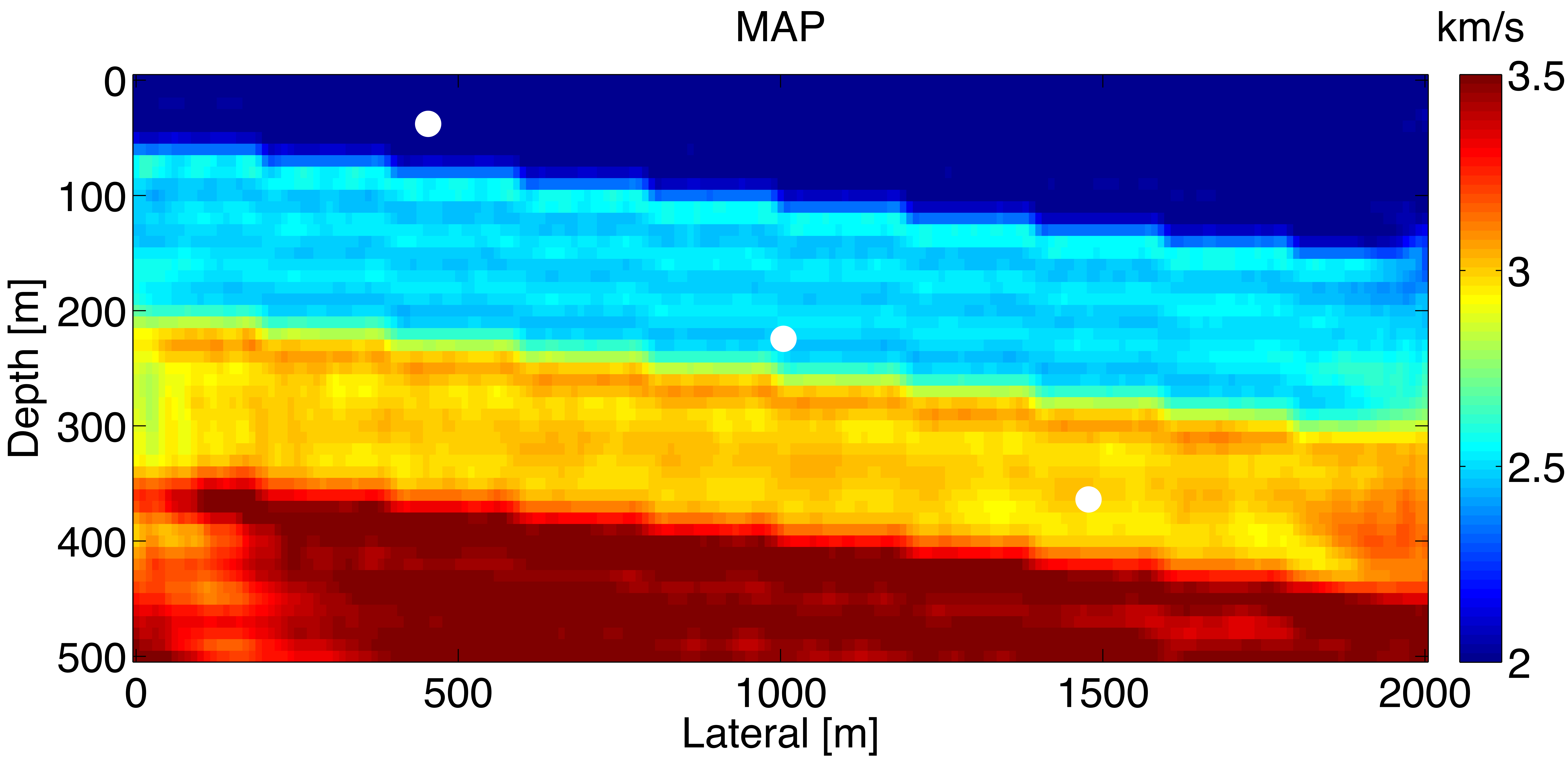
Numerical experiment (Full acquisition)



a) Maximum a posteriori estimate

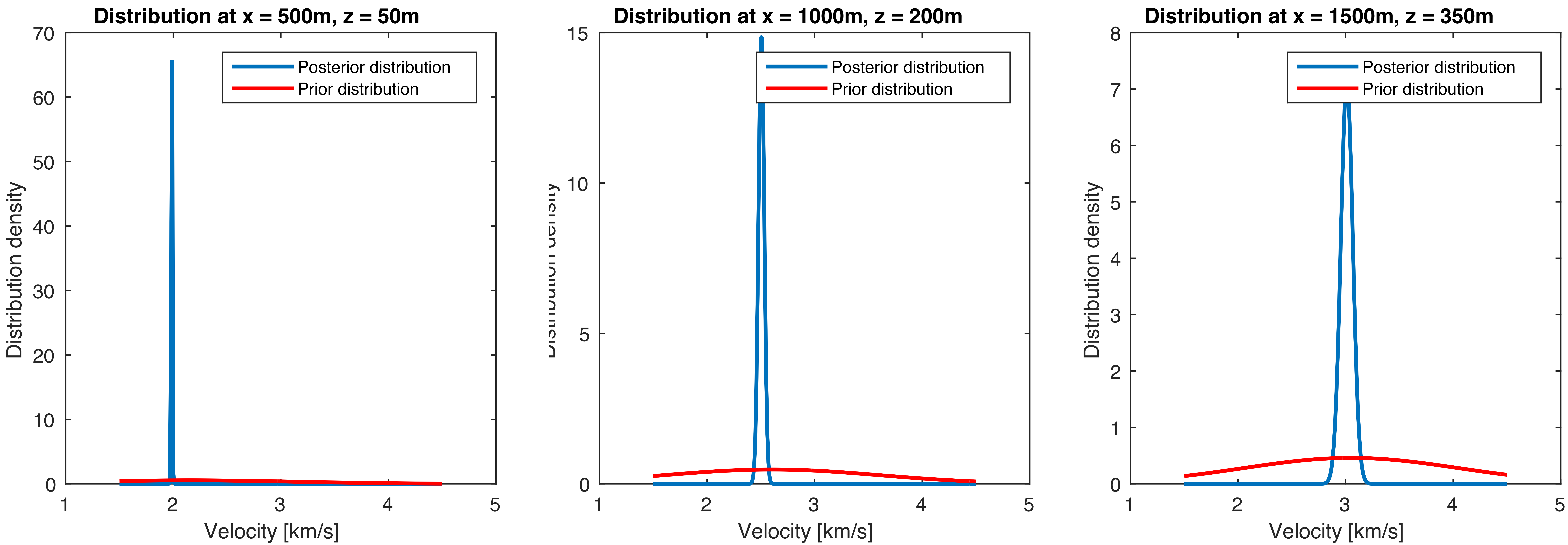
b) The standard deviation

Posterior distribution



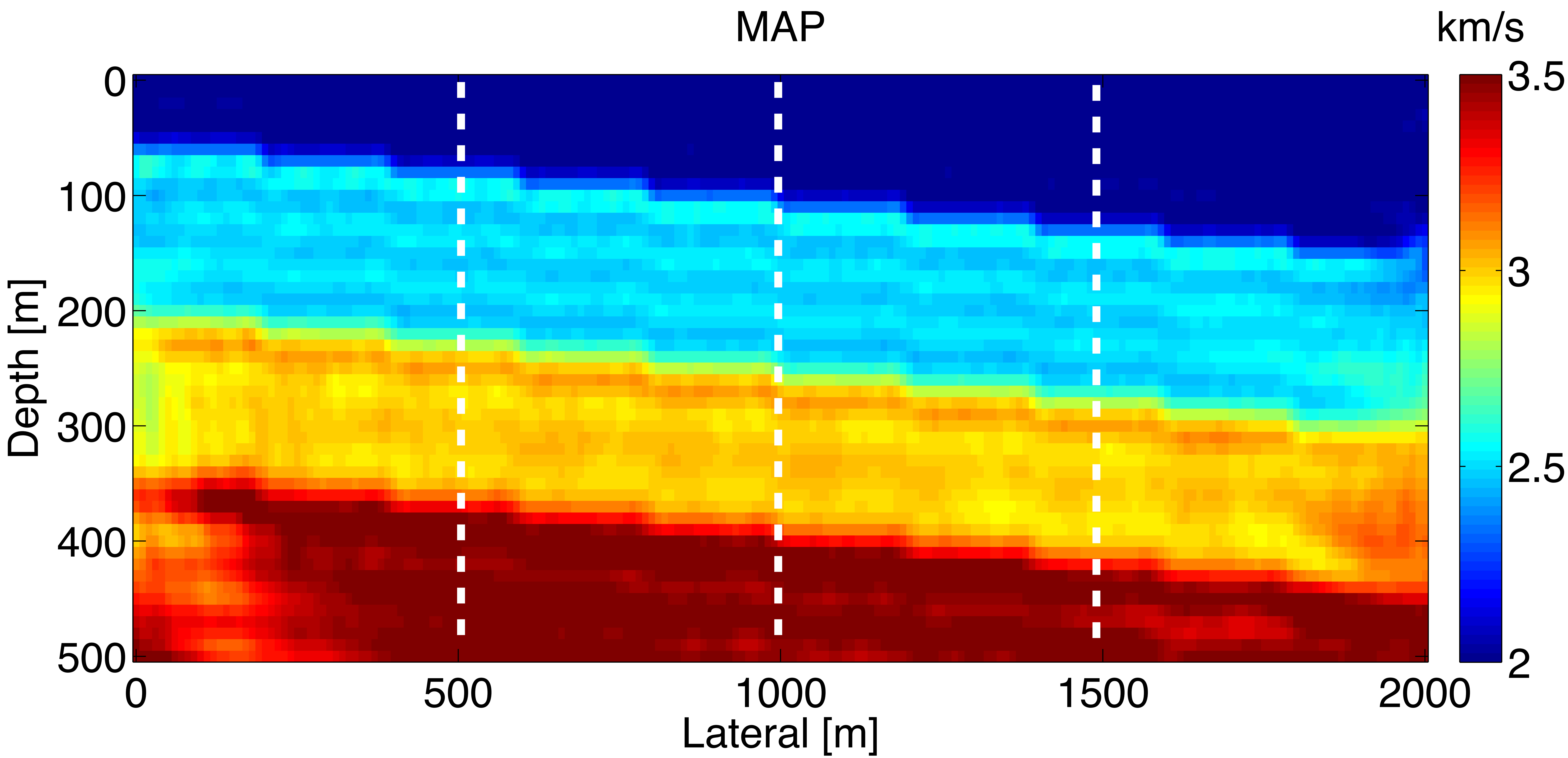
Maximum a posterior

Posterior distribution vs Prior distribution



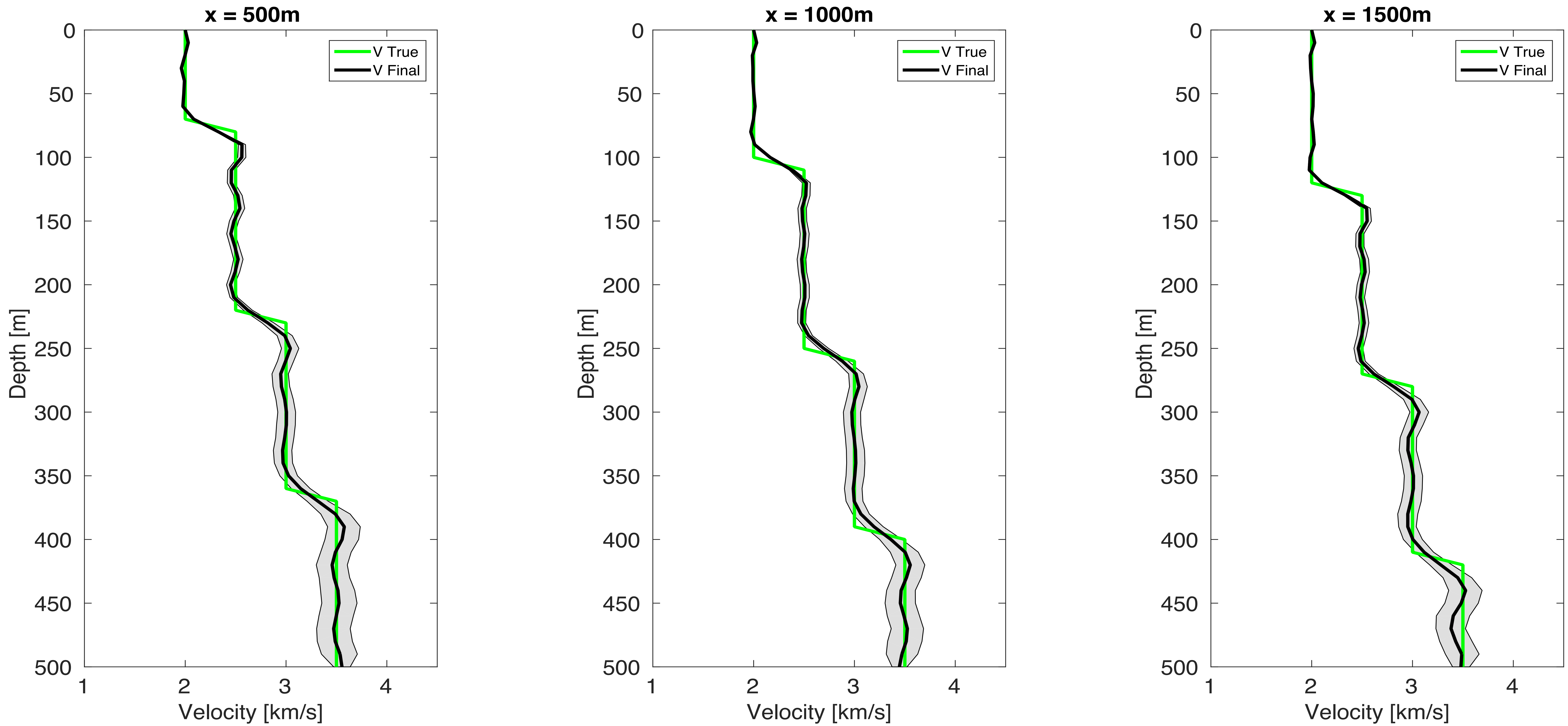
(a) **(b)** **(c)**

Confidence intervals (90%)



Maximum a posterior

Confidence intervals (90%)



(a) (b) (c)

Confidence intervals

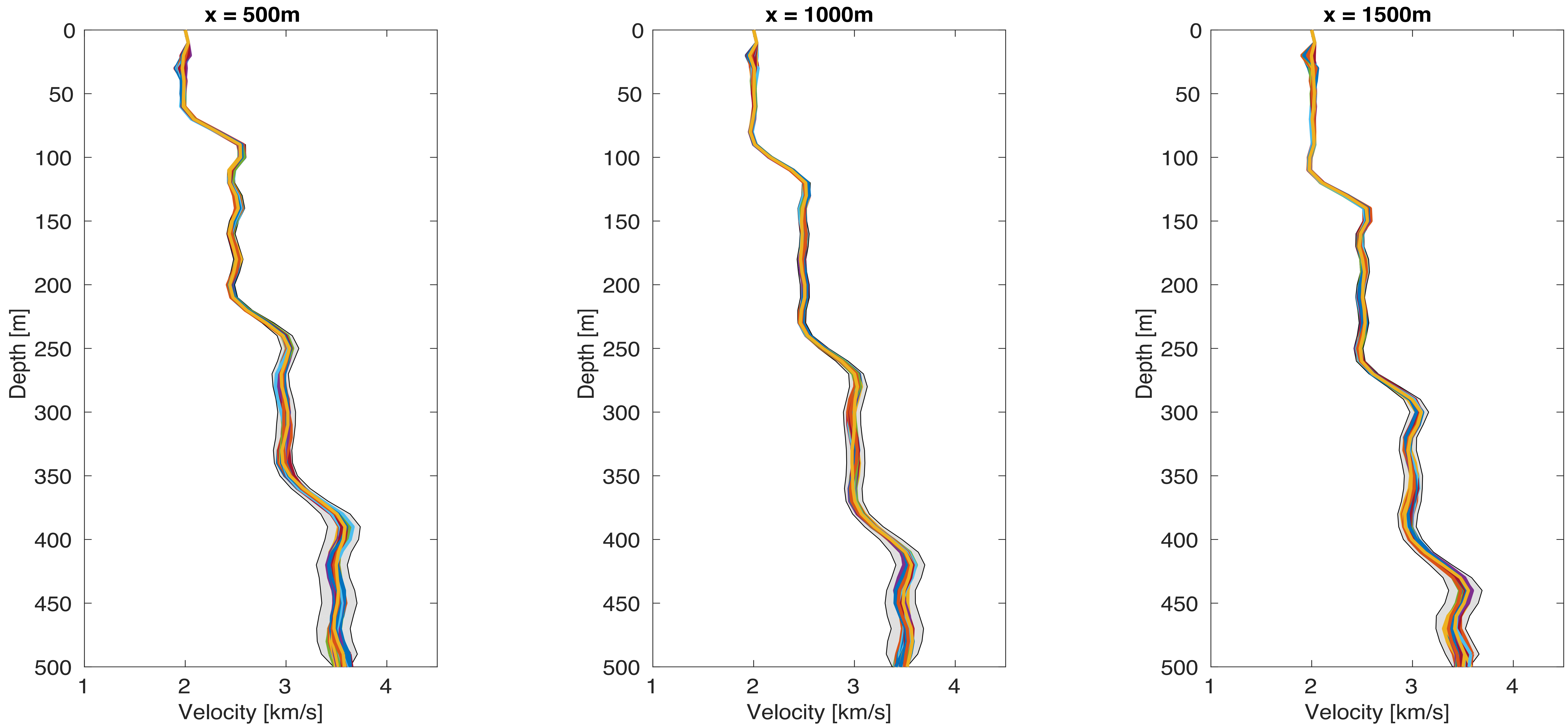
100 random realizations of data:

$$\mathbf{d}_i = \mathbf{F}(\mathbf{m}_{\mathrm{t}}) + \epsilon_i, \epsilon_i \sim \mathcal{N}(0, \sigma^2\mathbf{I})$$

Inversion results correspond to these data:

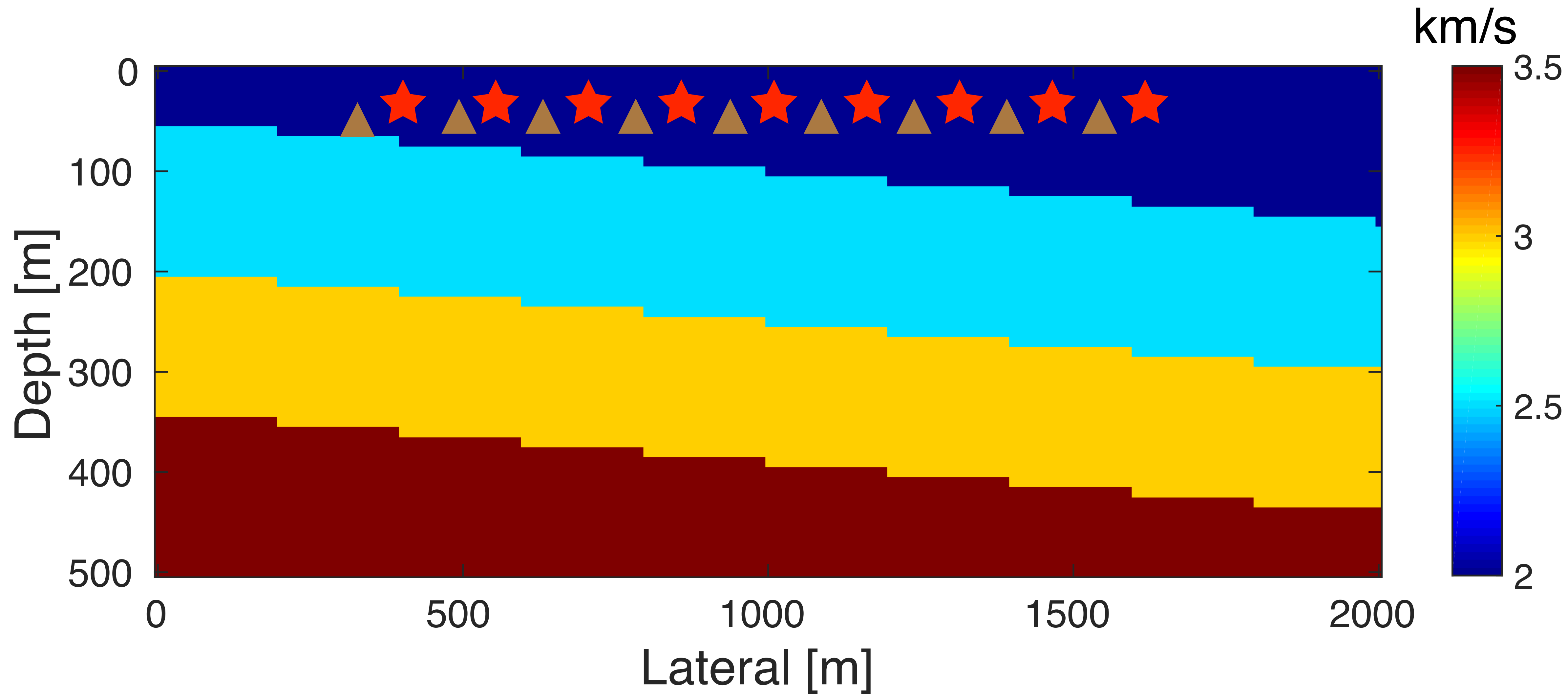
$$\mathbf{d}_i \rightarrow \mathbf{m}_{\mathrm{i}}$$

Confidence intervals (90%)
— Inversion results of 100 random realizations

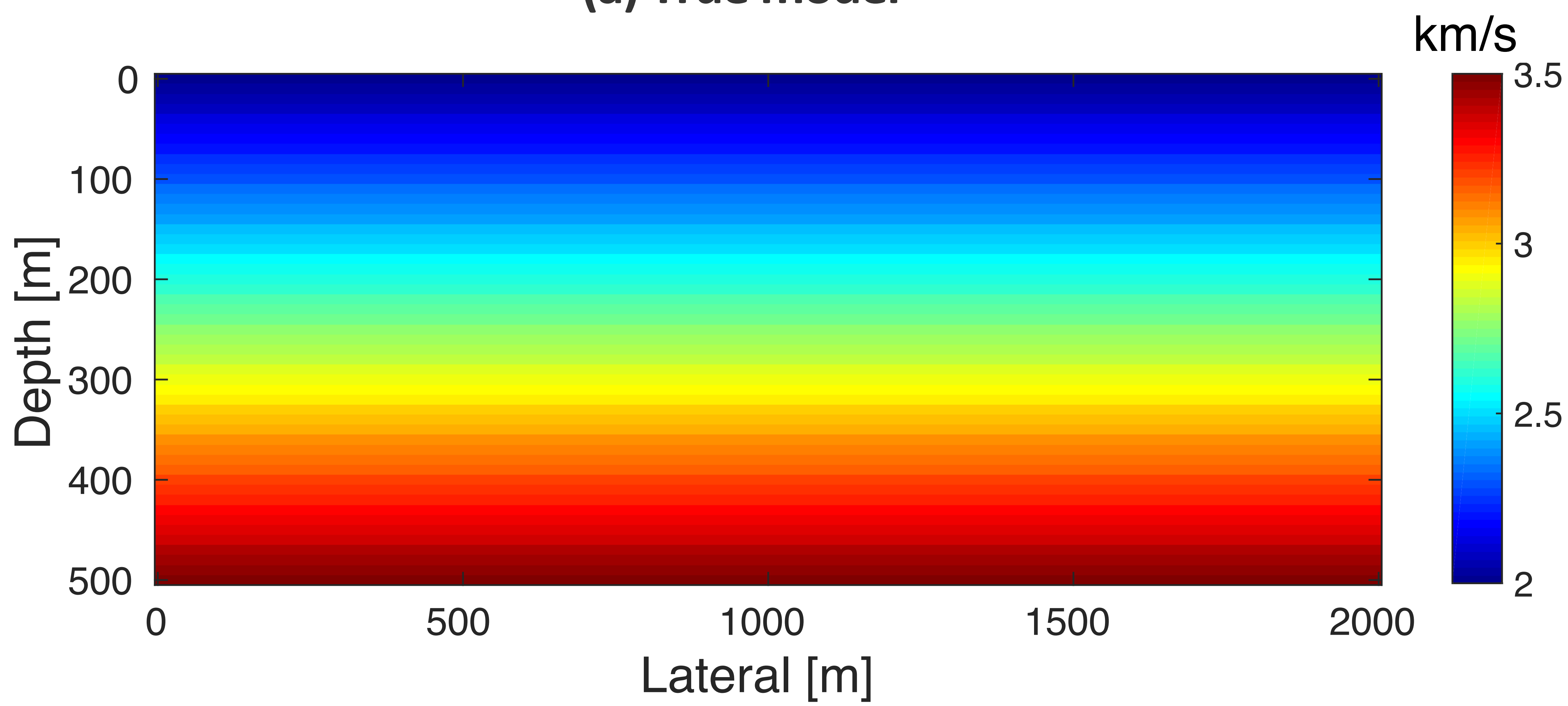


(a) (b) (c)

Numerical experiment (Partial acquisition)



(a) True model

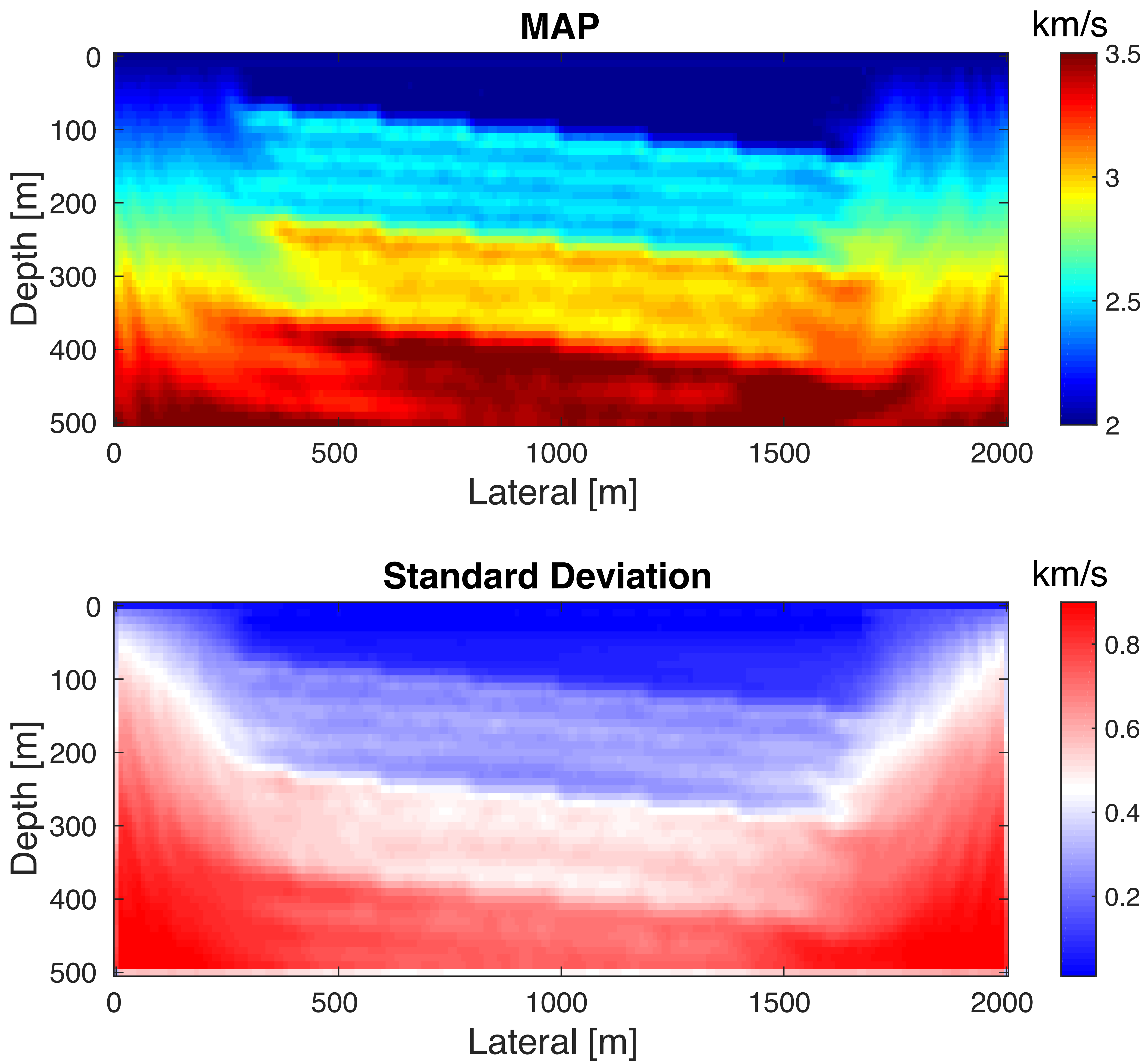


(b) Initial model

Model size: 500m x 2000m
Source spacing: 80m
Receiver spacing: 20m
Fixed spread 2km
Frequency : 10-31 Hz

Standard deviation of data noise: 0.1
Standard deviation of pde: 0.1
Standard deviation of model: 1
lambda: 1

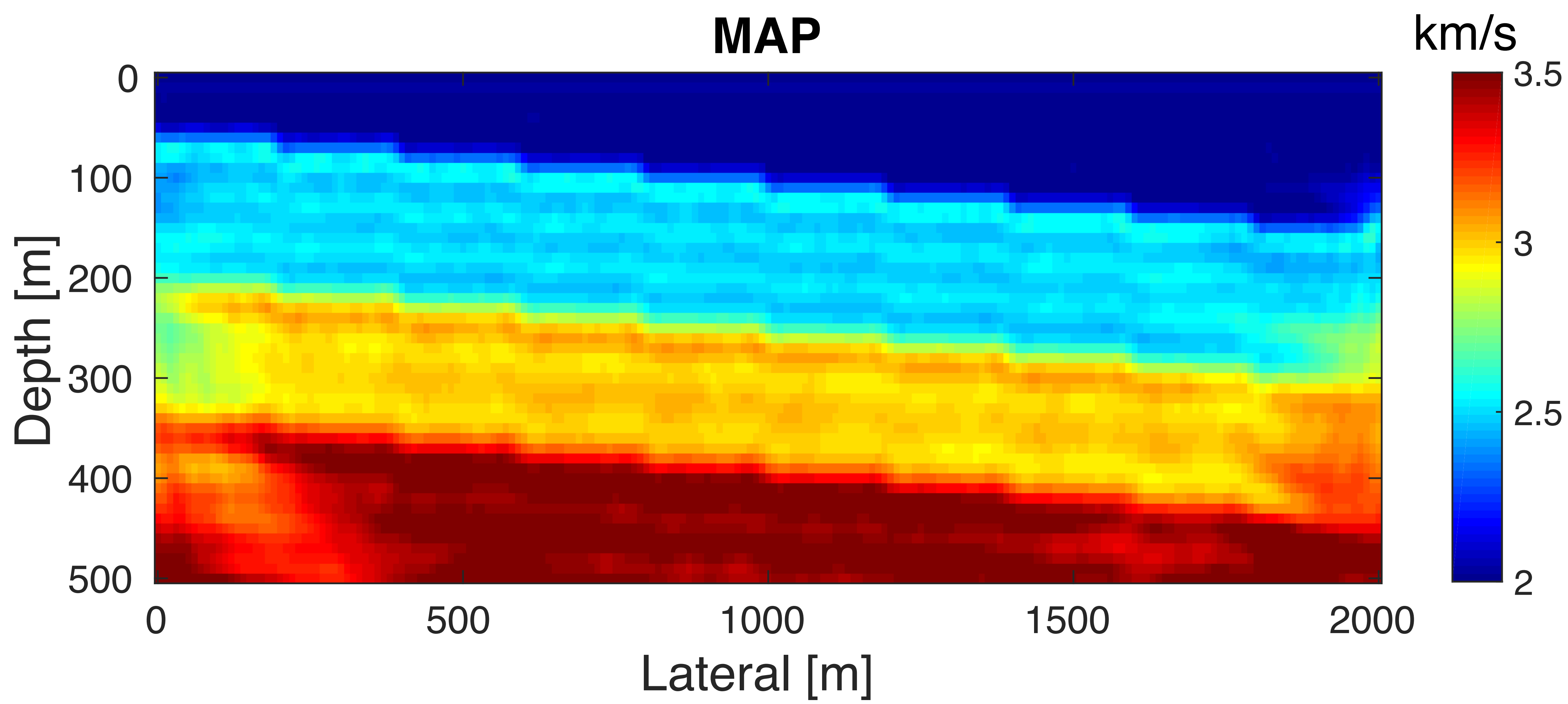
Numerical experiment (Partial acquisition)



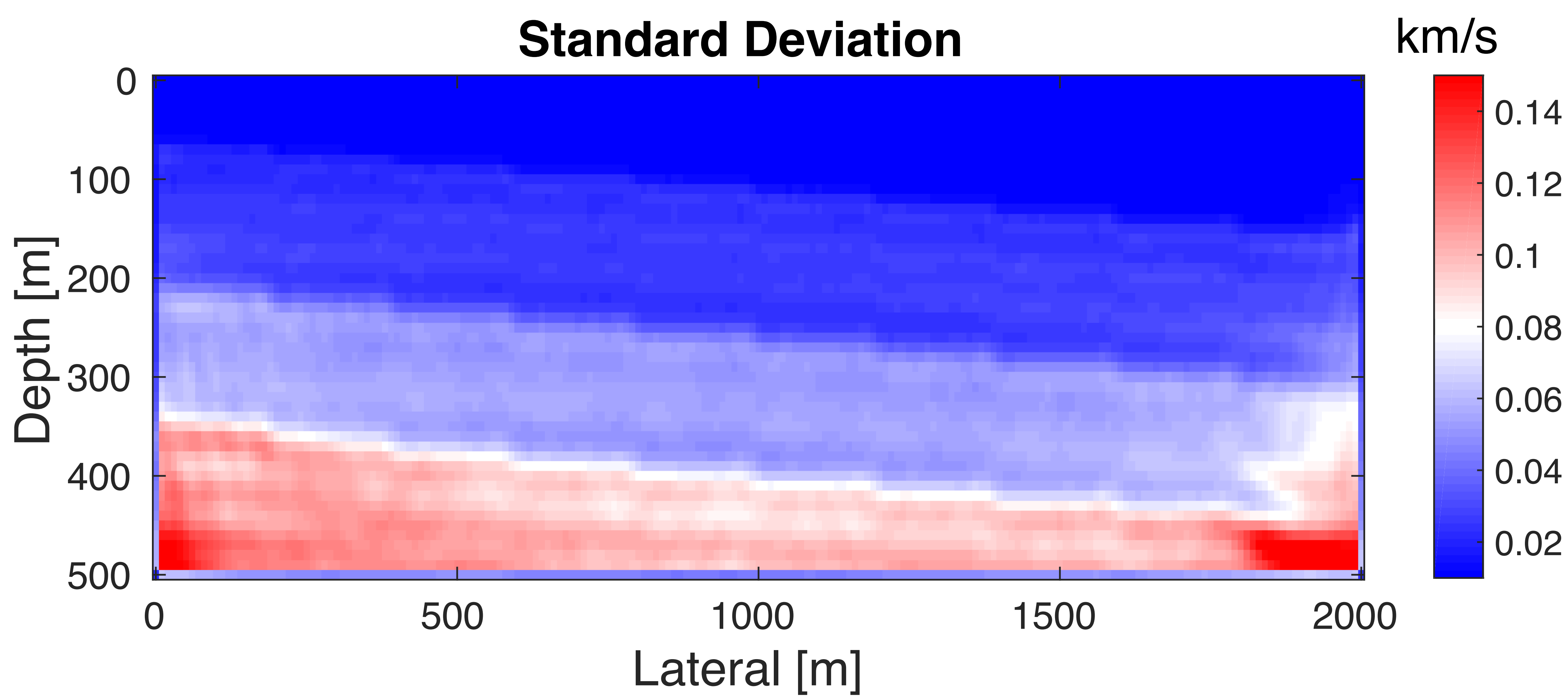
a) Maximum a posteriori estimate

b) The standard deviation

Full acquisition

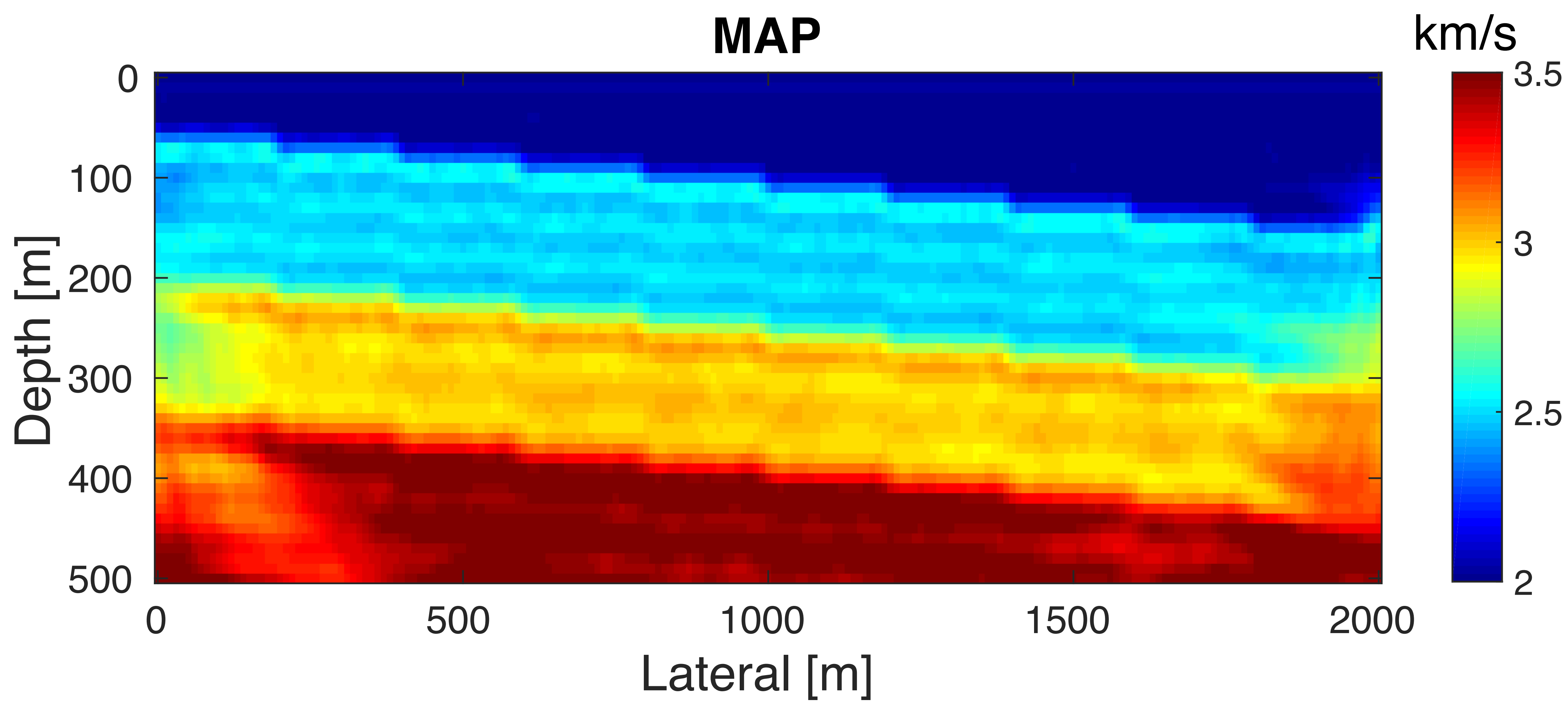


a) Maximum a posteriori estimate

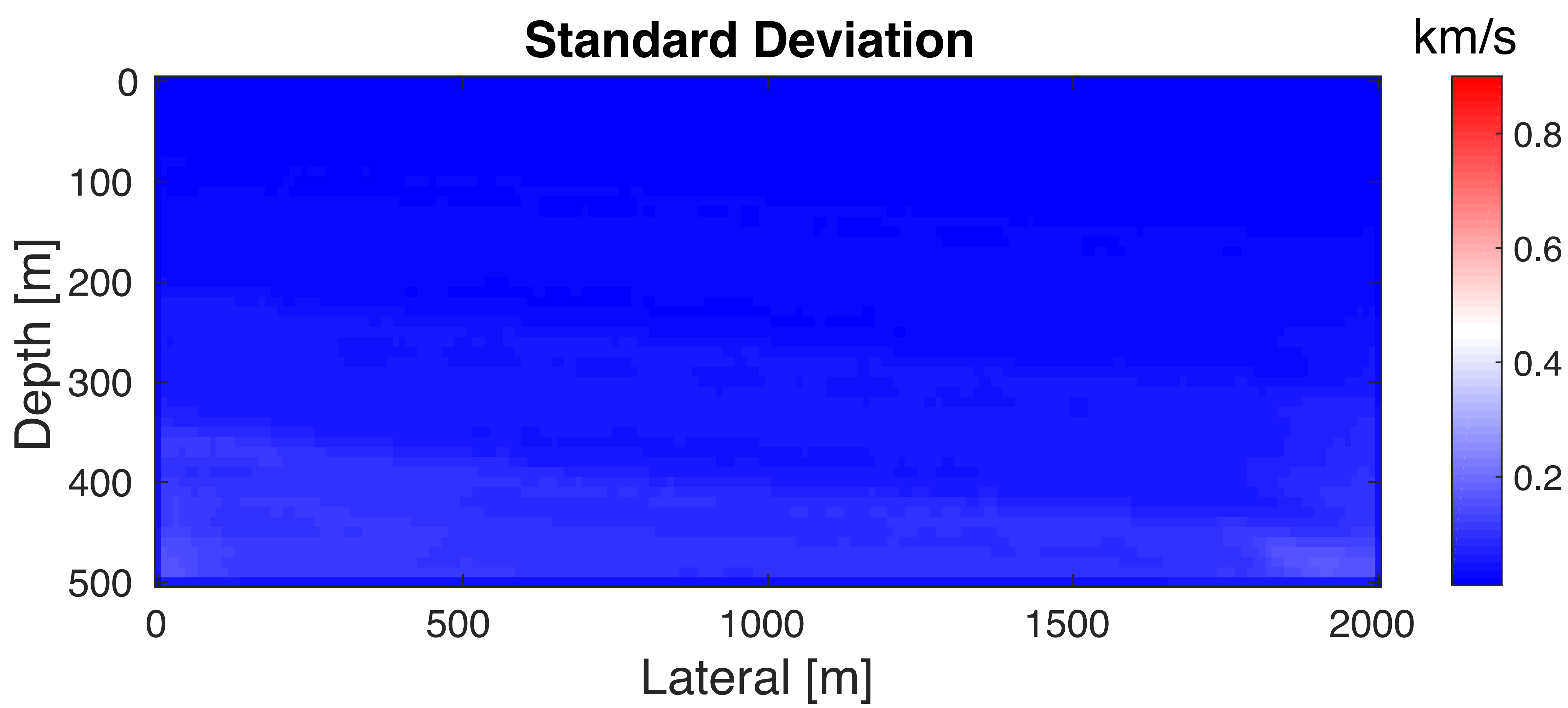


b) The standard deviation

Full acquisition

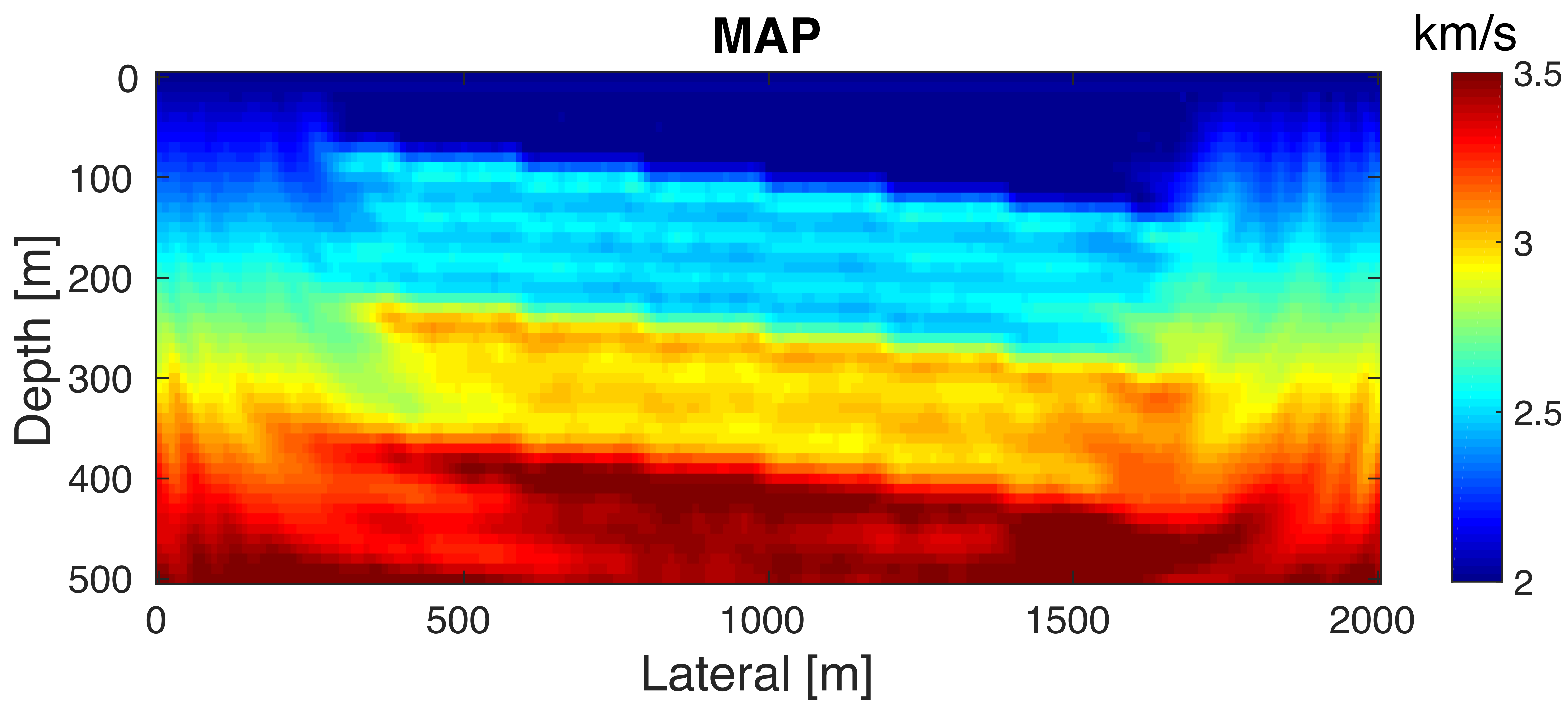


a) Maximum a posteriori estimate

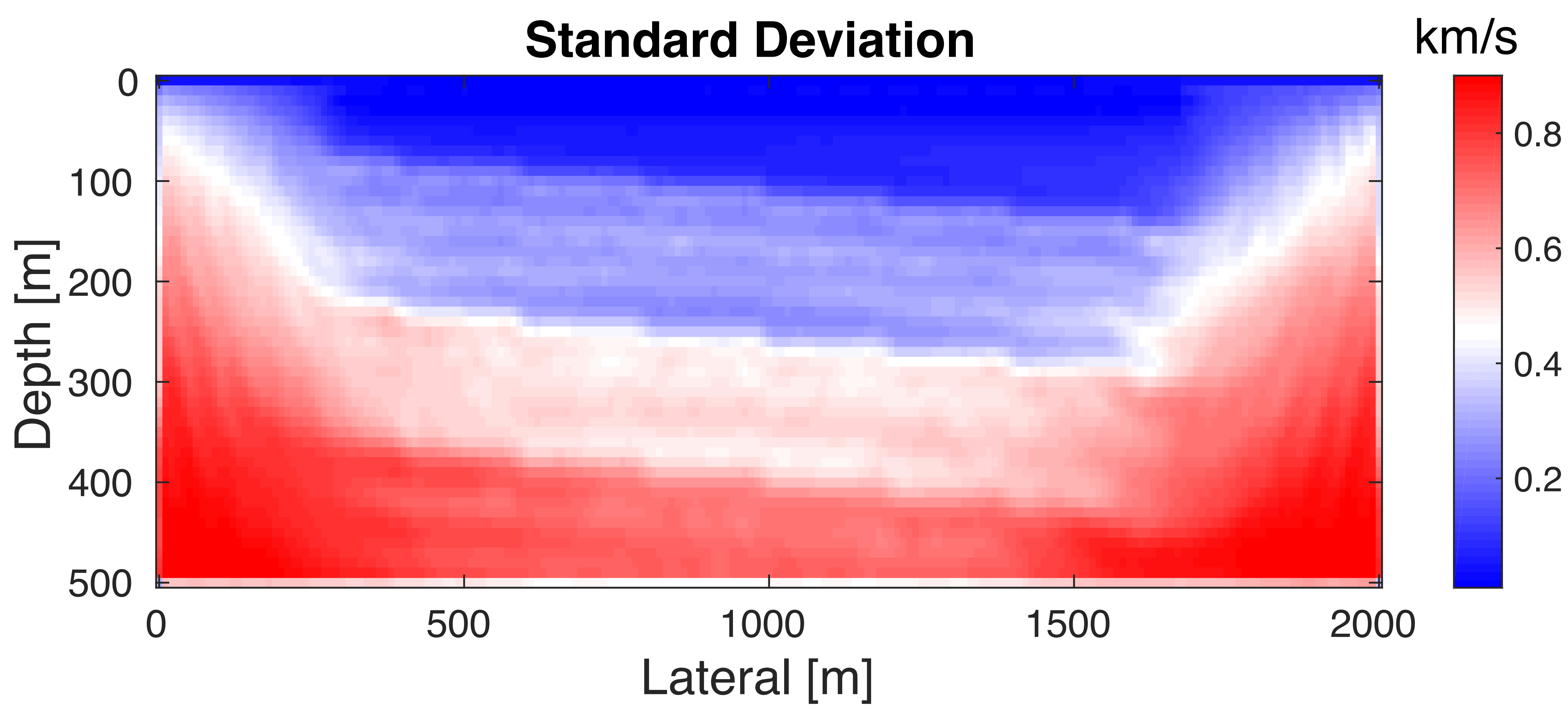


b) The standard deviation

Partial acquisition

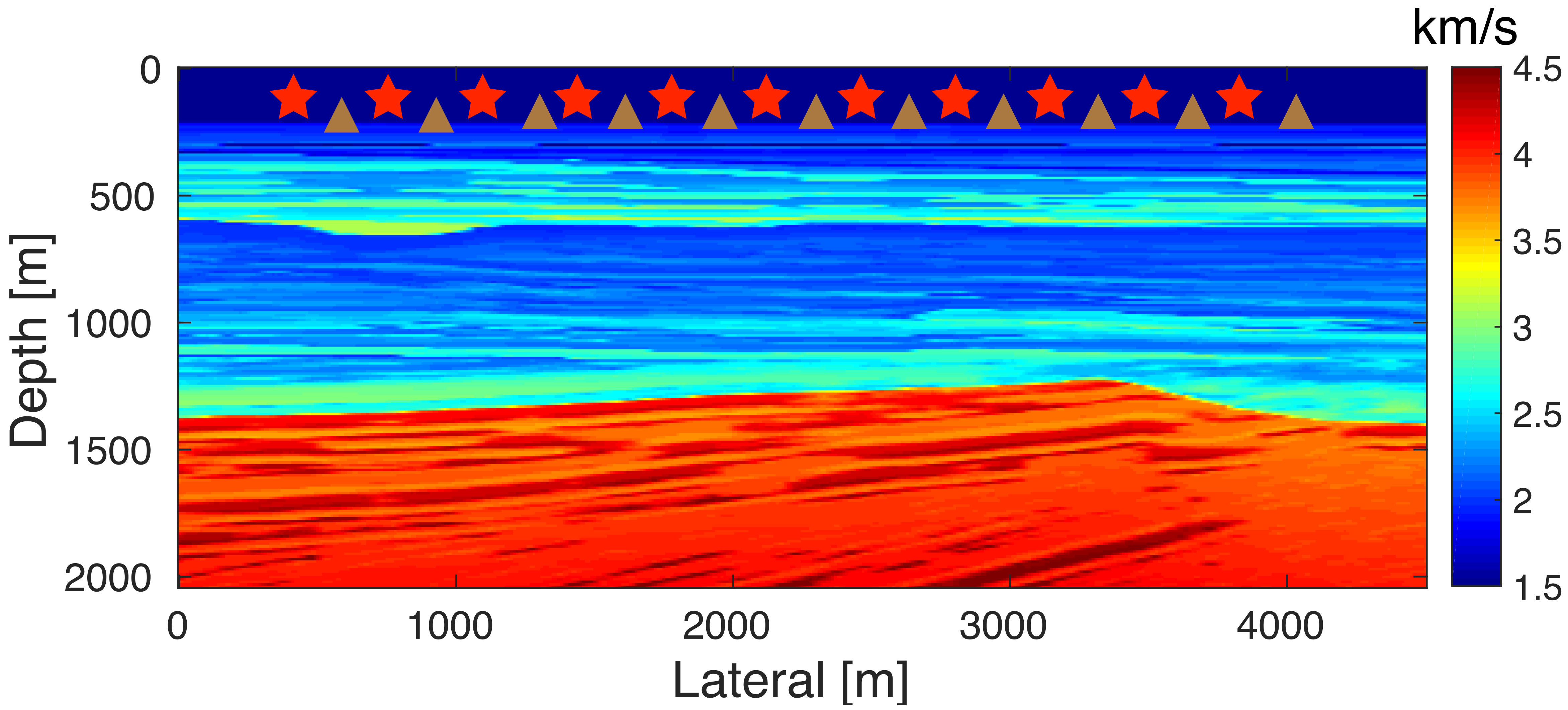


a) Maximum a posteriori estimate

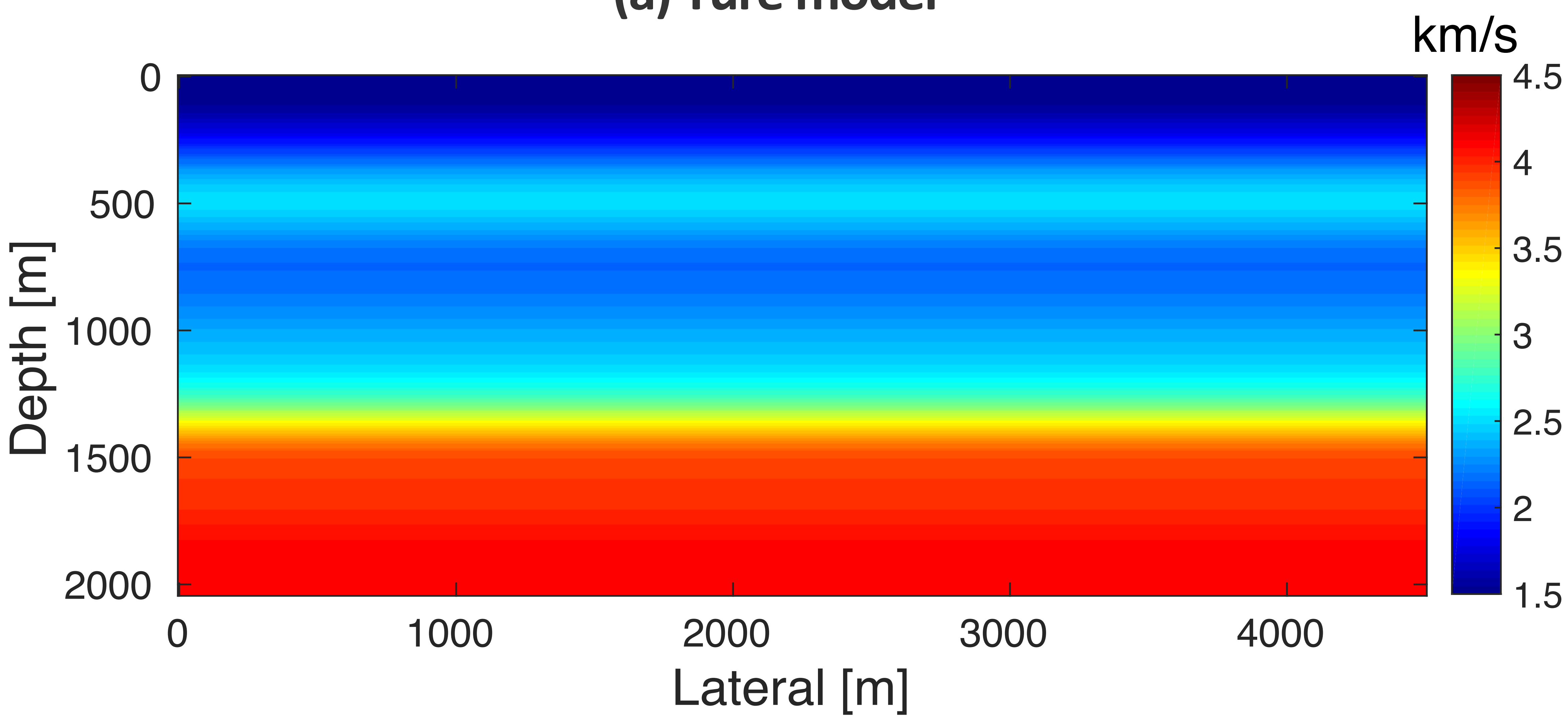


b) The standard deviation

BG model



(a) Ture model

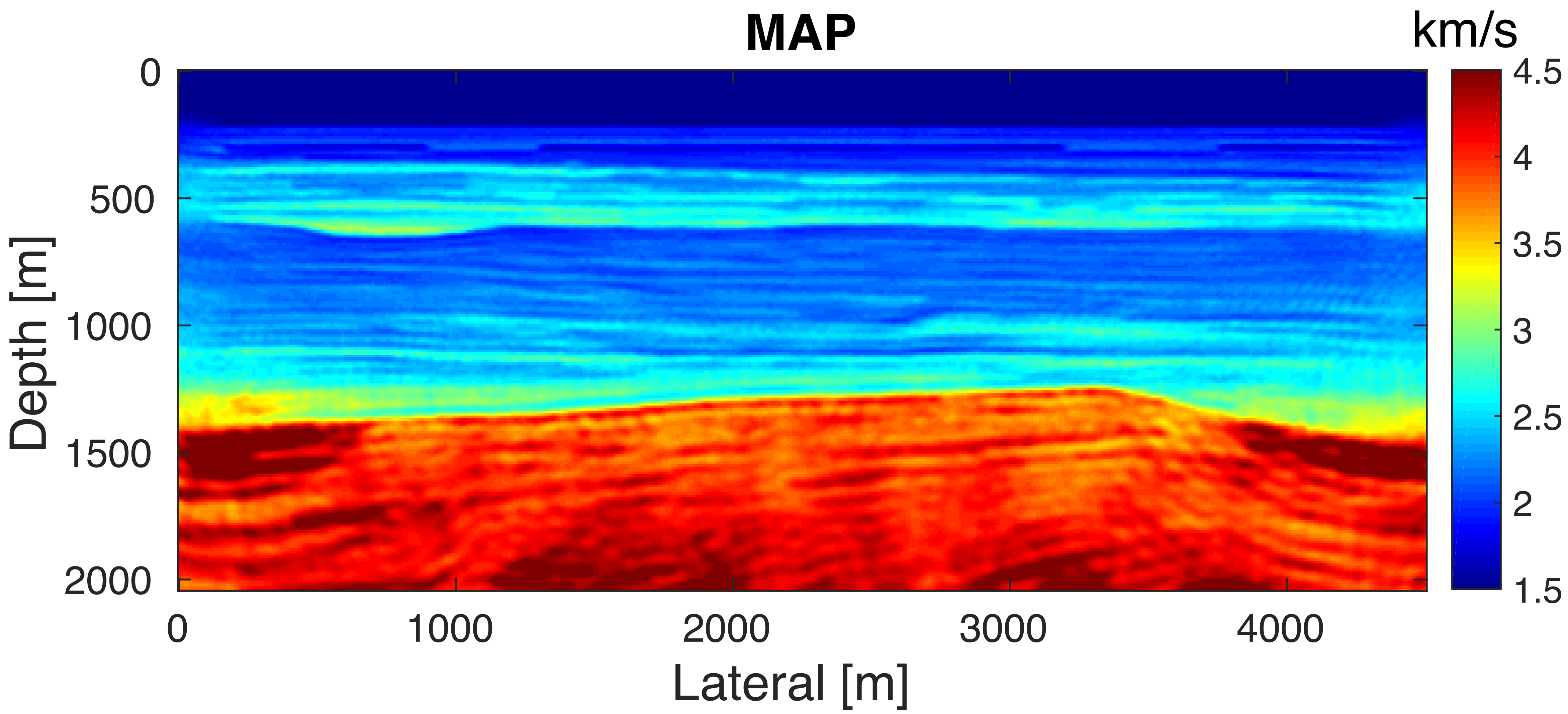


(b) Initial model

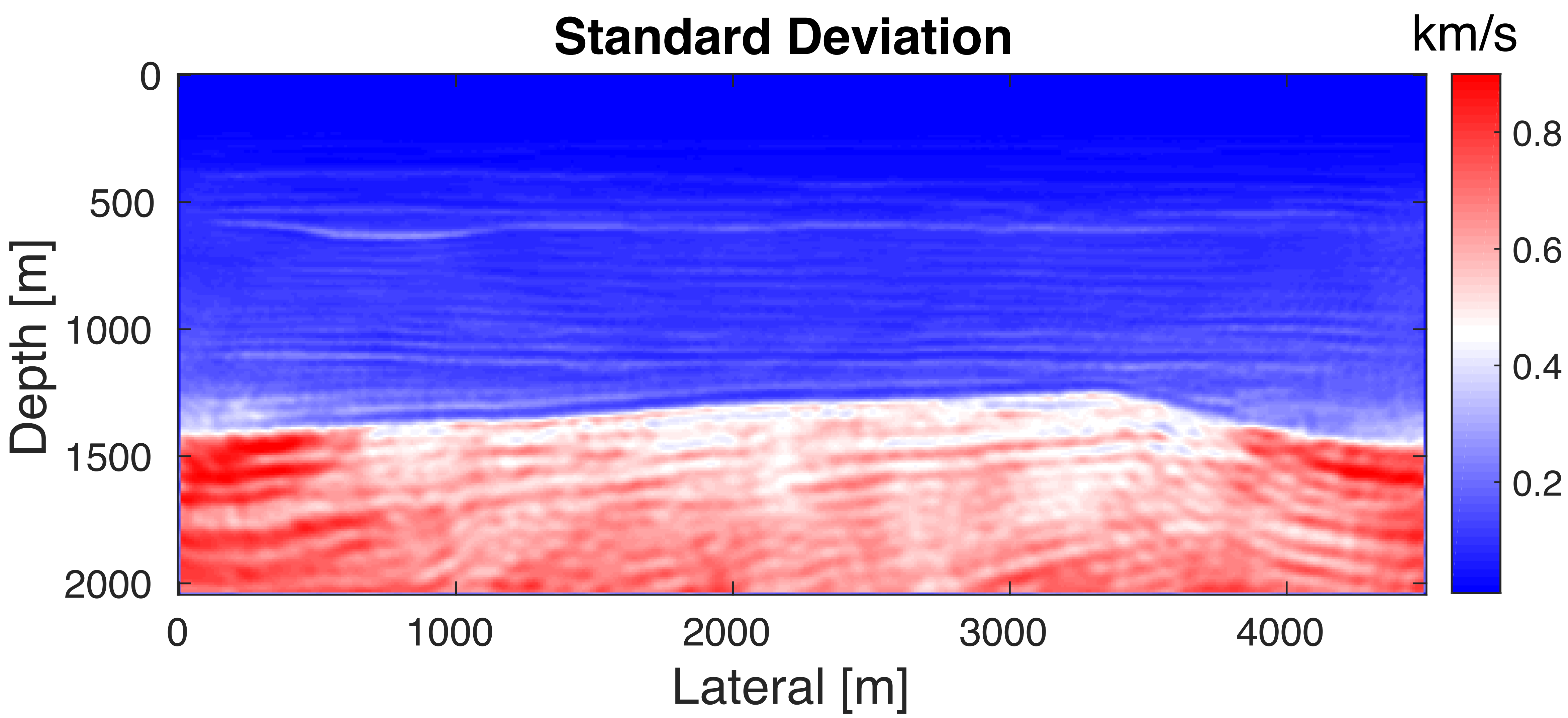
Model size: 2000m x 4500m
Source spacing: 50m
Receiver spacing: 10m
Fixed spread 4.5km
Frequency : 5~31 Hz

Standard deviation of data noise: 0.5
Standard deviation of pde: 0.5
Standard deviation of model: 1
lambda: 1

BG model

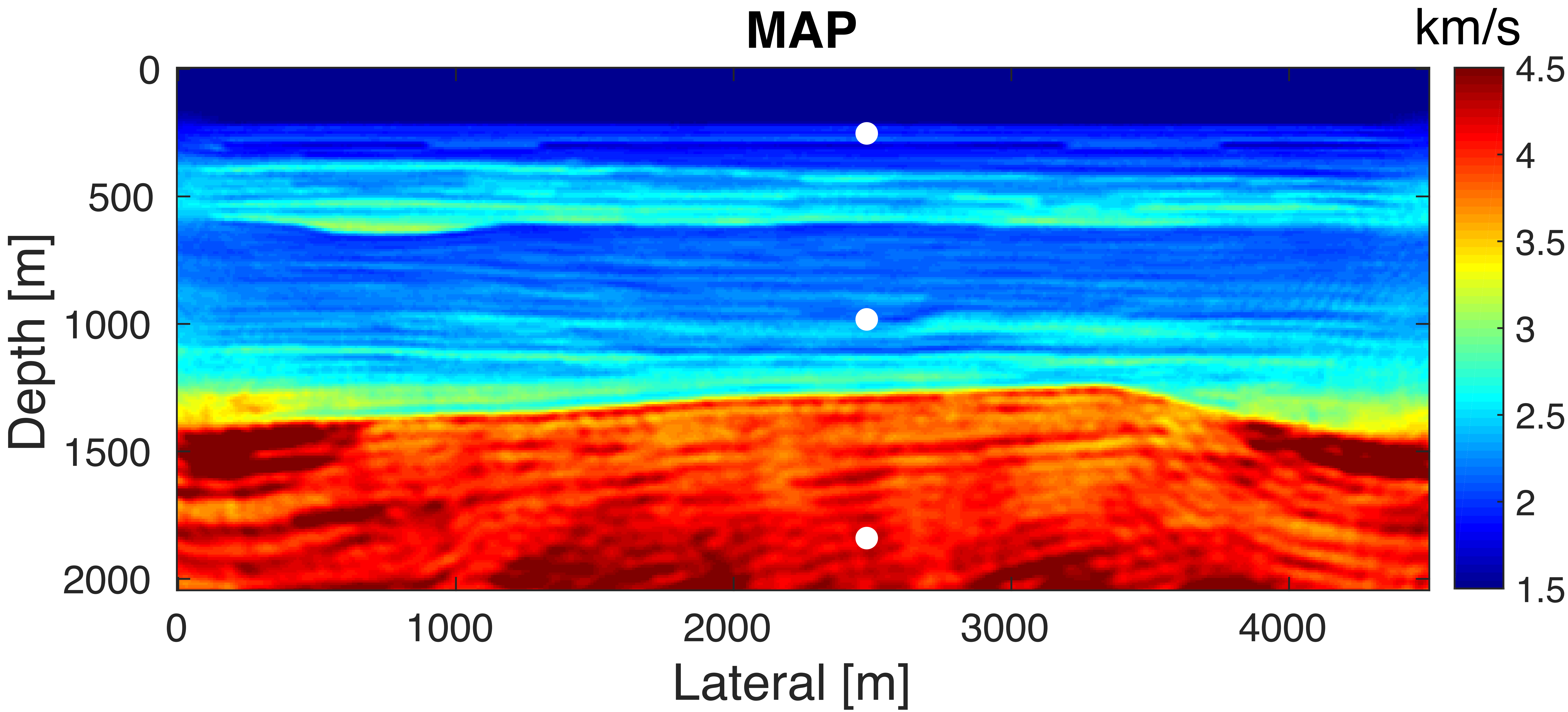


a) Maximum a posterior estimate



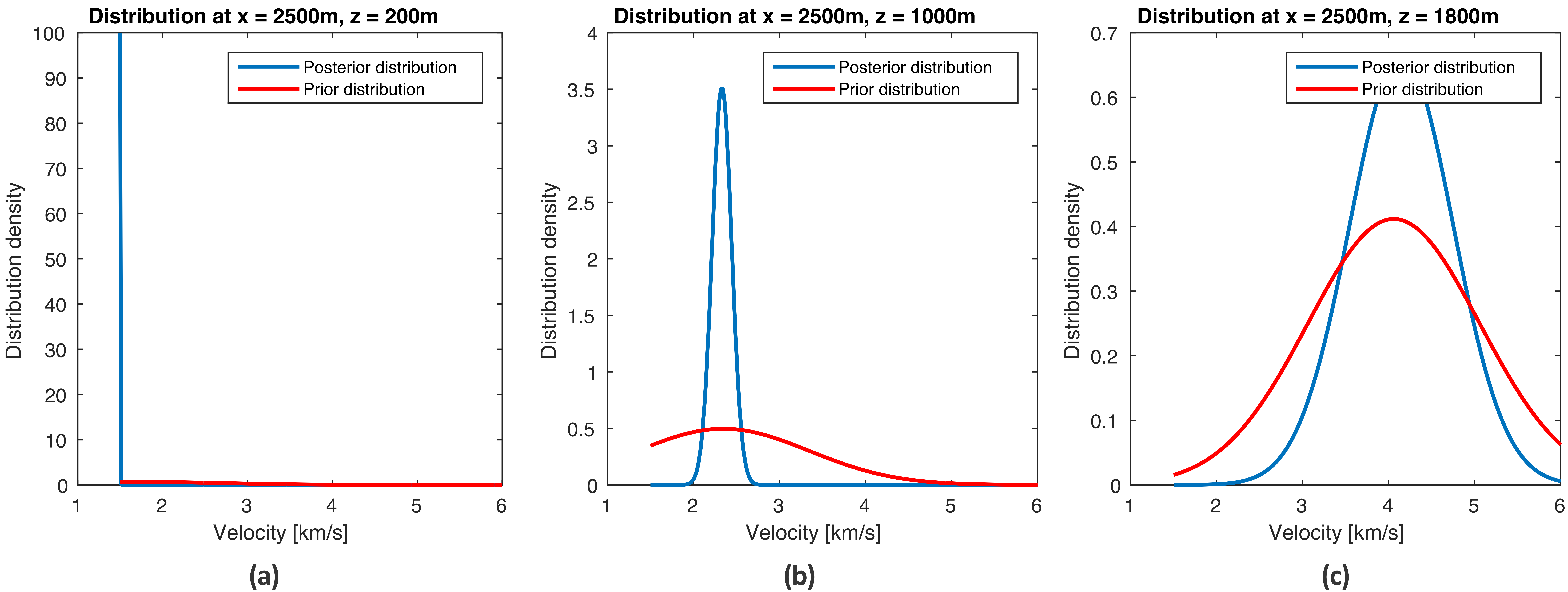
b) The standard deviation

Posterior distribution

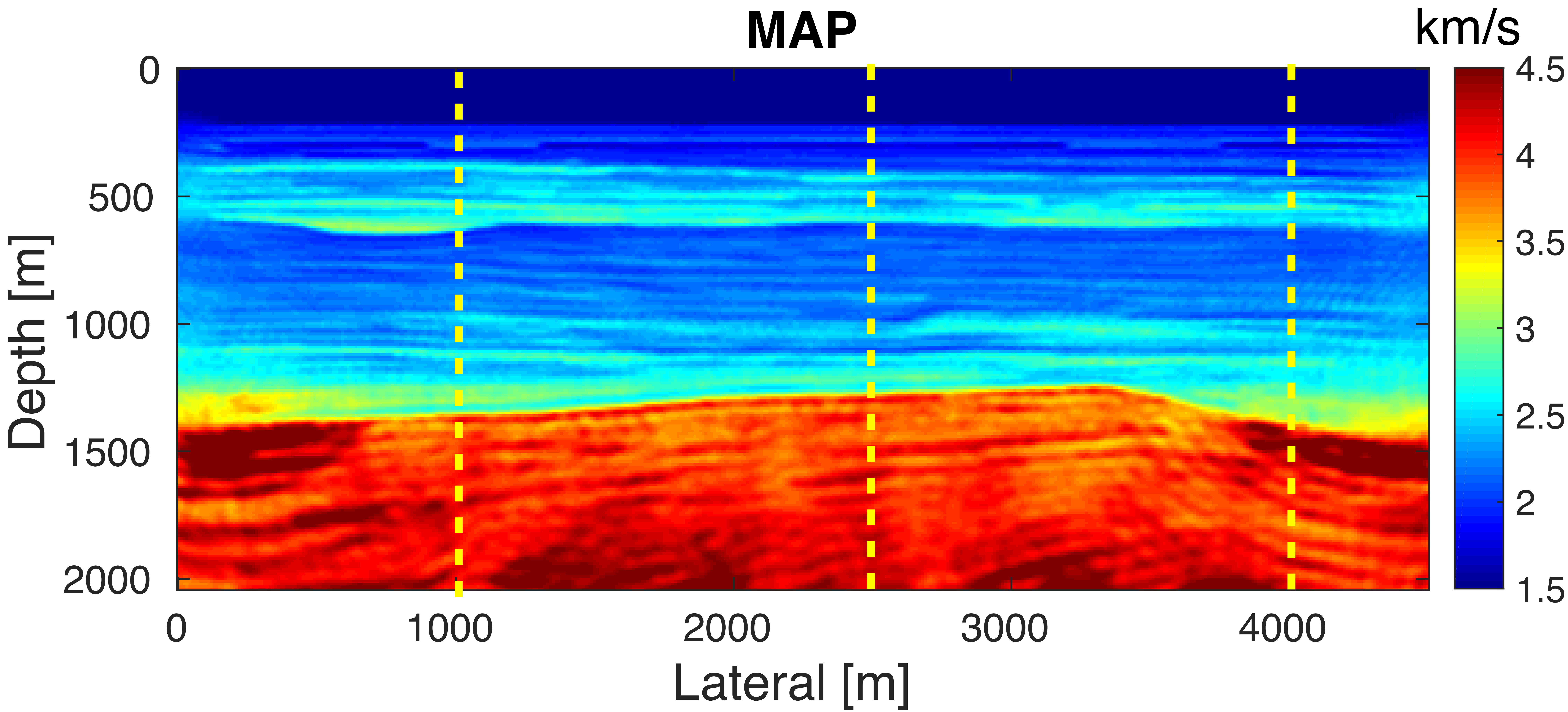


Maximum a posterior

Posterior distribution vs Prior distribution

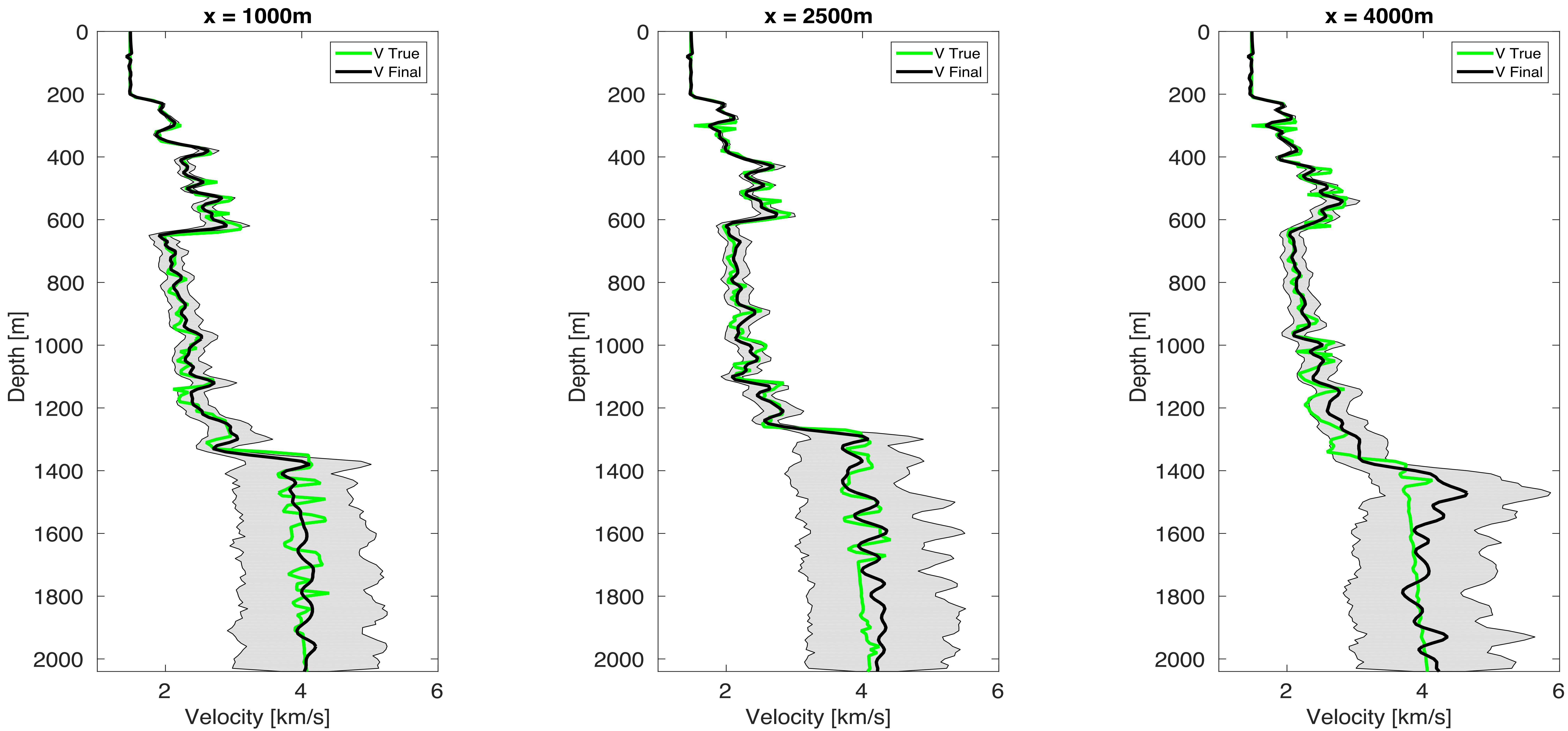


Confidence intervals (90%)



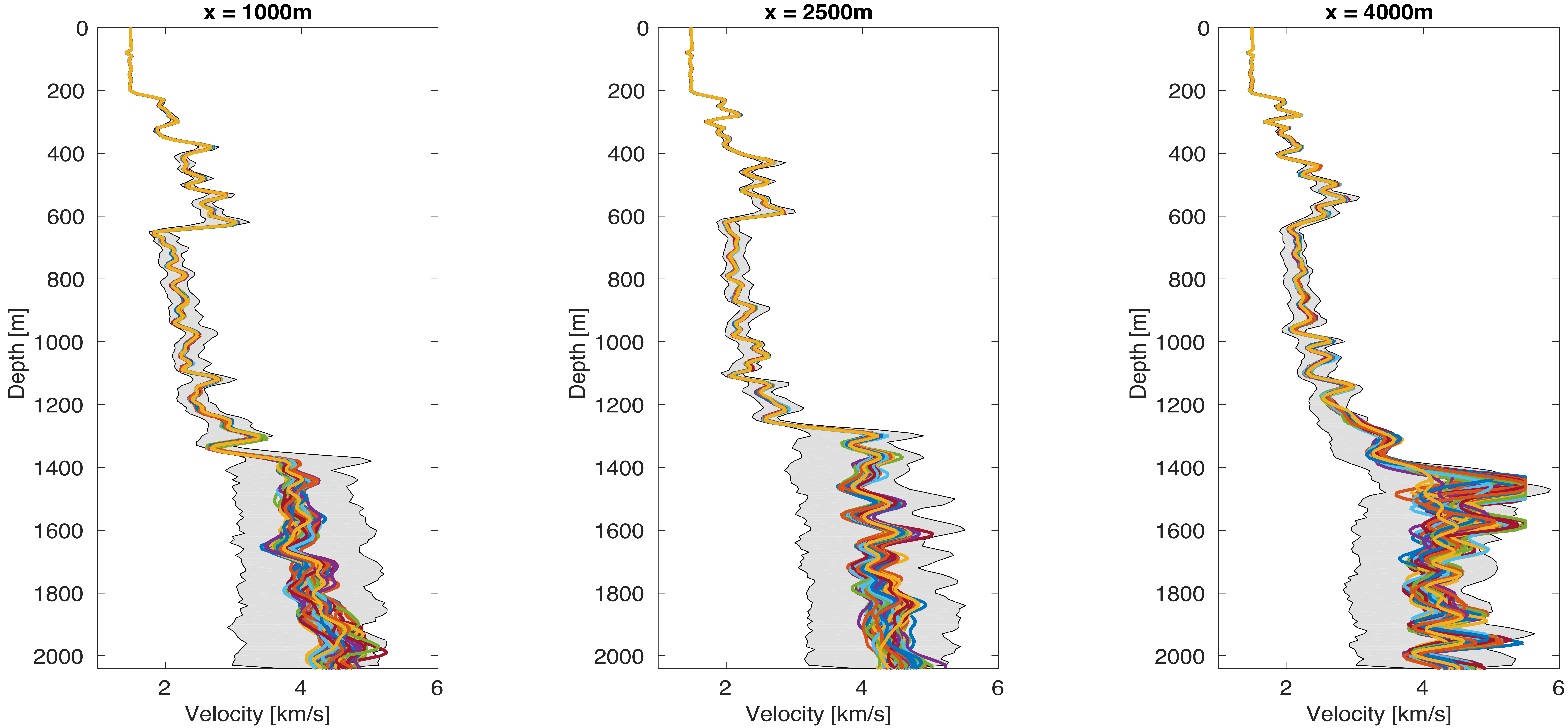
Maximum a posterior

Confidence intervals (90%)



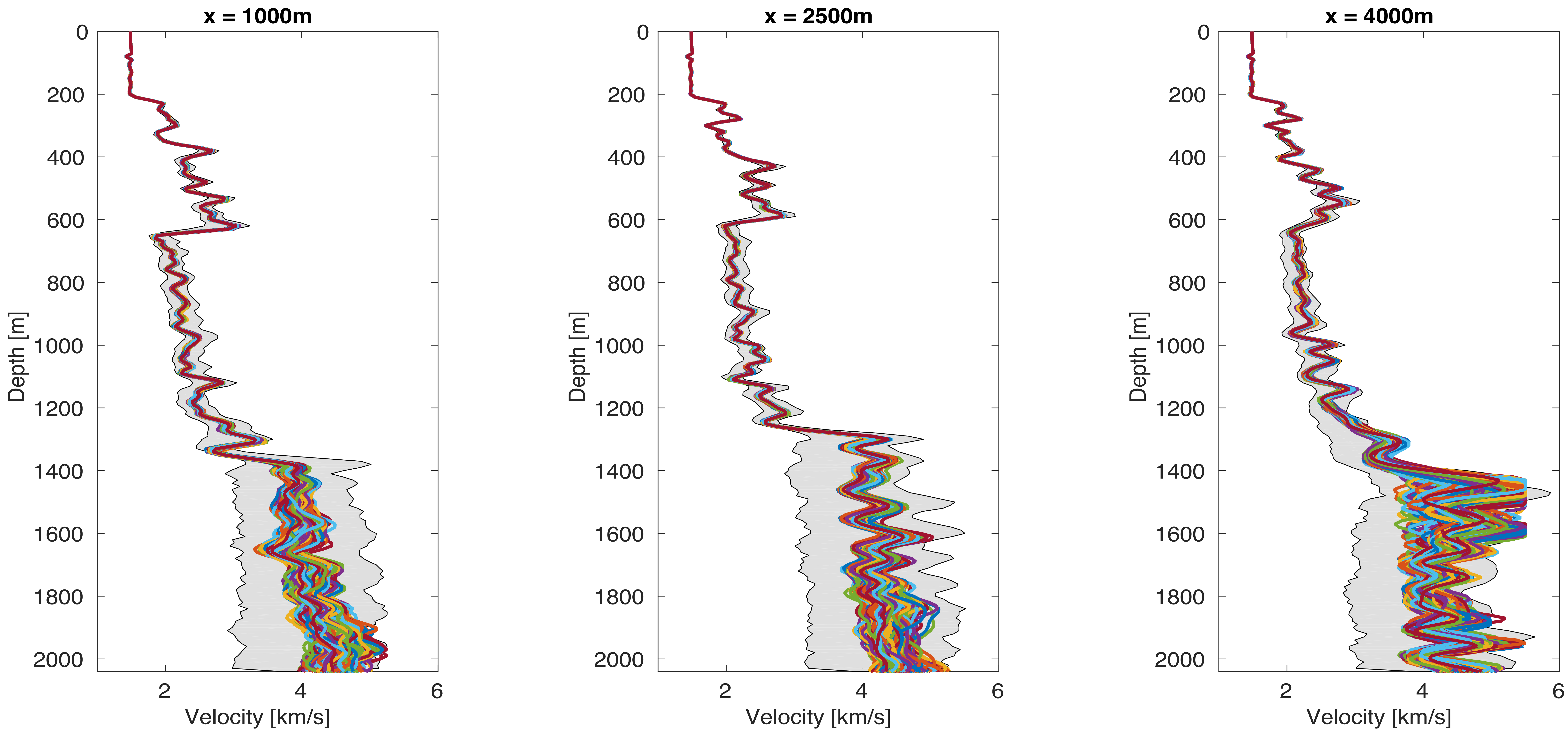
(a) (b) (c)

Confidence intervals (90%)
— Inversion results of 100 random realizations



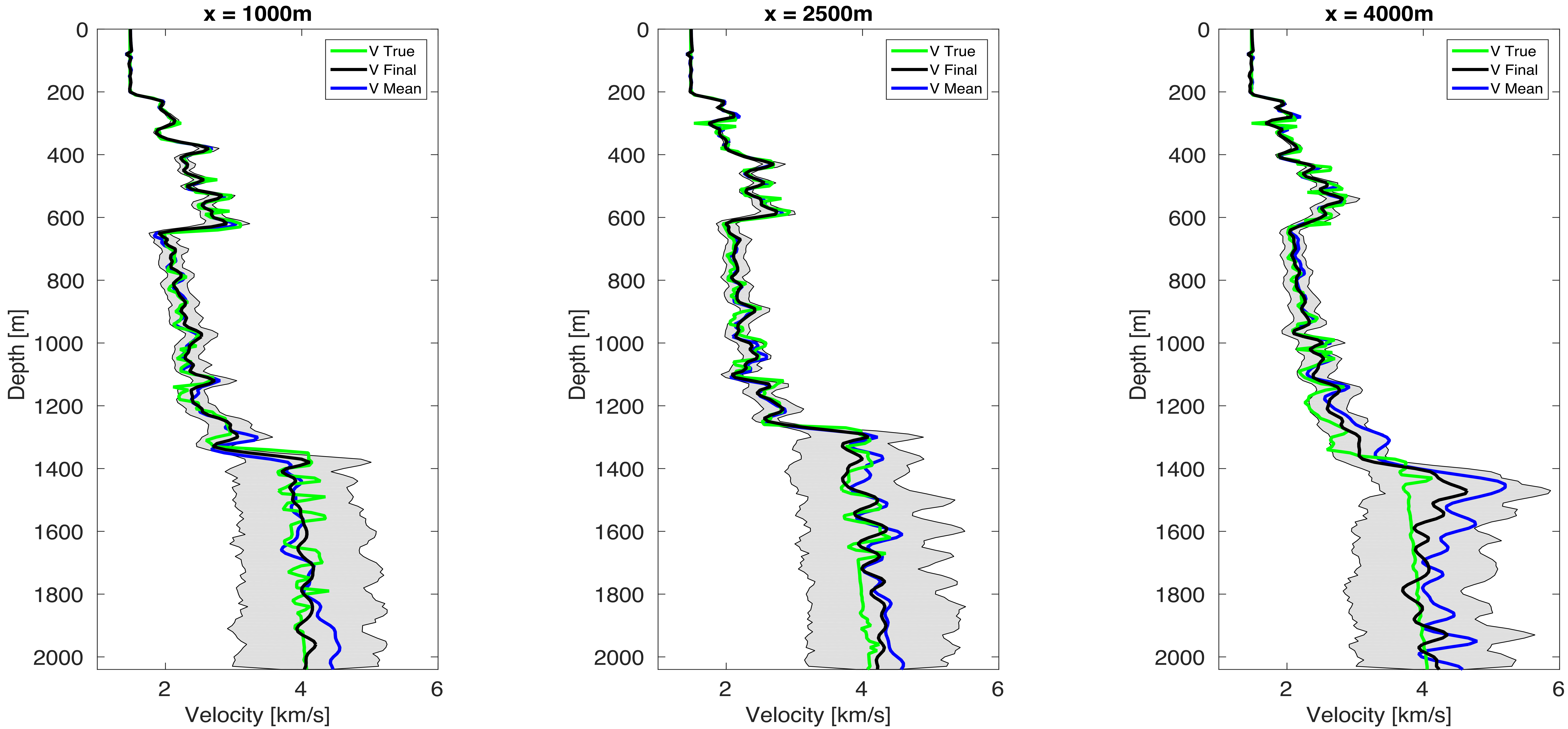
(a) (b) (c)

Confidence intervals (90%)
— Inversion results of 1000 random realizations



(a) (b) (c)

Confidence intervals (90%)
— Inversion results of 1000 random realizations



(a) (b) (c)

Conclusion

1. A positive-definite approximated Hessian of WRI is proposed
    - Matrix free
    - Easy to obtain the diagonal part

2. The quadratic function with the approximated Hessian is a good approximate of the original WRI misfit function

3. The results of inverting noisy data still lie in the confidence intervals
    - This gives us confidence in our confidence intervals

Future work

1. Apply the positive-definite approximated Hessian to the deterministic WRI.

2. Consider the density into the workflow and quantify the uncertainty of density.

3. Add the uncertainty quantification workflow into the soft-release of SINBAD.

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