Exploring applications of depth stepping in seismic inverse problems

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Motivation

- Downward wavefield extrapolation
- Matrix functions
- Low rank matrix compression (HSS)
- Combine to explore and develop efficient algorithms for modeling/imaging

Introduction

- In downward extrapolation, the goal is to solve Helmholtz equation

$$\frac{\partial^2 p(\mathbf{x}, z, \omega)}{\partial z^2} = -\left(\frac{\omega^2}{c^2(\mathbf{x}, z)} + \nabla_{\mathbf{x}}^2\right) p(\mathbf{x}, z, \omega)$$

  by stepping in depth from the initial data $p(\mathbf{x}, z = z_0, \omega)$.
- Main advantage: reduction in dimensionality of extrapolation problem
- Main difficulty: instability due evanescent modes

Introduction

Full wave equation depth extrapolation (Sandberg & Beylkin, 2009)

- Operator $\mathcal{H}_2 = \frac{\omega^2}{c^2(\mathbf{x},z)} + \nabla^2_{\mathbf{x}}$ is projected to its non-negative invariant subspace:

$$\mathcal{H}_2 \rightarrow \mathcal{P}\mathcal{H}_2\mathcal{P}$$

- Downward extrapolation equation:

$$\frac{\partial^2 p(\mathbf{x}, z, \omega)}{\partial z^2} = -\mathcal{P}\mathcal{H}_2\mathcal{P}p(\mathbf{x}, z, \omega) \qquad (1)$$

- Spectral projector is computed by:

$$\mathcal{P} = \frac{1}{2}(I + \mathsf{sign}(\mathcal{H}_2))$$

where sign$(\mathcal{H}_2)$ is found by recursion (e.g. Kenney & Laub, 1995)

$$S_0 = \frac{\mathcal{H}_2}{\|\mathcal{H}_2\|_2}, \qquad S_{k+1} = \frac{3}{2}S_k - \frac{1}{2}S_k^3$$

- Efficiency is achieved by low rank matrix compression (PLR, HSS), estimated cost $\sim O(N)$

Introduction

One way wave equation:

- Square root operator $\mathcal{H}_1 = \mathcal{H}_2^{1/2}$ can be computed by polynomial recursion
- Filtering of evanescent waves is still necessary
- Modeling of all propagating modes is possible

Other uses:

- Correct modeling of a volume injection (e.g. air gun) source as a initial condition for depth stepping
- Scattering operators **(e.g. Wapenaar, 1990)**
- These require computation of inverse square root $\mathcal{H}_1^{-1}$

One way wave equation

- The one way wave equation: factor $\mathcal{H}_2 = \frac{\omega^2}{c^2(\mathbf{x},z)} + \nabla^2_{\mathbf{x}}$; neglect scattering terms (e.g. Grimbergen et al., 1998; Wapenaar, 1990):

$$\frac{\partial p^{\pm}}{\partial z} = \mp i \mathcal{H}_1 p^{\pm}$$

where

p^+, p^- - down and up going fields
$\mathcal{H}_1$ - propagator, $\mathcal{H}_1 \mathcal{H}_1 p = \mathcal{H}_2 p$.

- Extrapolation is done by finite differences or matrix exponentiation (scaling and squaring)

One way wave equation

- $\mathcal{H}_1 = \mathcal{H}_2^{1/2}$ is non-local pseudo-differential
- Approximate sqrt by a polynomial or rational function $\Rightarrow$ paraxial wave equation
  - Efficient with finite differences and operator splitting
  - Propagating modes up to certain angle
- Modal decomposition **(e.g. Grimbergen et al., 1998; Margrave et al., 2002; Lin & Herrmann, 2007)**
  - All propagating modes
  - Requires eigenvalue decomposition
- Our goal:
  - Polynomial recursion with matrix compression
  - Benefits of modal decomposition without eigenvalue decomposition

Square root calculation

- Assume:
  - Absorbing boundary conditions in x are decoupled, H_2 is self-adjoint
  - Negative eigenvalues have been removed by spectral projector: $\tilde{H}_2 = \mathcal{P} H_2 \mathcal{P}$ - no evanescent modes
- Principal root of matrix H with no nonpositive eigenvalues can be computed by Shultz iteration (Higham, 2008):

$$
\begin{aligned}
Y_0 &= \frac{H}{\|H\|_2}, Z_0 = I \\
Y_{k+1} &= \frac{3}{2} Y_k - \frac{1}{2} Y_k Z_k Y_k \\
Z_{k+1} &= \frac{3}{2} Z_k - \frac{1}{2} Z_k Y_k Z_k
\end{aligned}
$$

- Derived by applying polynomial recursion for matrix sign function to $\begin{bmatrix} 0 & \frac{H}{\|H\|_2} \\ I & 0 \end{bmatrix}$ and Newton's method
- $Y_k \longrightarrow \left(\frac{H}{\|H\|_2}\right)^{1/2}$, $Z_k \longrightarrow \left(\frac{H}{\|H\|_2}\right)^{-1/2}$ quadratically

Square root calculation

- The square root polynomial recursion is poorly conditioned for $\tilde{H}_2 = \mathcal{P} H_2 \mathcal{P}$, has zeros eigenvalues (numerically they are very small complex numbers)
- $Y_k \longrightarrow \left(\frac{\tilde{H}_2}{\|\tilde{H}_2\|_2} \right)^{1/2}$ in $\sim O(10)$ with high accuracy.
- Z_k part causes the iteration eventually to diverge $\Rightarrow$ careful stopping criterion
- Stopping criterion we use: (a) difference between iterates $\|Y_{k+1} - Y_k\|$, (b) misfit $\|\tilde{H}_1^2 - \tilde{H}_2\|$, (c) update direction

Computation of pseudo inverse square root

Volume injection source as initial condition at $z = z_0$:

$$S^{\pm}(\mathbf{x}, z = 0, \omega) = \frac{i\omega^2}{2} \tilde{H}_1^{\dagger} I(\mathbf{x}, z = z_0, \omega)$$

(e.g. Wapenaar, 1990)

To compute pseudo inverse $\tilde{H}_1^{\dagger}$:

- Compute ${H_2}^{-1}$, e.g. by recursion (Ben Israel and Cohen, 1966) or HSS
- Apply spectral projector to ${H_2}^{-1}$: ${H_2}^{-1} \rightarrow \tilde{H}_2^{\dagger} = \mathcal{P} {H_2}^{-1} \mathcal{P}$
- Compute pseudo inverse of $\tilde{H}_1^{\dagger}$ by Shultz iteration from $\tilde{H}_2^{\dagger}$

Convergence

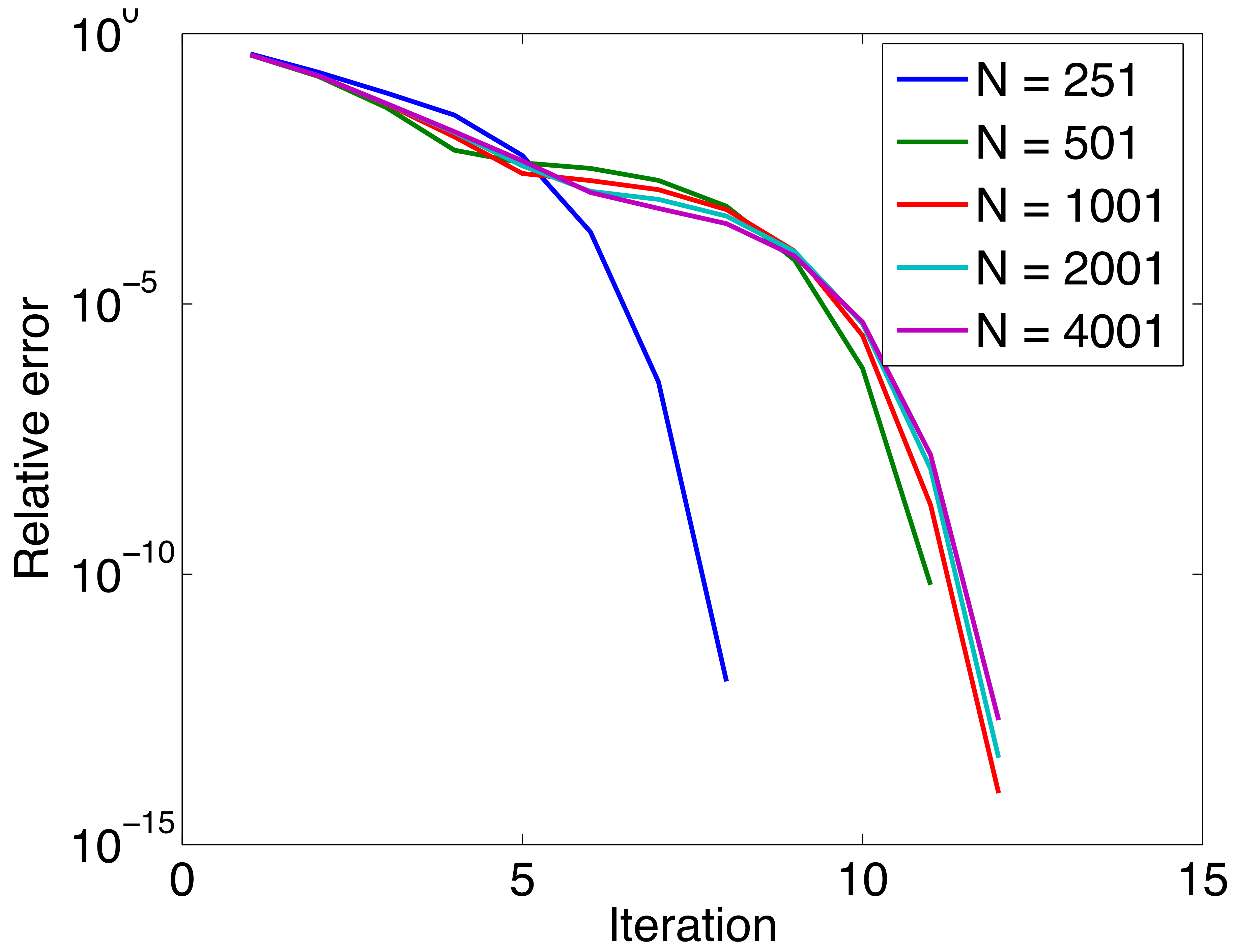


Figure: Convergence of iteration for the square root, relative error $= \frac{\|\tilde{H}_1^2 - \tilde{H}_2\|}{\|\tilde{H}_2\|}$

Convergence

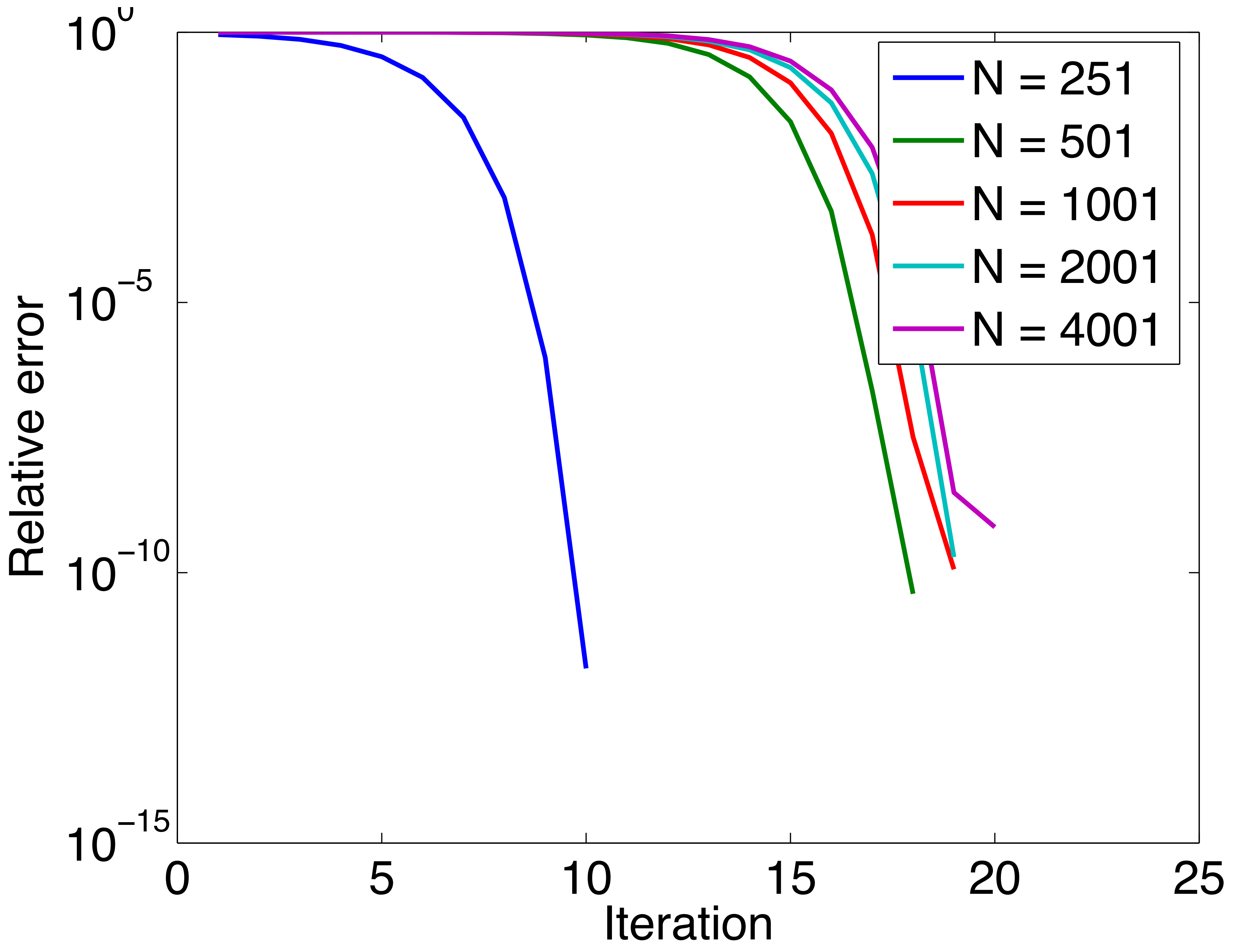
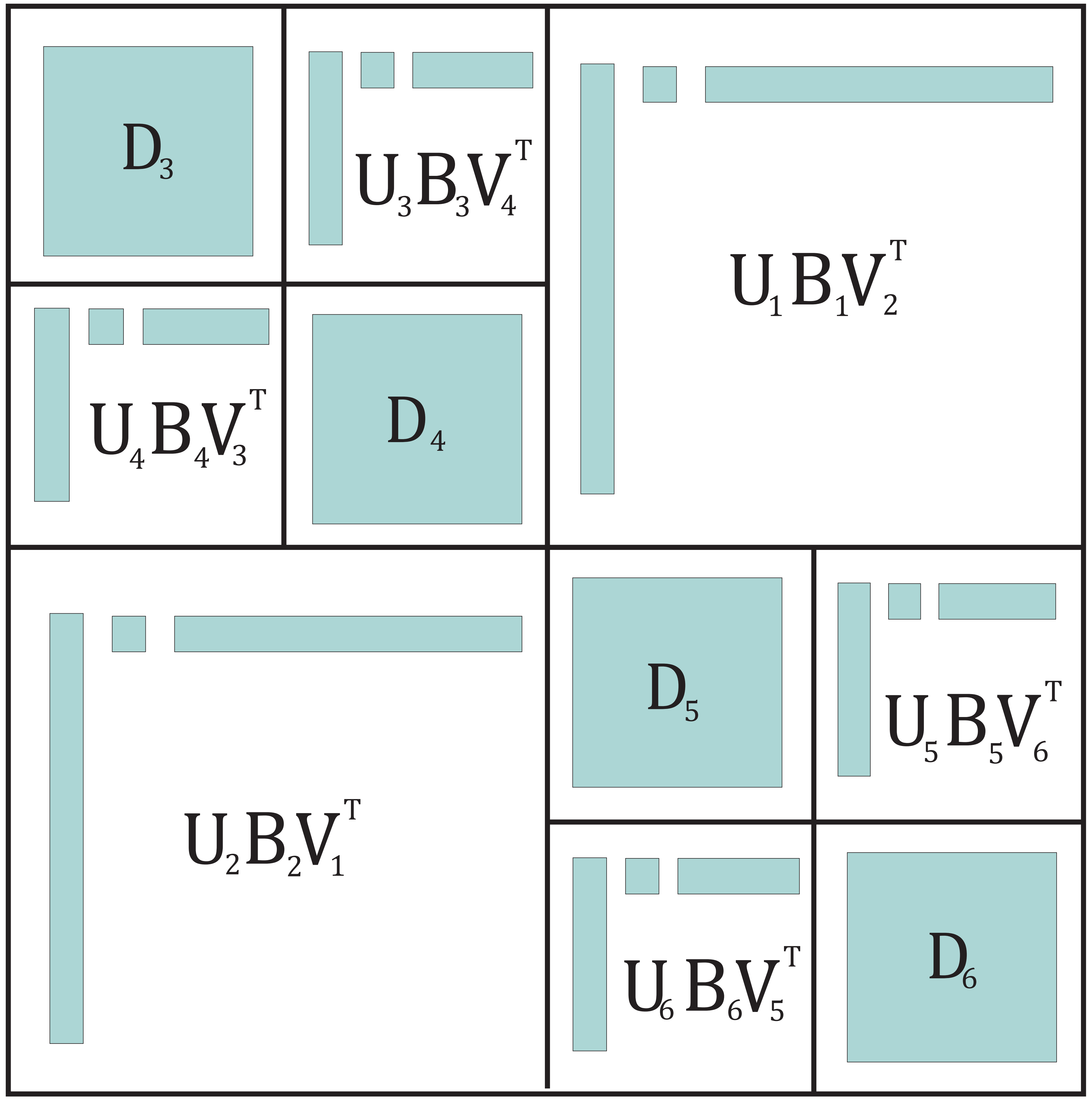


Figure: Convergence of iteration for the matrix inverse, relative error $= \frac{\|H_2^{-1} - \text{pinv}(H_2)\|}{\|\text{pinv}(H_2)\|}$

HSS compression



- Off-diagonal blocks have low numerical rank
- Each low rank approximation is a product of
  - a tall matrix
  - a small matrix and
  - a flat matrix
- The hierarchy is organized in a binary tree
- New developments: randomized parallel HSS **(e.g. Liu, et al., 2014)**

Examples

- Modeling and prestack one way wave equation migration
- True model: 2D SEG salt model
- Absorbing BC: taper wavefield **(Cerjan et al., 1985)**
- Model parameters:
    - Model size 840 $\times$ 3370 m
    - Grid: $\Delta x = \Delta z = 10$ m
    - Ricker wavelet 15 Hz
    - Sources: every 100 m, receivers: at every grid point
- Data is generated by the linearized constant density acoustic frequency domain forward modeling operator

Examples

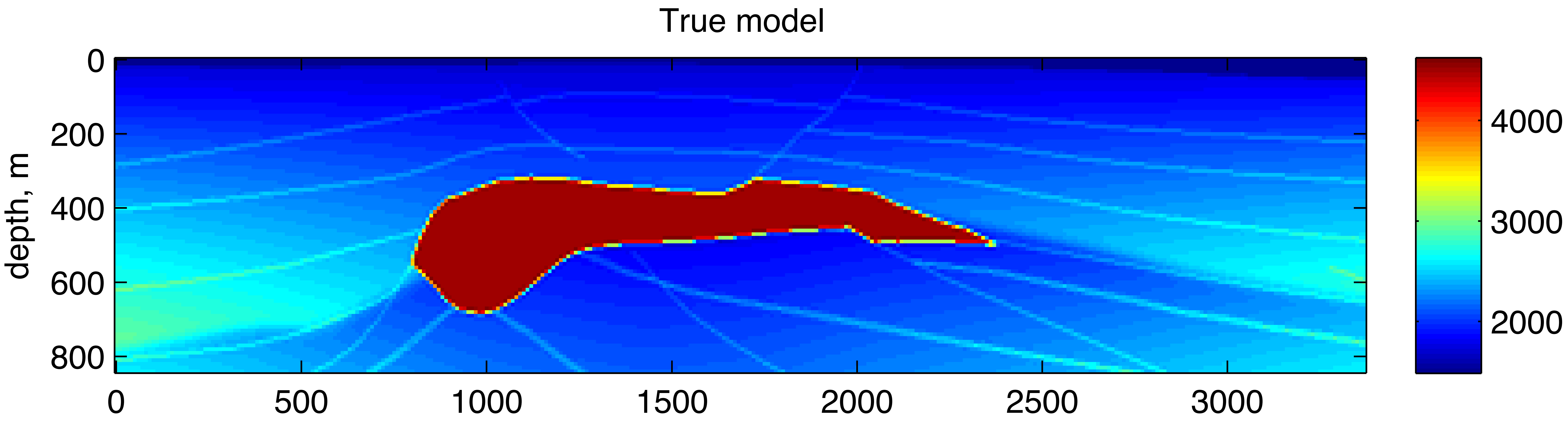


Figure: True velocity model

Examples

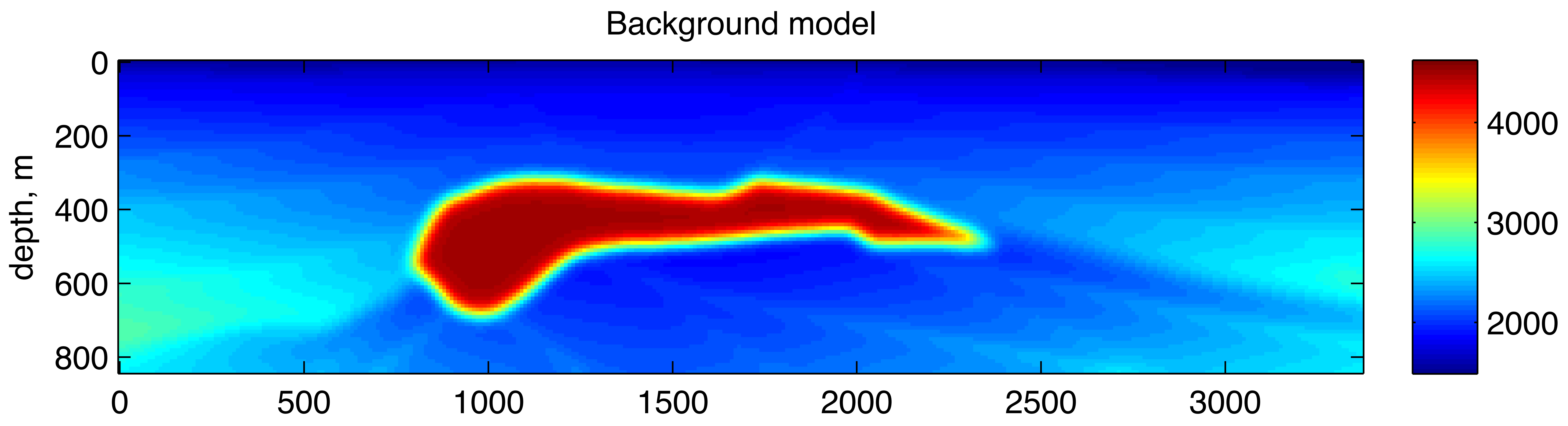


Figure: True velocity model

Examples

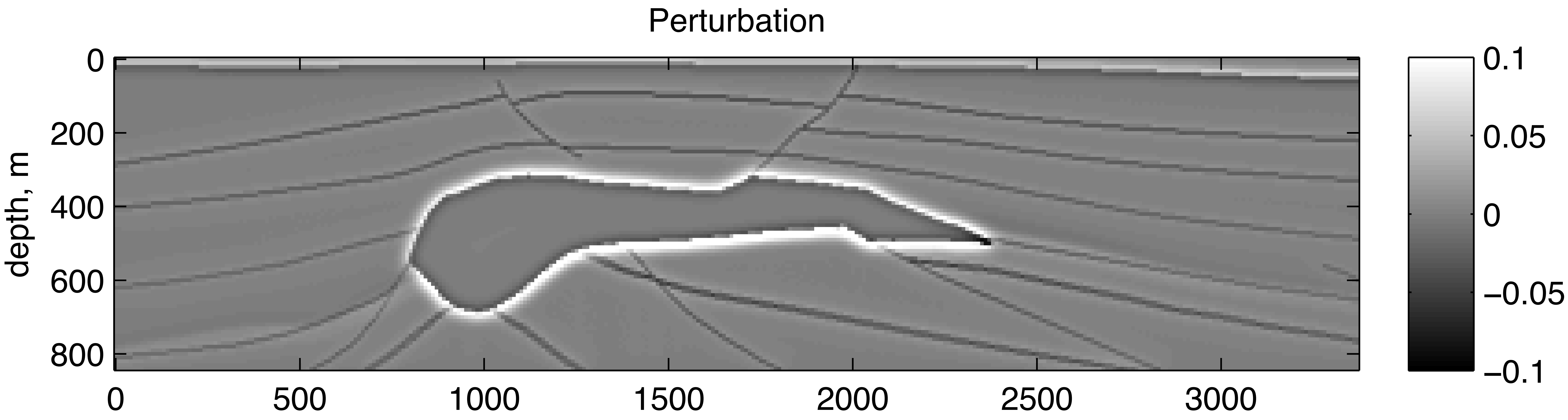


Figure: Model perturbation

Examples

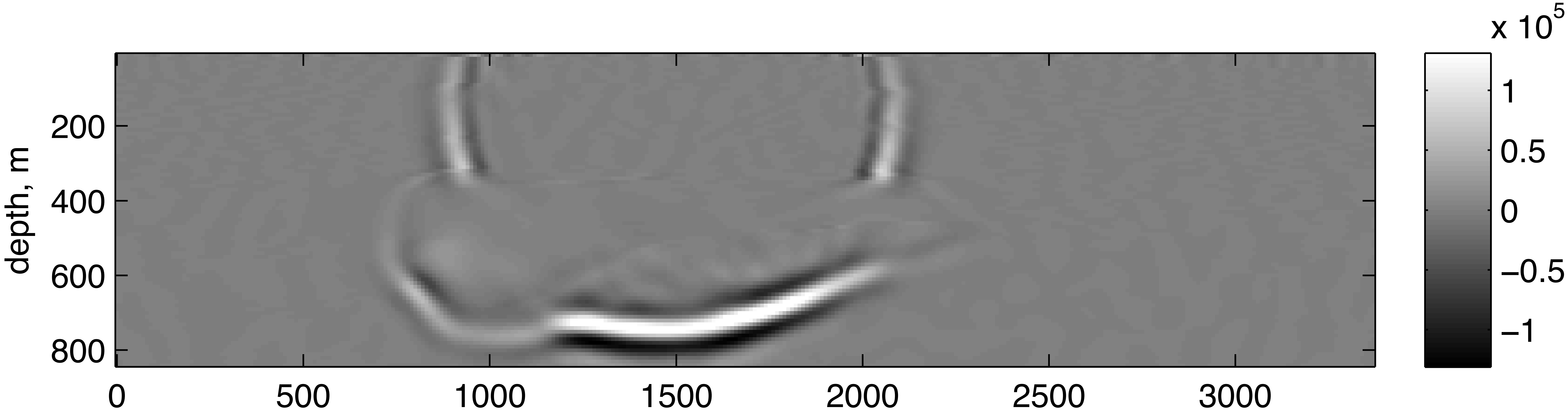


Figure: Wavefield time slice at $t = 0.35$ sec, source $x = 1500$ m

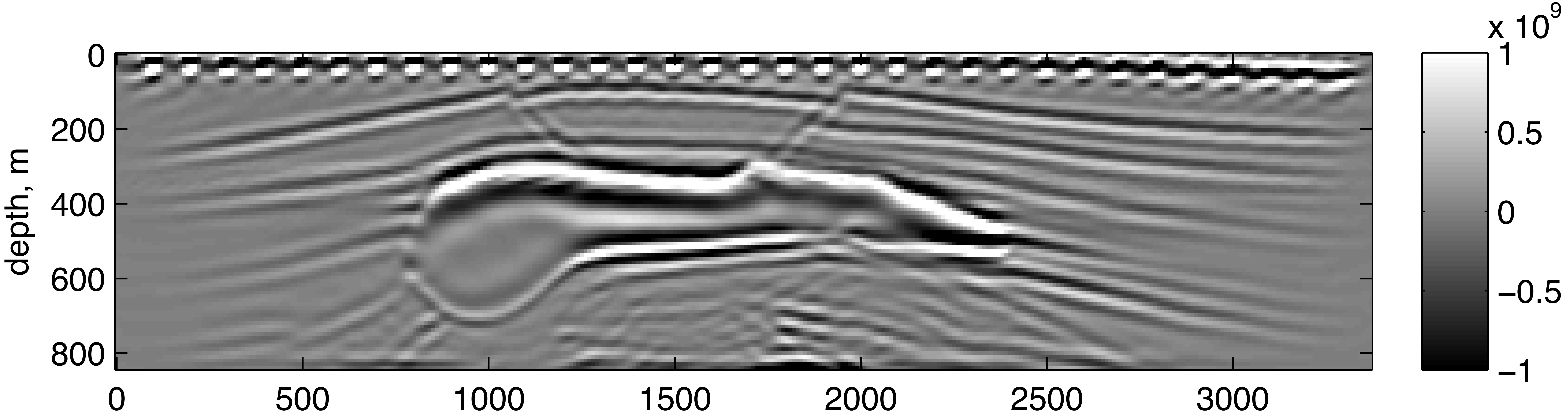


Figure: One way wave equation migration result

Examples

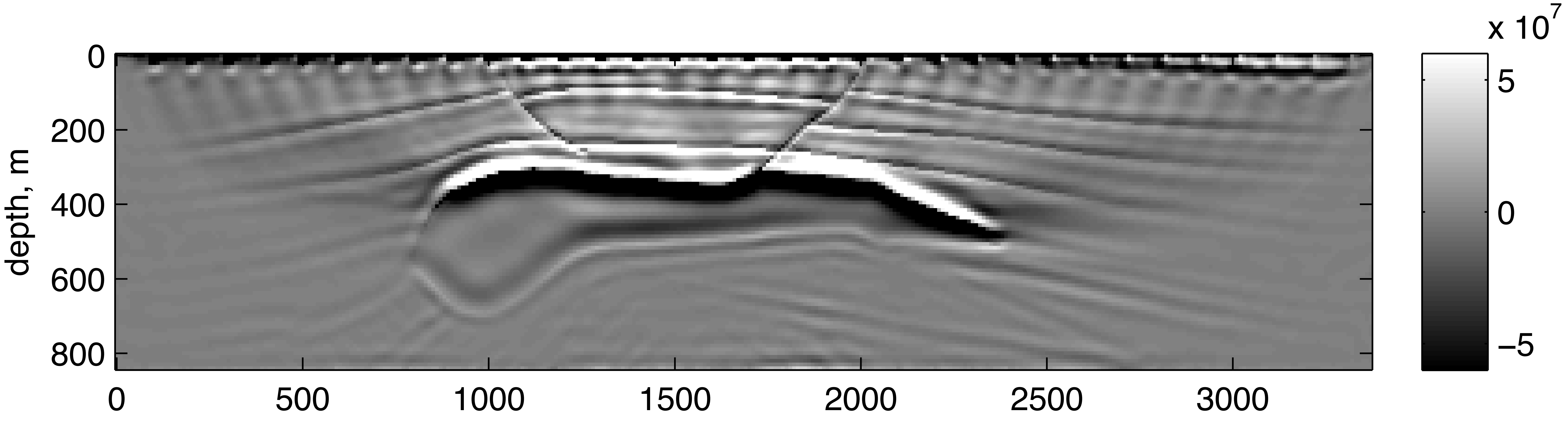


Figure: Reverse time migration result (for comparison)

Conclusions and future work

Conclusions:

- Matrix square root and inverse by polynomial recursion
- Model all propagating modes, avoid eigenvalue decomposition
- Correct volume injection source modeling

Future work:

- Good implementation of HSS matrix compression
- Other ways to compute matrix functions efficiently
- 3D

Beyond this: full wave equation

Full wave-equation depth stepping

Main ideas:

- Current work: approximating of Helmholtz system solution with depth stepping (preconditioner)
- Future work: using depth stepping for FWI gradient (e.g. enhance “rabbit ears”, update shallow part of the model, see **Sandberg & Beylkin, 2010)**
- This work is in early stage

Full wave-equation depth stepping

Look at the Helmholtz problem from the point of veiw of ODE theory:

- Discretize in x direction (method of lines)
- Then Helmholtz equation can be thought of as a *Two-point Boundary Value Problem (BVP)* in z
- Depth extrapolation tries to solve the corresponding *Initial Value Problem (IVP)*

Full wave-equation depth stepping

Helmholtz (BVP1):

$$u_{zz} + H_2 u = q$$

Boundary conditions:

periodic in x

$$u_z = \frac{i\omega}{c} u, \quad z = z_0$$

$$u_z = -\frac{i\omega}{c} u, \quad z = z_{max}$$

Projected Helmholtz (BVP2):

$$u_{zz} + \mathcal{P} H_2 \mathcal{P} u = q$$

Boundary conditions:

periodic in x

$$u_z = \frac{i\omega}{c} u, \quad z = z_0$$

$$u_z = -\frac{i\omega}{c} u, \quad z = z_{max}$$

Depth extrapolation (IVP):

$$u_{zz} = -\mathcal{P} H_2 \mathcal{P} u + q$$

Boundary conditions:

periodic in x

Initial conditions:

$$u(z_0) = v$$

$$u_z(z_0) = v_z$$

$$H_2 = \partial_{xx} + \frac{\omega^2}{c(x,z)^2}, \quad \mathcal{P} - \text{spectral projector onto the propagating modes}$$

Full wave-equation depth stepping

Helmholtz (BVP1):

$$u_{zz} + H_2 u = q$$

Boundary conditions:

periodic in x

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Depth extrapolation (IVP):

$$u_{zz} = -\mathcal{P} H_2 \mathcal{P} u + q$$

Boundary conditions:

periodic in x

Initial conditions:

$$u(z_0) = v$$

$$u_z(z_0) = v_z$$

Questions:

1. Can solution to BVP2 be used e.g. as a preconditioner for BVP1?
2. Can we get approximation to BVP2 using IVP?

Full wave-equation depth stepping

- Apply the 3 methods to a simple velocity profile
- Slow velocity layer on top of fast velocity layer
- One frequency
- Source at the center top of the model

Full wave-equation depth stepping

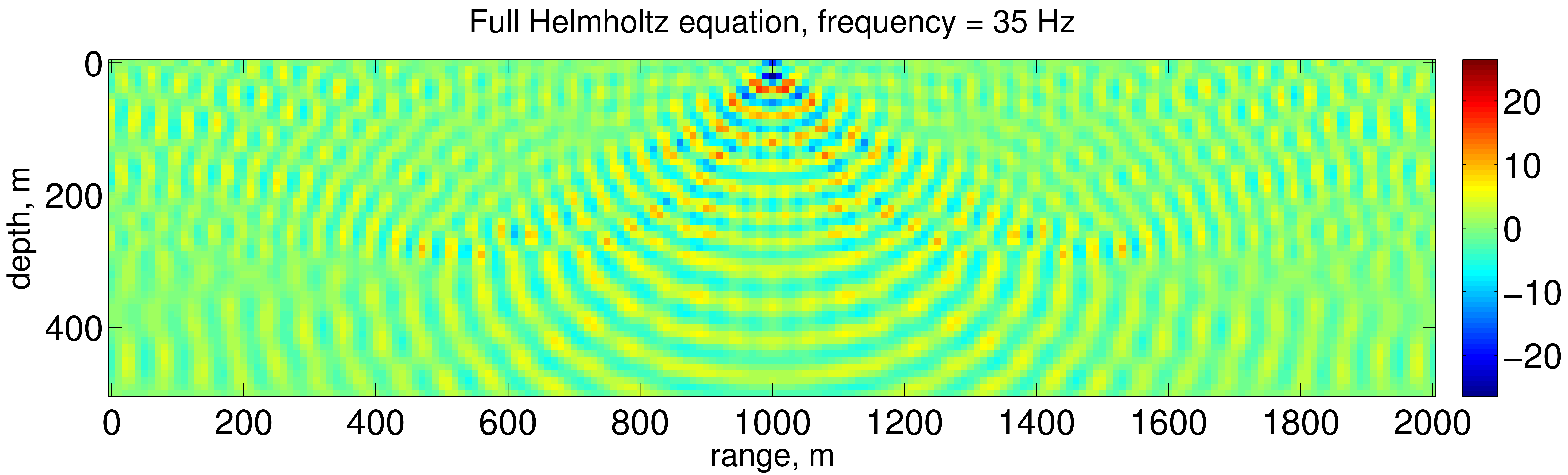

Figure: Solution of BVP1 (Helmholtz)

Full wave-equation depth stepping

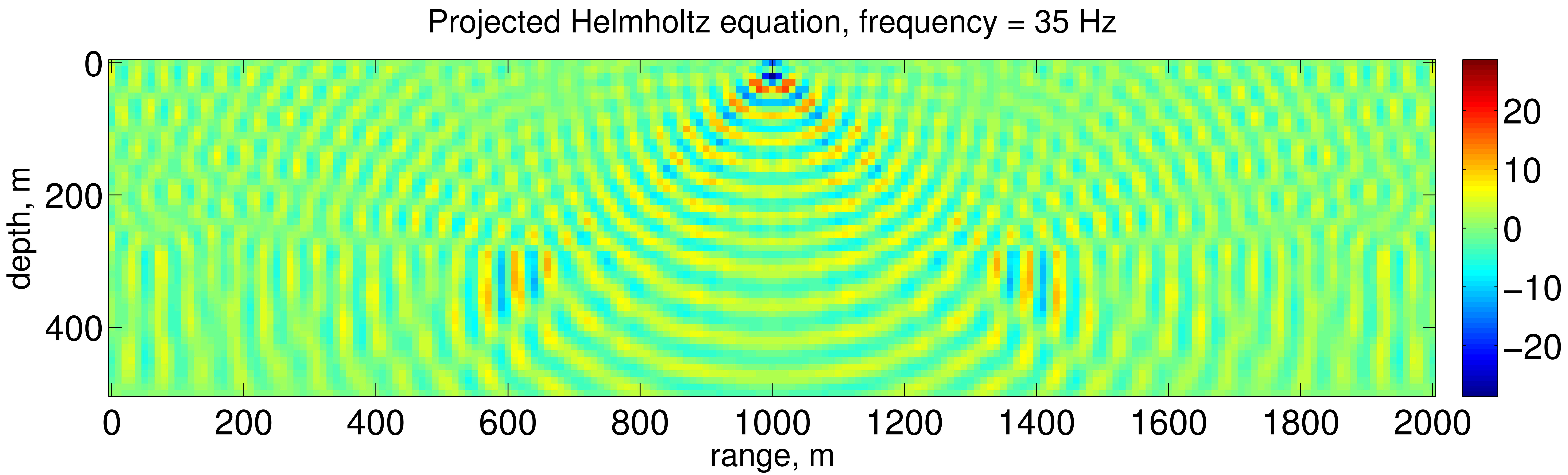

Figure: Solution of BVP2 (Projected Helmholtz)

Full wave-equation depth stepping

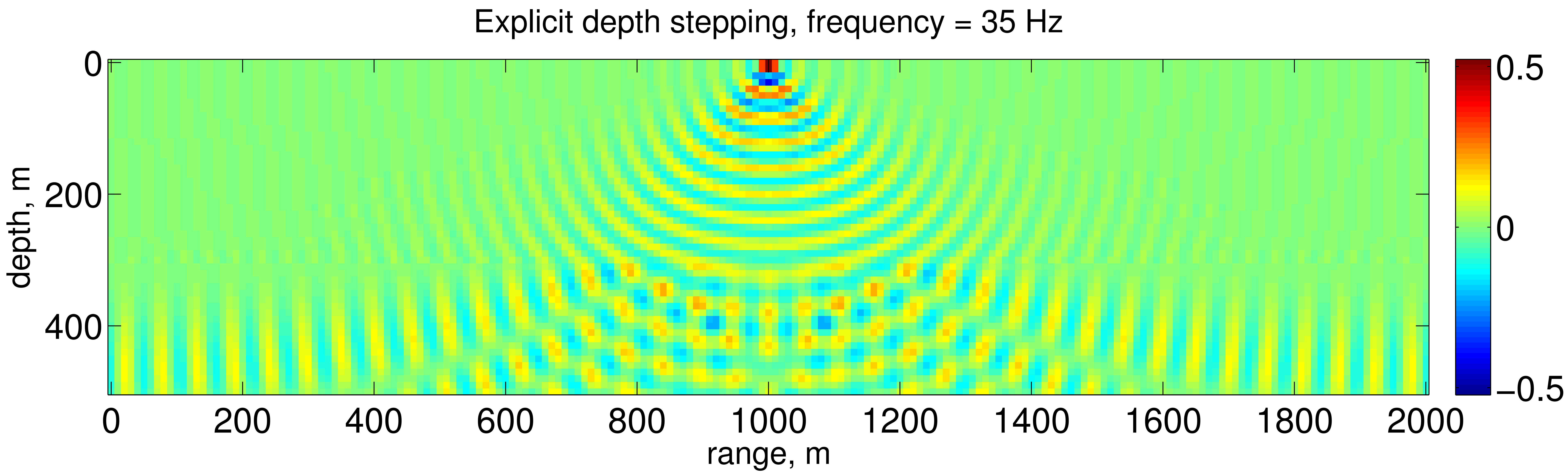

Figure: Solution of IVP (Depth extrapolation)

Full wave-equation depth stepping

Observations:

- Solutions of BVP1 and BVP2 are qualitatively similar
- Solution to IVP only contains downgoing wavefield - no reflected wave
- Initial conditions for the IVP contain only source and no reflected field
- Downward continue top layer of the solution to e.g. BVP2 with IVP

Full wave-equation depth stepping

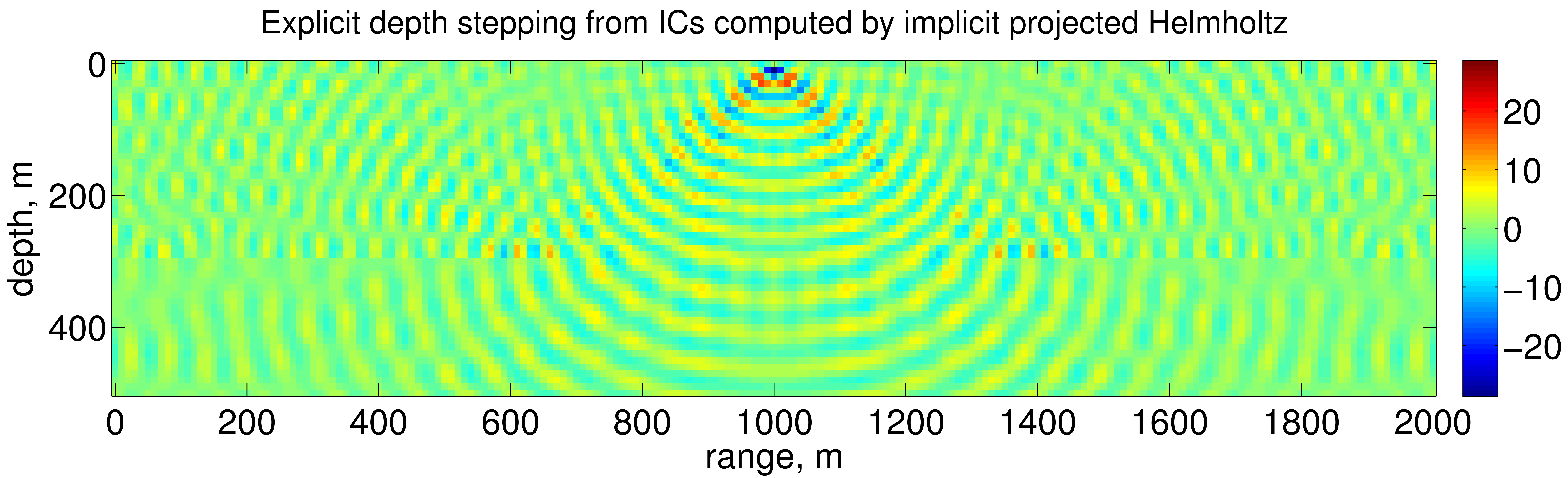

Figure: Solution of IVP with "correct" initial conditions

Full wave-equation depth stepping

Observations:

- "Correct" initial conditions for the IVP should contain source field and reflected field
- Linear ODE theory: invert the fundamental solution matrix of the IVP to recover the correct initial conditions
- Our case: fundamental solution is not invertible: no evanescent modes
- Q: can we invert the fundamental solution in generalized sense to recover the propagating modes in the initial data?

Full wave-equation depth stepping

Future work:

- Preconditioning: implement iterartive solver using BVP2 as a preconditioner
- Solution of BVP2 using IVP

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