Rank Minimization via Alternating Optimization

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Motivation

Acquisition challenges

- highly subsampled data

Data exhibits *low-rank* structure

- SVD-free matrix completion

Benefits of Alternating Optimization procedures

- problems become tractable

Outline

Alternating Optimization

Nuclear Norm Minimization
- factorized formulation
- can we benefit from alternating optimization?

Motivation: Variable Decomposition

$$x = x_1 \cdot x_2 \cdot \ldots \cdot x_s$$

Benefits:

- memory efficiency
- computational efficiency

Motivation: Variable Decomposition

$$\boxed{x} = \boxed{x_1} \cdot \boxed{x_2} \cdot \ldots \cdot \boxed{x_s}$$

Benefits:

- memory efficiency
- computational efficiency
- flexibility

SLIM

Xu, Yin. "A Block Coordinate Descent Method for Regularized Multiconvex Optimization with Appications to Nonnegative Tensor Factorization and Completion". SIAM 2013

Alternating Optimization

Want to solve:

$$\min_{x \in \mathcal{X}} f(x_1, x_2, \ldots, x_s) + \sum_{i=1}^{s} r_i(x_i)$$

f is multi-convex

each r_i is convex

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Alternating Optimization

Algorithm :

1.Initialization: choose initial point$(x_1^0, x_2^0, \ldots, x_s^0)$

2. for $k = 1, 2, \ldots, T$ do

3. for $i = 1, 2, \ldots, s$ do

4. $x_i^k \leftarrow \arg\min_{x_i \in \mathcal{X}_i^k} f_i^k(x_i) + r_i(x_i)$

5. end for

6. end for

Output: $(x_1^T, x_2^T, \ldots, x_s^T)$

Alternate optimization between factors

Conclusion of Alternating Minimization

Problem becomes tractable

- solve main problem via simpler subproblems

Computationally efficient

- "shown to be superior than other procedures in both speed and quality" **- Xu, Yin**

Outline

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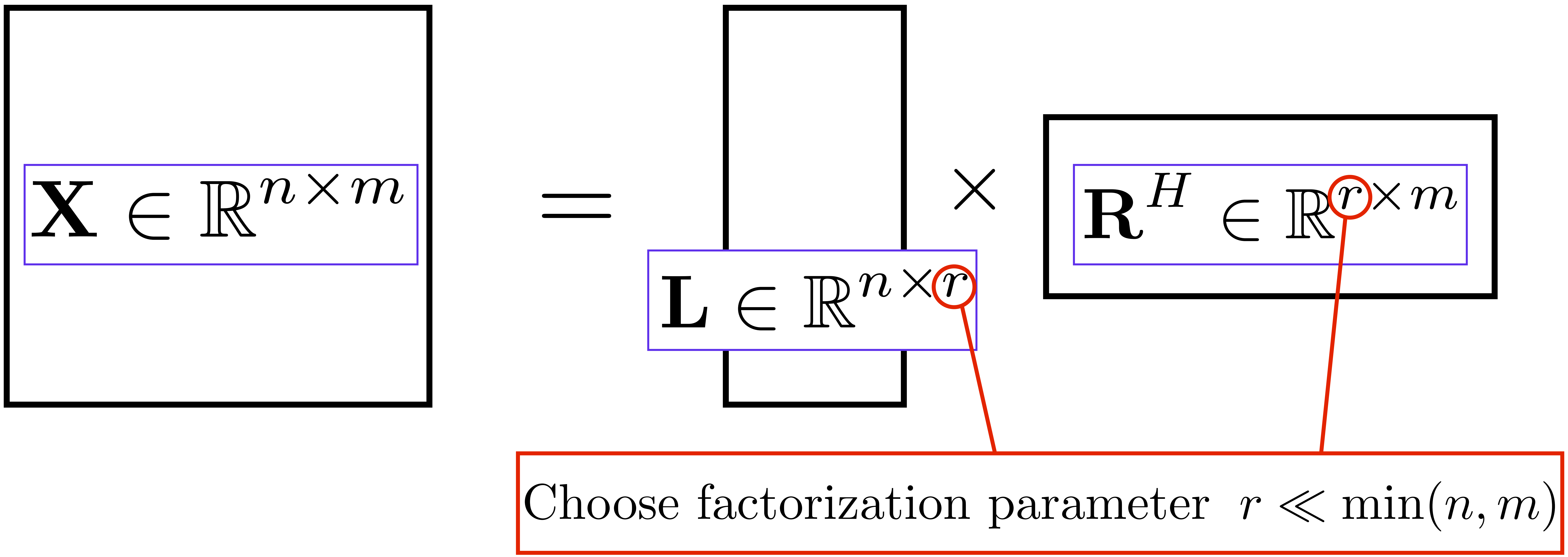
Matrix Factorization

$$\mathbf{X} \in \mathbb{R}^{n \times m} \quad = \quad \mathbf{L} \in \mathbb{R}^{n \times r} \quad \times \quad \mathbf{R}^H \in \mathbb{R}^{r \times m}$$

$$\mathbf{X} = \mathbf{L}\mathbf{R}^{\mathbf{H}}$$

Matrix Factorization



$$\mathbf{X} = \mathbf{L}\mathbf{R}^{\mathbf{H}}$$

[Rennie and Srebro 2005]

Nuclear Norm Minimization- Factorized Form

$$\mathbf{X} = \mathbf{L}\mathbf{R}^H$$

- Nuclear norm is bounded by

$$||\mathbf{X}||_* \leq \frac{\mathbf{1}}{\mathbf{2}}(||\mathbf{L}||^{\mathbf{2}}_{\mathbf{F}} + ||\mathbf{R}||^{\mathbf{2}}_{\mathbf{F}})$$

where $\|\cdot\|_F^2$ is sum of squares of all entries

- choose r *explicitly* & avoid costly *SVD's*

Nuclear Norm Minimization- Factorized Form

We want to solve

$$\min_{\mathbf{L}\in\mathbb{R}^{n\times r},\mathbf{R}\in\mathbb{R}^{m\times r}} \frac{1}{2}(||\mathbf{L}||_F^2 + ||\mathbf{R}||_F^2) \;\; \text{s.t.} \;\; ||\mathcal{A}(\mathbf{L}\mathbf{R}^H) - \mathbf{b}||_{\mathbf{F}}^{\mathbf{2}} \leq \sigma$$

Nuclear Norm Minimization- Factorized Form

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convex functions

multi-convex function

Nuclear Norm Minimization- Factorized Form

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convex functions

multi-convex function

Let's Alternate!

Alternating Nuclear Norm Minimization

1. Input: $\mathcal{A}, \mathbf{b}$
2. Initialize: $\mathbf{L}^0$ to be the top-r left singular vectors of $\mathbf{b}$
3. for $t = 0, ..., T-1$ do
4.
$$\mathbf{R}^{t+1} = \min_{\mathbf{R} \in \mathbb{R}^{m \times r}} \frac{1}{2}||\mathbf{R}||_F^2 \quad \text{s.t.} \quad ||\mathcal{A}(\mathbf{L}^t \mathbf{R}^H) - \mathbf{b}||_F^2 \leq \sigma_t$$
5.
$$\mathbf{L}^{t+1} = \min_{\mathbf{L} \in \mathbb{R}^{n \times r}} \frac{1}{2}||\mathbf{L}||_F^2 \quad \text{s.t.} \quad ||\mathcal{A}(\mathbf{L}(\mathbf{R}^{t+1})^H) - \mathbf{b}||_F^2 \leq \sigma_t$$
6. end for
7. Return $\tilde{\mathbf{X}} = \mathbf{L}^T (\mathbf{R}^T)^H$

Implementation

$$\min_{\mathbf{R}\in\mathbb{R}^{m\times r}} \frac{1}{2}||\mathbf{R}||_F^2 \quad \text{s.t. } ||\mathcal{A}(\mathbf{L}^t\mathbf{R}^H) - \mathbf{b}||_F^2 \leq \sigma_t$$

- approximately solve a series of $LASSO_\tau$ formulation

$$v(\tau) = \min_{\mathbf{R}} ||\mathcal{A}(\mathbf{L}^t\mathbf{R}^H) - \mathbf{b}||_F^2 \quad \text{s.t. } \frac{1}{2}||\mathbf{R}||_F^2 \leq \tau$$

where τ is the regularization parameter

Experiments: Nelson 2-D seismic line

1024 x 401 x 401 matrix

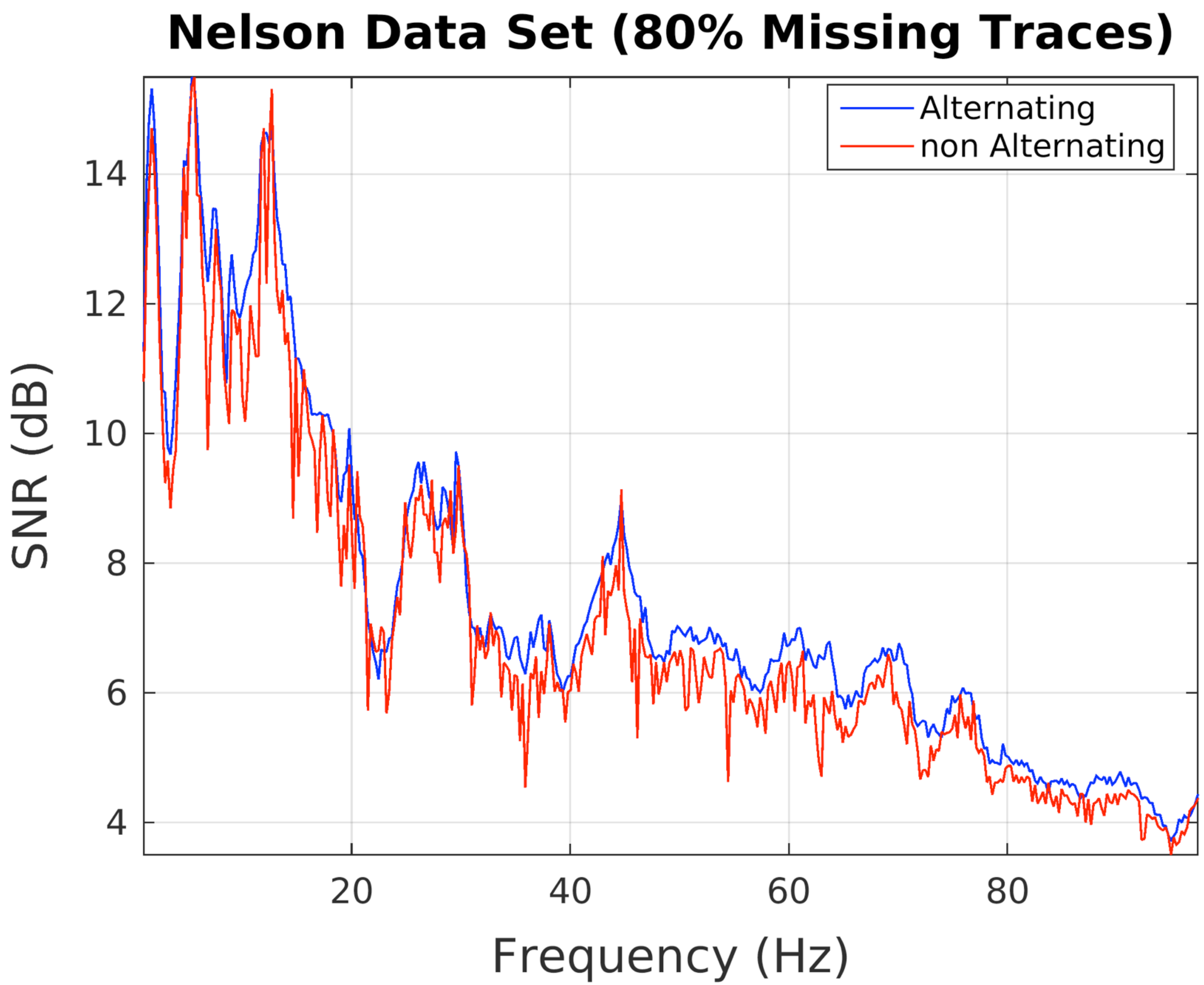
- 80% missing traces
- Factorization parameter adjusted from low to high frequency
- Jittered subsampling
- solve with SPGL1

Compare: Alternating vs Non Alternating

- 6 alternations, 15 iterations per alternation
- 180 iterations total for both

Experiments

Average Improvement
.5 dB



Experiments

Average Improvement
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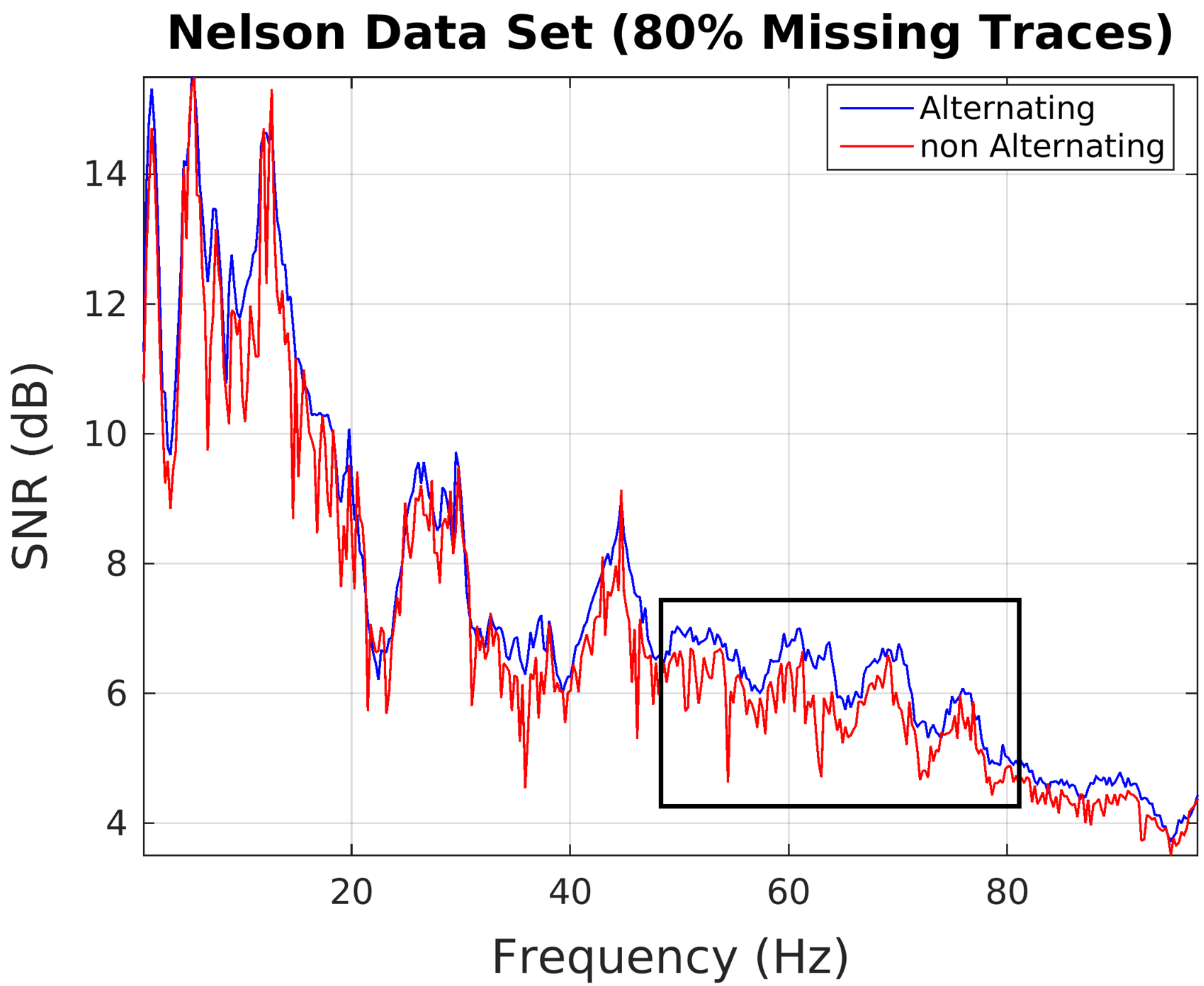

Experiments

Average Improvement
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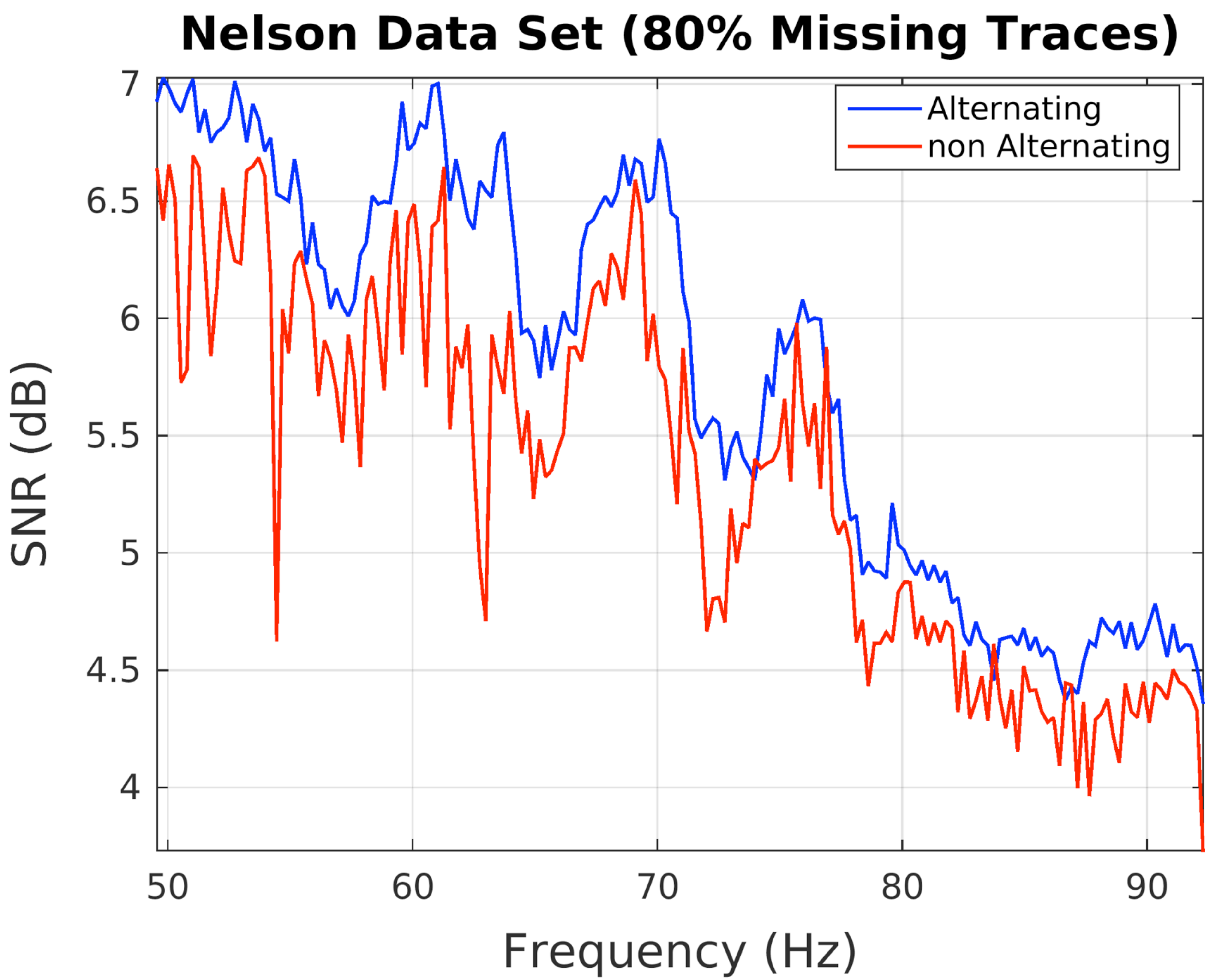
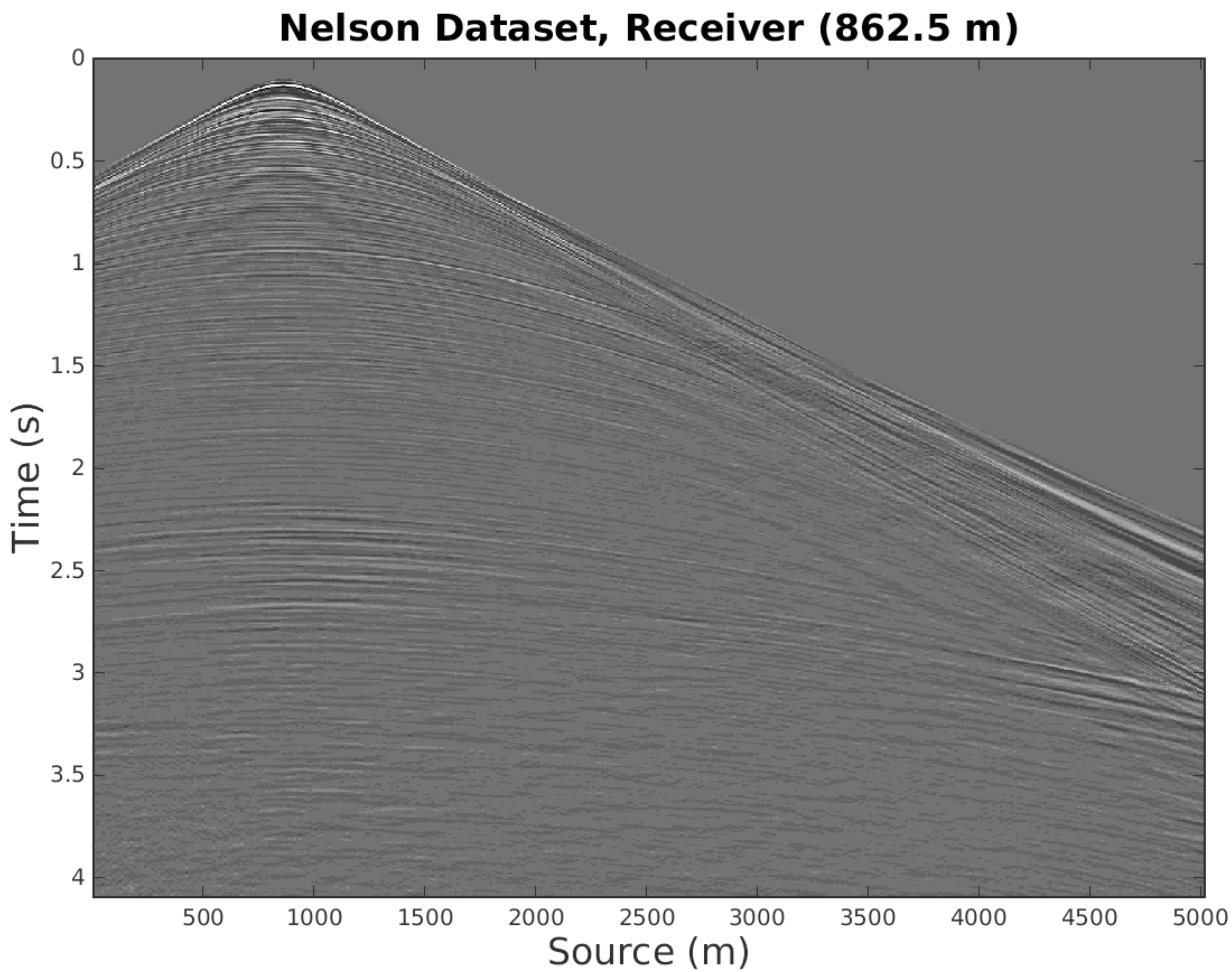
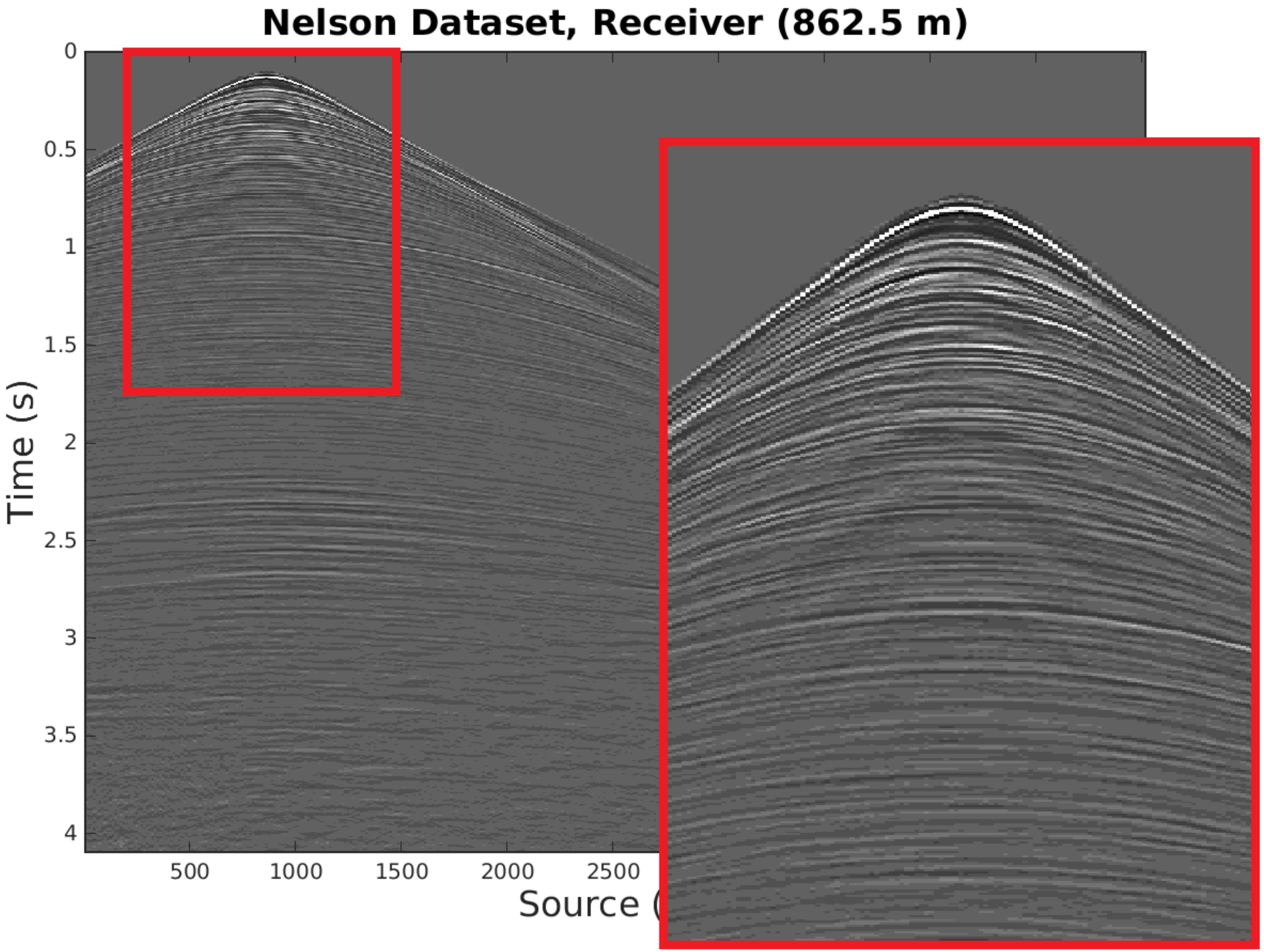



Nelson Dataset, Receiver (862.5 m)
Time (s)
0
0.5
1
1.5
2
2.5
3
3.5
4
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
Source (m)


Nelson Dataset, Receiver (862.5 m)
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1
1.5
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2.5
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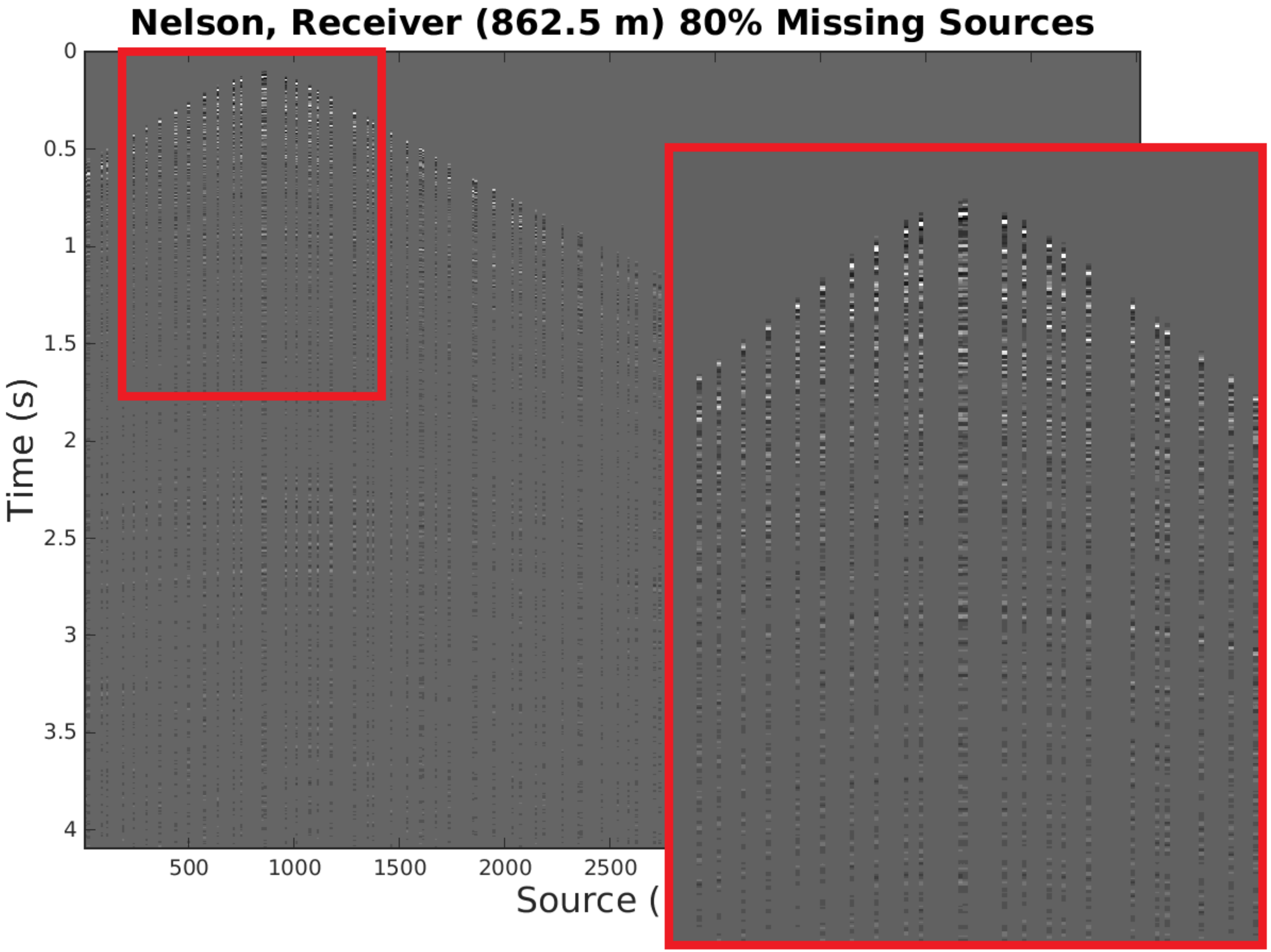
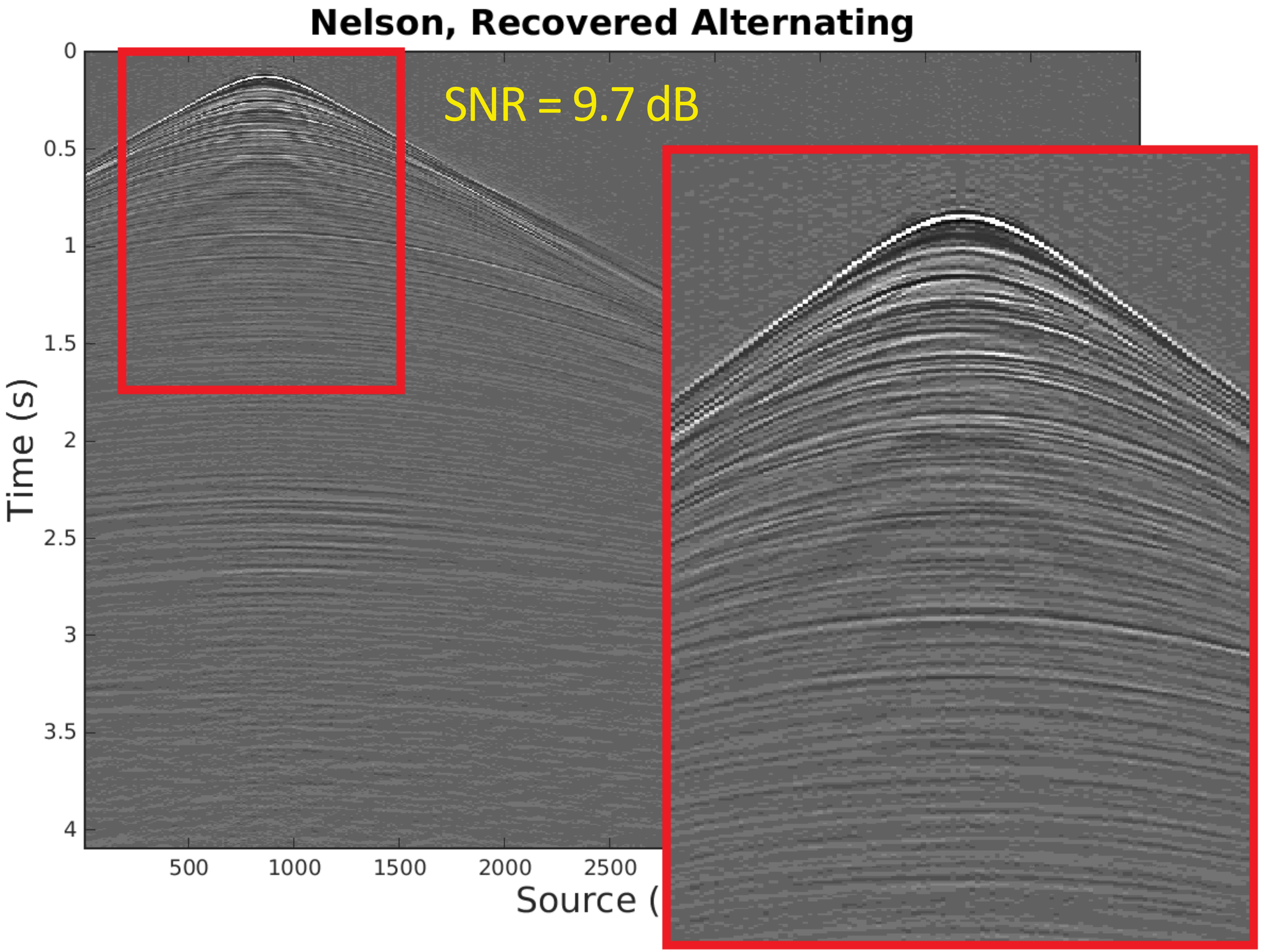
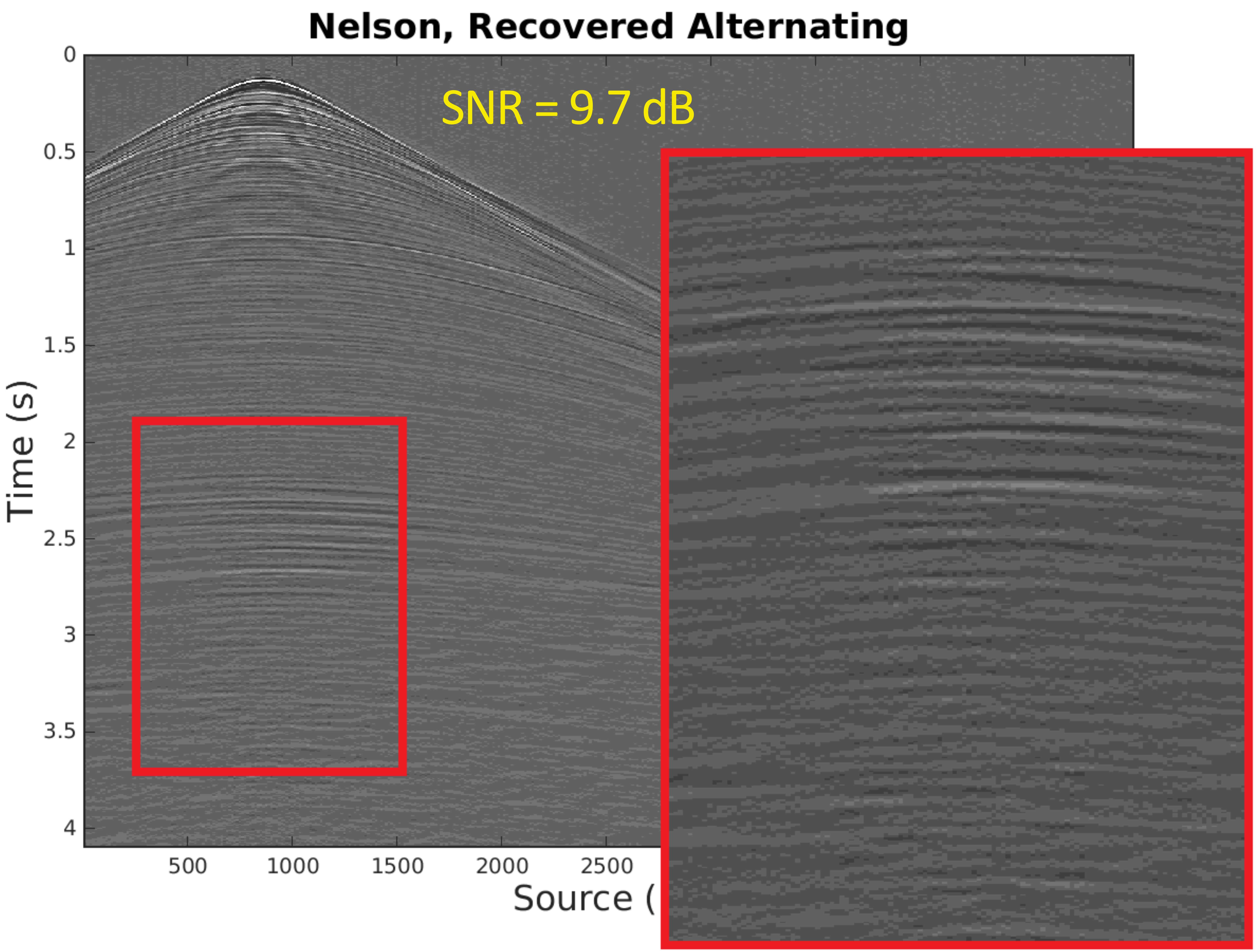
500
1000
1500
2000
2500
Source (



Nelson, Receiver (862.5 m) 80% Missing Sources
Time (s)
Source (
0
0.5
1
1.5
2
2.5
3
3.5
4
500
1000
1500
2000
2500


Nelson, Recovered Alternating
SNR = 9.7 dB
Time (s)
0
0.5
1
1.5
2
2.5
3
3.5
4
500
1000
1500
2000
2500
Source (


Nelson, Recovered Alternating
SNR = 9.7 dB
Time (s)
Source (
0
0.5
1
1.5
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2.5
3
3.5
4
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1000
1500
2000
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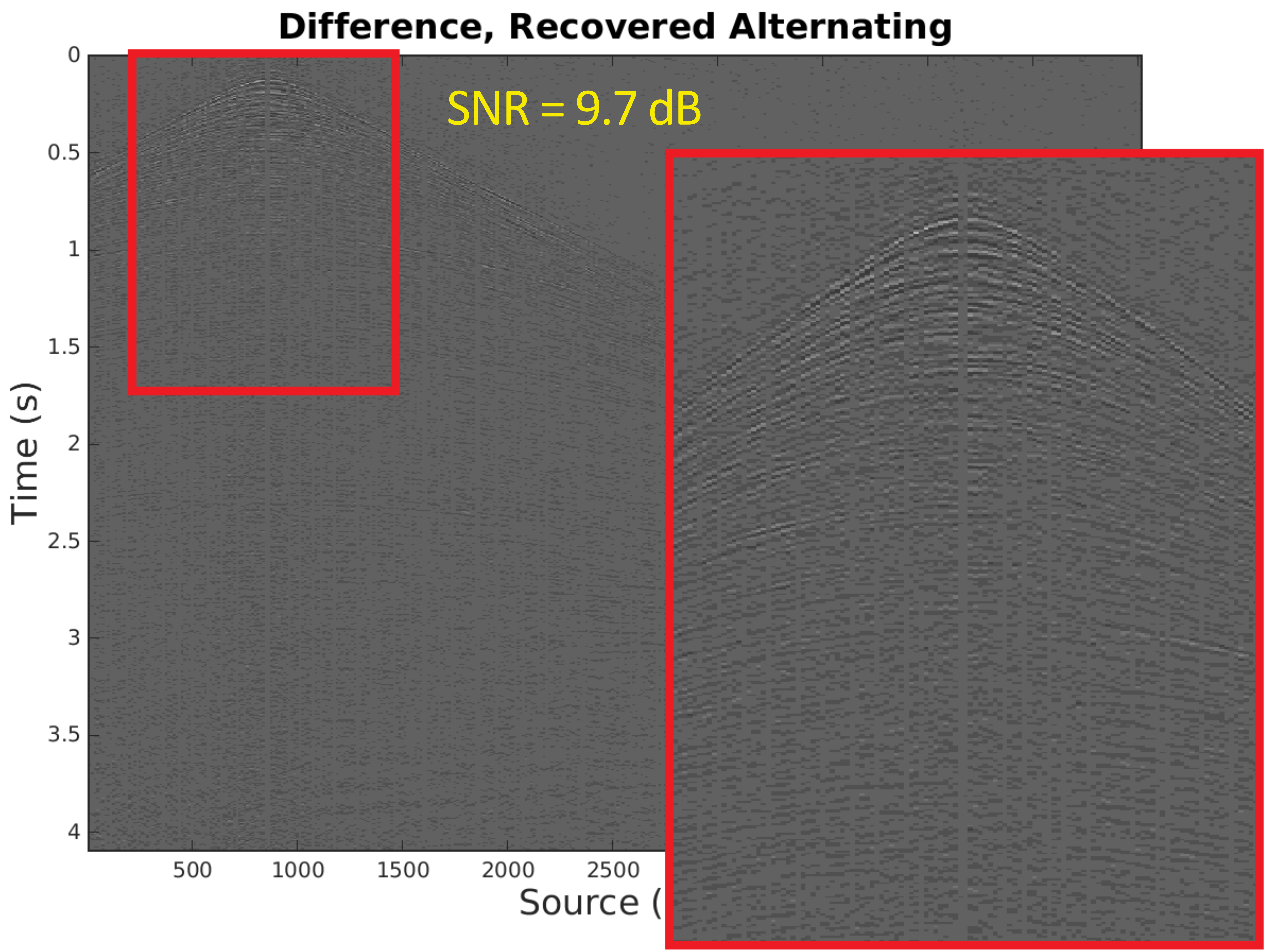
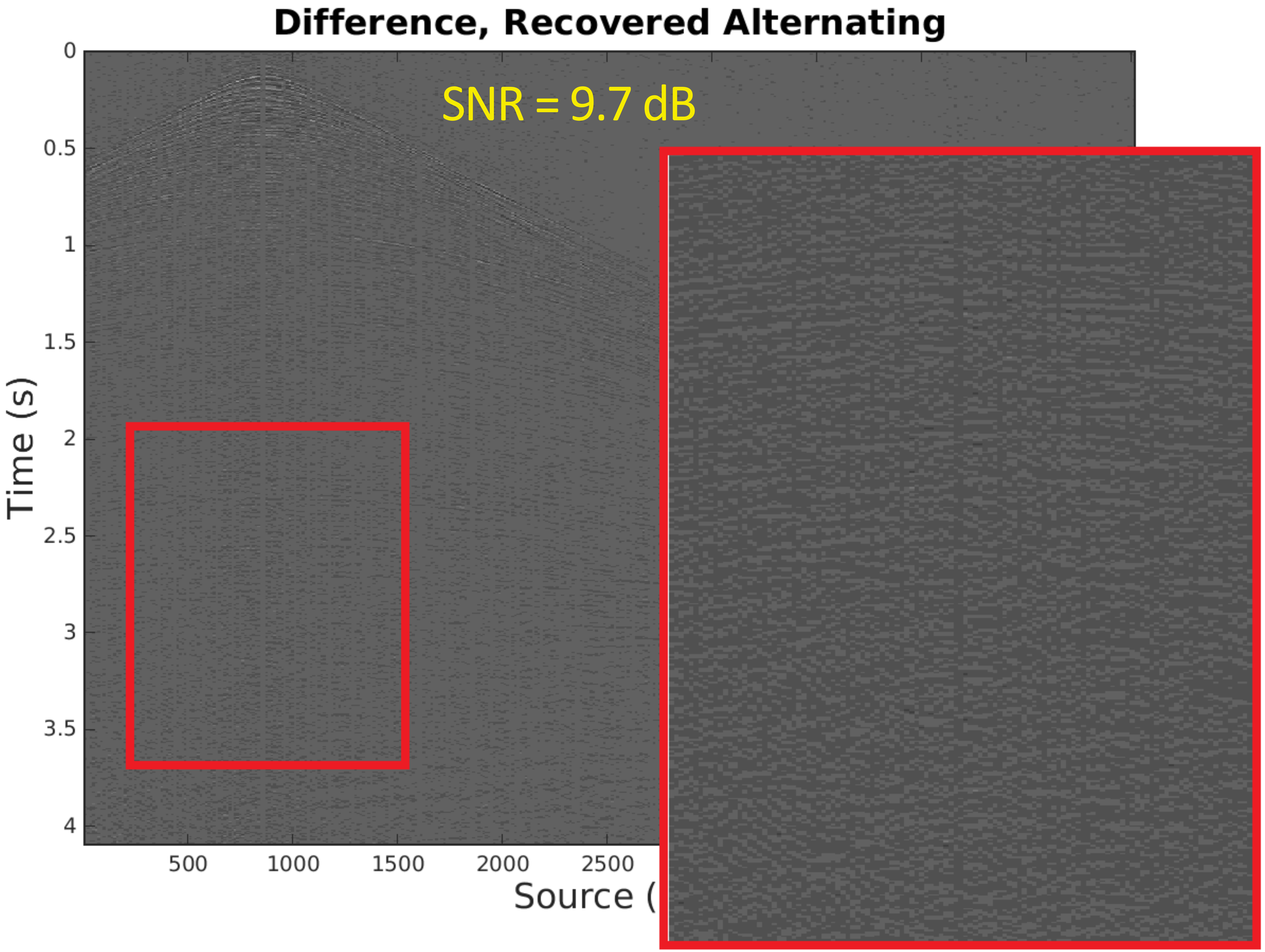

Difference, Recovered Alternating
SNR = 9.7 dB
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (s)
500
1000
1500
2000
2500
Source (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Time (s)
0
0.5
1
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4
500
1000
1500
2000
2500
Source (

Experiments: Gulf Of Mexico 7Hz Slice

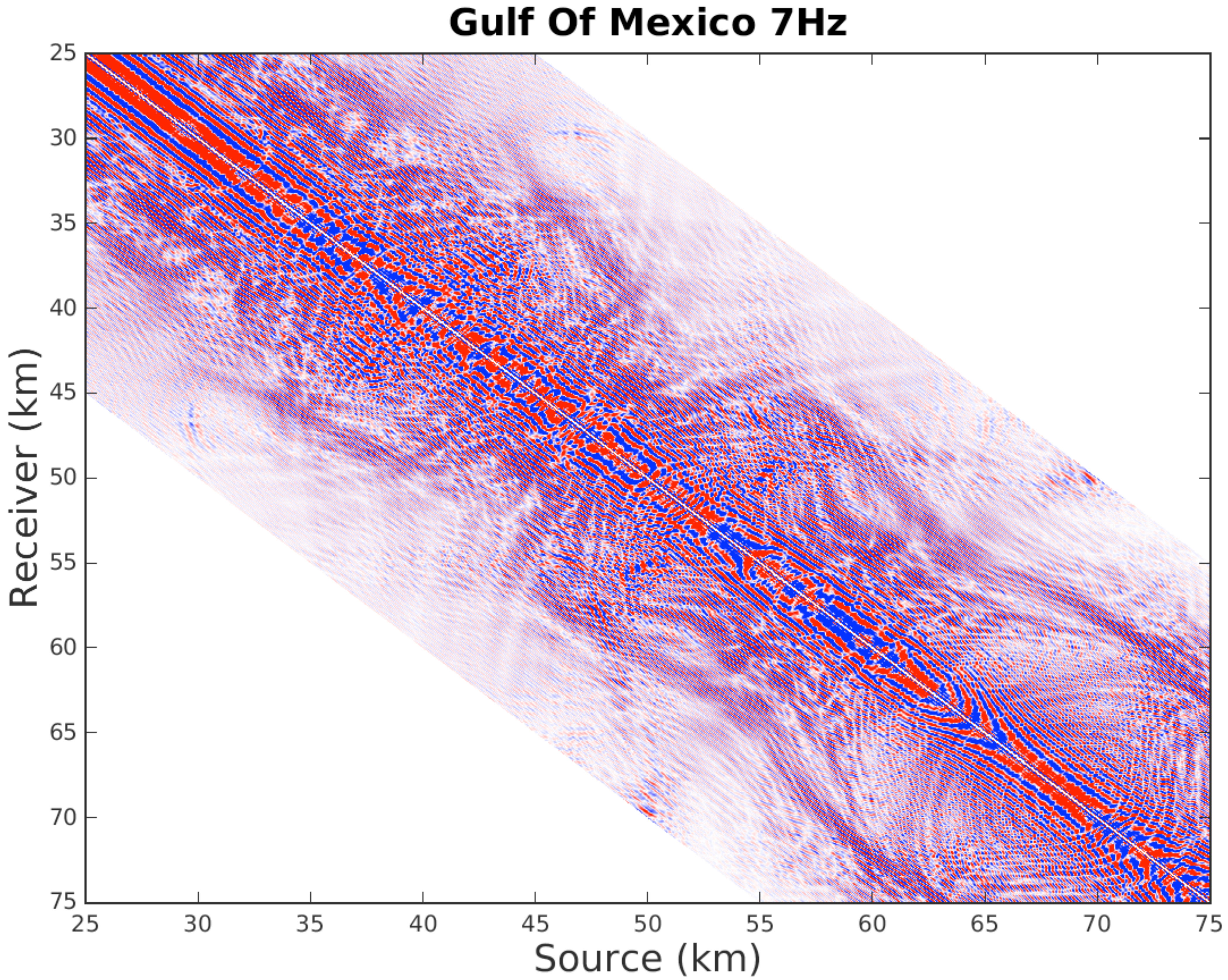
4001 x 4001 matrix (factorization parameter = 80)

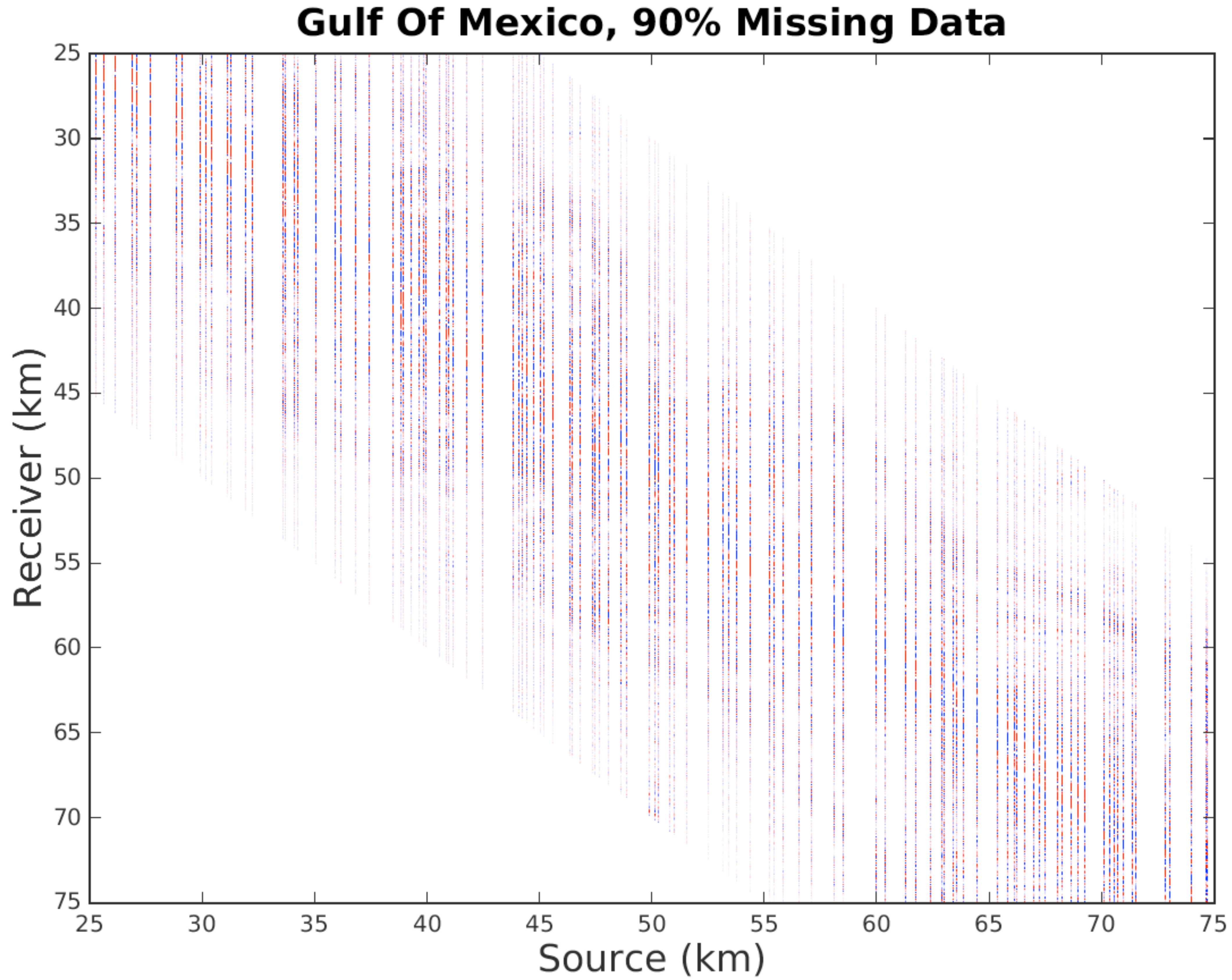
- 90% missing traces
- JIttered subsampling
- solve with SPGL1

Compare: Alternating vs Non Alternating

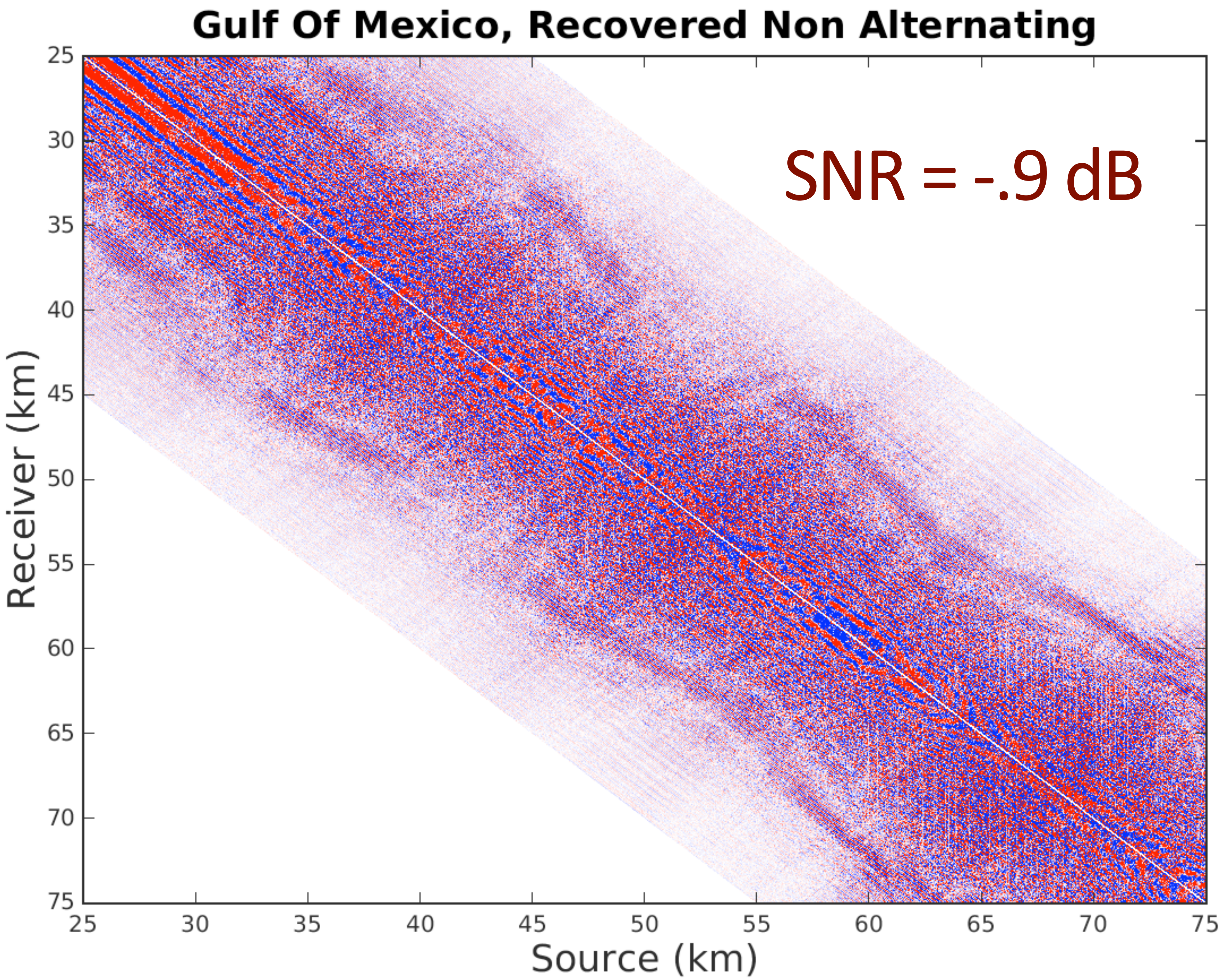
- 10 alternations, 15 iterations per alternation
- 300 iterations total for both

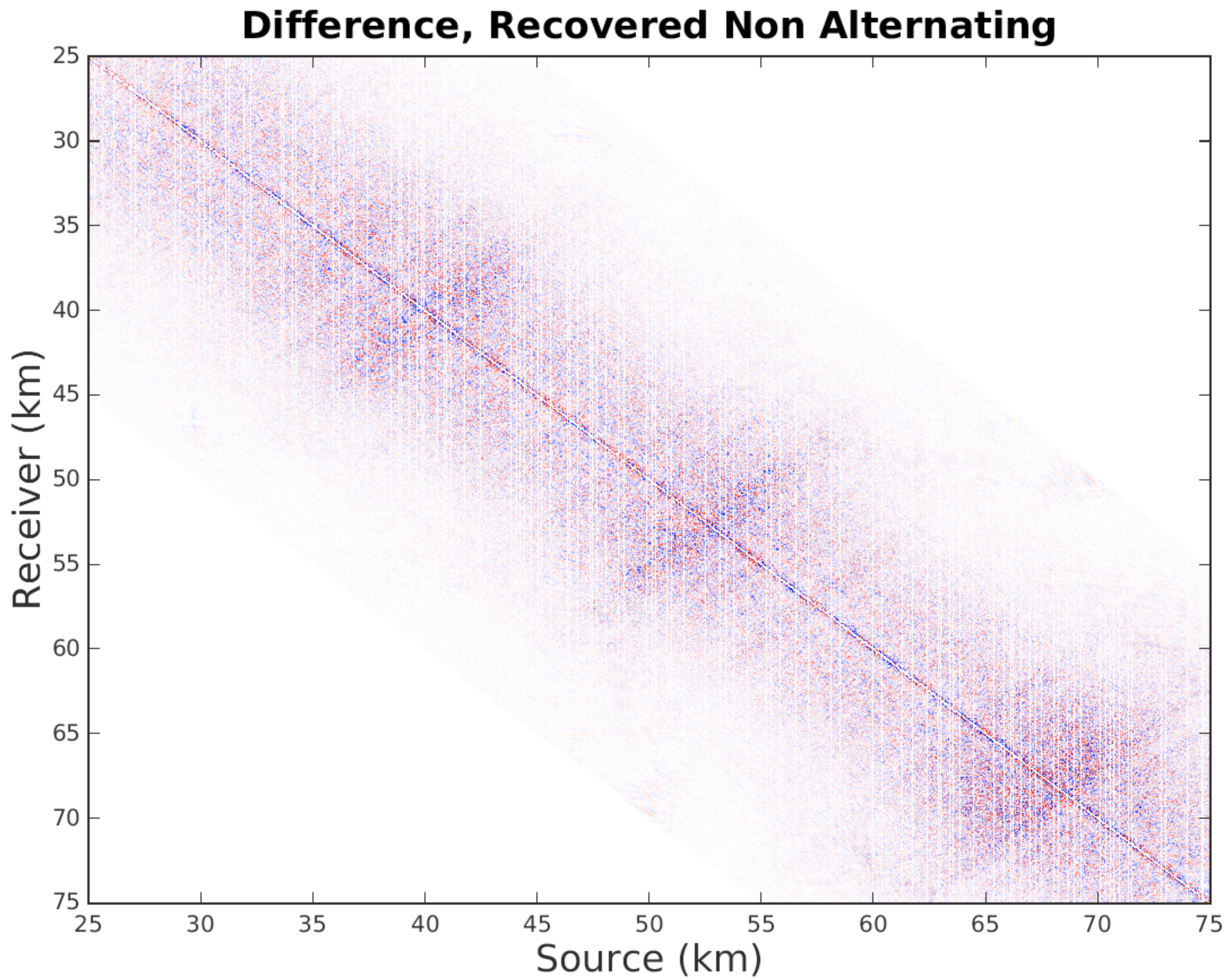
Experiments: Gulf Of Mexico 7Hz (90% missing)



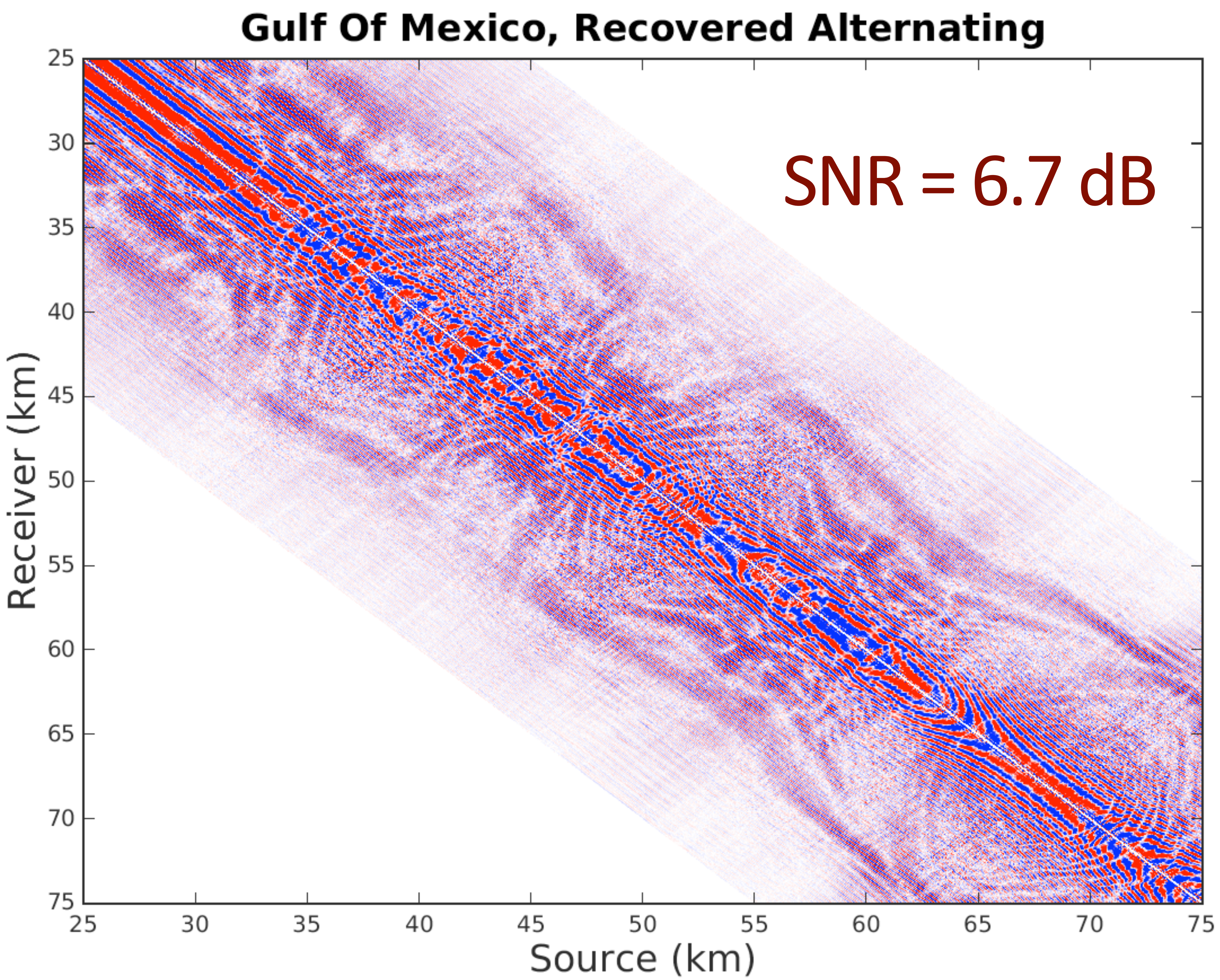


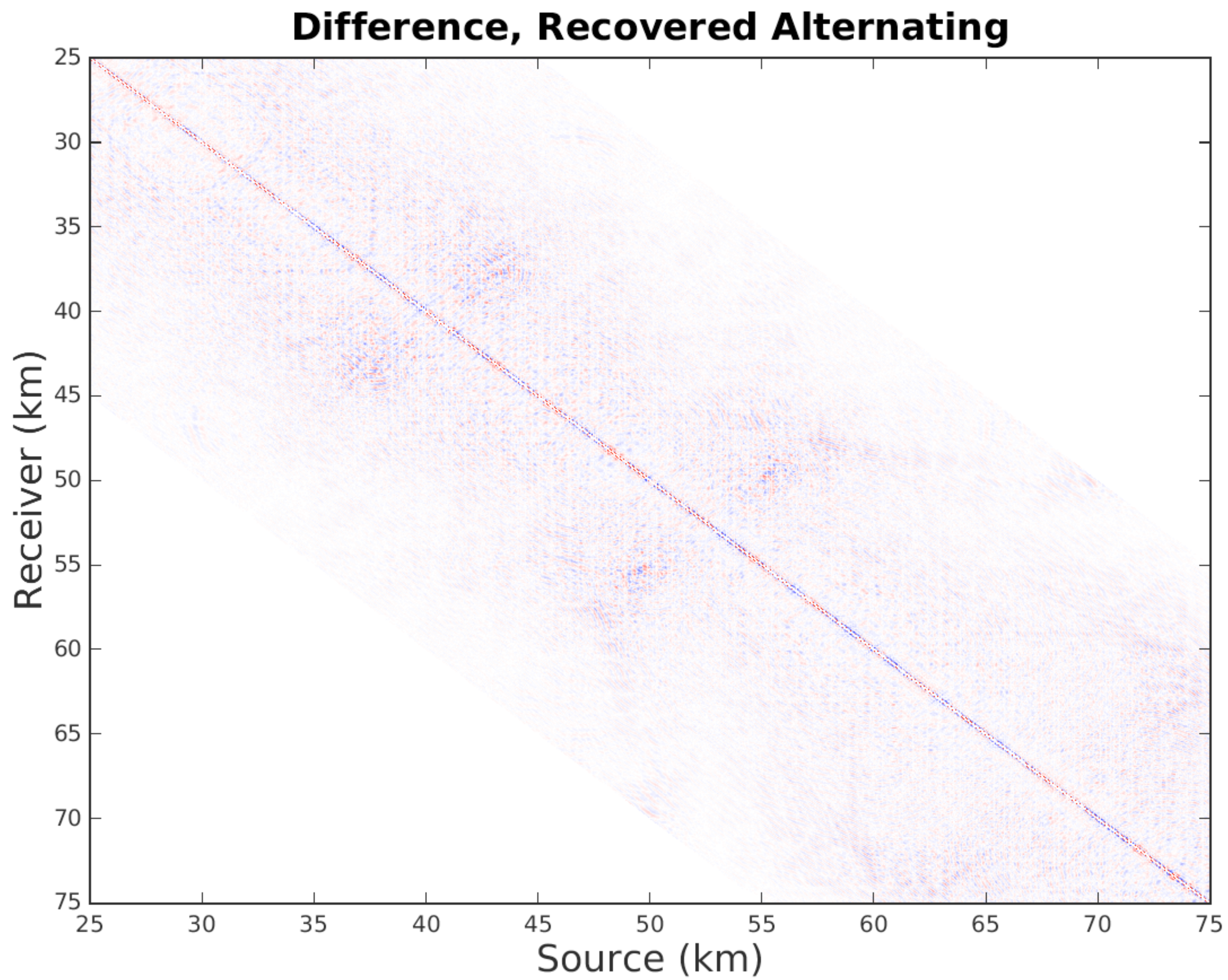
Experiments: Recovered Non Alternating



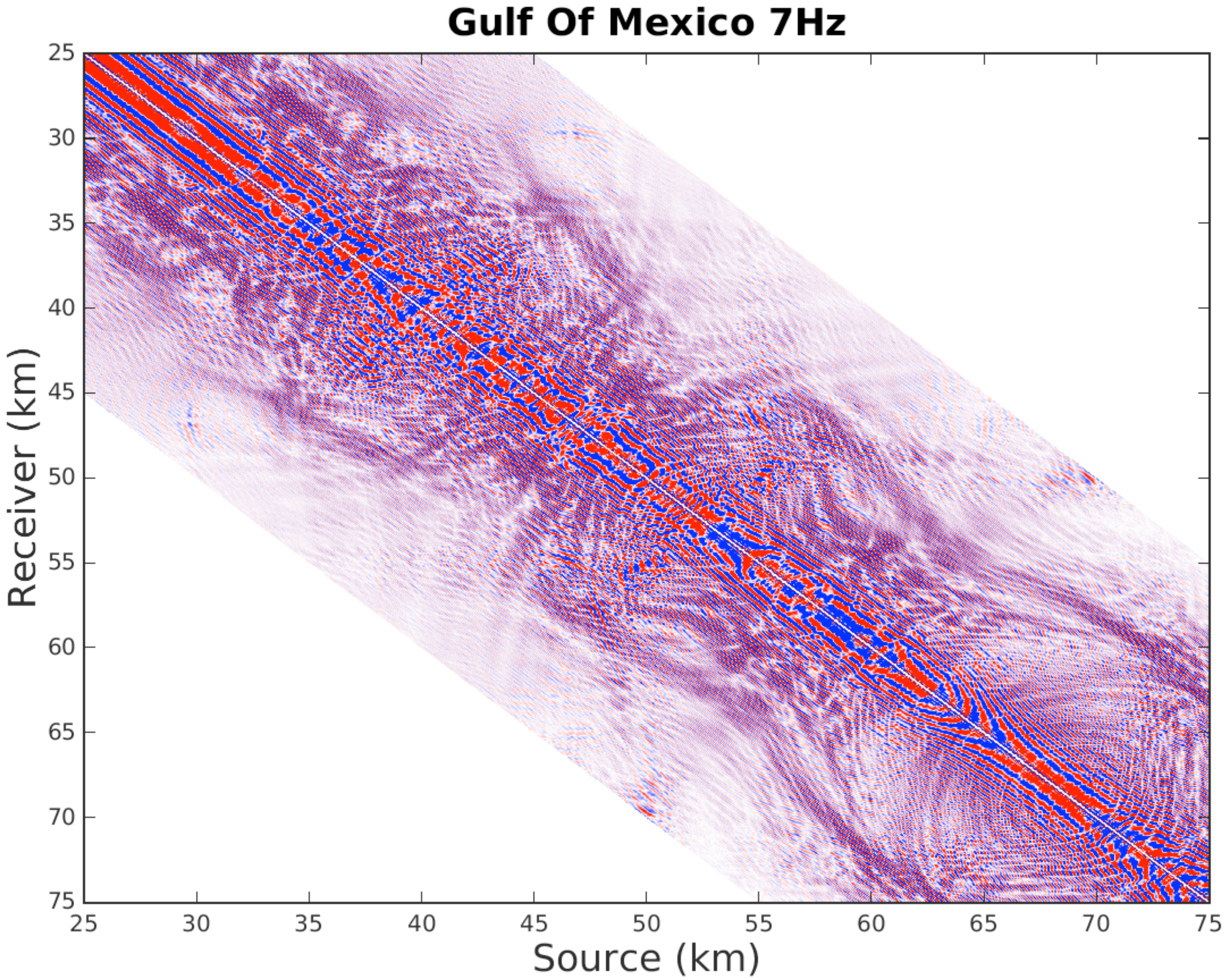


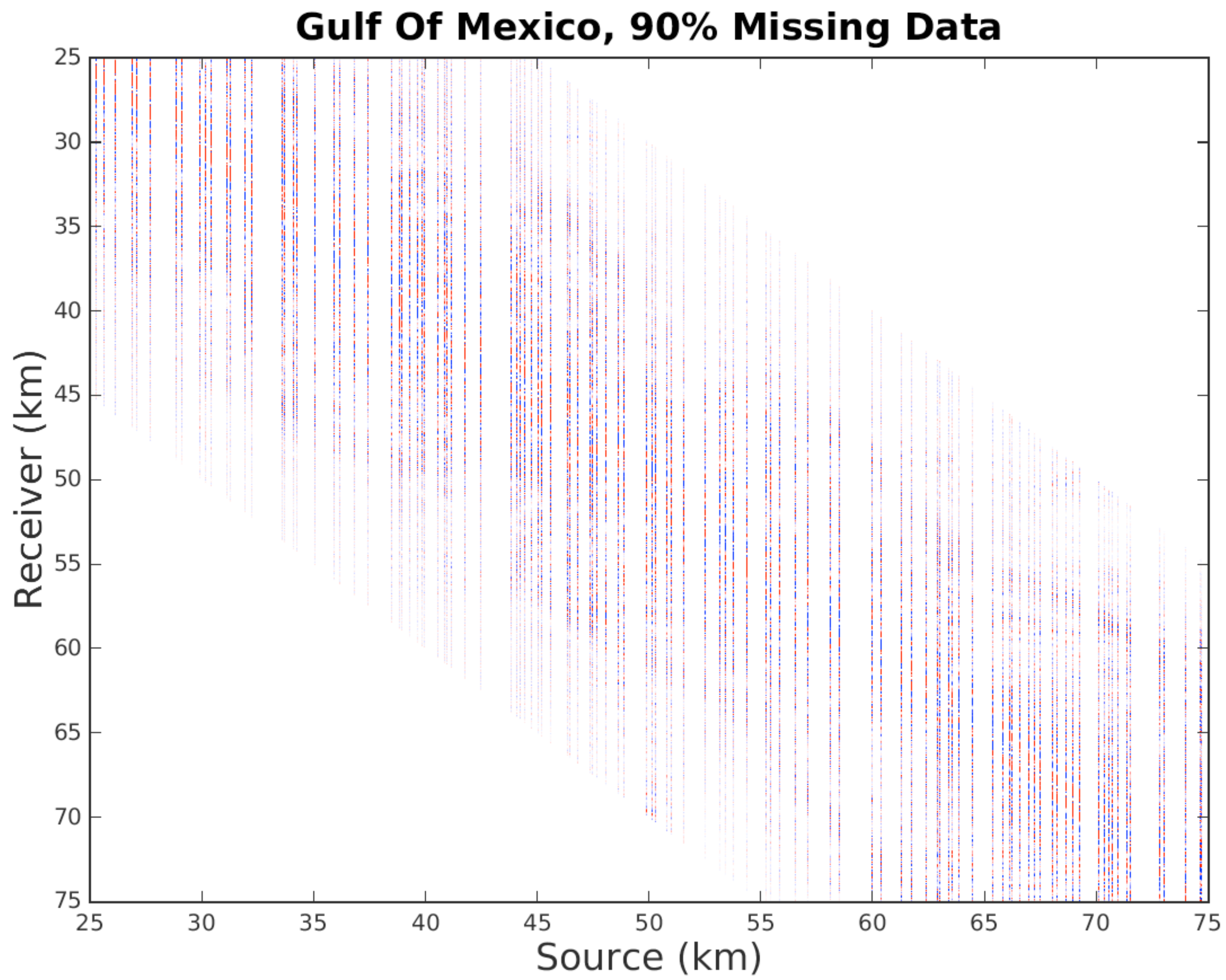
Experiments: Recovered Alternating





Experiments: Gulf Of Mexico 7Hz (90% missing)





Experiments: Regularization + Interpolation

Gulf Of Suez 10 hz slice - 354 x 354 matrix

- irregular data
- varying % missing sources
- Jittered subsampling

Compare: Alternating vs Non Alternating

- 6 alternations, 15 iterations per alternation
- 180 iterations total for both

Experiments: Regularization + Interpolation

solve:

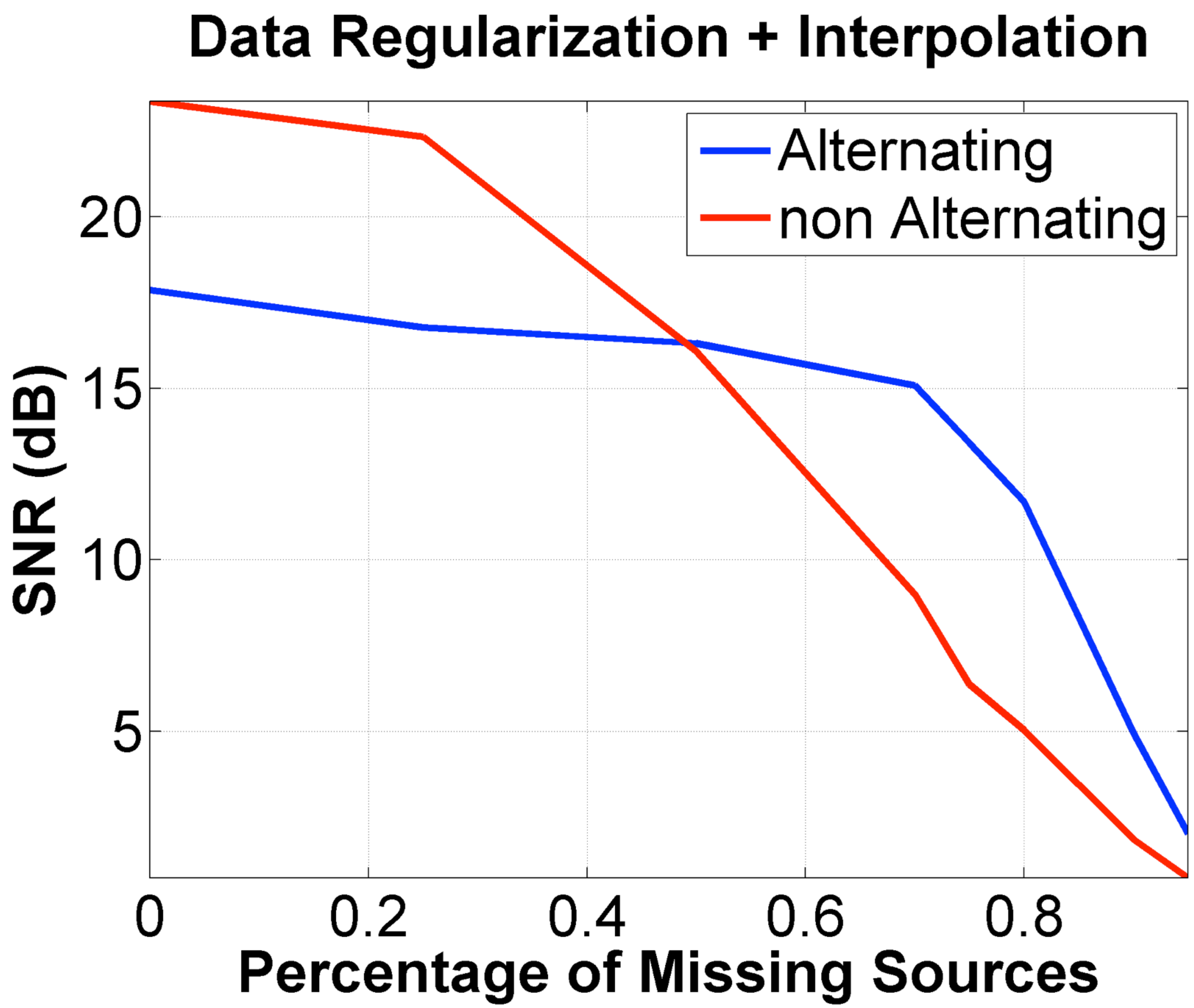
$$\min_{\mathbf{L}\in\mathbb{R}^{n\times r},\mathbf{R}\in\mathbb{R}^{m\times r}} \frac{1}{2}(||\mathbf{L}||_F^2 + ||\mathbf{R}||_F^2) \;\; \text{s.t.} \;\; ||\mathcal{A}(\mathbf{L}\mathbf{R}^H) - b||_F^2 \leq \sigma$$

$$\mathcal{A} = \mathbf{RMN}\mathcal{S}^H$$

where

$\mathbf{R}$: restriction operator
$\mathbf{M}$: measurement operator
$\mathbf{N}$: regularization operator
$\mathcal{S}^H$: transform operator

Experiments: Regularization + Interpolation



Computational Cost
with and without SVD

Percentage missing sources		50.0%		75.0%	
σ		0.1	0.1	0.1	0.1
Matrix completion w/ SVD	SNR (dB)	17.3	18.3	11.6	11.5
	time (sec)	**812**	**937**	**790**	**765**
Matrix completion w/o SVD	SNR (dB)	17.6	18.4	12.6	13.3
	time (sec)	**8**	**10**	**8**	**7**

Computational Cost
Matrix Completion vs Curvelet-based methods

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	time (sec)	**8**	**10**	**8**	**7**
Curvelet-based sparsity promotion	SNR (dB)	17.4	18.6	12.5	12.8
	time (sec)	**879**	**989**	**817**	**1010**

Conclusions

- Alternating Optimization can better handle complicated cases
  - Highly sub sampled data (missing > 80%)
  - Complex data (high frequencies)

- Alternating Optimization does not increase time complexity

Future Work

- Analysis of method
  - How many alternations are needed?

- Consider method for other applications
  - Tensor Completion
  - Source Separation
  - Parallel Matrix Completion

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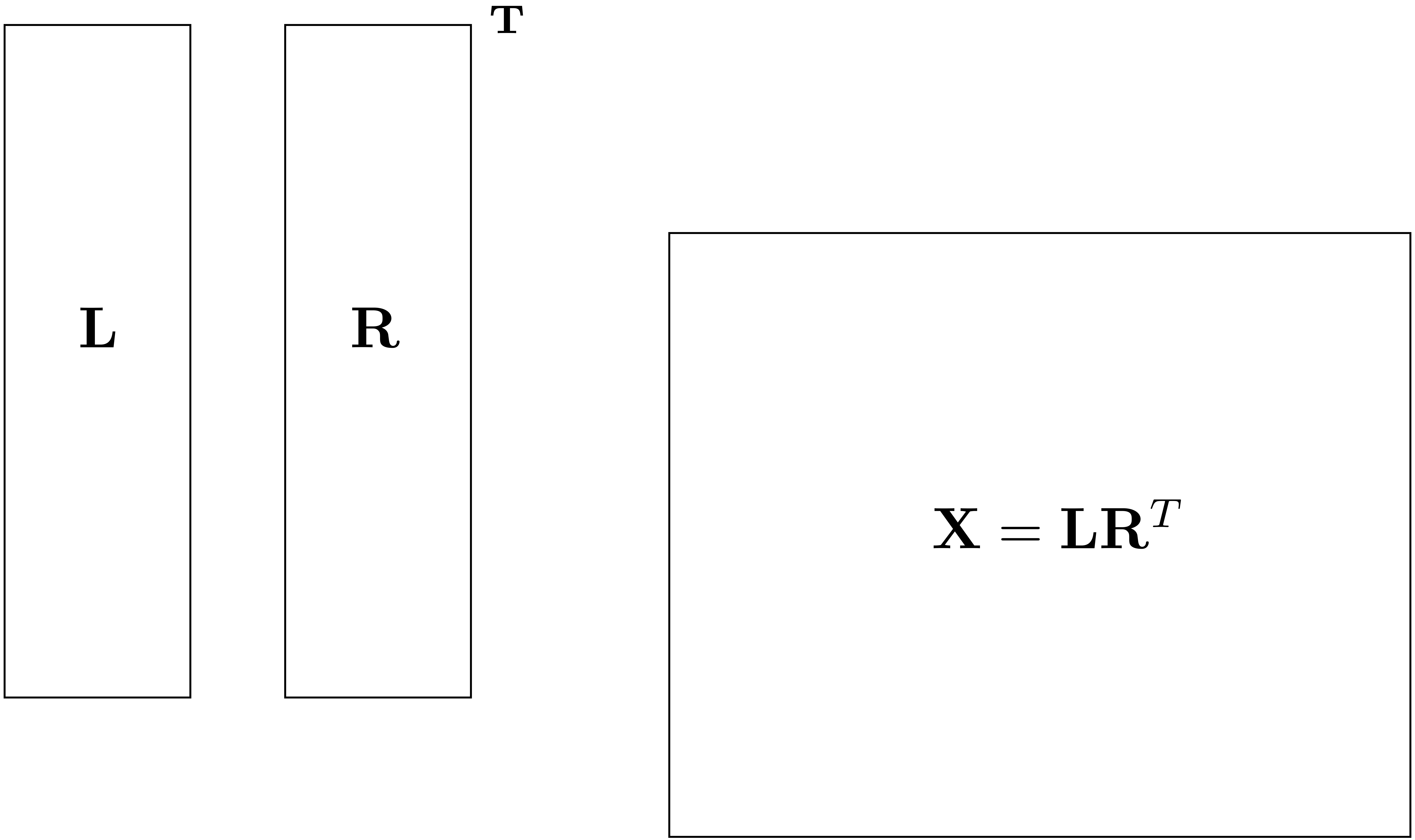
LR parallel matrix multiplication



LR parallel matrix multiplication

	$\mathbf{L}$	$\mathbf{R}^{\mathbf{T}}$
Worker 1	$\mathbf{L_1}$	$\mathbf{R_1}$
Worker 2	$\mathbf{L_2}$	$\mathbf{R_2}$
Worker 3	$\mathbf{L_3}$	$\mathbf{R_3}$
Worker 4	$\mathbf{L_4}$	$\mathbf{R_4}$
Worker 5	$\mathbf{L_5}$	$\mathbf{R_5}$

$\mathbf{L_1R_1^T}$	$\mathbf{L_1R_2^T}$	$\mathbf{L_1R_3^T}$	$\mathbf{L_1R_4^T}$	$\mathbf{L_1R_5^T}$
$\mathbf{L_2R_1^T}$	$\mathbf{L_2R_2^T}$	$\mathbf{L_2R_3^T}$	$\mathbf{L_2R_4^T}$	$\mathbf{L_2R_5^T}$
$\mathbf{L_3R_1^T}$	$\mathbf{L_3R_2^T}$	$\mathbf{L_3R_3^T}$	$\mathbf{L_3R_4^T}$	$\mathbf{L_3R_5^T}$
$\mathbf{L_4R_1^T}$	$\mathbf{L_4R_2^T}$	$\mathbf{L_4R_3^T}$	$\mathbf{L_4R_4^T}$	$\mathbf{L_4R_5^T}$
$\mathbf{L_5R_1^T}$	$\mathbf{L_5R_2^T}$	$\mathbf{L_5R_3^T}$	$\mathbf{L_5R_4^T}$	$\mathbf{L_5R_5^T}$

Acknowledgements

Software Release Coming Soon

Thank you for your attention

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