Off-The-Grid Matrix Completion

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Motivation

- acquisition challenges
    - missing data
    - irregular acquisition grid

- fully sampled data at regular grid
    - EPSI, FWI, RTM

- exploit *low-rank* structure of seismic data
    - *SVD-free* matrix factorization

Outline

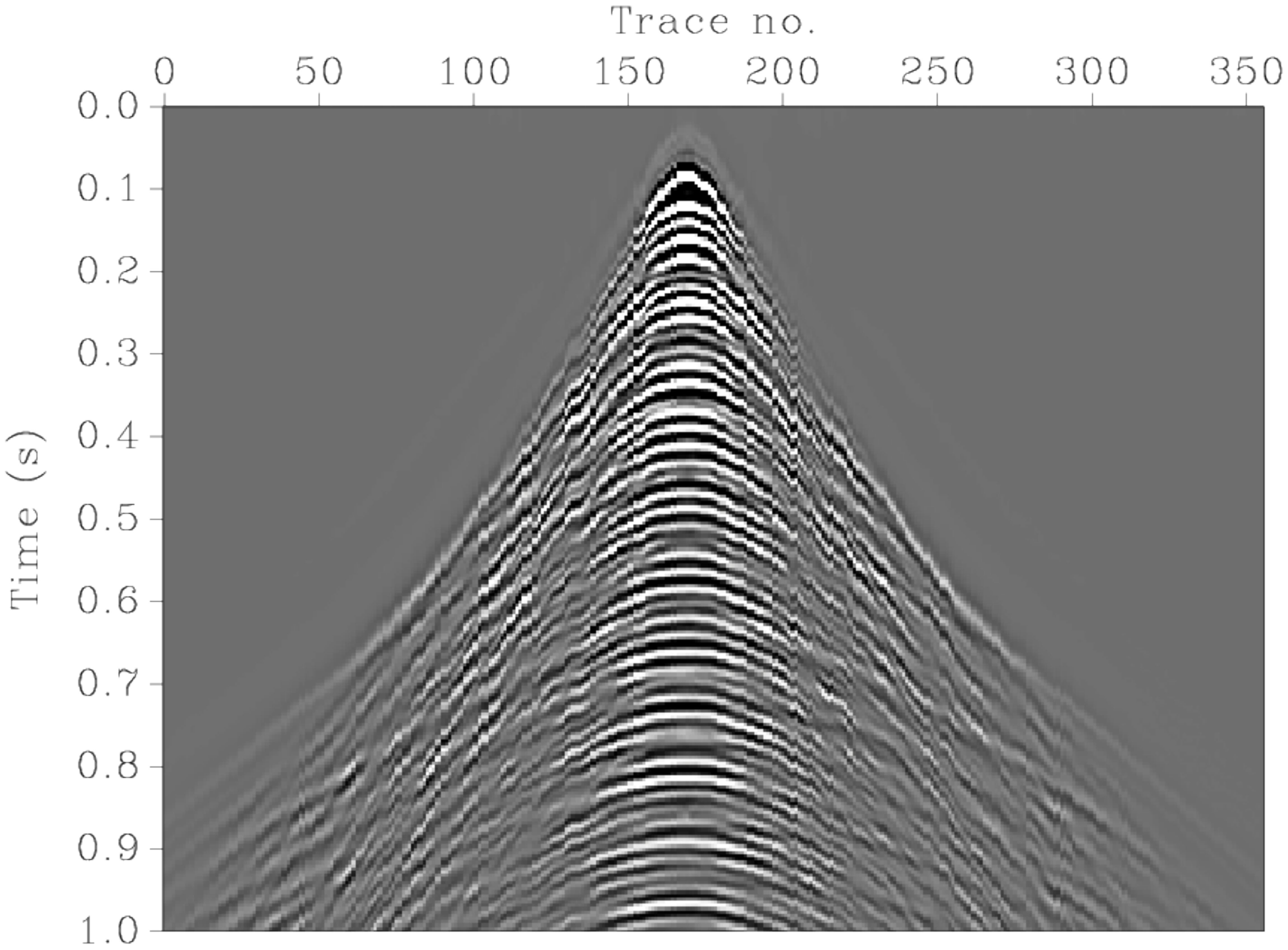
- Regularization
  - – is binning the right approach?
  - – exploit low rank structure

- Can we quantify acquisition design?

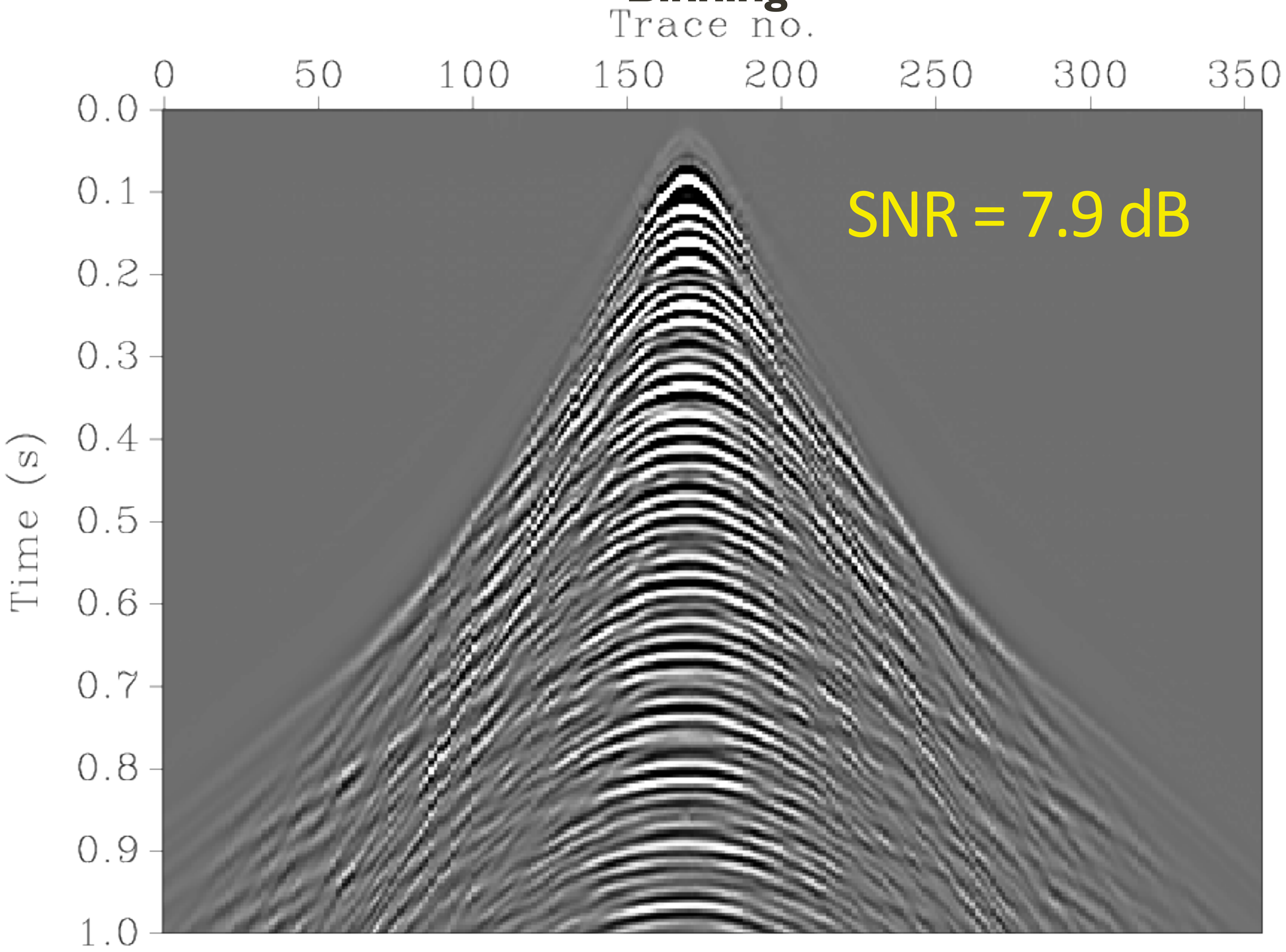
Motivation

Binning

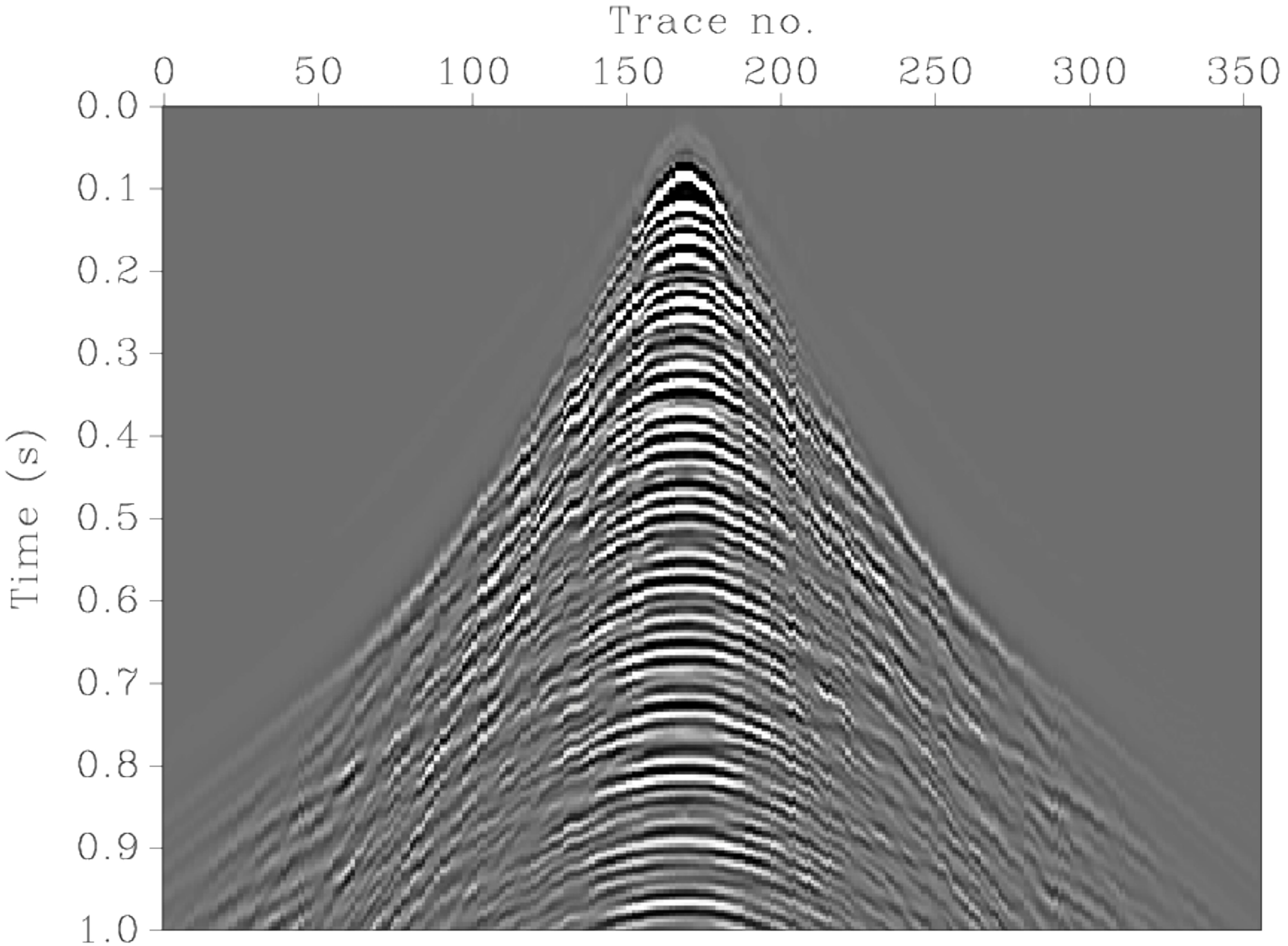
Observed irregular data

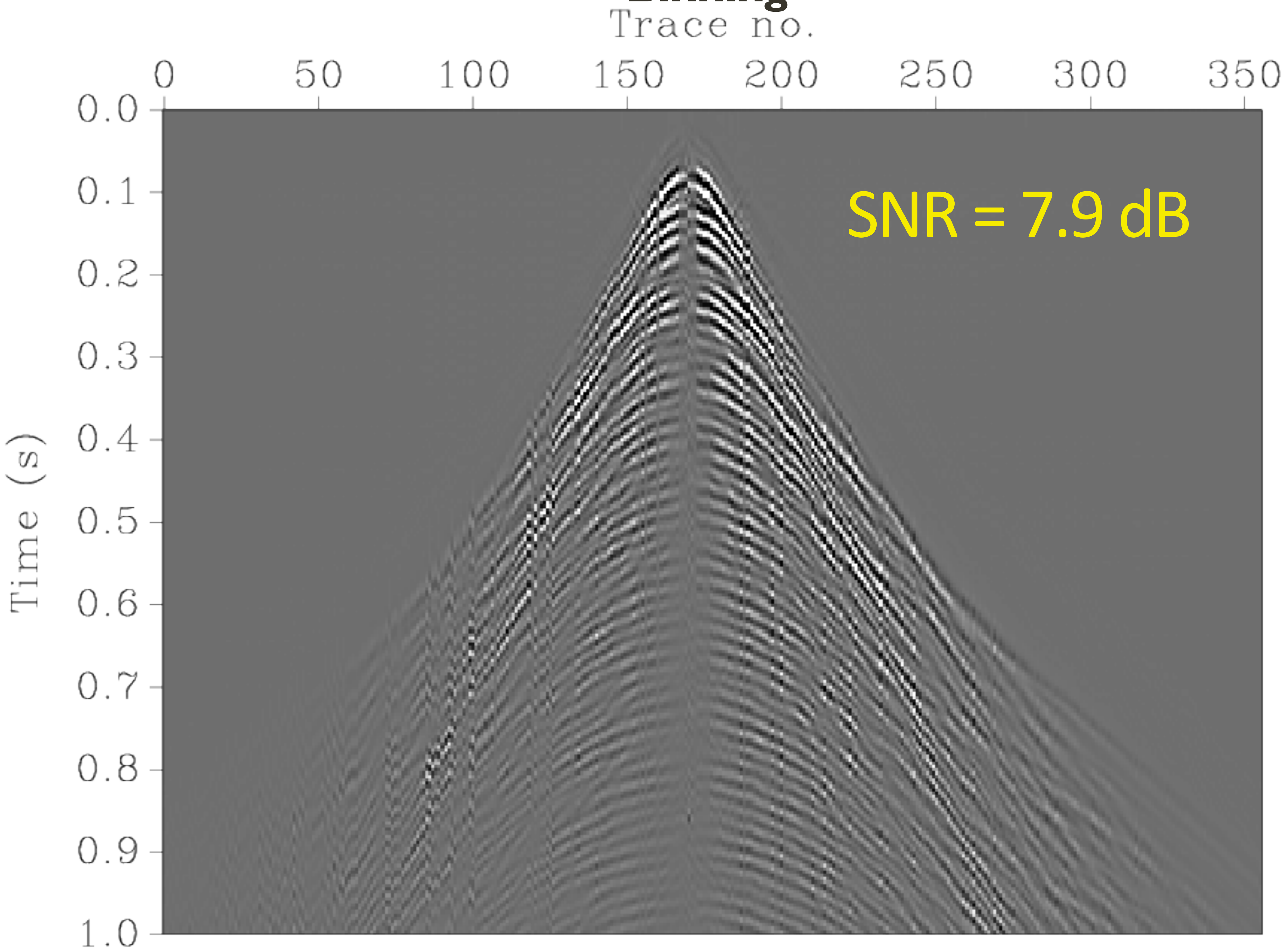


Recovery
Binning



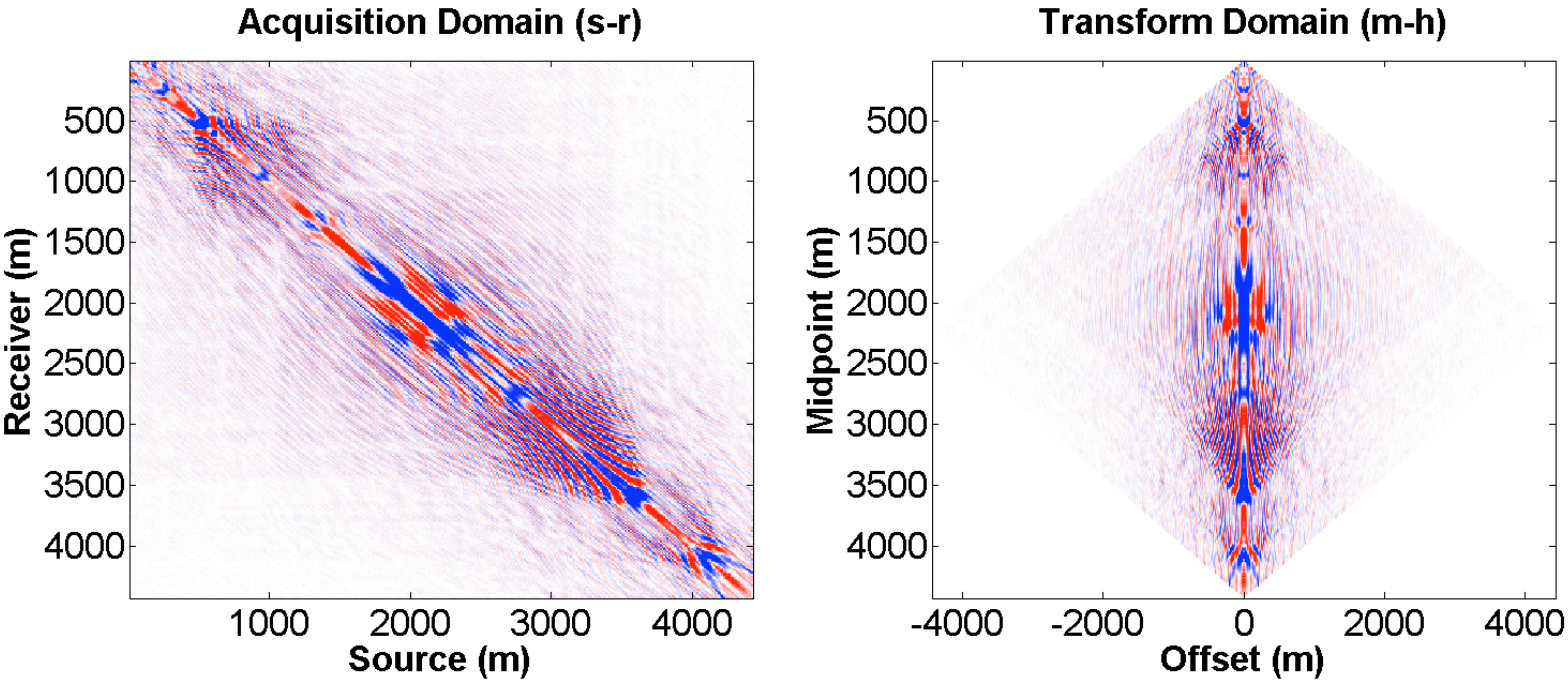
Motivation
Binning


Observed irregular data


Residual Binning

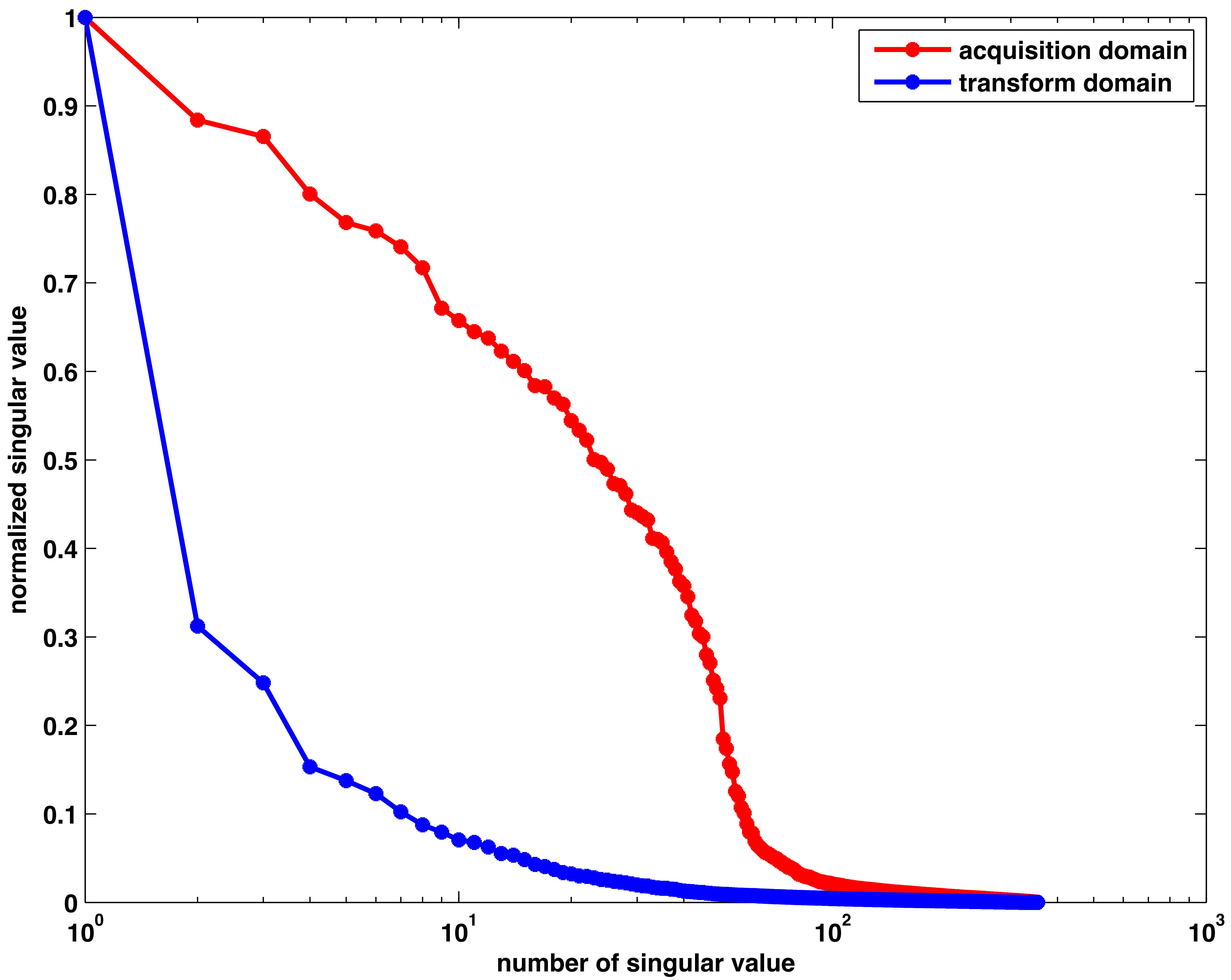
Low-rank structure

2-D acquisition

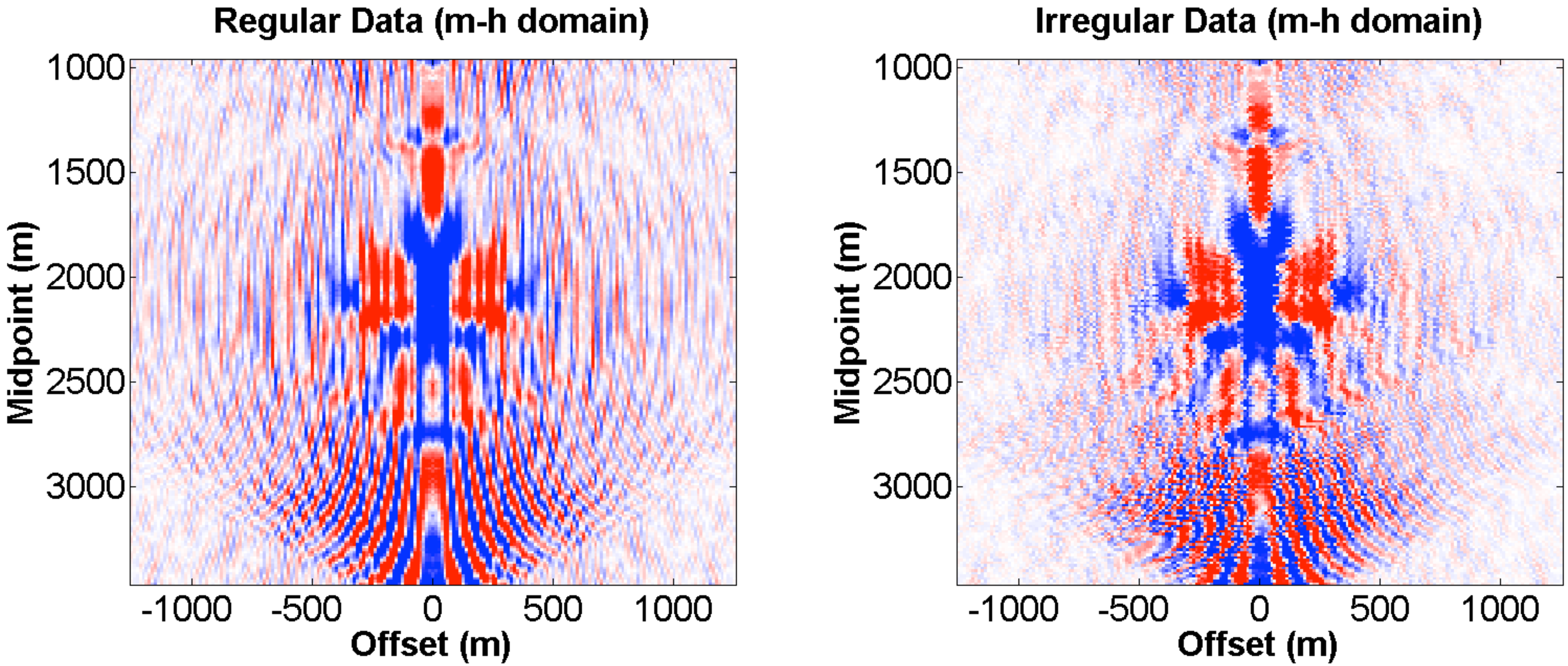


Singular value decay

s-r domain vs m-h domain



Low-rank structure
midpoint-offset domain

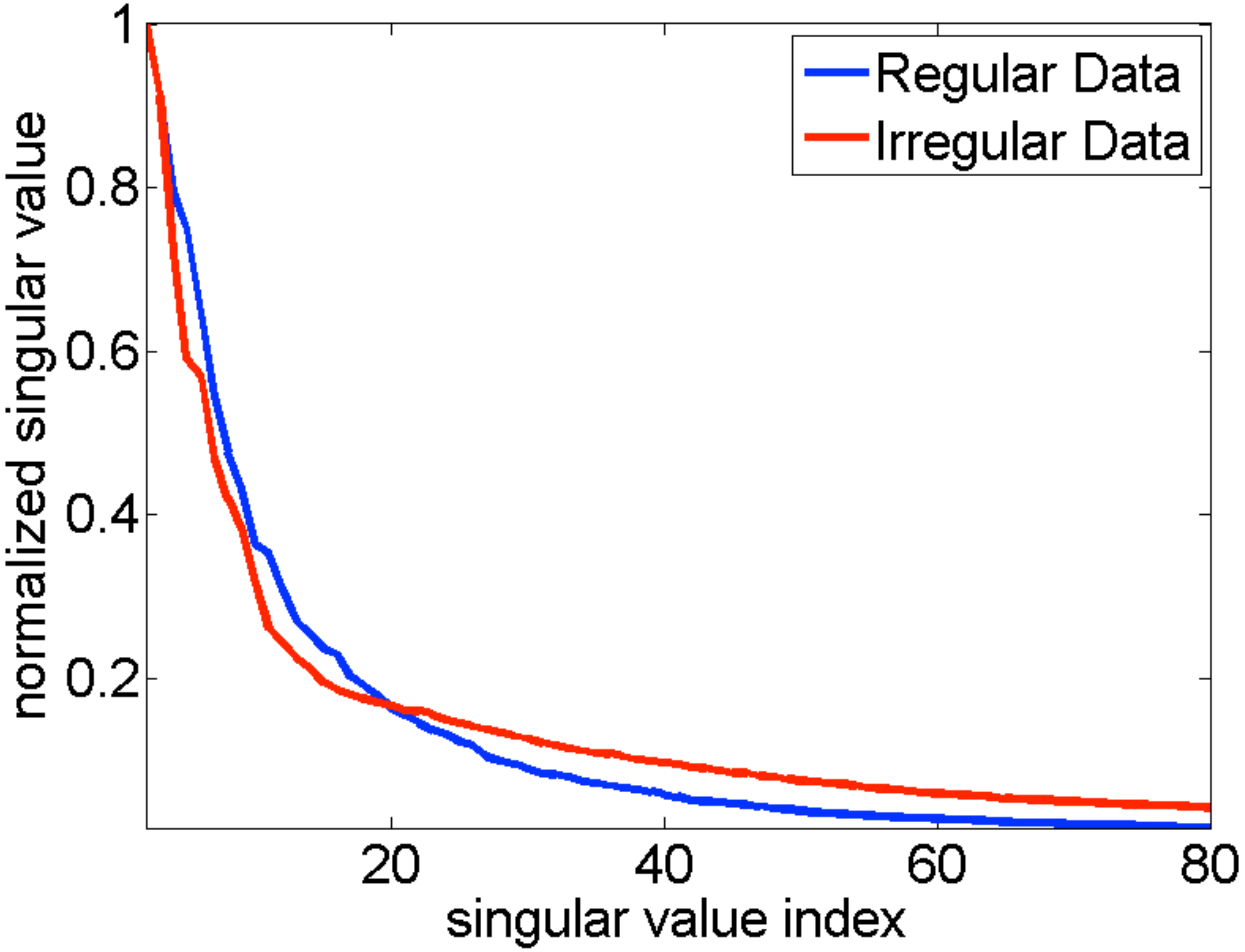


Regular Data (m-h domain)

Irregular Data (m-h domain)

Singular value decay

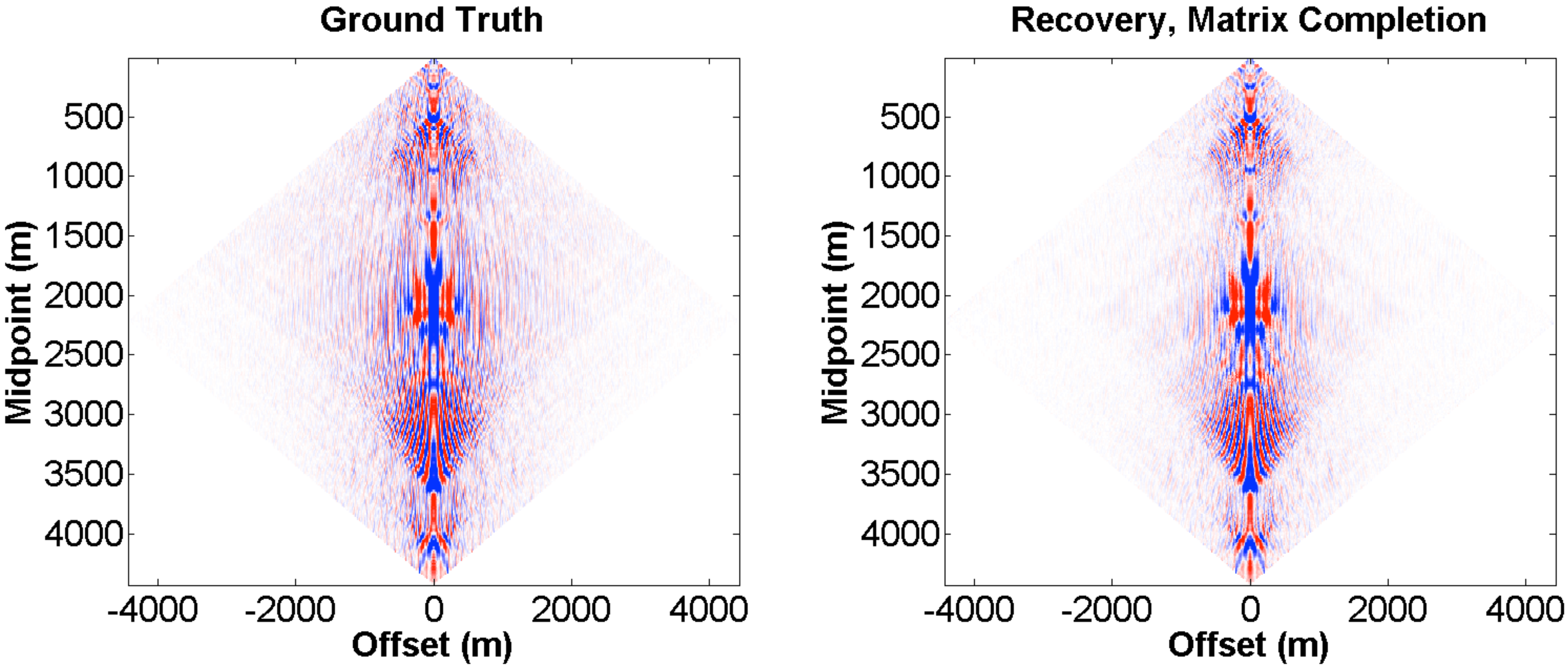
regular vs irregular



Irregularity affects decay of singular values

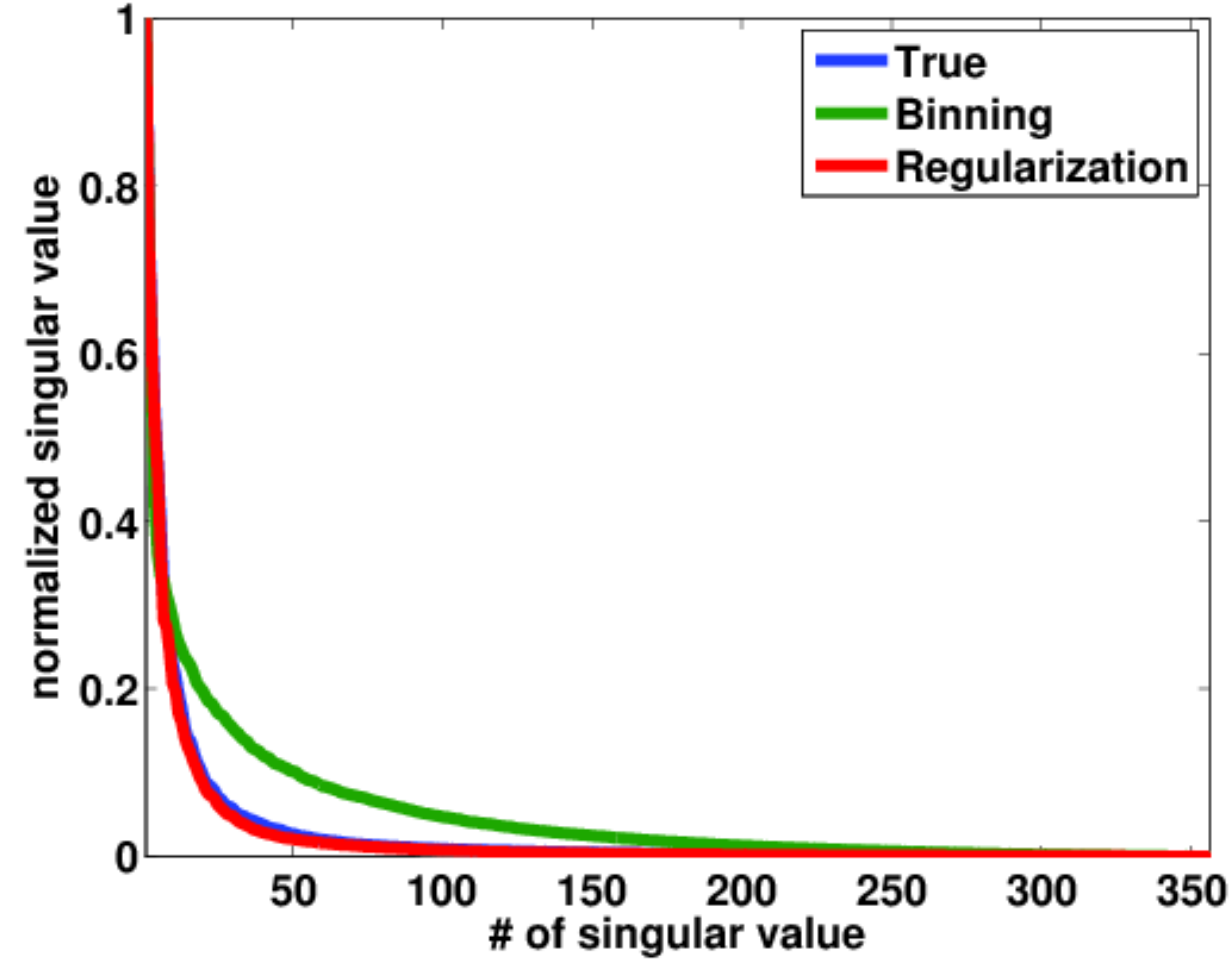
Use Nuclear Norm Minimization Techniques

Regularization
matrix completion, midpoint-offset domain



Singular value decay

regularization v/s binning



[Recht et. al. 2010]

Methodology

- want to minimize nuclear norm

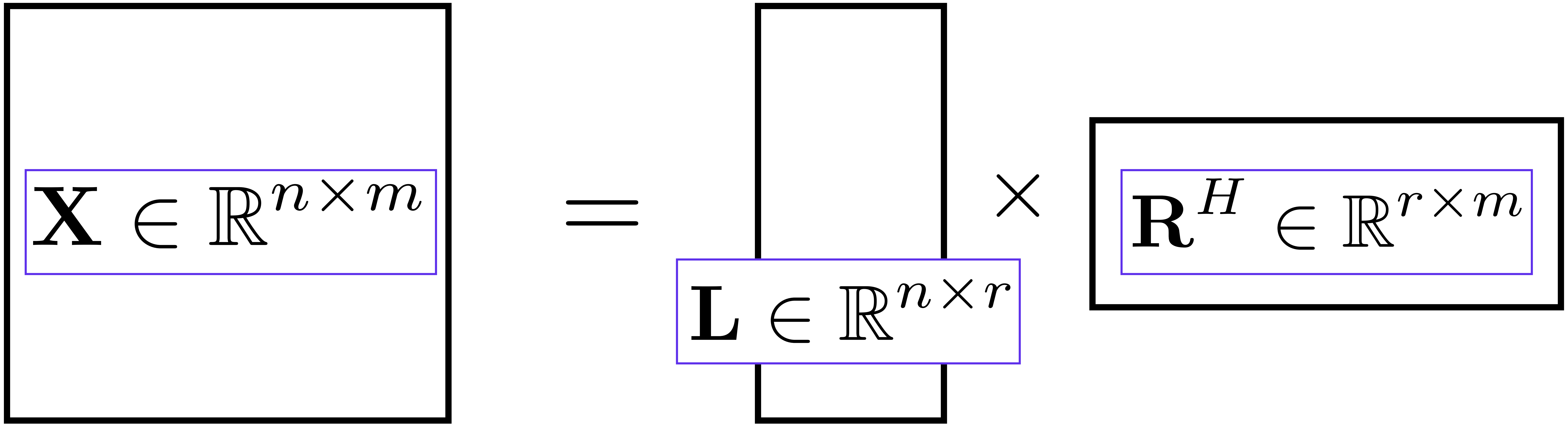
$$(BPDN_\sigma) \qquad \min_{\mathbf{X}} ||\mathbf{X}||_* \quad \text{s.t.} \; ||\mathcal{A}(\mathbf{X}) - \mathbf{b}||_2^2 \leq \sigma$$

where

$$\|\mathbf{X}\|_* = \sum_{i=1}^{m} \lambda_i = \|\lambda\|_1$$

where λ_i are the *singular* values

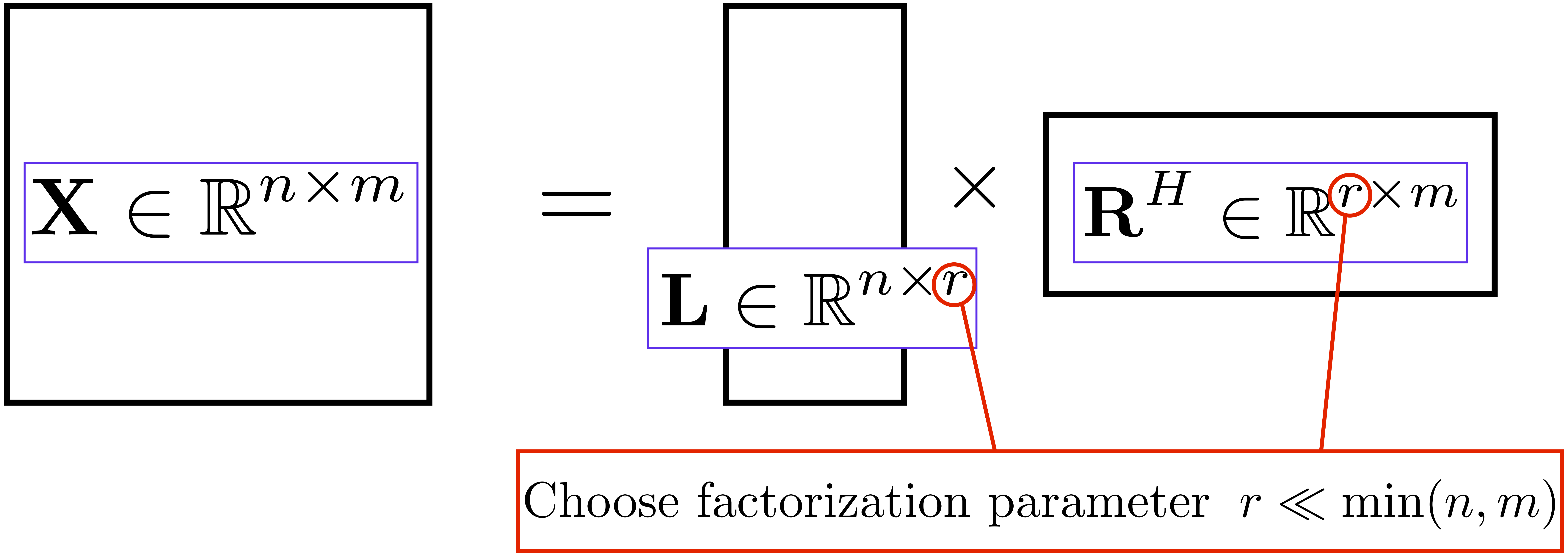
Factorized formulation



$$\mathbf{X} \in \mathbb{R}^{n\times m} = \mathbf{L} \in \mathbb{R}^{n\times r} \times \mathbf{R}^{H} \in \mathbb{R}^{r\times m}$$

$$\mathbf{X} = \mathbf{L}\mathbf{R}^{\mathbf{H}}$$

Factorized formulation



$\mathbf{X} \in \mathbb{R}^{n \times m}$ $=$ $\mathbf{L} \in \mathbb{R}^{n \times r}$ $\times$ $\mathbf{R}^H \in \mathbb{R}^{r \times m}$

Choose factorization parameter $r \ll \min(n, m)$

$$\mathbf{X} = \mathbf{L}\mathbf{R}^{\mathbf{H}}$$

[Rennie and Srebro 2005]

Factorized formulation

$$\mathbf{X} = \mathbf{L}\mathbf{R}^H$$

- Nuclear norm is bounded by

$$||\mathbf{X}||_* \leq \frac{\mathbf{1}}{\mathbf{2}}(||\mathbf{L}||^{\mathbf{2}}_{\mathbf{F}} + ||\mathbf{R}||^{\mathbf{2}}_{\mathbf{F}})$$

where $\|\cdot\|_F^2$ is sum of squares of all entries

Nuclear Norm Minimization- Factorized Form

- want to solve

$$\min_{\mathbf{L}\in\mathbb{R}^{n\times r},\mathbf{R}\in\mathbb{R}^{m\times r}} \frac{1}{2}(||\mathbf{L}||_F^2 + ||\mathbf{R}||_F^2) \ \text{ s.t. } \ ||\mathcal{A}(\mathbf{L}\mathbf{R}^H) - b||_F^2 \leq \sigma$$

with sampling operator:

$$\mathcal{A} = \mathbf{R}\mathbf{M}\mathcal{S}^H$$

where

$\mathbf{R}$: restriction operator
$\mathbf{M}$: measurement operator
$\mathcal{S}^H$: transform operator

Nuclear Norm Minimization- Factorized Form

- want to solve

$$\min_{\mathbf{L}\in\mathbb{R}^{n\times r},\mathbf{R}\in\mathbb{R}^{m\times r}} \frac{1}{2}(||\mathbf{L}||_F^2 + ||\mathbf{R}||_F^2) \;\; \text{s.t.} \;\; ||\mathcal{A}(\mathbf{L}\mathbf{R}^H) - b||_F^2 \leq \sigma$$

with sampling operator:

$$\boxed{\mathcal{A} = \mathbf{RM}\mathcal{S}^H}$$

Does not incorporate unstructured grid

where

$\mathbf{R}$: restriction operator
$\mathbf{M}$: measurement operator
$\mathcal{S}^H$: transform operator

Methodology
matrix completion

- introduce regularization operator

$$\mathbf{N} : \mathbb{C}^{n\times m} \rightarrow \mathbb{C}^{n\times m} \quad \text{so that} \quad \mathbf{N}(\mathbf{X}_r) = (\mathbf{X}_{ir})$$

defined as:

$$\mathbf{N} := (\mathcal{NF})(\mathcal{F})^H$$

where

$\mathcal{F}$: Fourier Transform
$\mathcal{NF}$: Nonuniform Fourier Transform (incorporate irregular grid)

Methodology

matrix completion

$$\mathbf{N} : \mathbb{C}^{n \times m} \rightarrow \mathbb{C}^{n \times m} \quad \text{so that} \quad \mathbf{N}(\mathbf{X}_r) = (\mathbf{X}_{ir})$$

redefine sampling operator as:

$$\mathcal{A} = \mathbf{RMN}\mathcal{S}^H$$

where

$\mathbf{R}$: restriction operator
$\mathbf{M}$: measurement operator
$\mathbf{N}$: regularization operator
$\mathcal{S}^H$: transform operator

Theorem

matrix completion

Let $\mathbf{X_r} \in \mathbb{C}^{n\times m}, \mathbf{b} = \mathbf{RM}(\mathbf{X}_{ir}) + e$ with $\|e\| \le \eta$, and $||\mathbf{N}(\mathbf{X_r}) - \mathbf{X_{ir}}||_\mathbf{F} \le \epsilon.$

Let $\tilde{\mathbf{X}}$ be the solution of $\mathrm{BPDN}_{\epsilon+\sigma},$ then

$$||\mathbf{X}_r - \tilde{\mathbf{X}}||_F \le \underbrace{\frac{C_1}{\sqrt{k}} \sum_{i=k+1}^{l} \sigma_i(\mathbf{X}_r)}_{\text{rank approx error}} + \underbrace{C_2\epsilon}_{\text{regularization operator error}} + \underbrace{C_2\eta}_{\text{noise}}$$

where

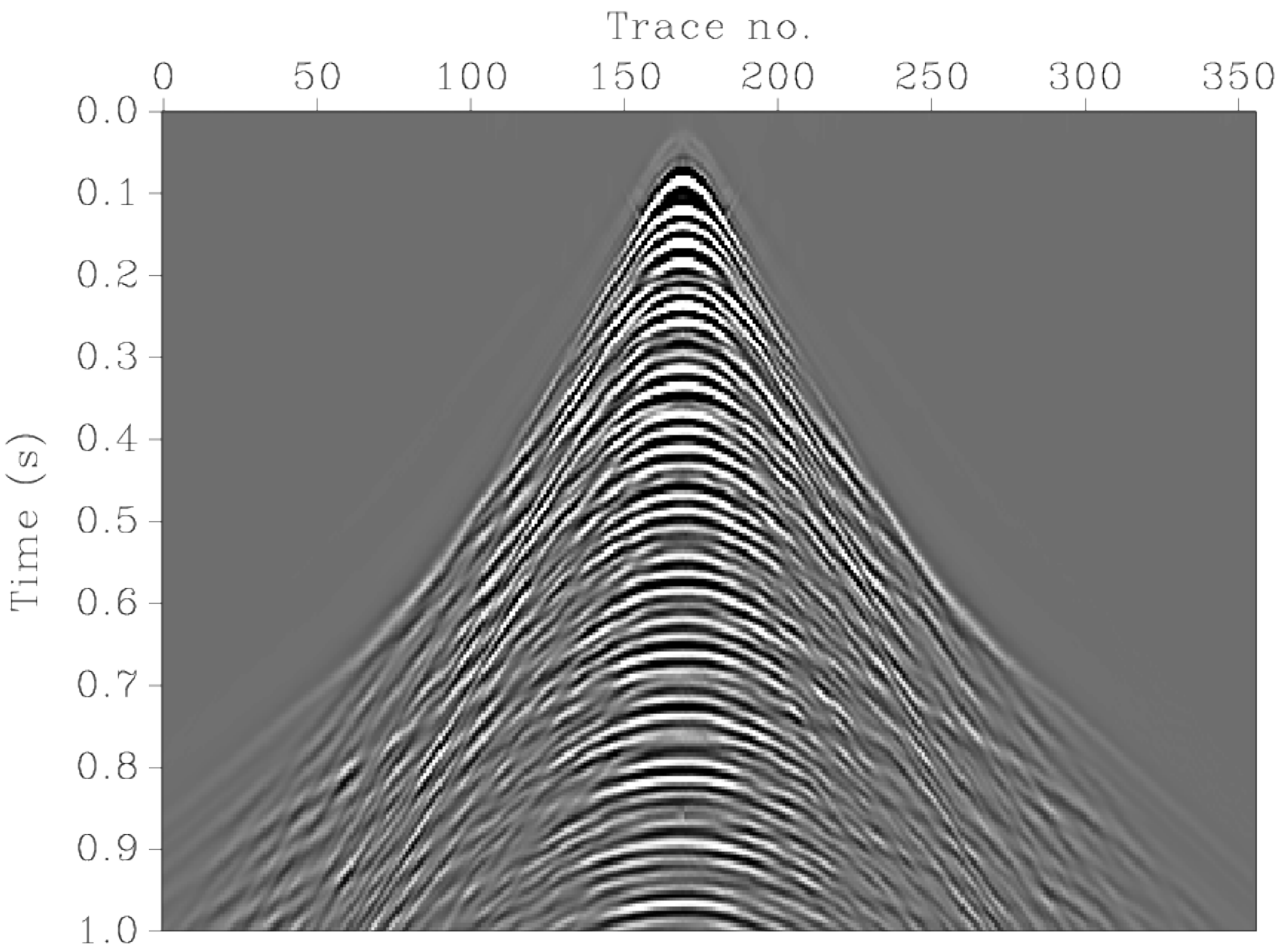
$$l = min\{n, m\}$$

$$C_1 \text{ and } C_2 > 0$$

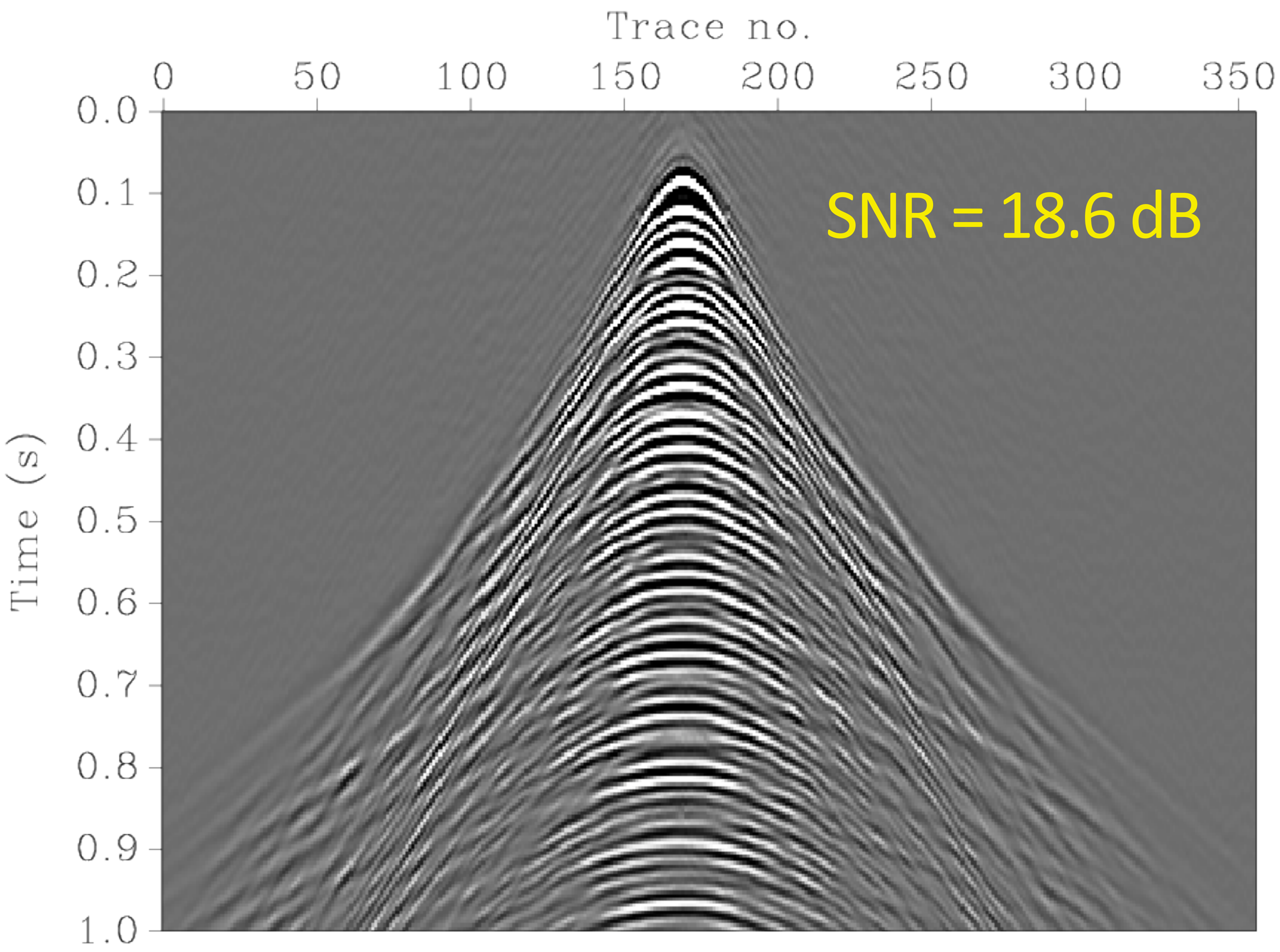
Regularization

matrix completion

Ground truth

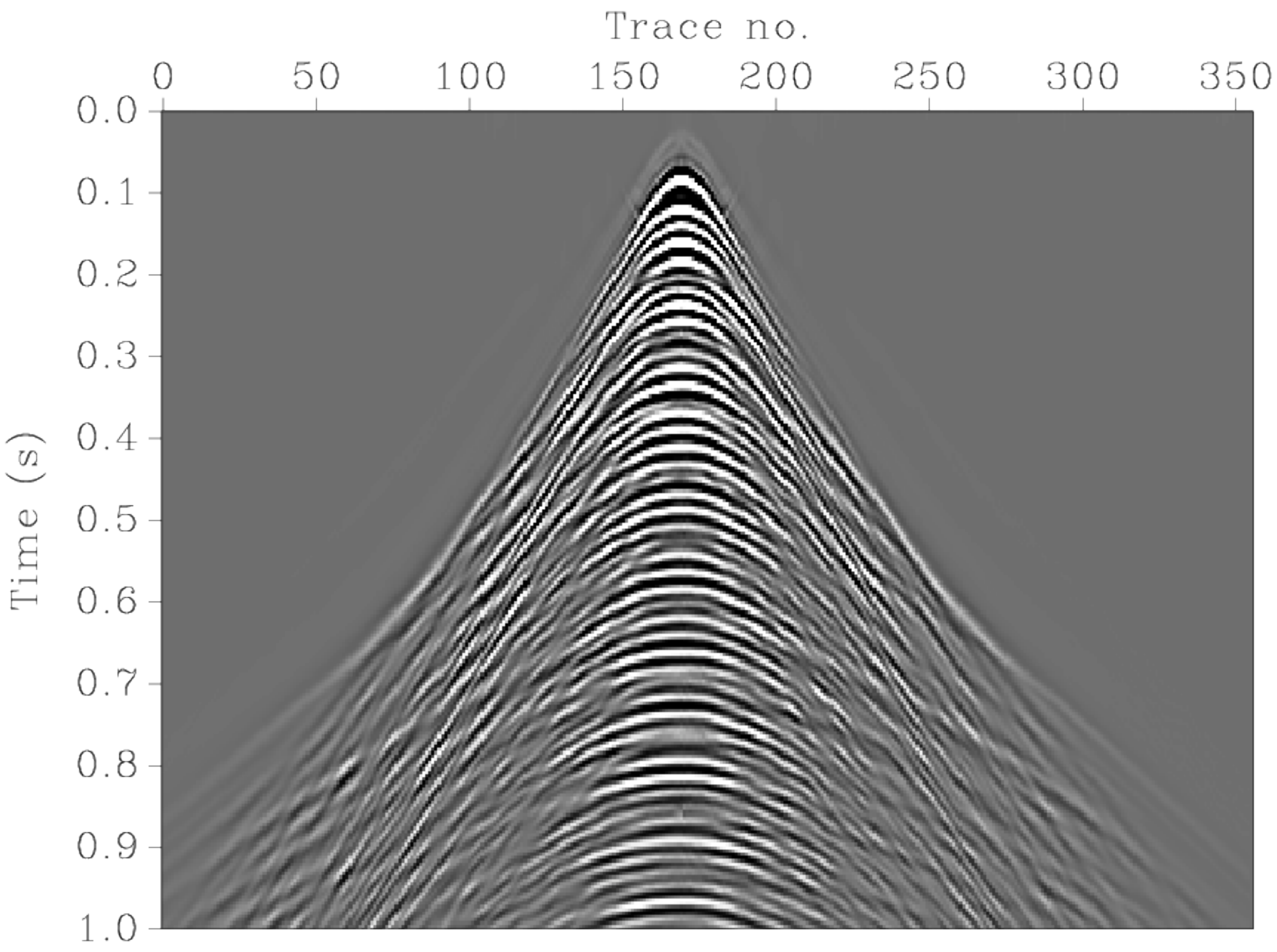


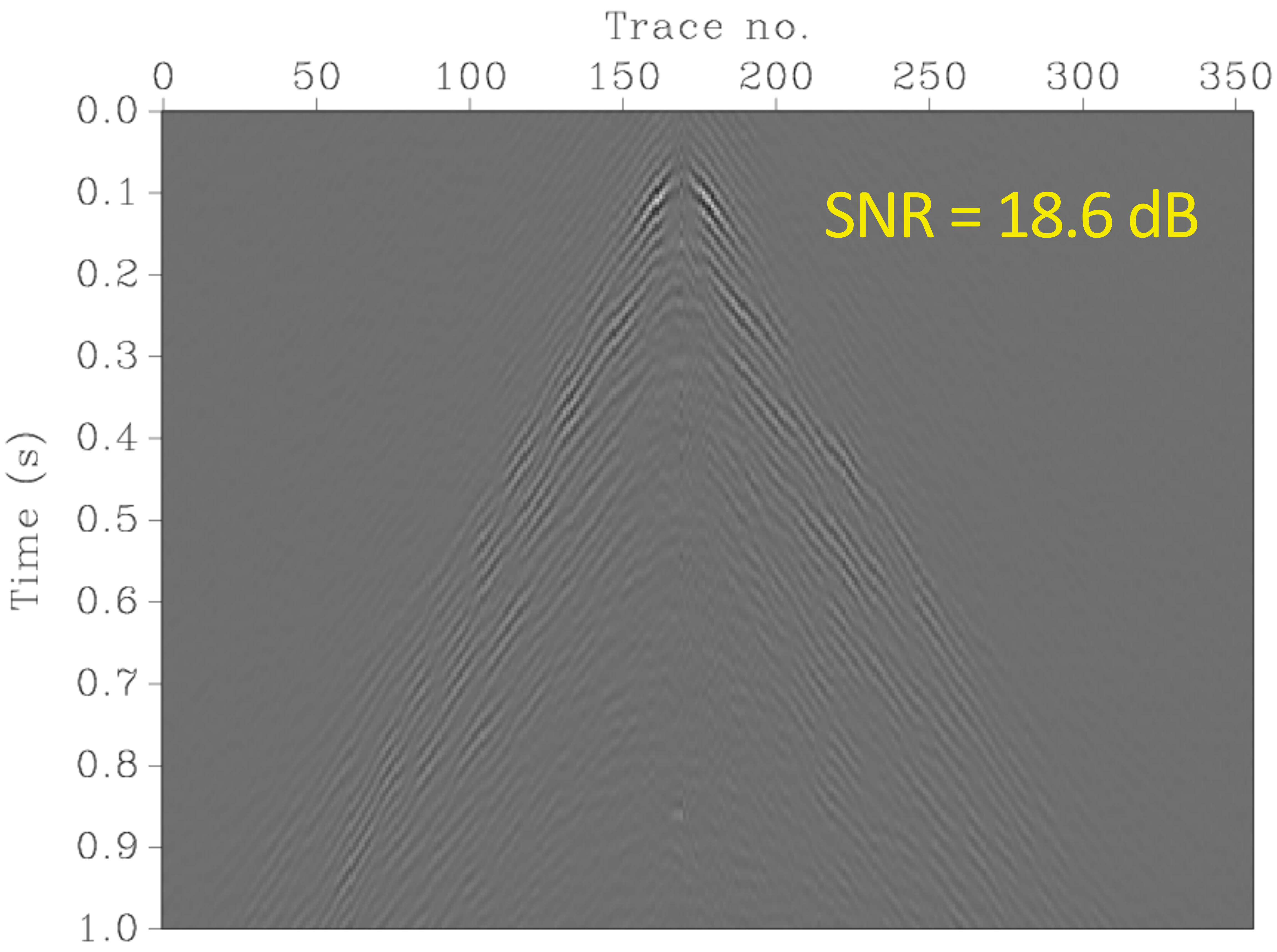
Recovery
Matrix Completion



SNR = 18.6 dB

Regularization
matrix completion

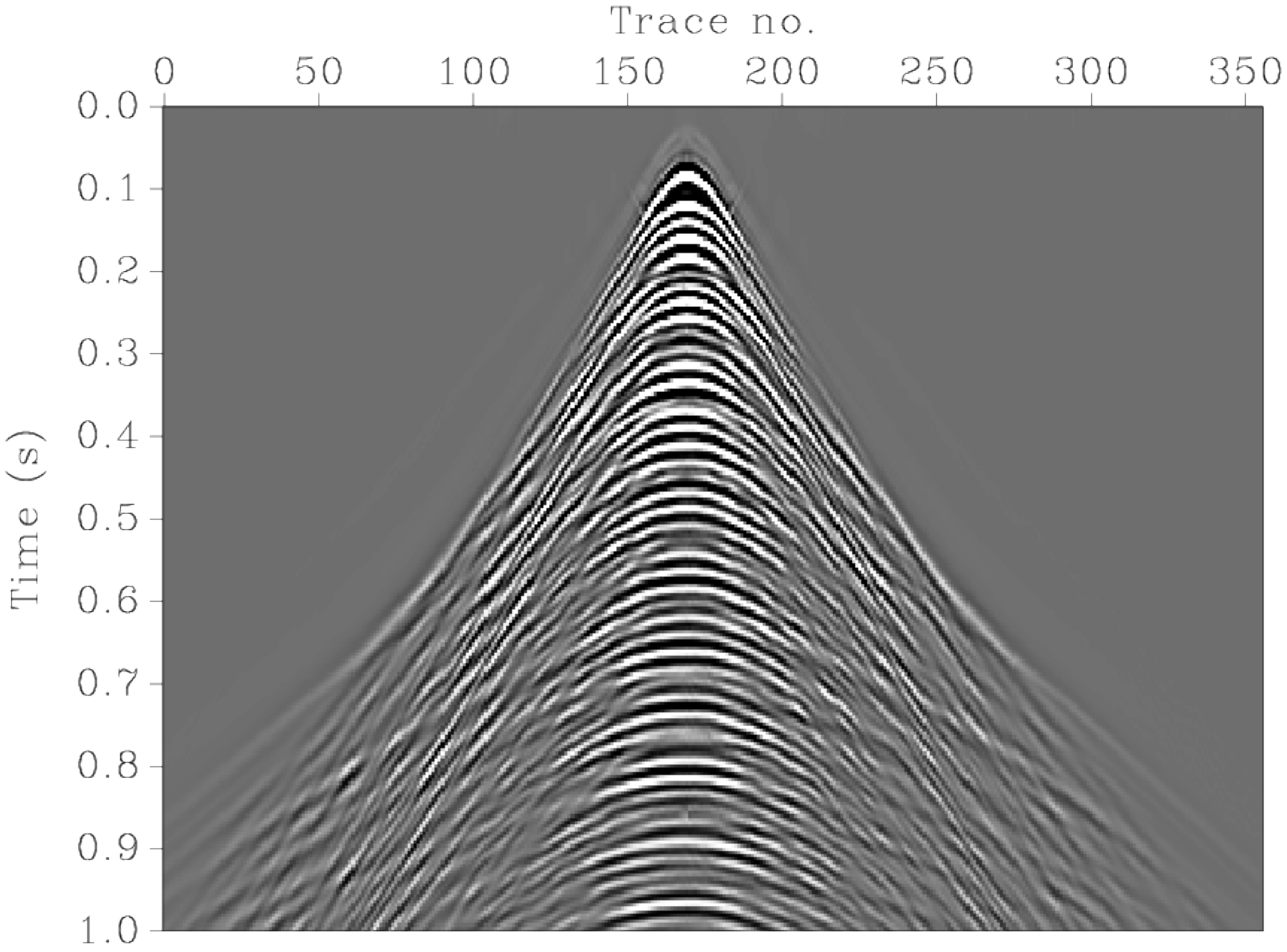

Ground truth


Residual
Matrix Completion

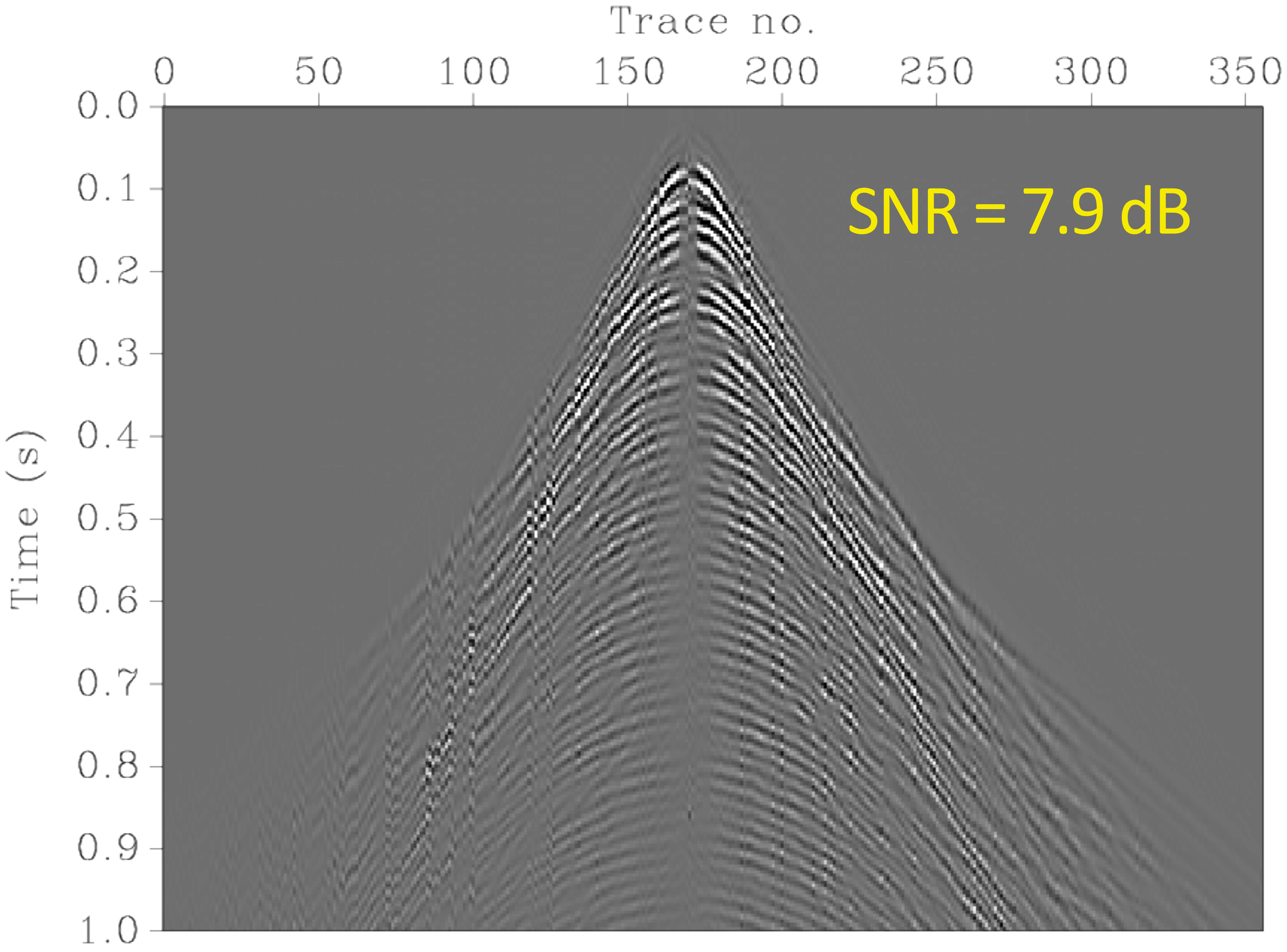
SNR = 18.6 dB

Regularization
binning

Ground truth

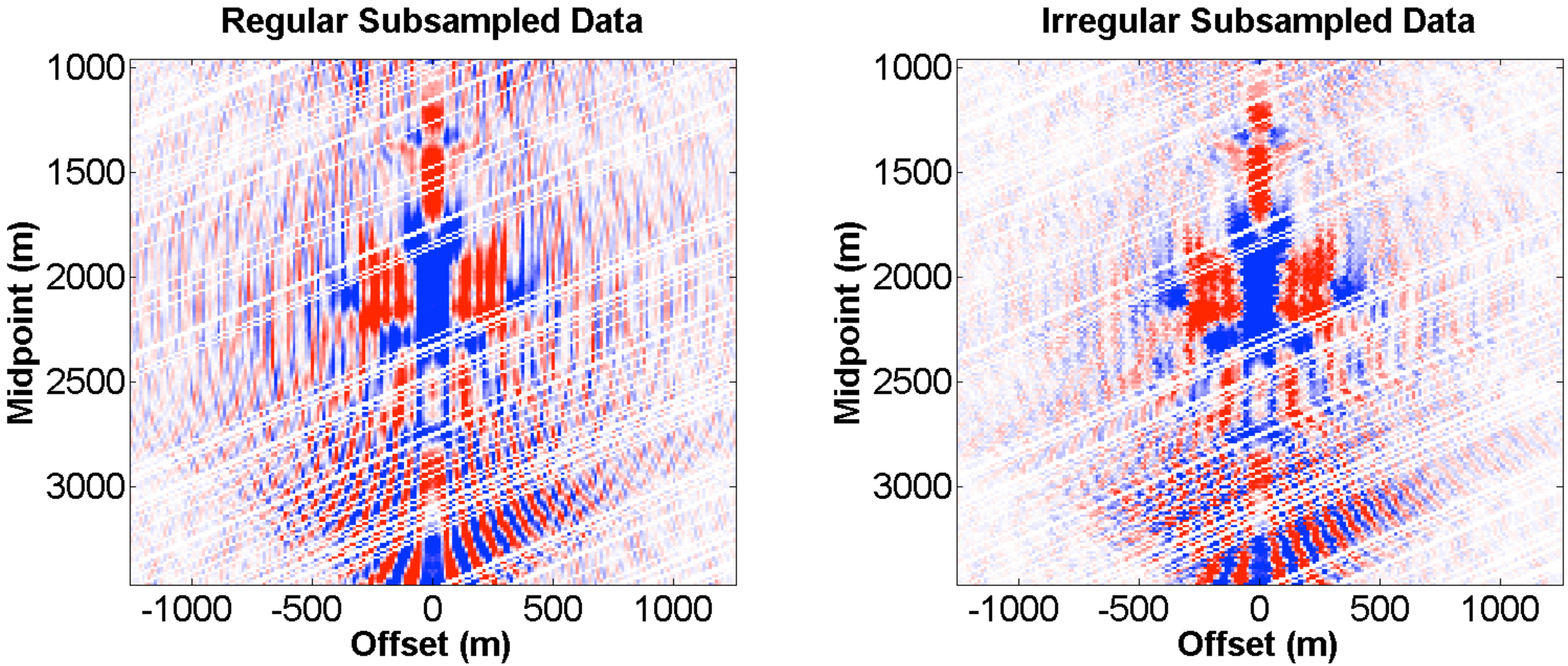


Residual Binning



Irregularity + Missing Sources

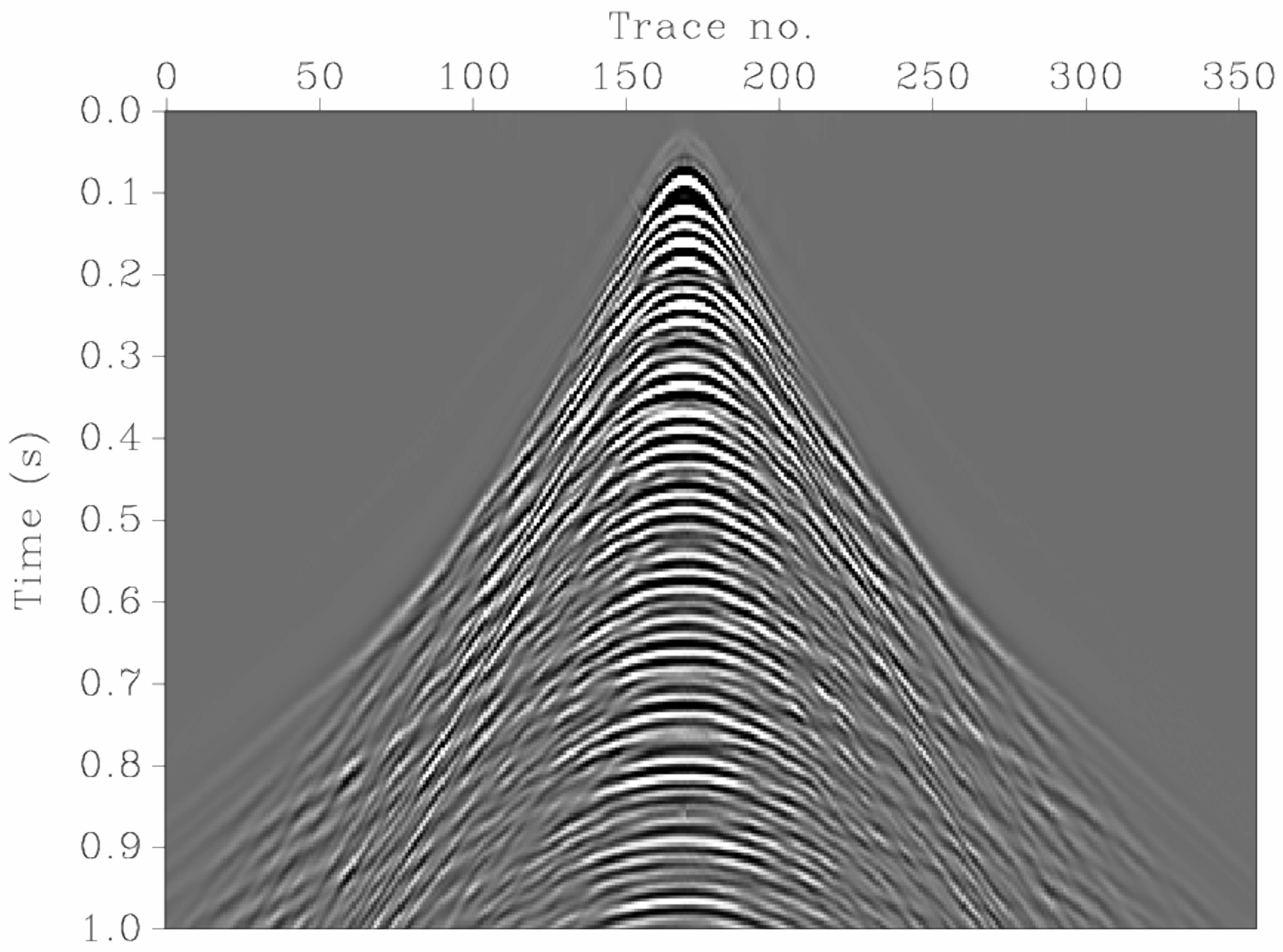
midpoint-offset domain



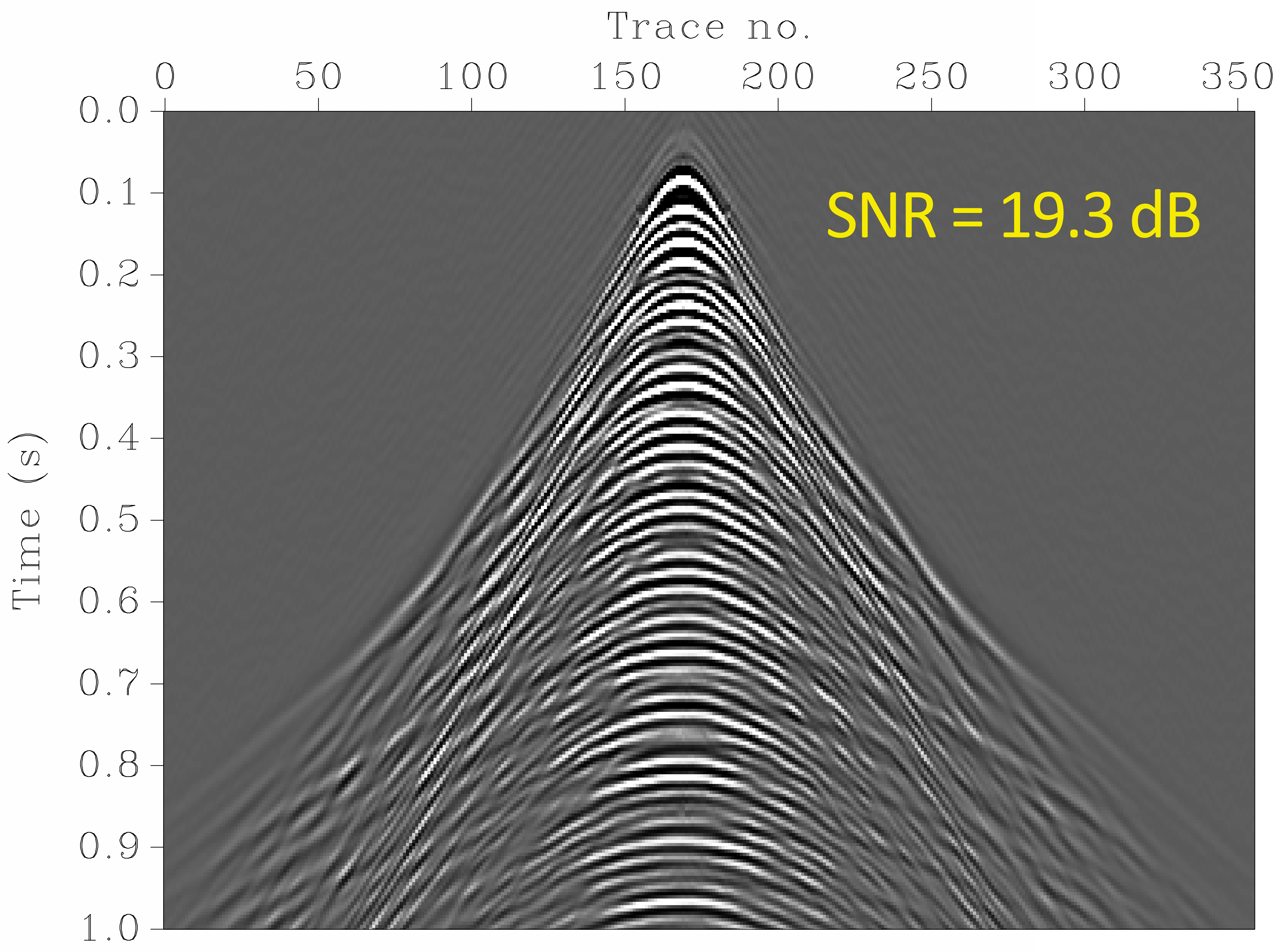
Regularization + Interpolation

matrix completion

50% missing sources



Ground truth

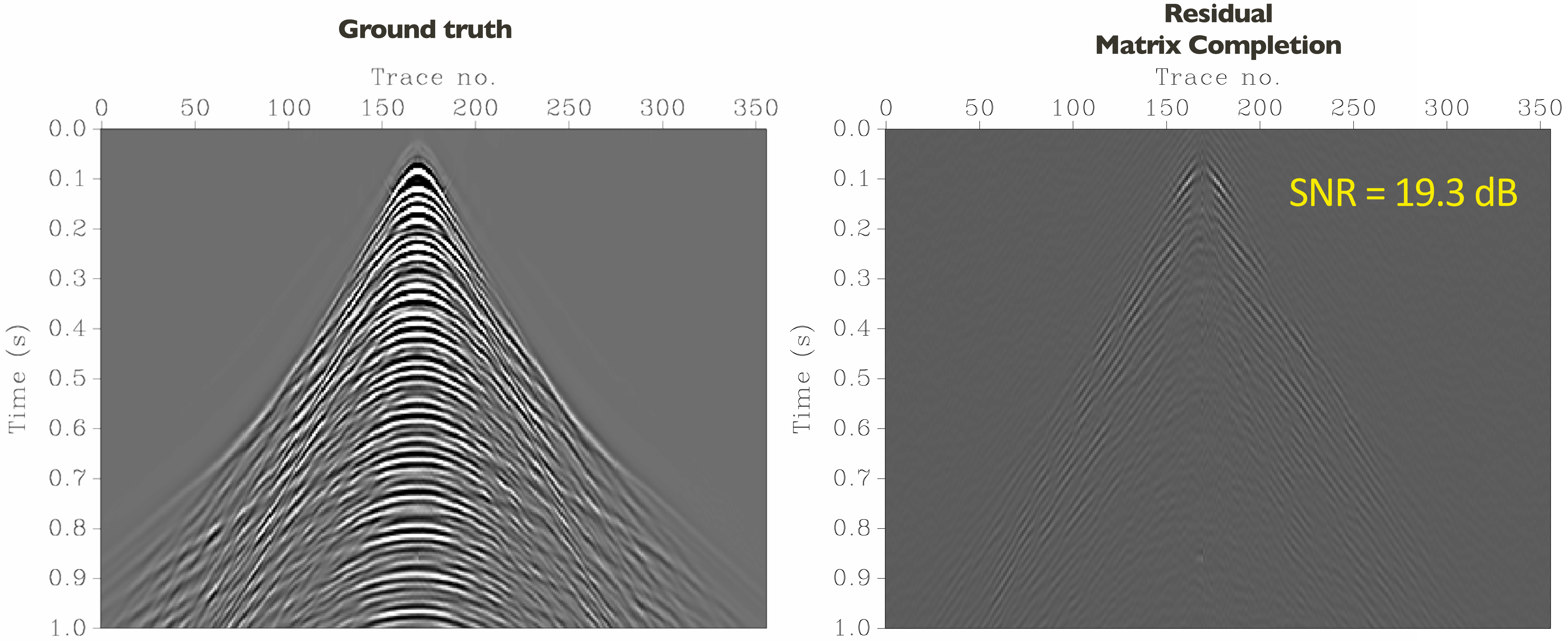


Recovery
Matrix Completion

Regularization + Interpolation

matrix completion

50% missing sources



Outline

- Regularization
  - is binning the right approach?
  - exploit low rank structure

- Can we quantify acquisition design?

How should we subsample?

Consider our sampling mask

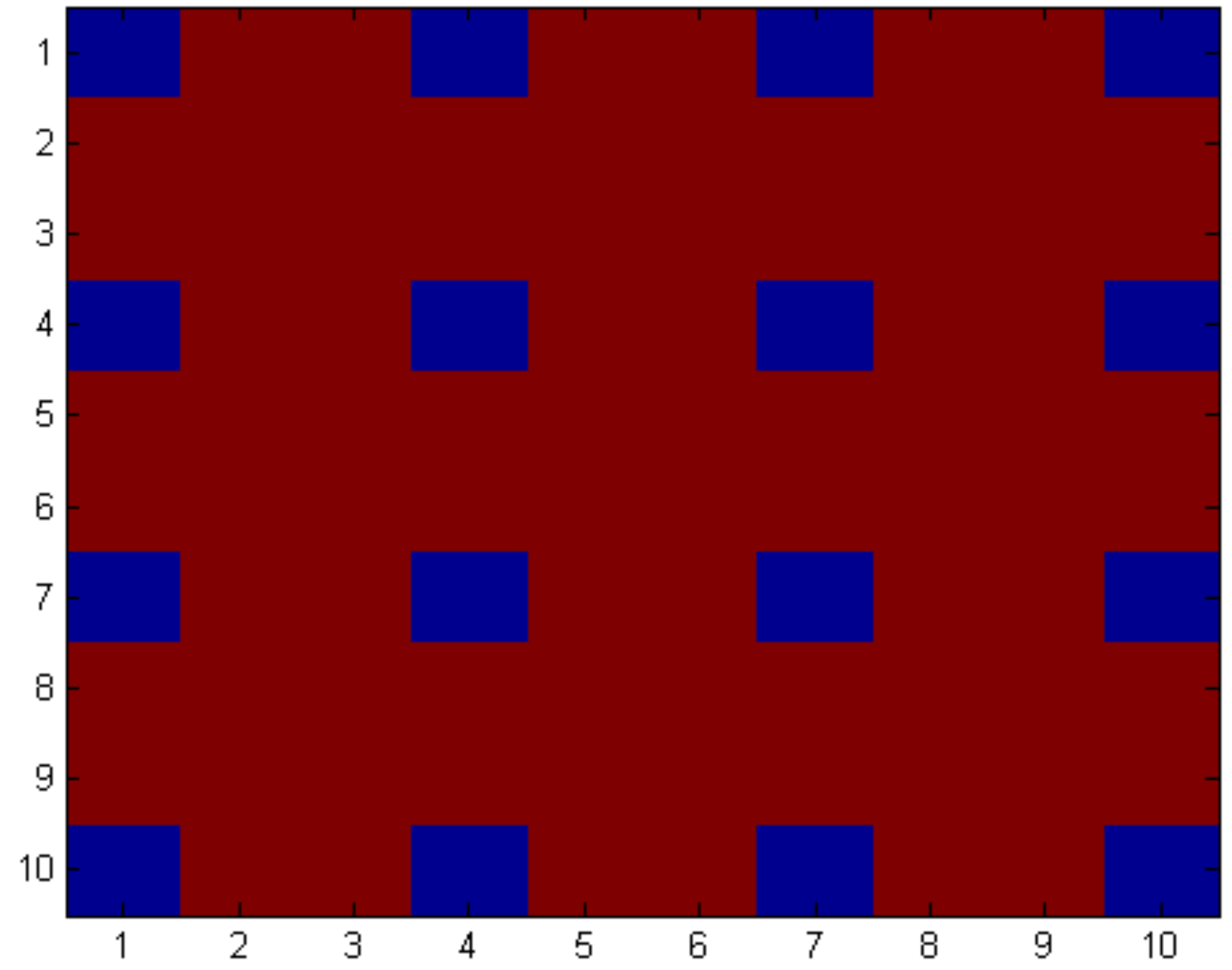
$$A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

What determines if A is good for matrix completion?

Example: Bad Sampling Mask

Periodic subsampling

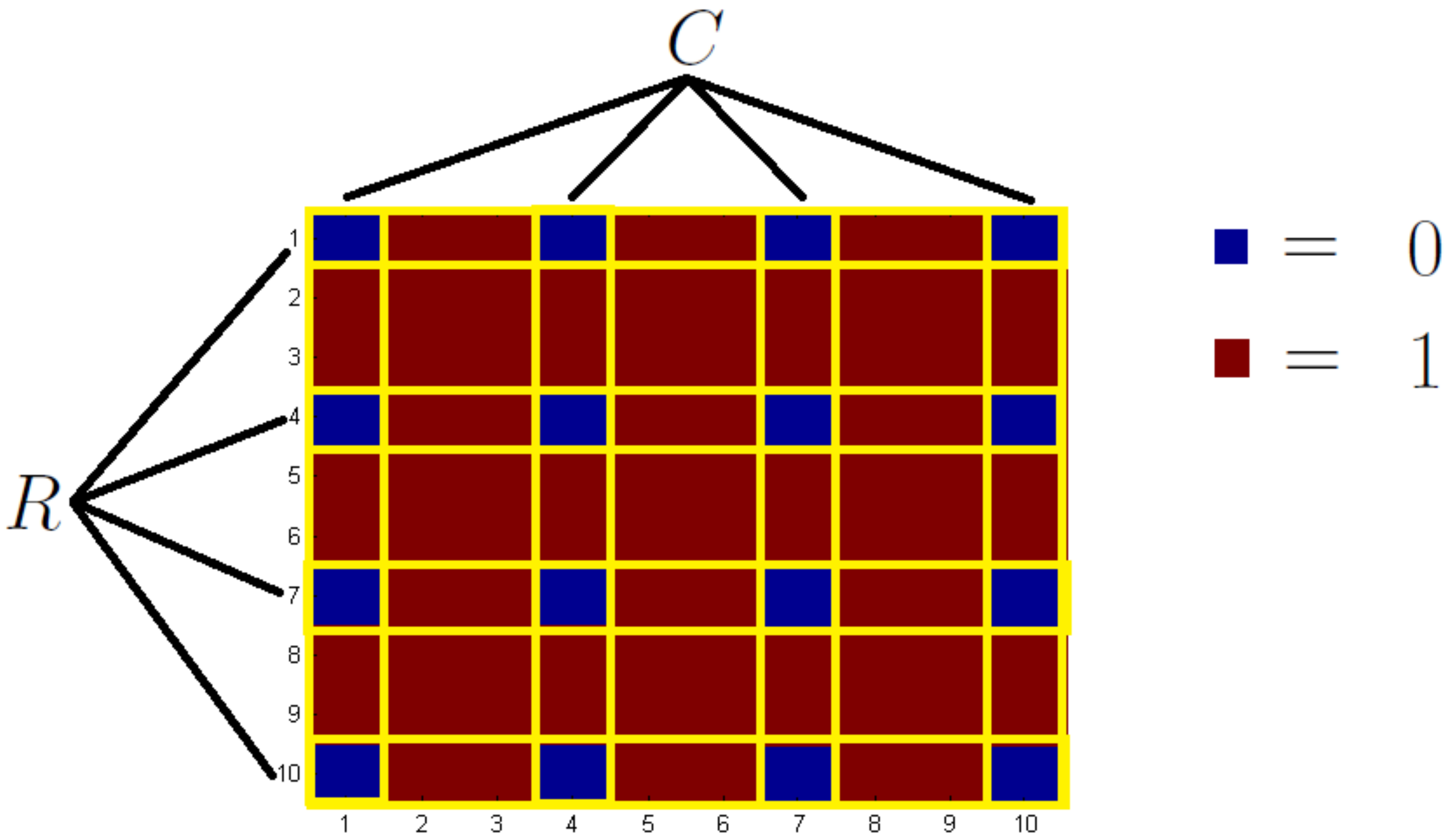
$A =$



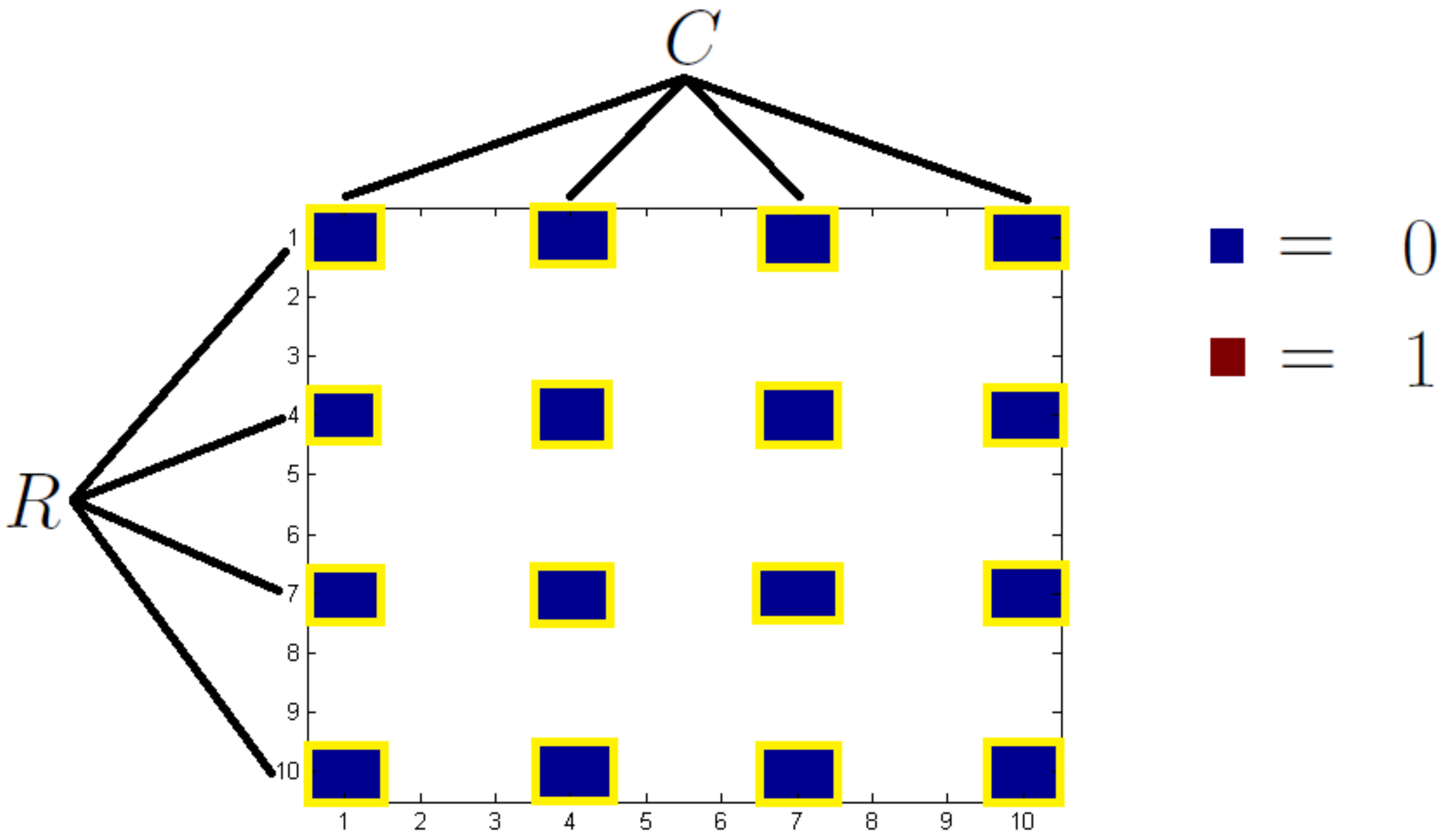
🟦 $= 0$

🟥 $= 1$

Example: Bad Sampling Mask



Example: Bad Sampling Mask

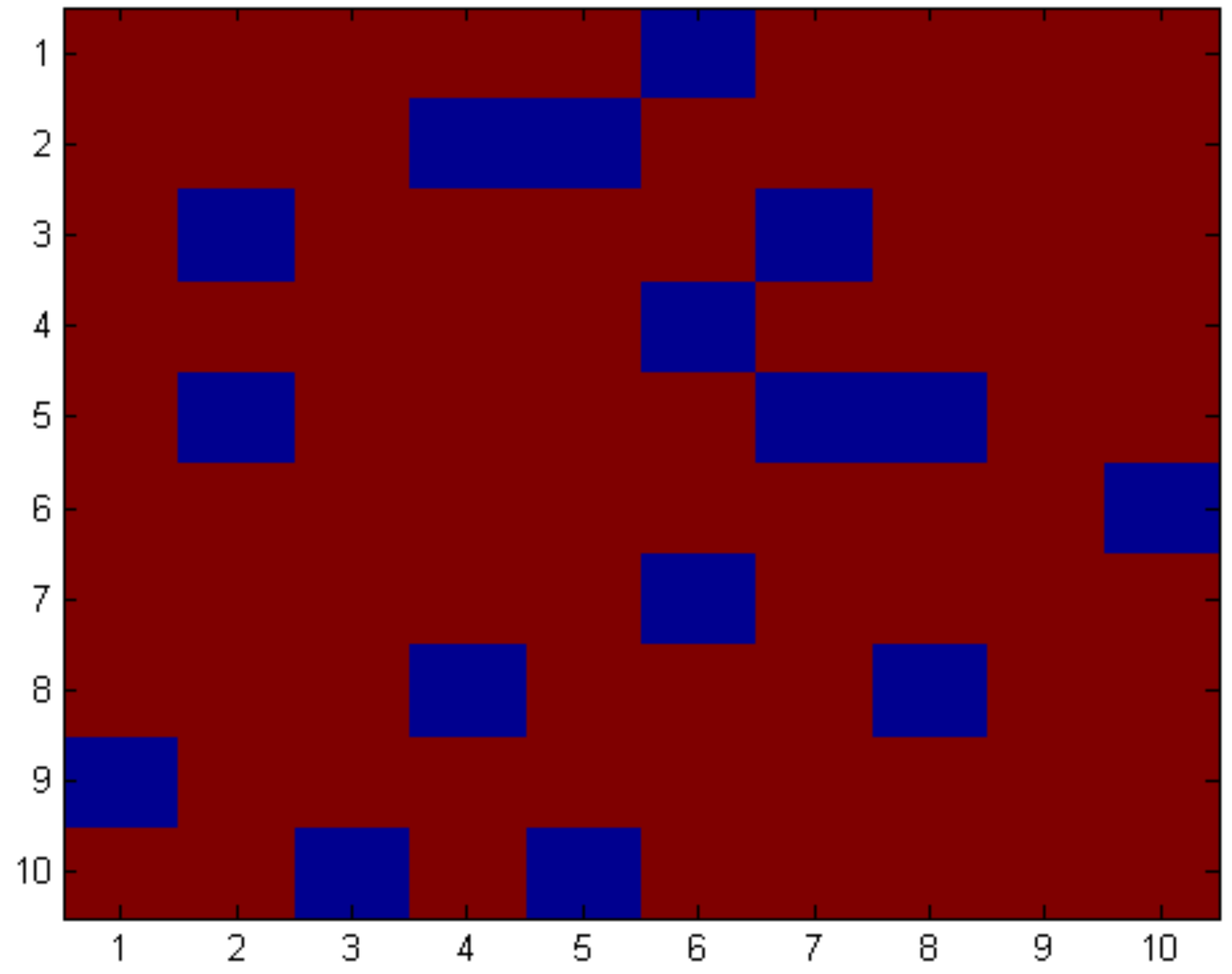


Did not sample this critical sub matrix

Example: Good Sampling Mask

Entries chosen independently at random

$A =$

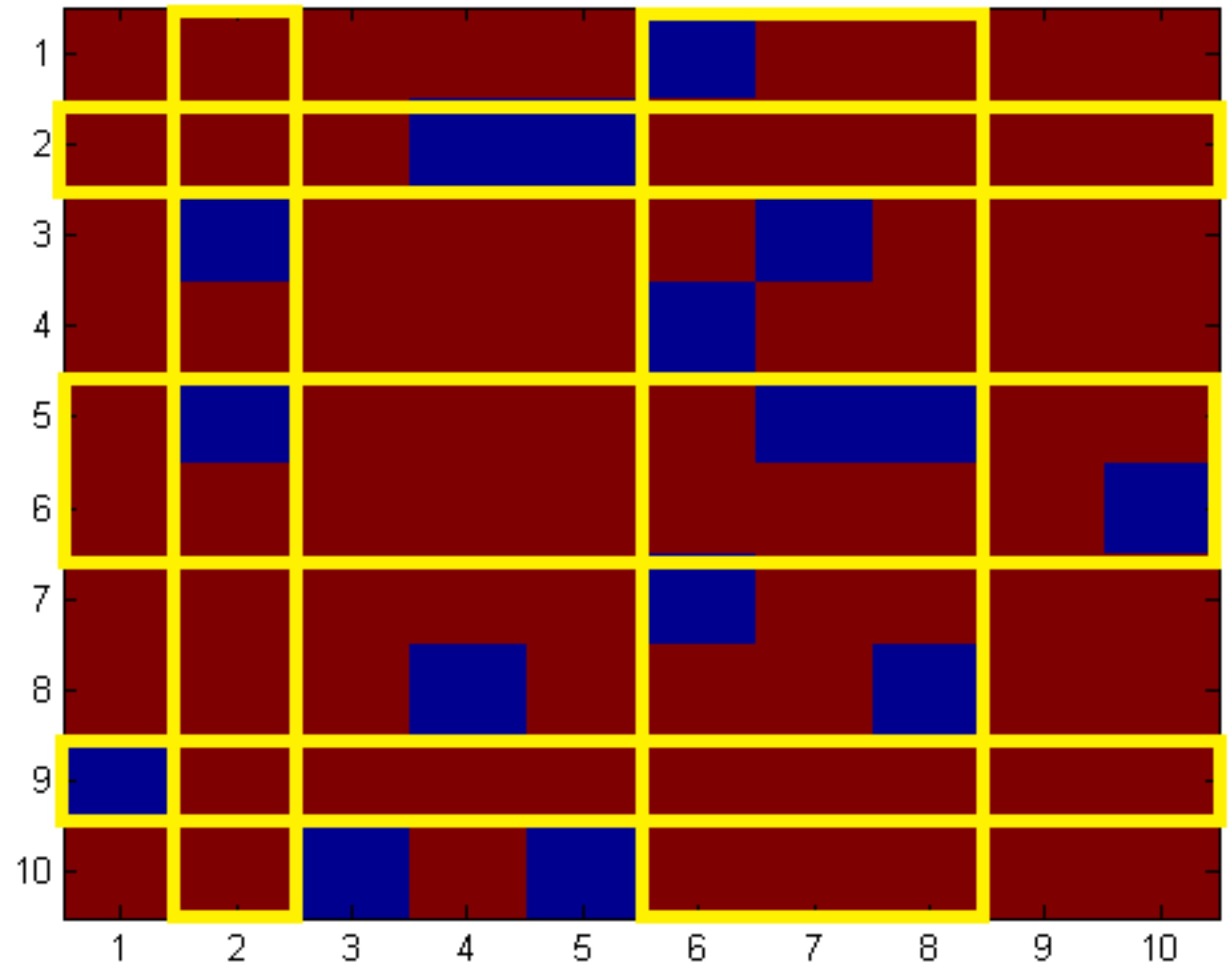


🟦 $= 0$

🟥 $= 1$

Example: Good Sampling Mask

$A =$

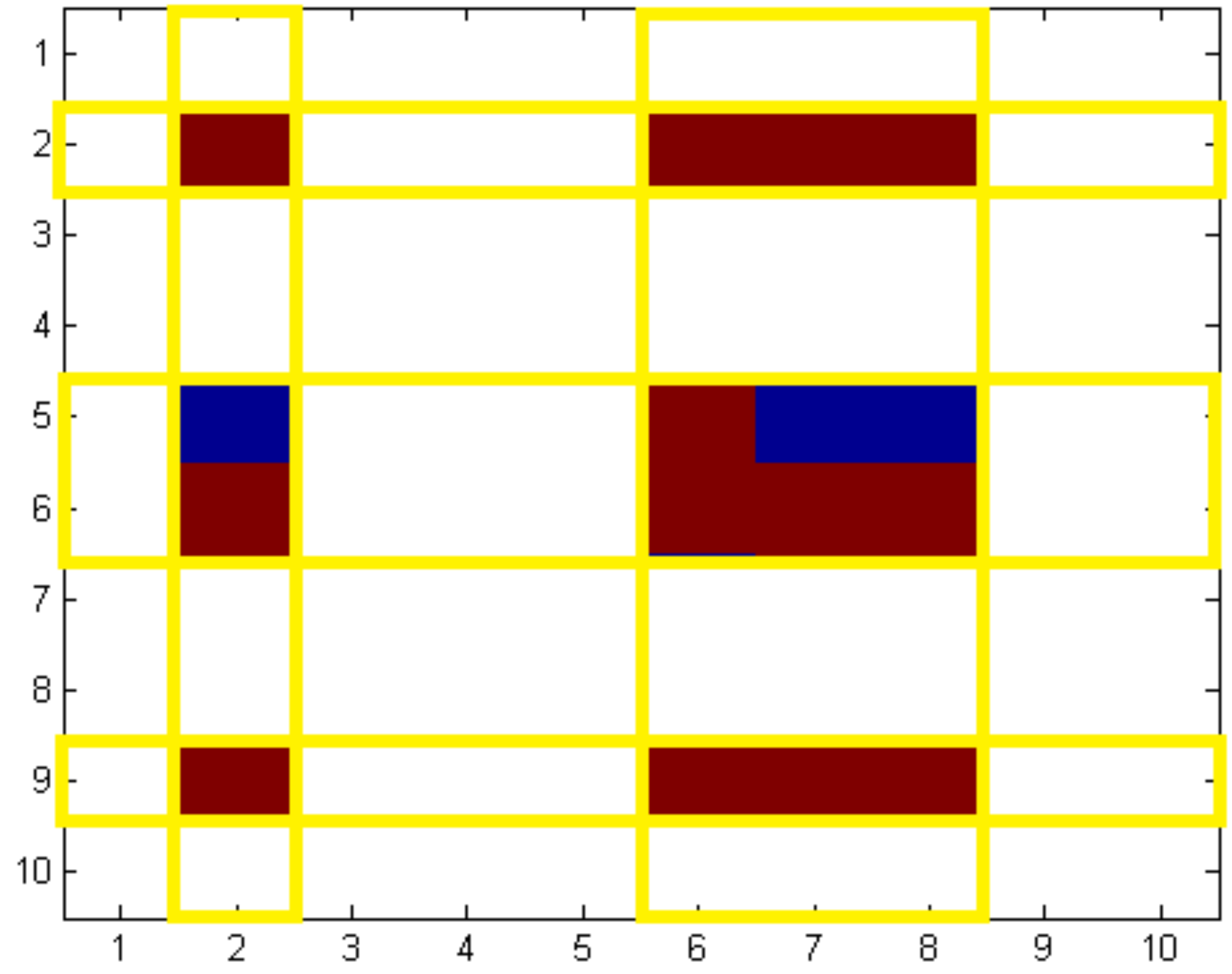


🟦 $= 0$

🟥 $= 1$

Example: Good Sampling Mask

$A =$



🟦 $= 0$

🟥 $= 1$

All sub matrices are nicely sampled!

Bhojanapalli, Jain. “Universal Matrix Completion” ICML 2014.

Spectral Gap

Consider the gap between the two largest singular values of A

$$\frac{\sigma_2}{\sigma_1} = \begin{cases} \approx 1 & \text{if small spectral gap} \\ << 1 & \text{if large spectral gap} \end{cases}$$

where σ_i is the i-th largest singular value of A

Bhojanapalli, Jain. “Universal Matrix Completion” ICML 2014.

Spectral Gap

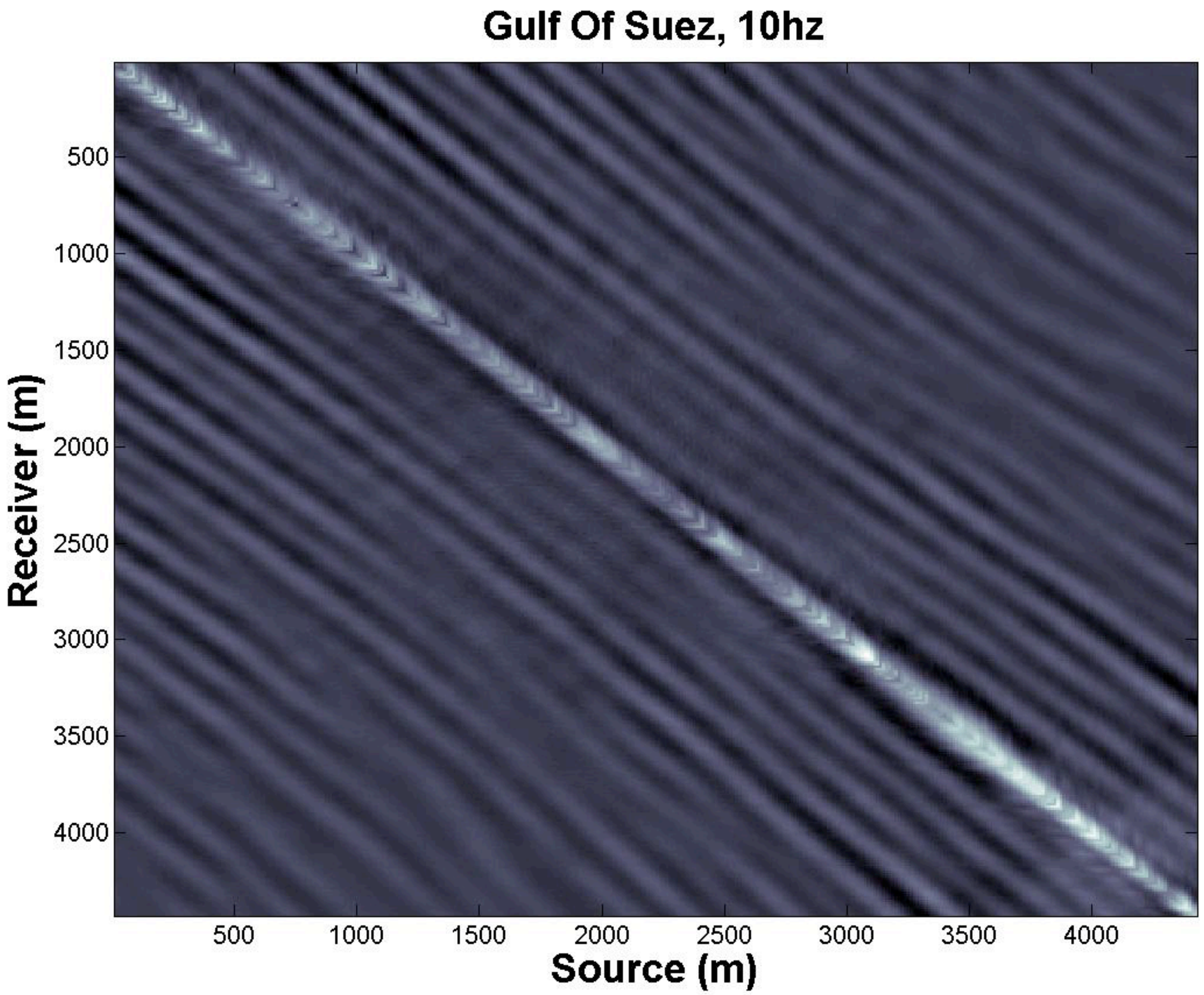
$$\frac{\sigma_2}{\sigma_1} = \begin{cases} \approx 1 \text{ if small spectral gap} \\ << 1 \text{ if large spectral gap} \end{cases}$$

From graph theory literature:

A with Large Spectral Gap $\Rightarrow$ all “sub matrices” are nicely sampled

$\Rightarrow$ better results for matrix completion

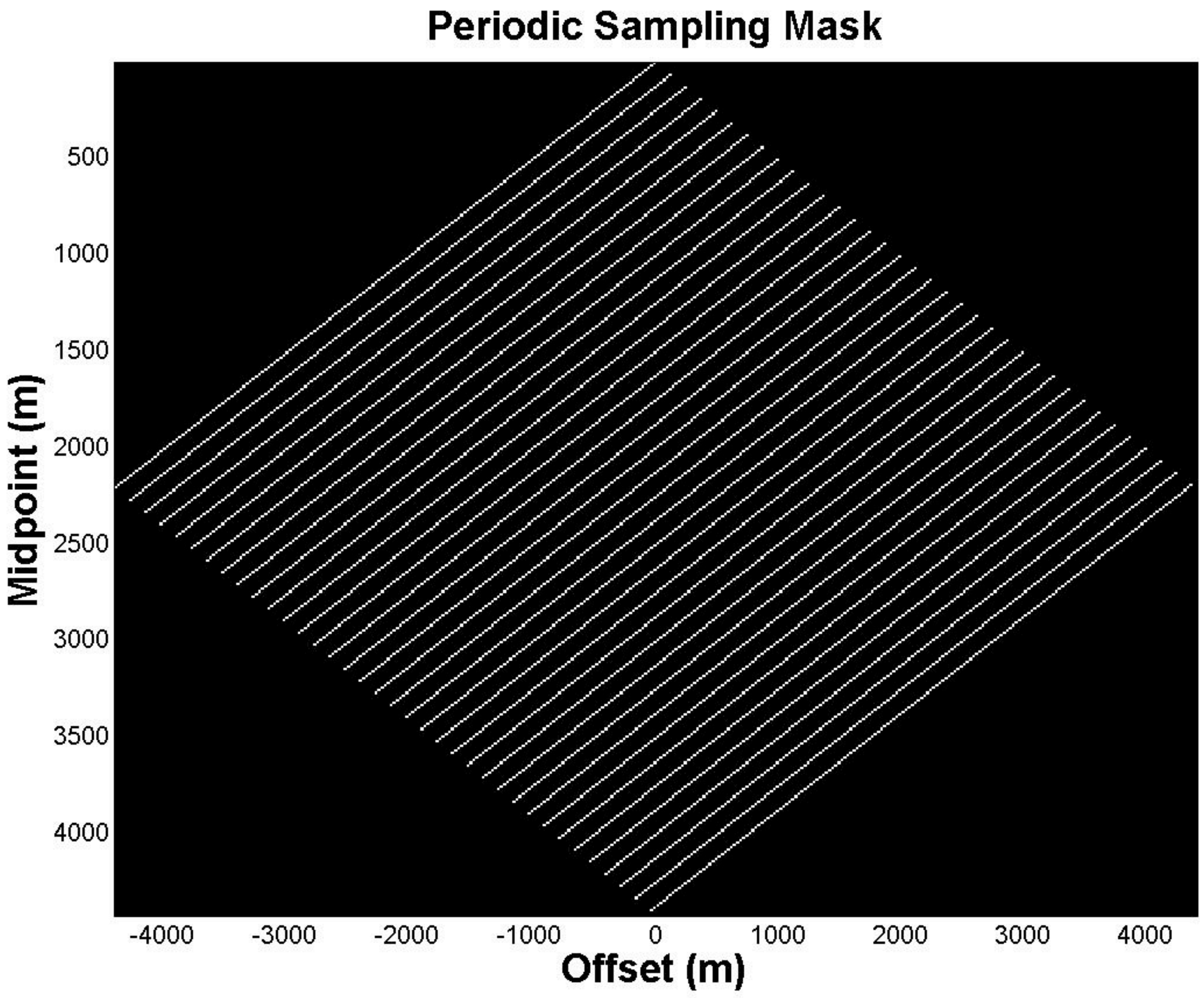
Periodic vs. Uniform Random vs. Jittered



Interpolate with 100 different random masks

75% missing sources

Periodic Sampling (75% Missing)

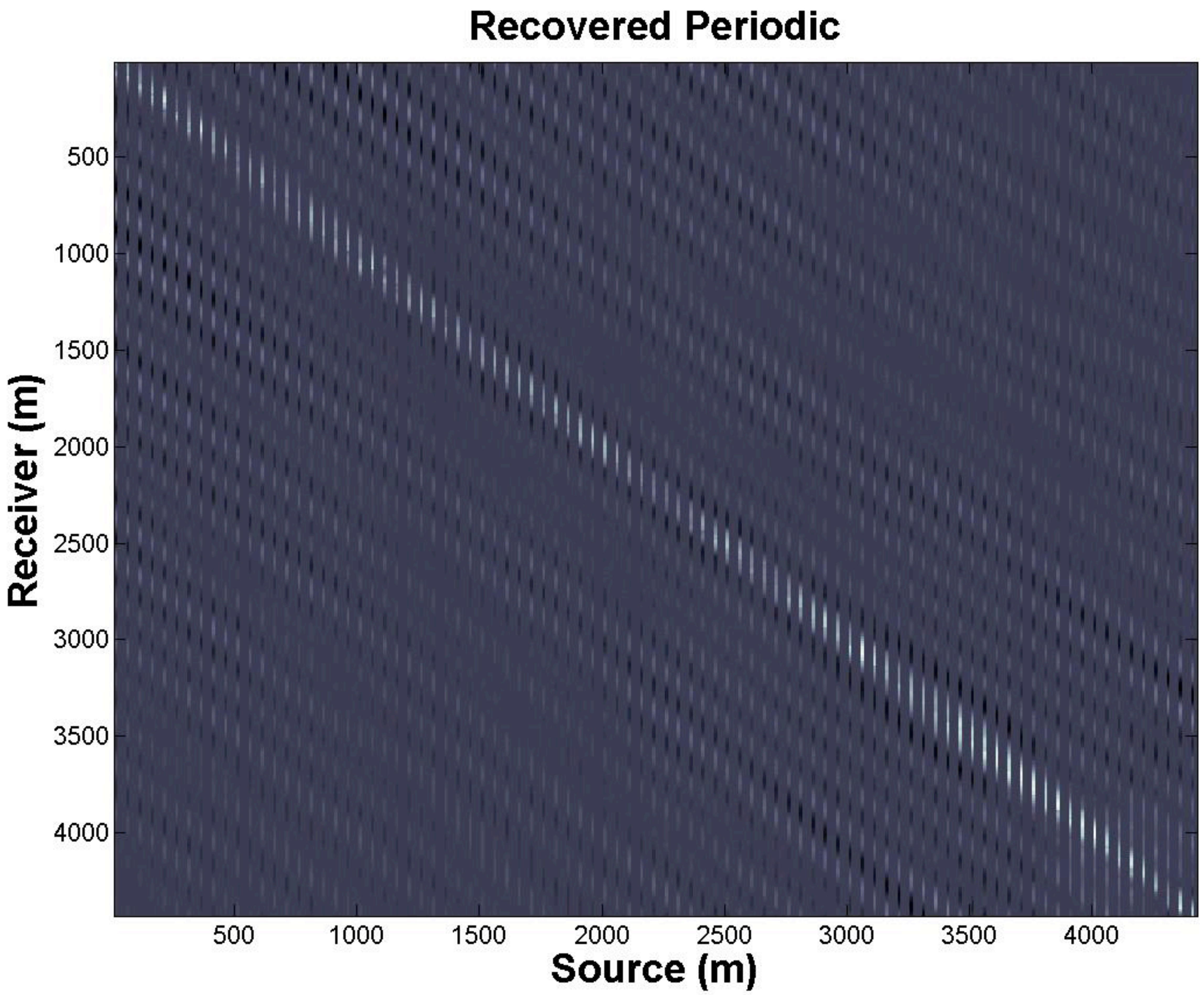


■ $= 0$

□ $= 1$

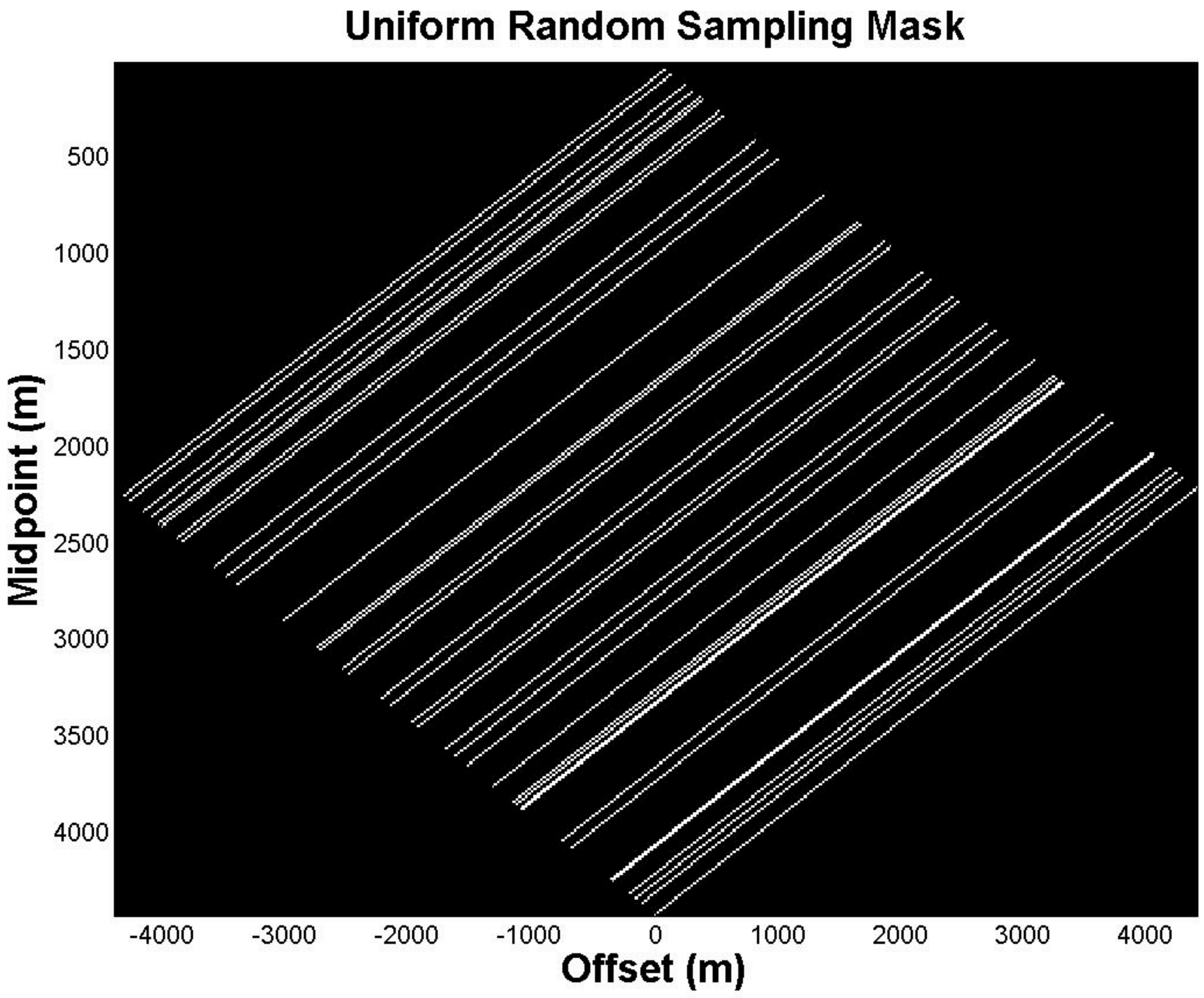
$$\frac{\sigma_2}{\sigma_1} = .9943$$

Periodic Sampling (75% Missing) Reconstruction



SNR = 1.2 dB

Uniform Random Sampling (75% Missing)



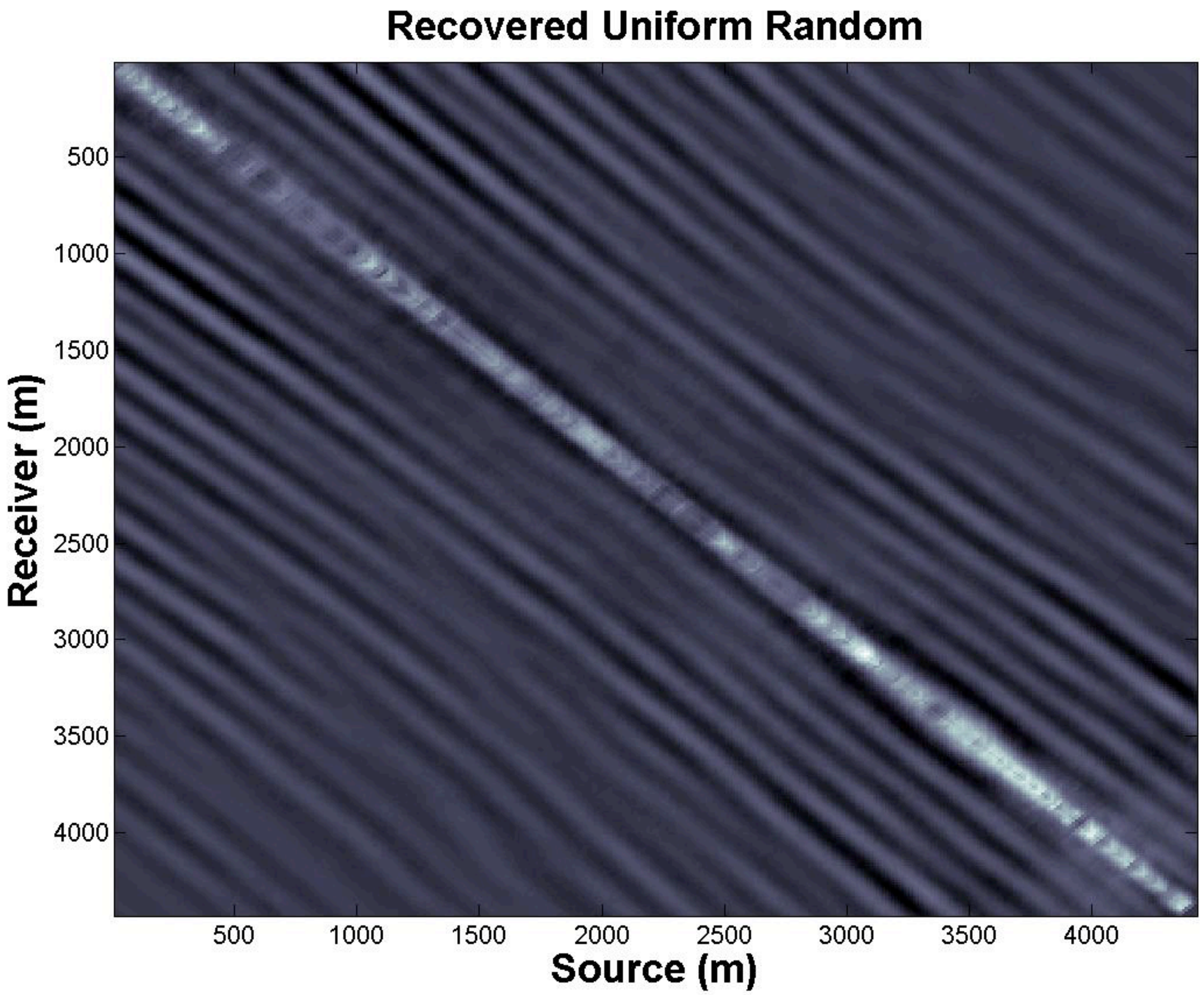
■ $= 0$

□ $= 1$

Mean

$$\frac{\sigma_2}{\sigma_1} = .3640$$

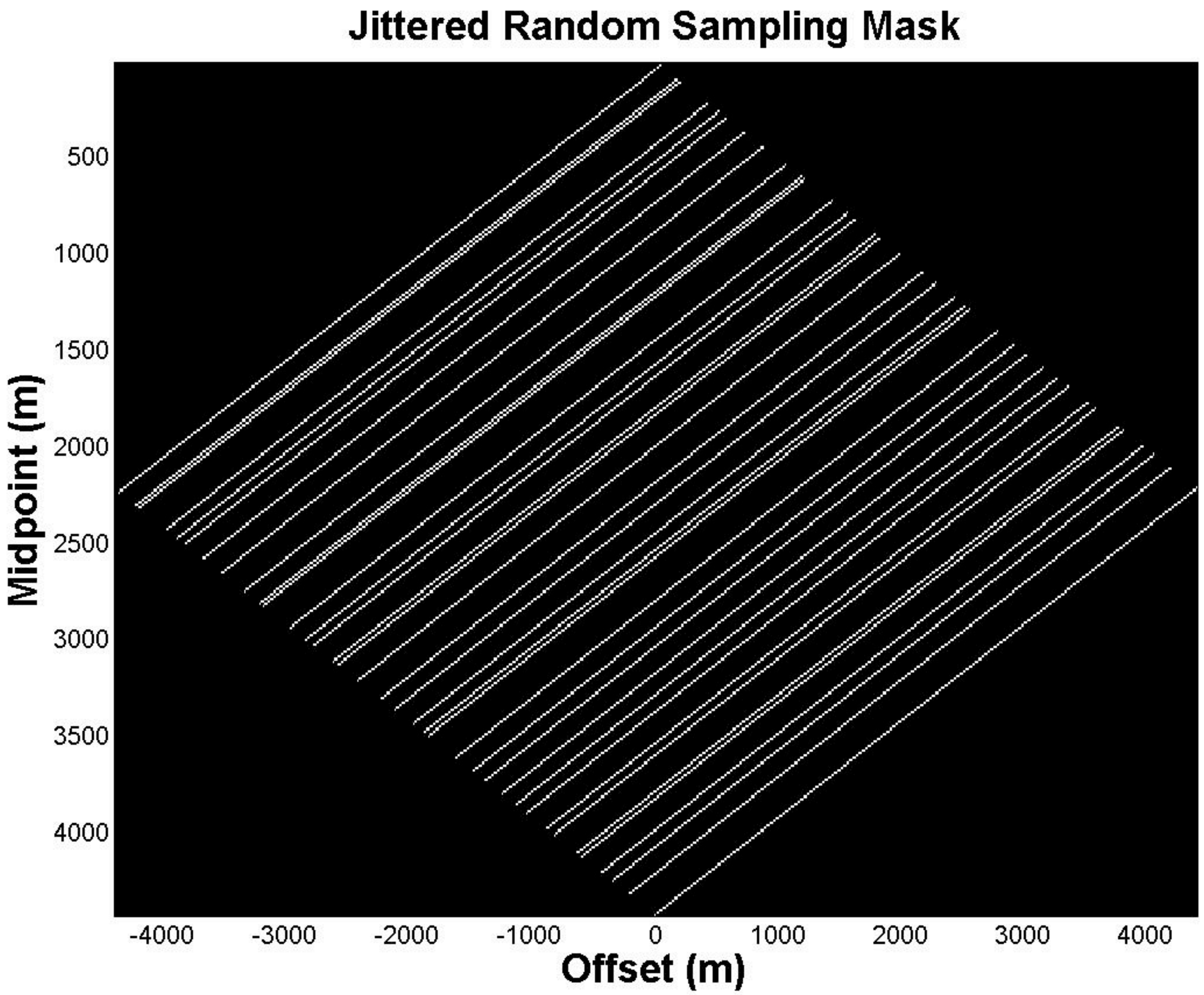
Uniform Random Sampling (75% Missing) Reconstruction



Mean

SNR = 12 dB

Jittered Random Sampling (75% Missing)



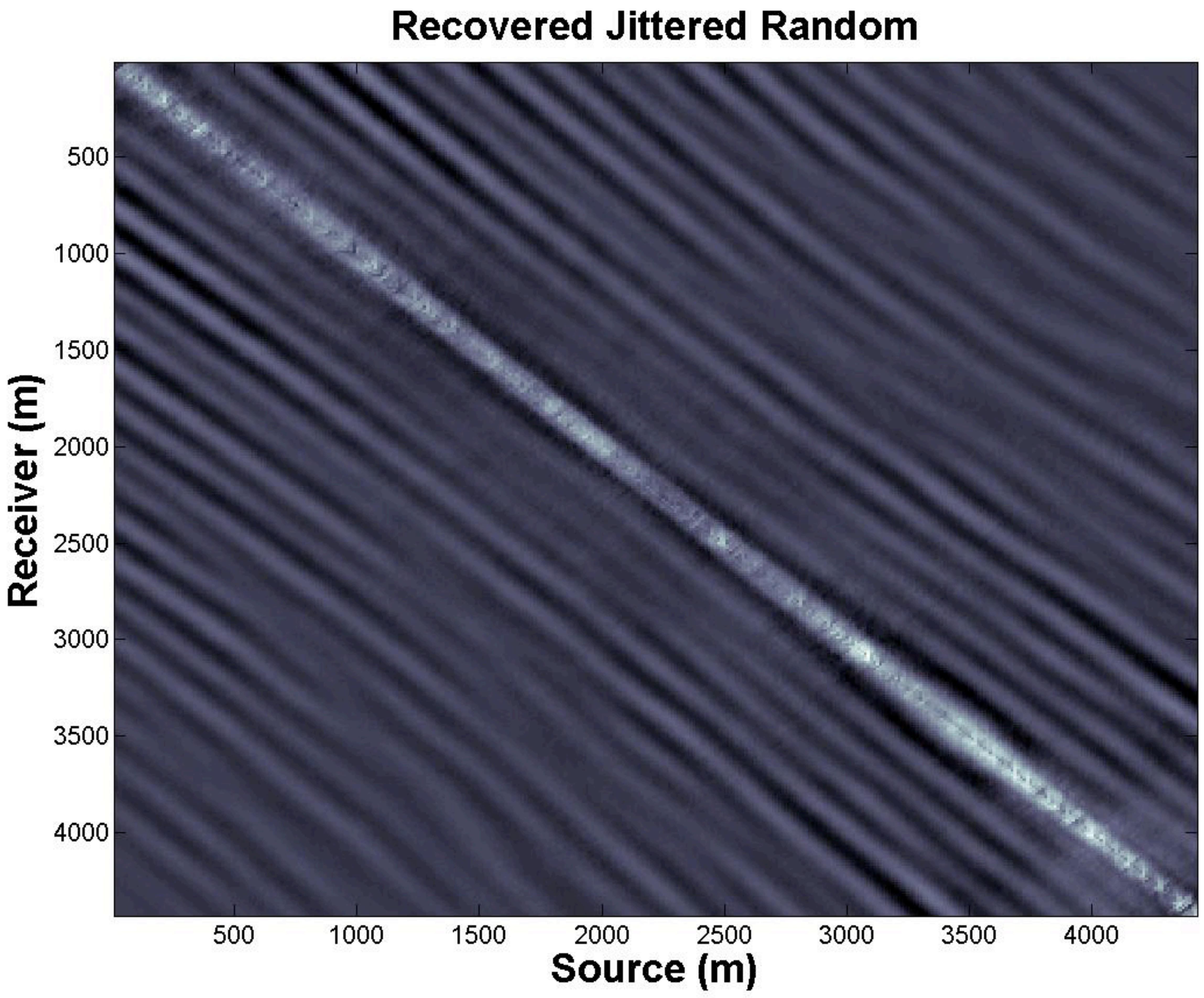
$\blacksquare = 0$

$\square = 1$

Mean

$$\frac{\sigma_2}{\sigma_1} = .3350$$

Jittered Random Sampling (75% Missing) Reconstruction



Mean

SNR = 13 dB

Conclusion

- Can exploit *low-rank* structure to regularize + interpolate data.

- Simple procedure to quantify acquisition design
  - compute only σ_1, σ_2 of sampling mask

Future work

- Incorporate irregularity along both sources & receivers coordinates
- Consider other regularization operators
- Further analysis of spectral gap quantification
  - Reconstruction Error bounds
  - Use theory to design optimal acquisition schemes

Acknowledgements

We need Real data set

Thank you for your attention !
https://www.slim.eos.ubc.ca/

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