Why the modified Gauss-Newton method?

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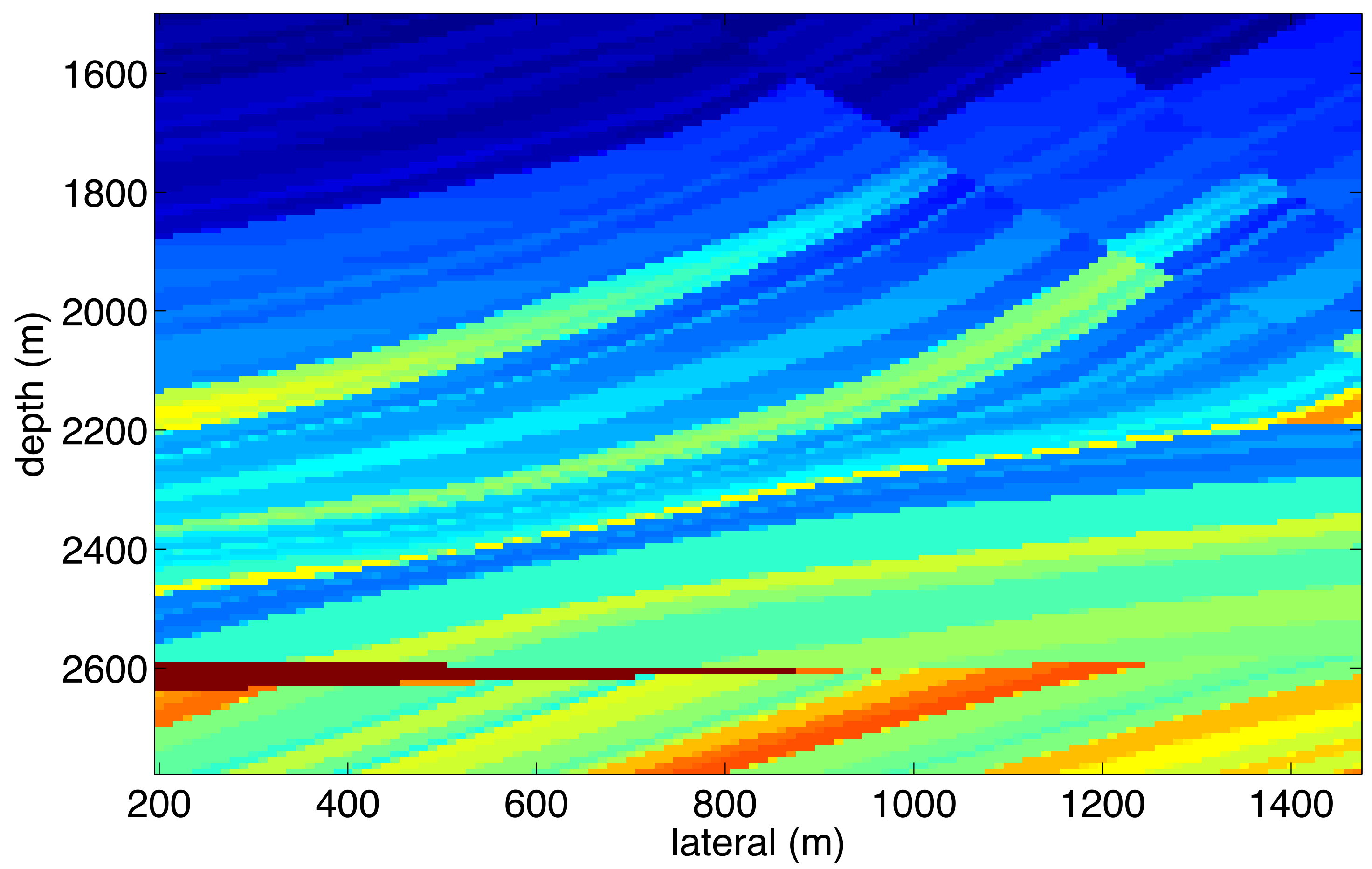
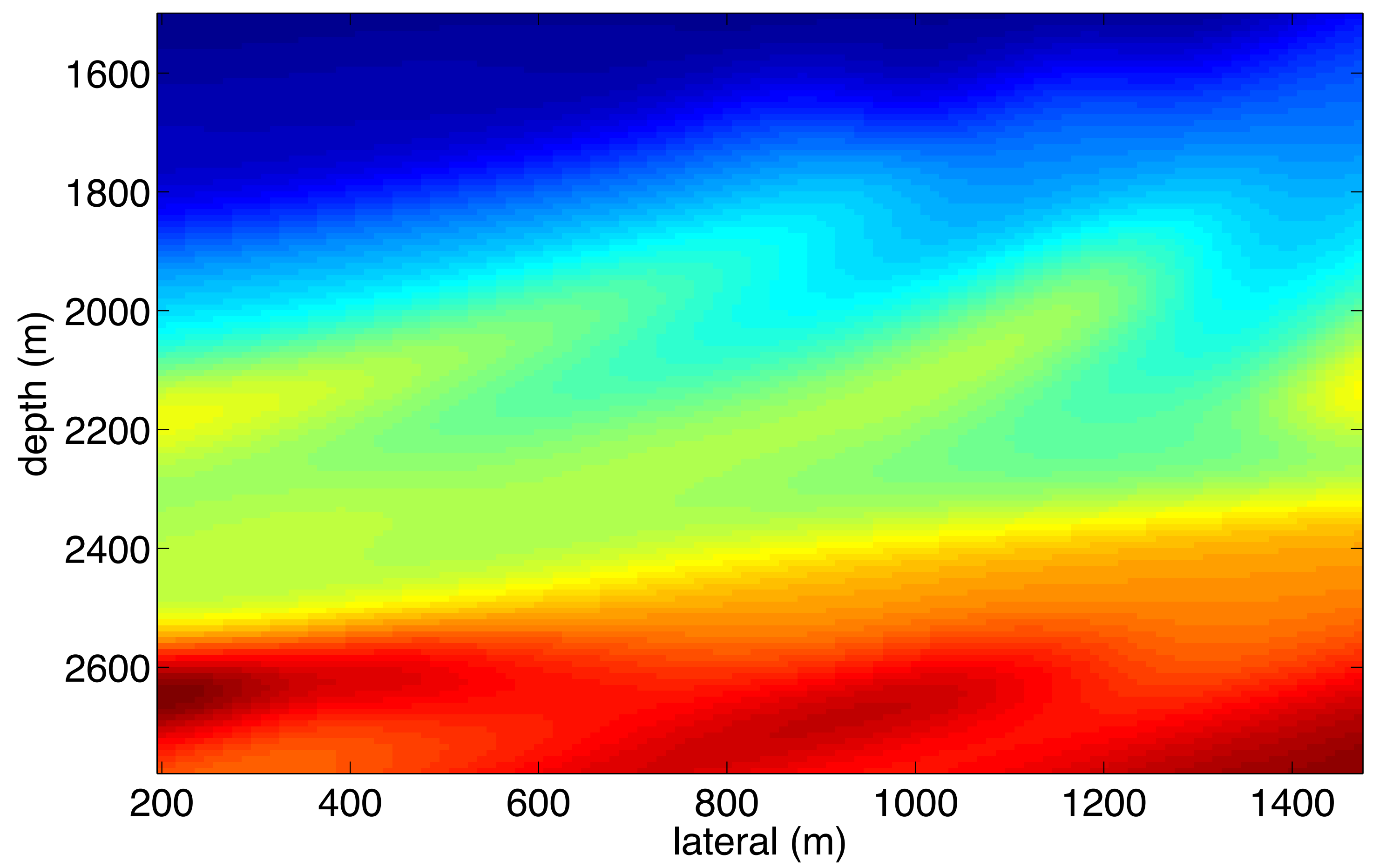
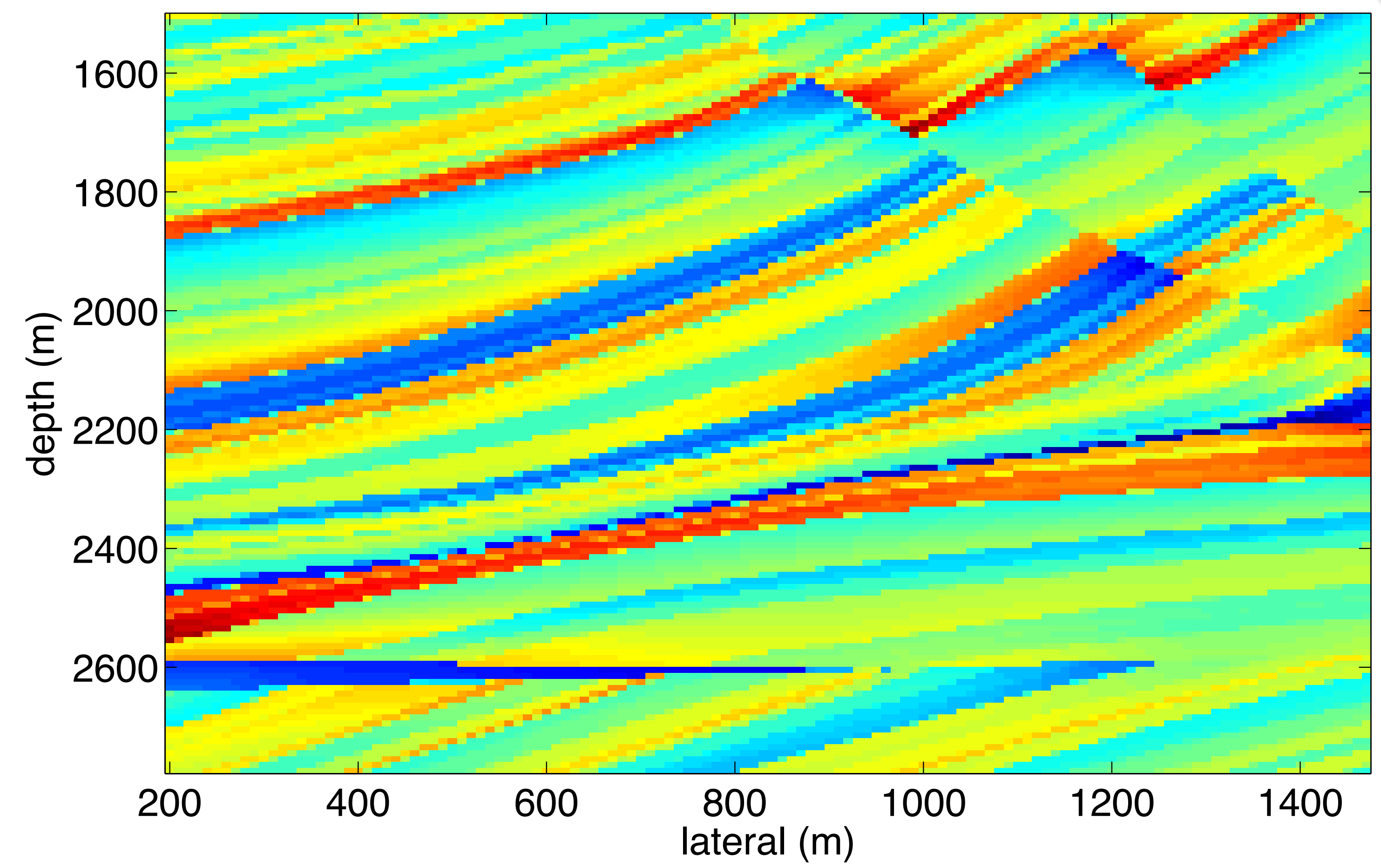
Motivation



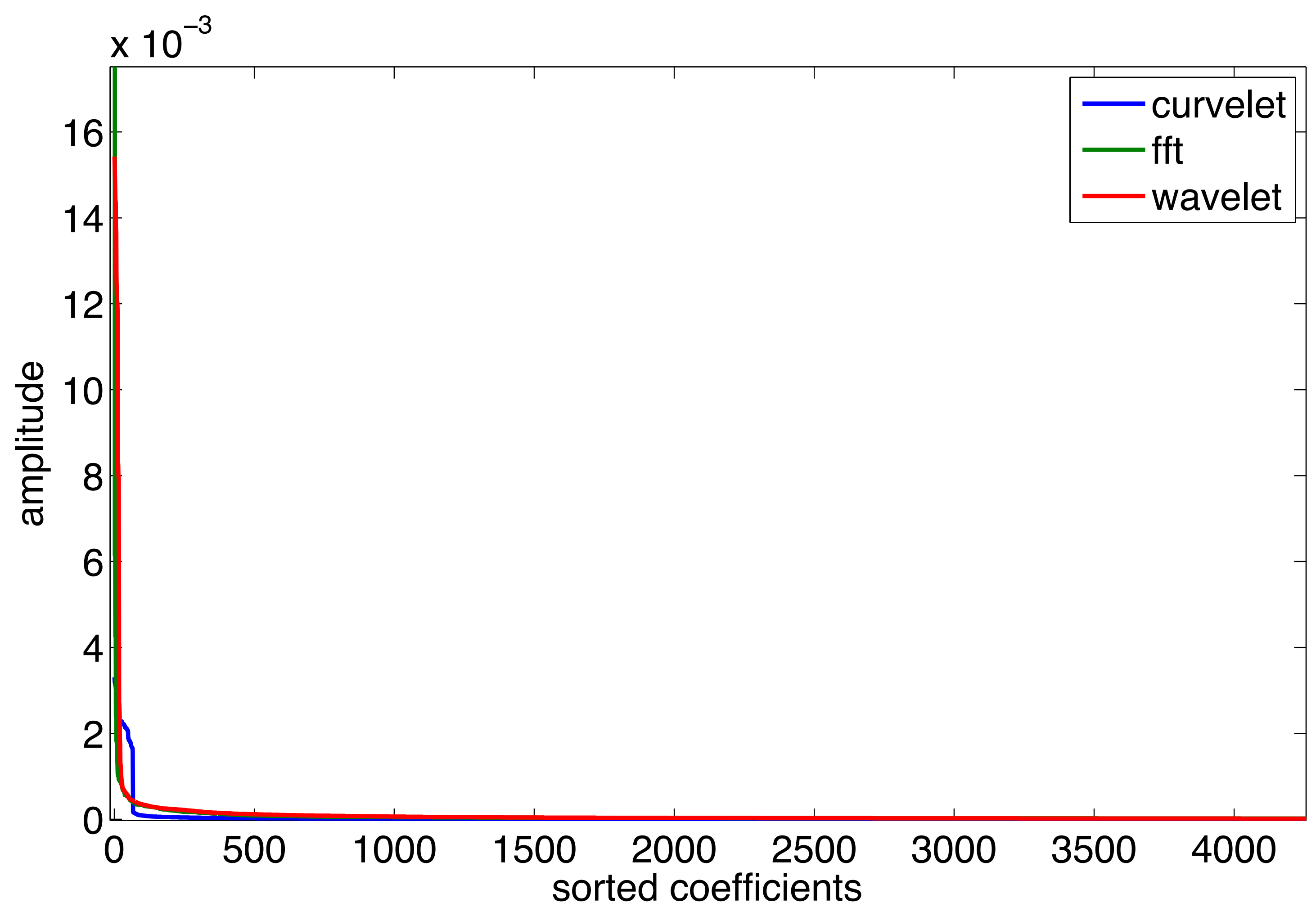
true model



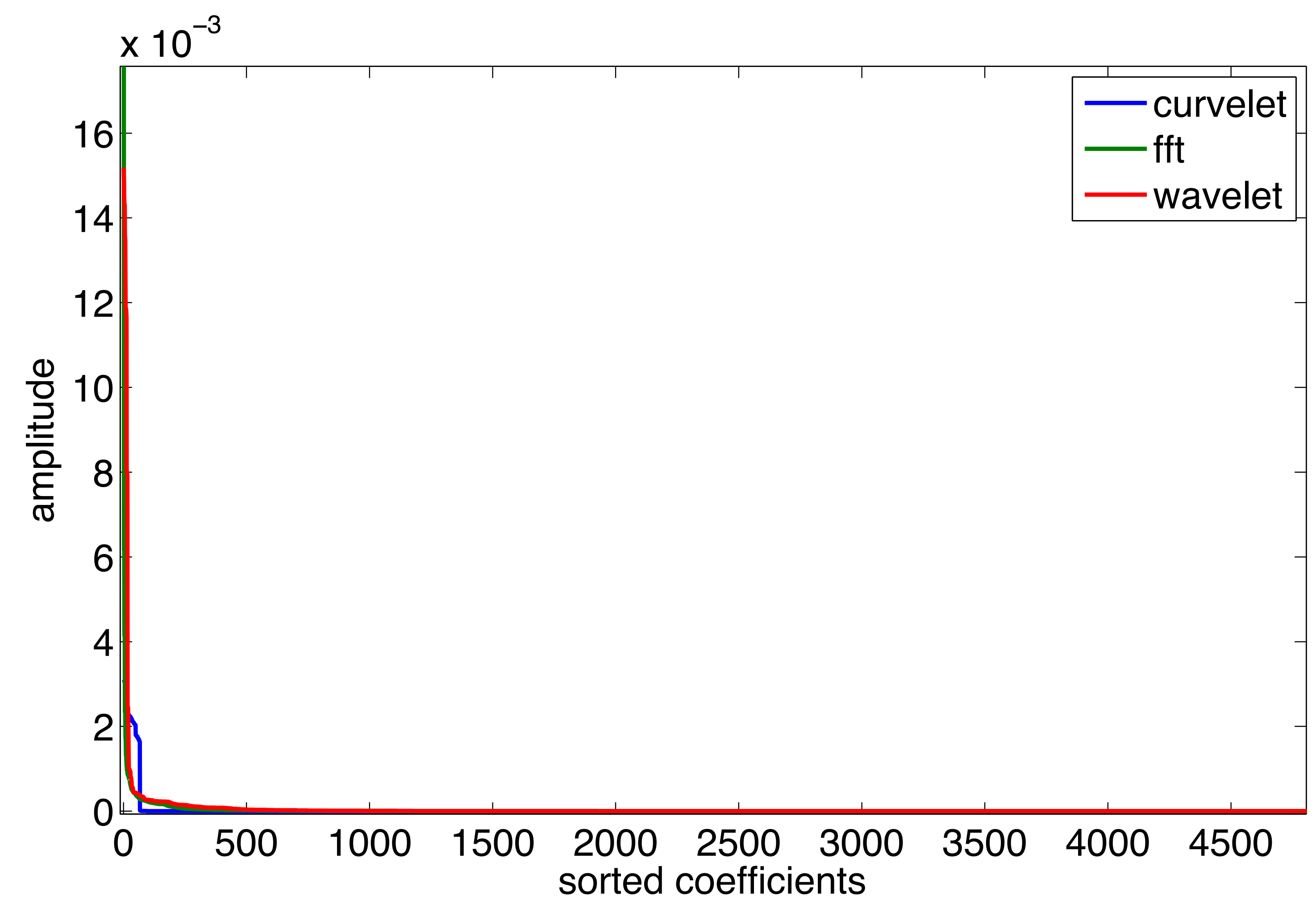
initial model



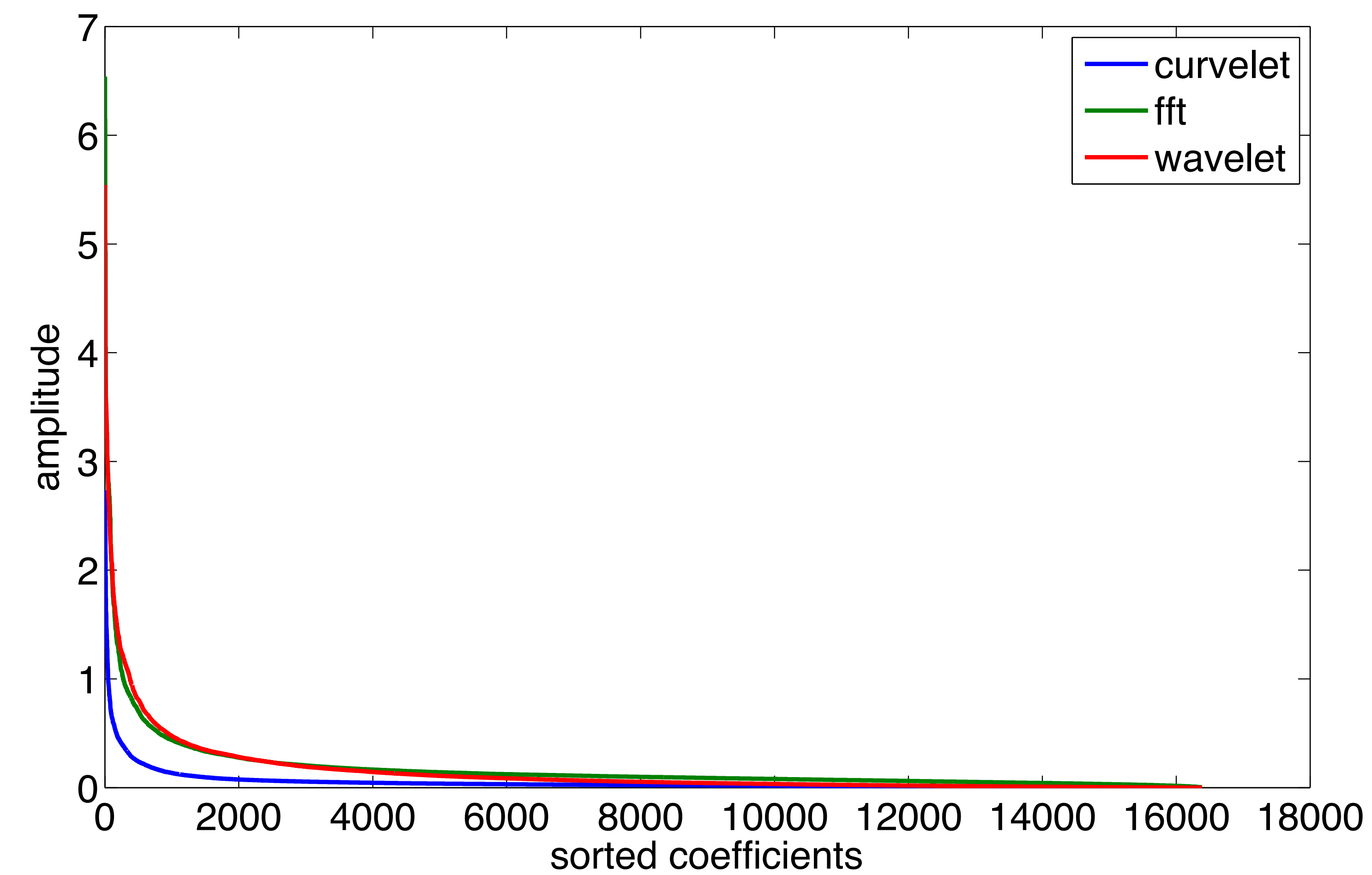
perturbation



transform domain coefficients



transform domain coefficients



transform domain coefficients

What is the modified Gauss-Newton?

FWI objective

$$\Phi(\mathbf{m}) := \frac{1}{2}\|\underbrace{\mathbf{D} - \mathbf{P}_r\mathbf{H}(\mathbf{m})^{-1}\mathbf{Q}}_{\delta\mathbf{D}}\|_F^2$$

gradient (action of Jacobian adjoint or RTM operator)

$$\nabla\Phi(\mathbf{m}) := \mathbf{J}^{\mathrm{T}}\delta\mathbf{D} = \mathrm{conj}\left(\frac{\partial\mathbf{H}}{\partial\mathbf{m}}(\mathbf{H}^{-1}\mathbf{Q})\right) \odot \left((\mathbf{H}^{\mathrm{T}})^{-1}(\mathbf{P}_r^{\mathrm{T}}\delta\mathbf{D})\right)$$

Gauss-Newton update

$$\delta\mathbf{m} := \arg\min_{\delta\mathbf{m}} \|\delta\mathbf{D} - \mathbf{J}\delta\mathbf{m}\|_F^2$$

Gauss-Newton analytic solution: $\delta\mathbf{m} = (\underbrace{\mathbf{J}^{\mathrm{T}}\mathbf{J}}_{\text{Gauss-Newton Hessian of FWI objective}})^{-1}\underbrace{\mathbf{J}^{\mathrm{T}}\delta\mathbf{D}}_{\text{RTM}}$

the modified Gauss-Newton update

$$\delta\mathbf{m} := \mathbf{S}^{\mathrm{T}}\arg\min_{\mathbf{x}} \|\delta\mathbf{D} - \mathbf{J}\mathbf{S}^{\mathrm{T}}\mathbf{x}\|_F^2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

The modified Gauss-Newton algorithm

Algorithm 1: the modified Gauss-Newton method for FWI

Result: Output estimate for the model $\mathbf{m}$

$\mathbf{m} \longleftarrow \mathbf{m}_0;\ k \longleftarrow 0; \epsilon$; `// initial model`

while $\frac{1}{2}\|\mathbf{D} - \mathbf{P}_r\mathbf{H}(\mathbf{m})^{-1}\mathbf{Q}\|_F^2 \geq \epsilon$ **do**

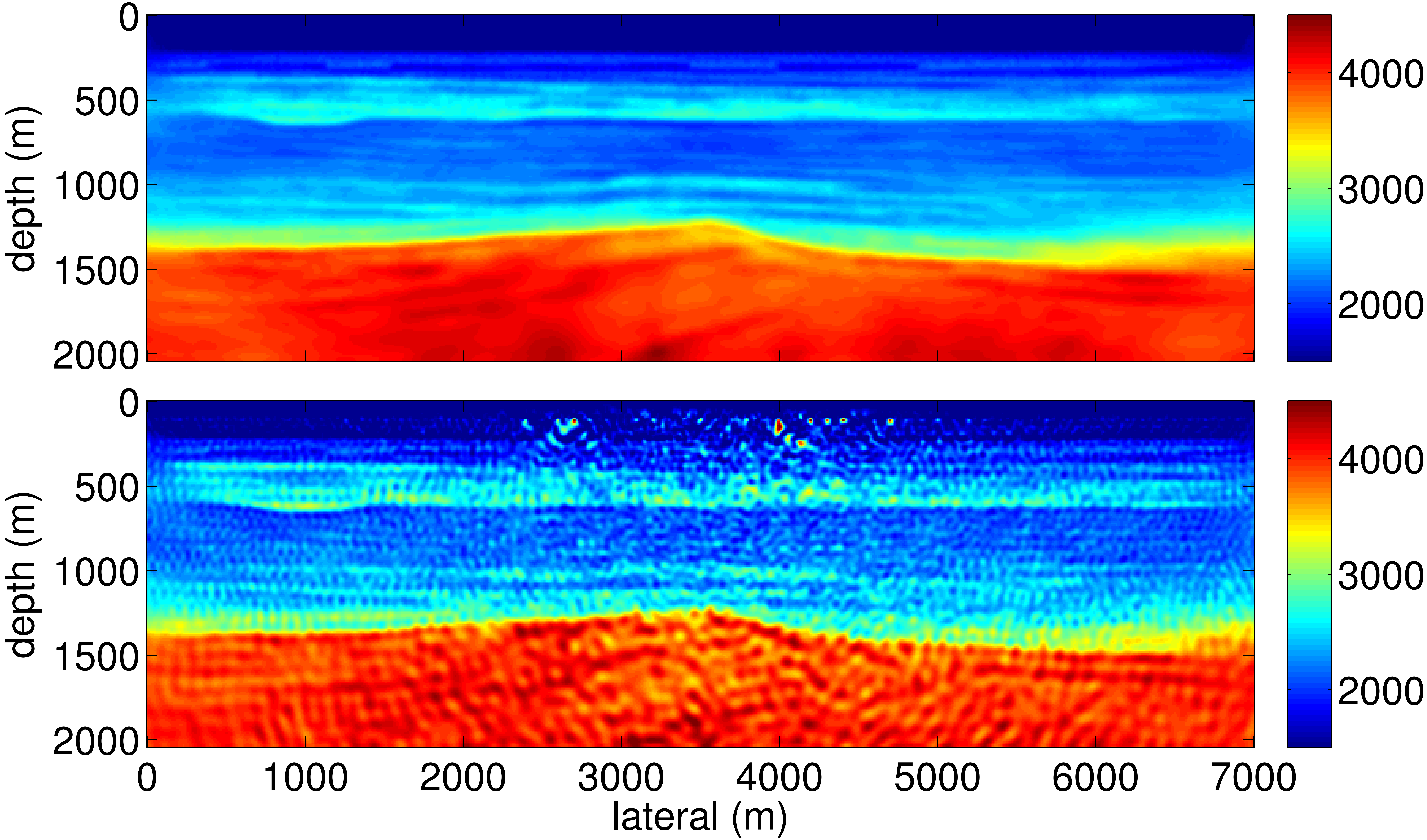
$\quad \delta\mathbf{m}^k := \mathbf{S}^{\mathrm{T}} \arg\min_{\mathbf{x}} \|\delta\mathbf{D} - \mathbf{J}\mathbf{S}^{\mathrm{T}}\mathbf{x}\|_F^2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau^k$

$\quad \mathbf{m}^{k+1} \longleftarrow \mathbf{m}^k + \alpha^k\delta\mathbf{m}^k$; `// update with linesearch`

$\quad k \longleftarrow k+1;$

end

Inversion results



the modified Gauss-Newton method with L1 constraint

the modified Gauss-Newton method with L2 constraint

Observation & question

- modified Gauss-Newton only promotes sparsity on individual updates
- modified Gauss-Newton does NOT change the FWI objective function

why the *sum* of all modified Gauss-Newton updates is sparse?

SLIM

Least-squares optimization problem

unconstrained objective function

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{m}]\|_2^2 \right\}$$

Gauss-Newton update

$$\delta\mathbf{m} = \arg\min_{\delta\mathbf{m}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\delta\mathbf{m}\|_2$$

the modified Gauss-Newton update

$$\delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x}\|_{\ell_1} \leq \tau$$

lasso problem

Least-squares optimization with sparse constraint

objective function with sparse constraint

$$\min_{\mathbf{x}} \Phi(\mathbf{x}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{S}^{\mathrm{H}}\mathbf{x}]\|_2^2 \right\} \quad \text{subject to} \quad \|\mathbf{x} - \mathbf{x}_0\|_{\ell_1} \leq \tau$$

Gauss-Newton update

$$\delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \nabla\boldsymbol{\mathcal{F}}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x} + \mathbf{x}_k - \mathbf{x}_0\|_{\ell_1} \leq \tau$$

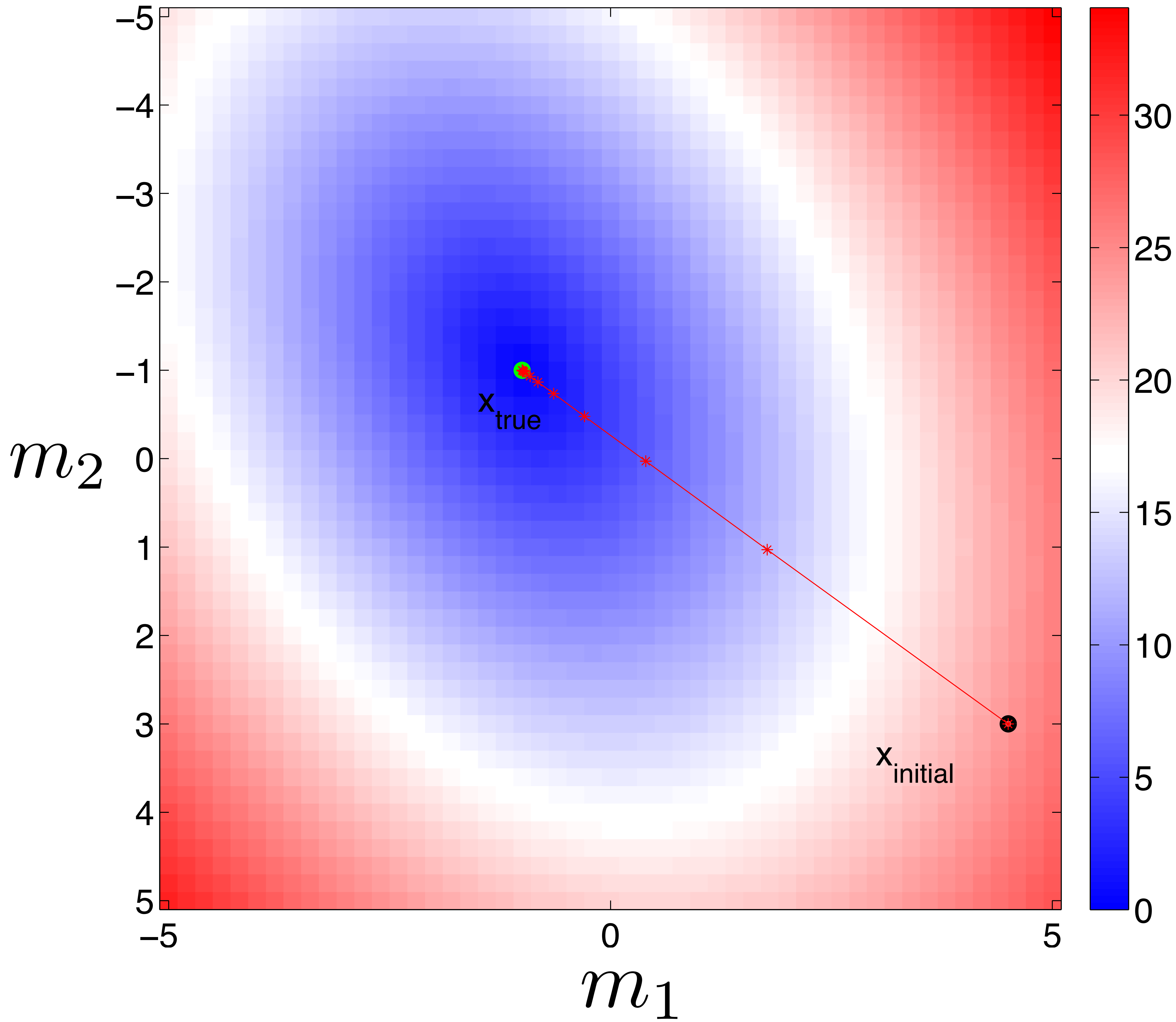
Convex problem with unique solution

$$\Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \mathbf{A}\mathbf{m}\|_2^2 \right\}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & -3 \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$$

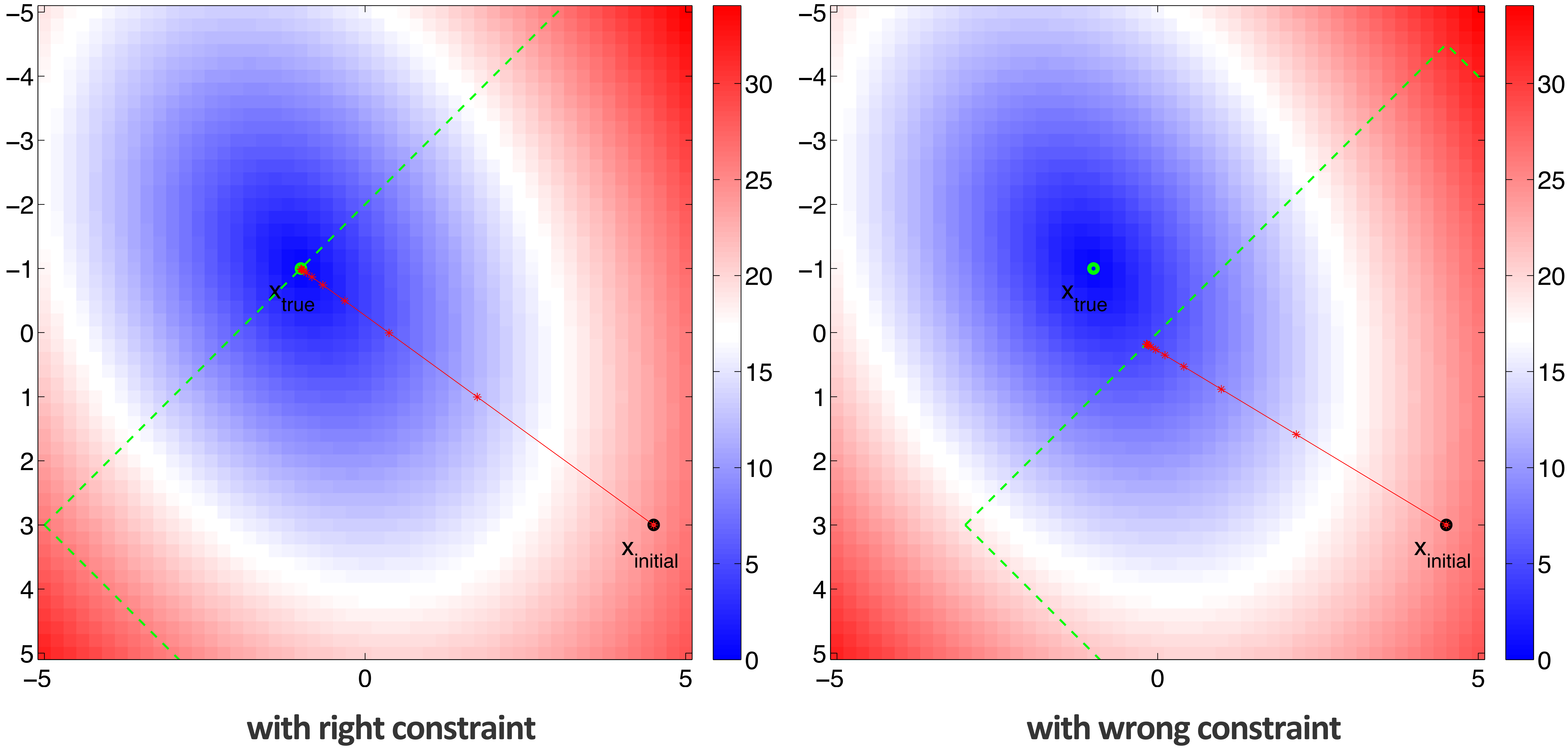
$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$



GN with unconstrained objective

GN with sparse constrained objective function

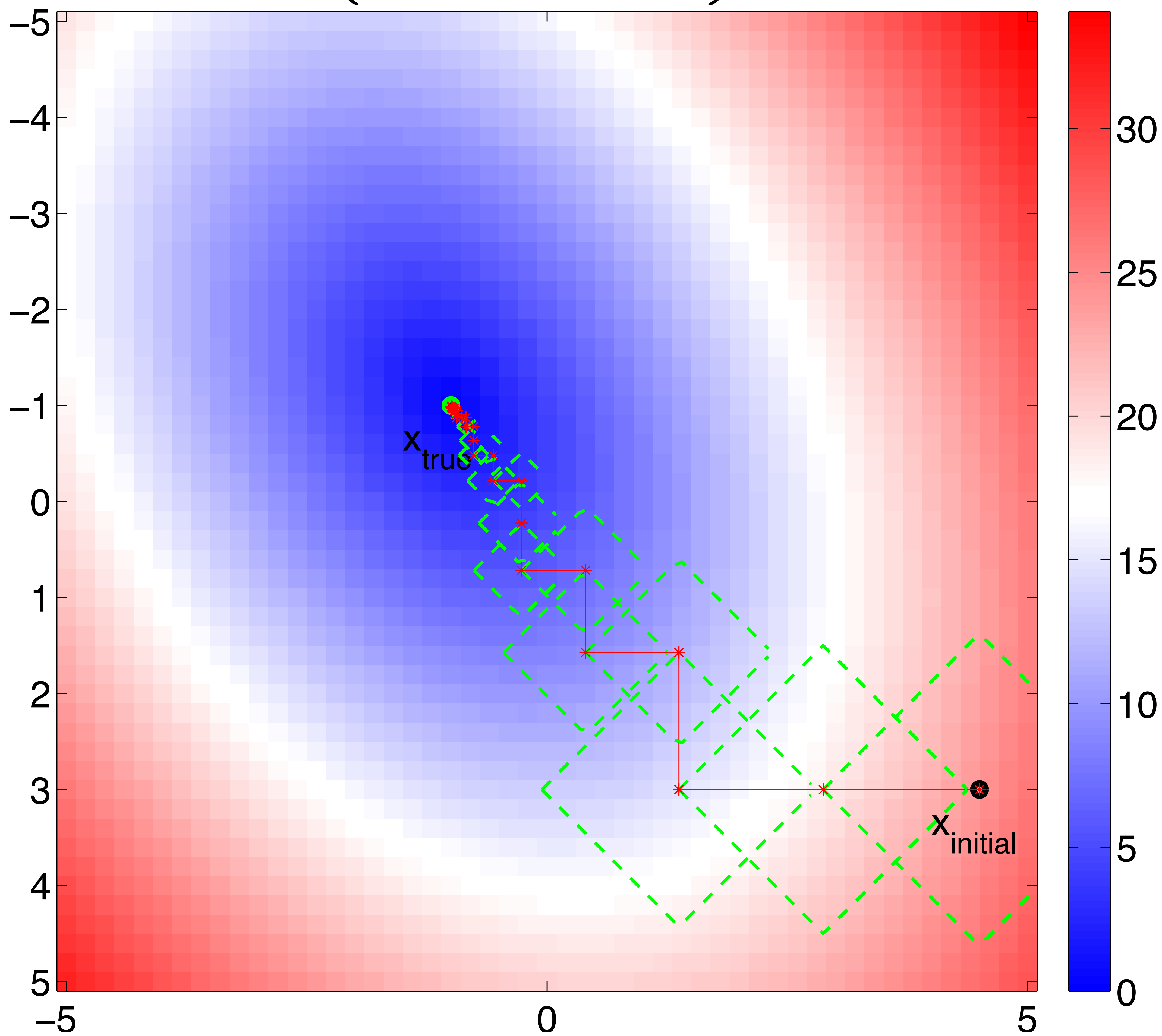
$$\min_{\mathbf{x}} \Phi(\mathbf{x}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{S}^{\mathrm{H}}\mathbf{x}]\|_2^2 \right\} \quad \text{subject to} \quad \|\mathbf{x} - \mathbf{x}_0\|_{\ell_1} \leq \tau$$



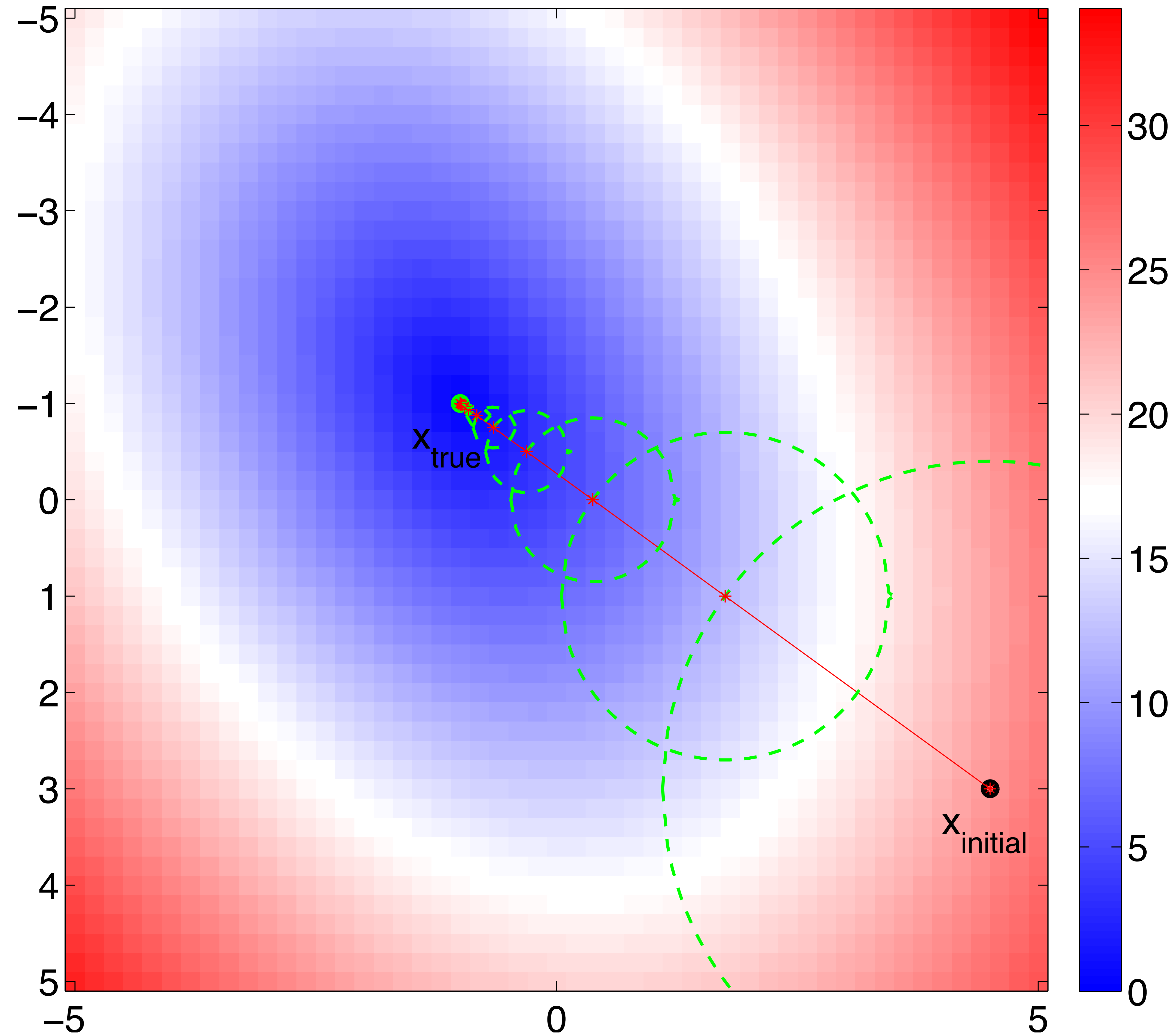
The modified Gauss-Newton

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{m}]\|_2^2 \right\}$$

$$\delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x}\|_{\ell_1} \leq \tau$$



with ℓ_1 constraint



with ℓ_2 constraint

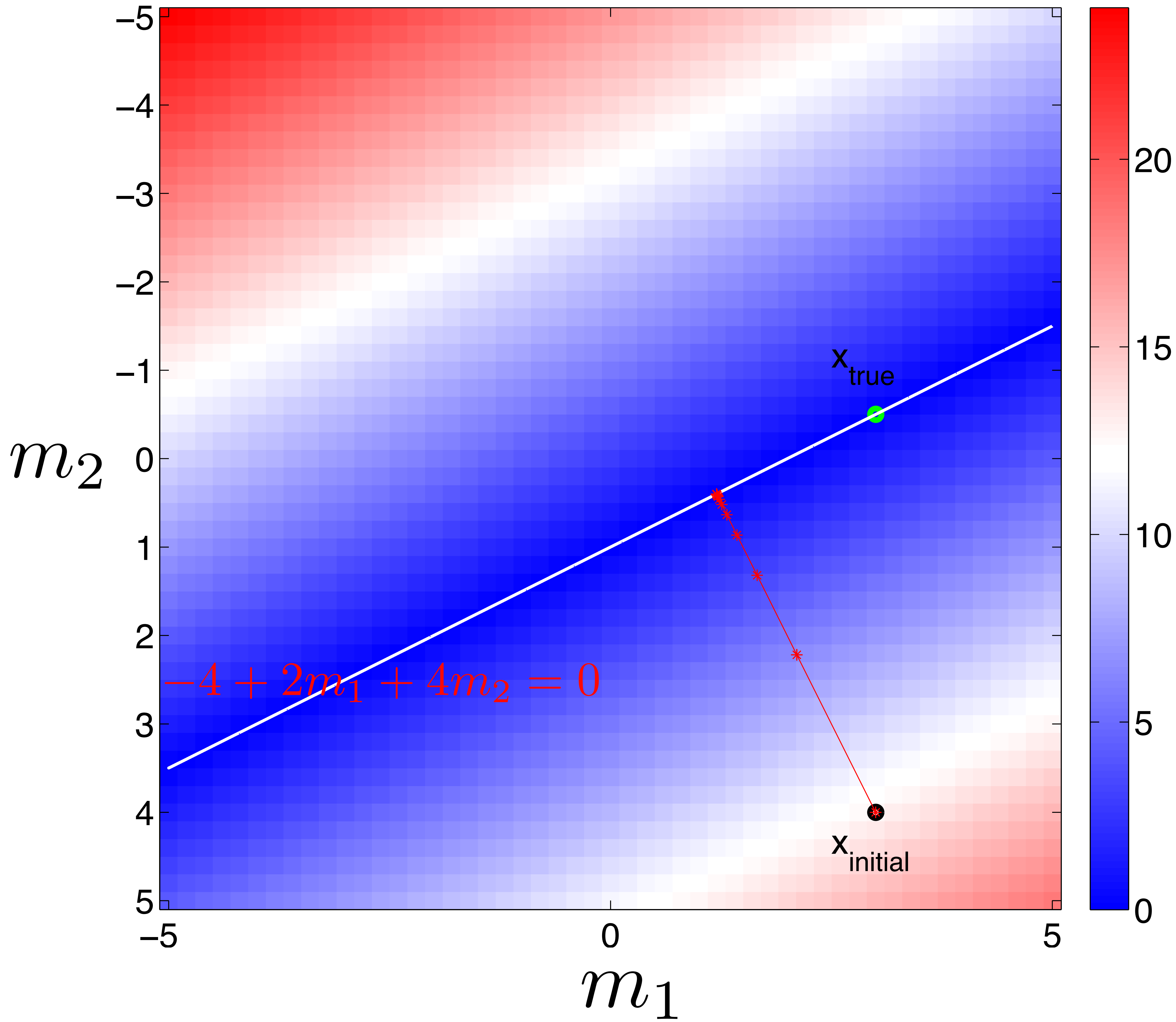
Linear example with multiple solutions (underdetermined)

$$\Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \mathbf{A}\mathbf{m}\|_2^2 \right\}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

$$\mathbf{d} = -4$$

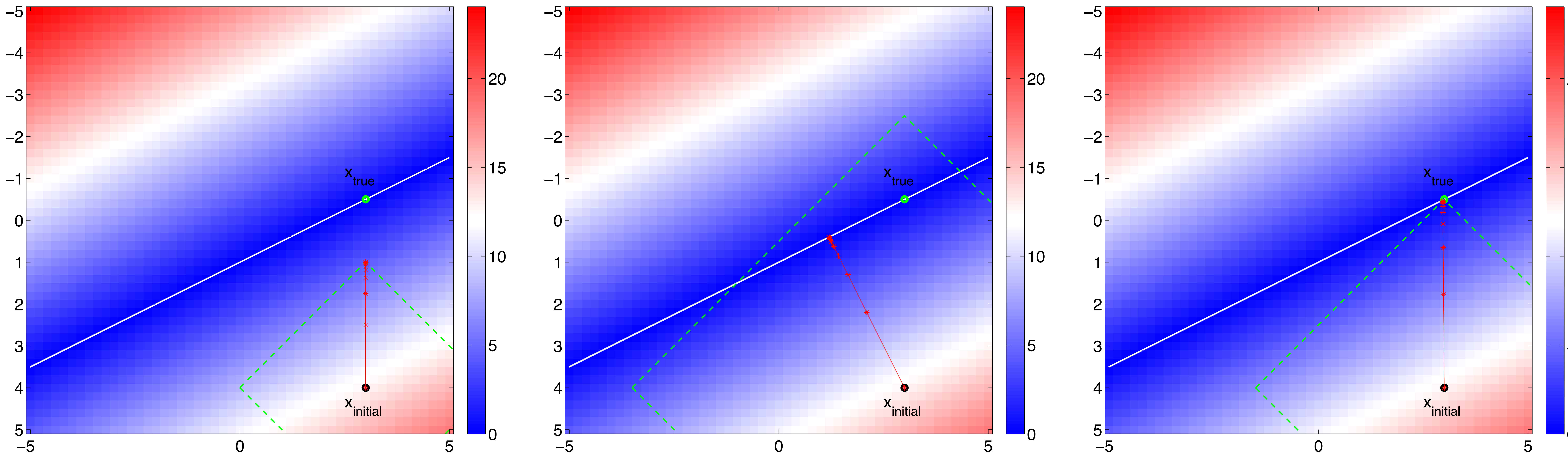
$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$



GN with unconstrained objective

GN with sparse constrained objective function

$$\min_{\mathbf{x}} \Phi(\mathbf{x}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{S}^{\mathrm{H}}\mathbf{x}]\|_2^2 \right\} \quad \text{subject to} \quad \|\mathbf{x} - \mathbf{x}_0\|_{\ell_1} \leq \tau$$



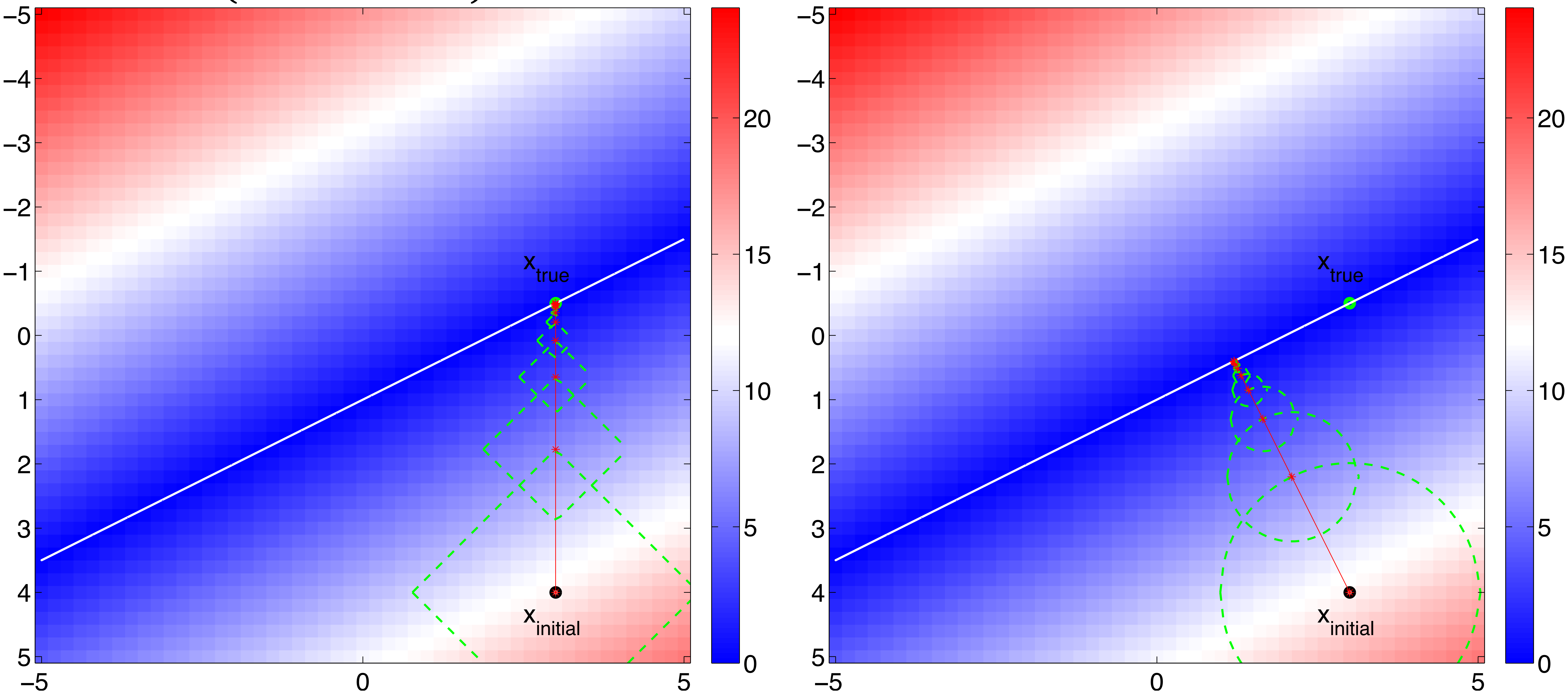
with wrong constraint — $\tau < \tau_{true}$

with wrong constraint — $\tau > \tau_{true}$

with right constraint — $\tau = \tau_{true}$

the modified Gauss-Newton

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{m}]\|_2^2 \right\} \qquad \delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x}\|_{\ell_1} \leq \tau$$



with ℓ_1 constraint **with ℓ_2 constraint**

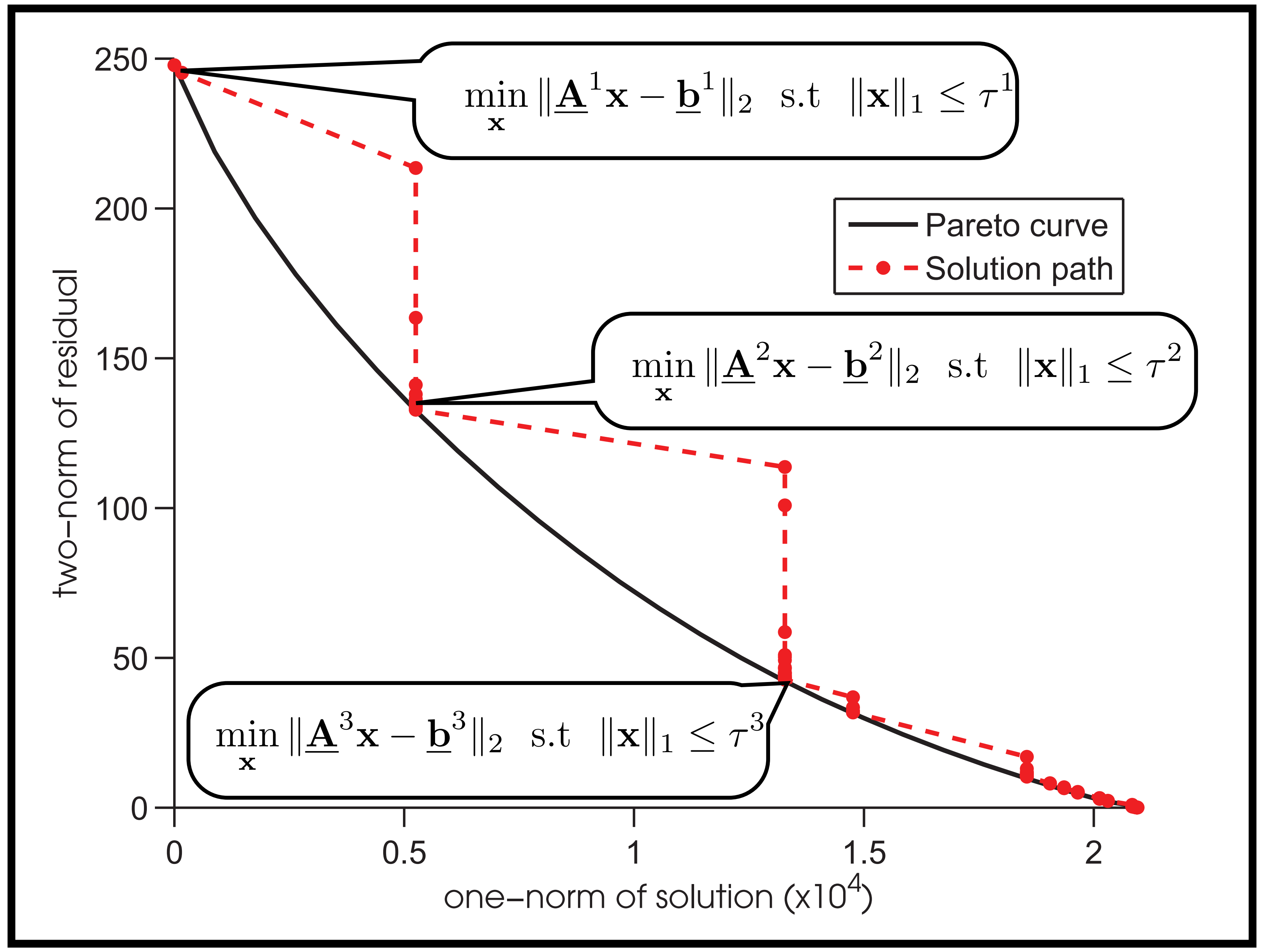
How to choose sparsity level for the modified GN

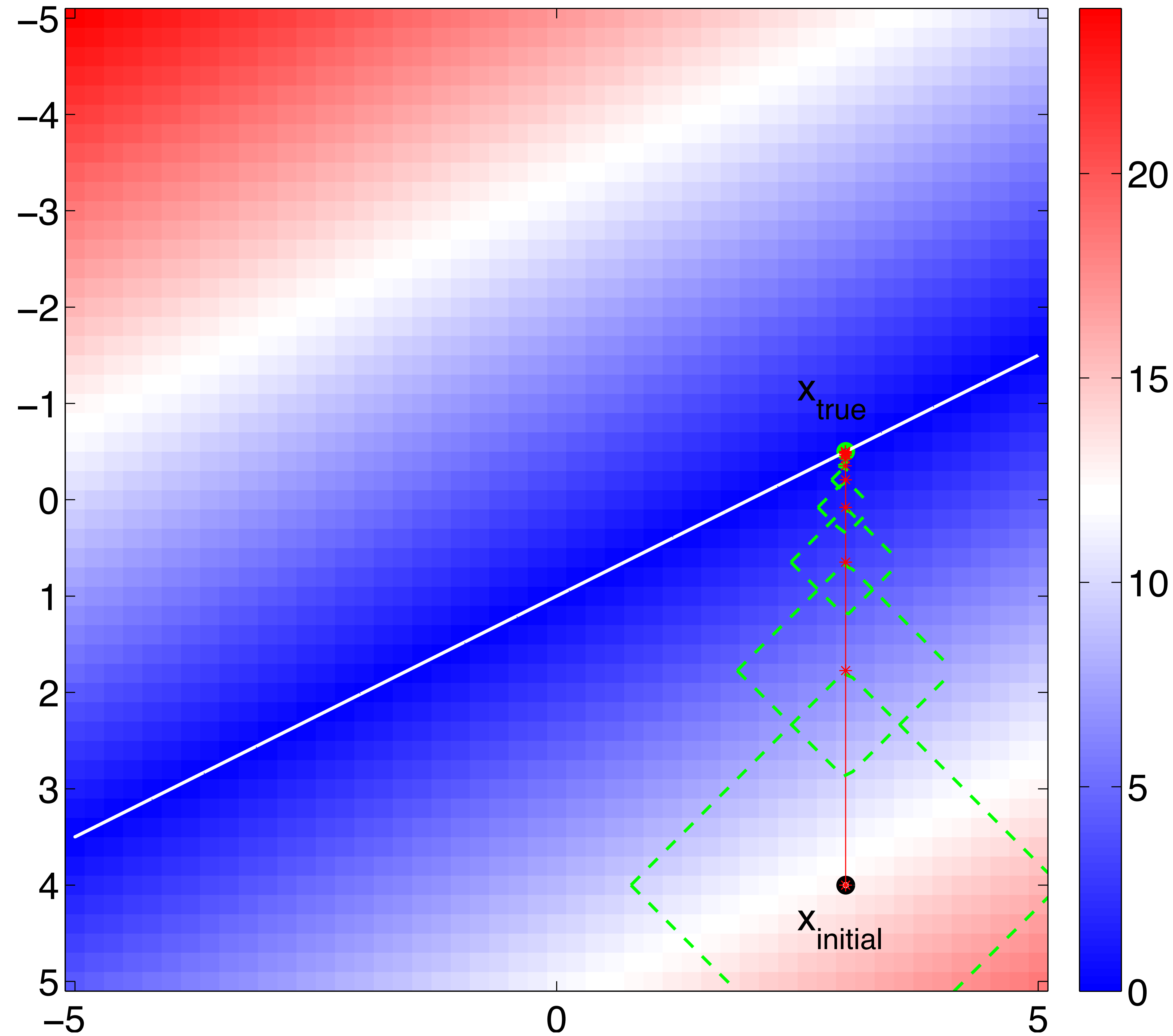
$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := \left\{\frac{1}{2}\|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{m}]\|_2^2\right\} \qquad \delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x}\|_{\ell_1} \leq \tau$$

BPDN problem

$$\delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{x}\|_1$$

$$\text{subject to} \quad \|\delta\mathbf{d} - \mathbf{J}\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \leq \sigma$$





Pareto reference

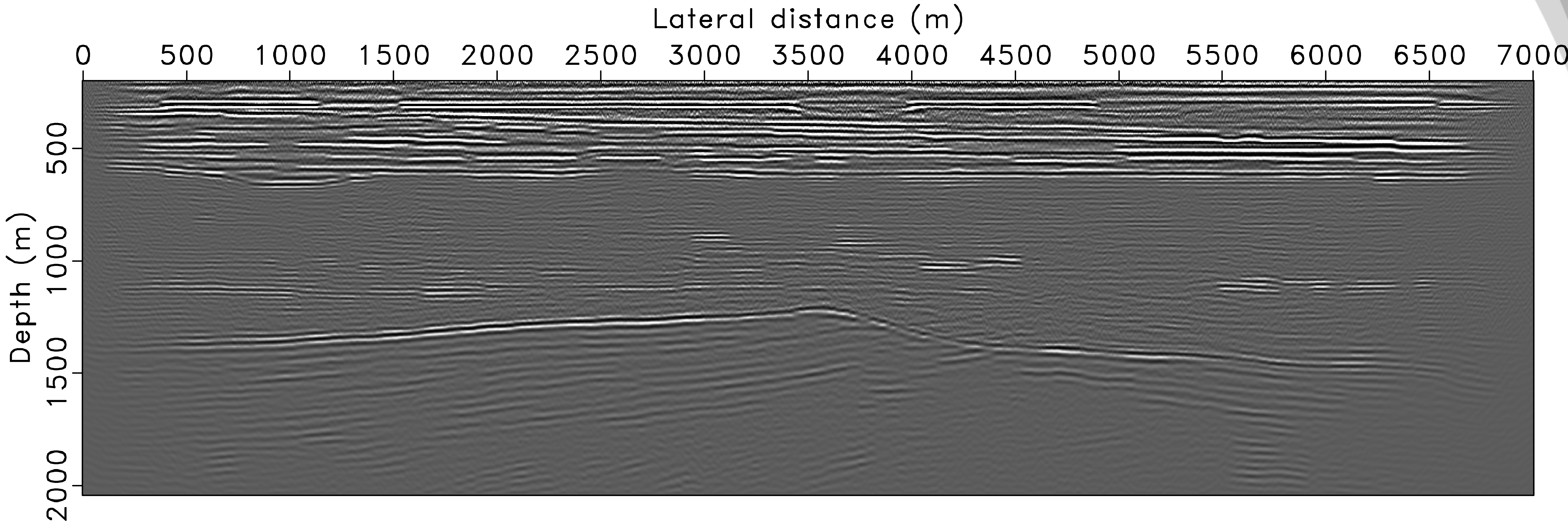
Least-squares migration

$$\delta \mathbf{m} = \arg\min_{\delta \mathbf{m}} \|\delta \mathbf{u} - \mathbf{J}\delta \mathbf{m}\|_2^2$$

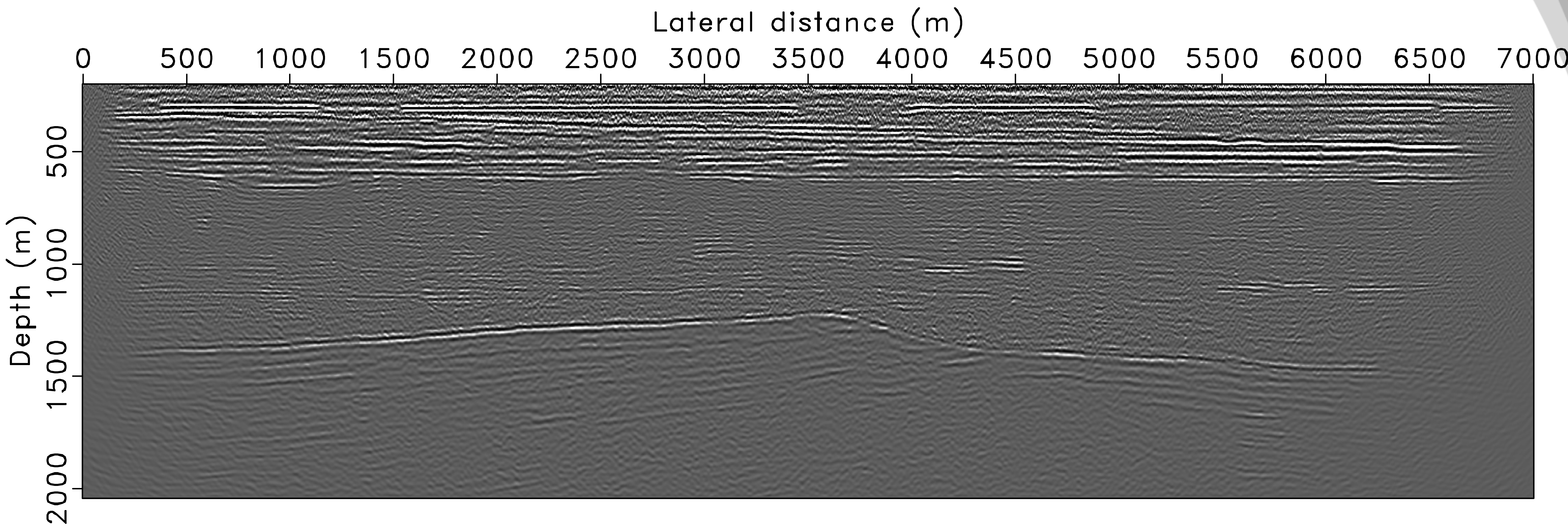
- 10 random frequencies (20Hz-50Hz)
- 17 randomly selected shots out of 350 shots
- LASSO problems determined by SPGL1

See "*Efficient least-squares imaging with sparsity promotion and compressive sensing*"

The modified GN with L1 constraint



The modified GN with L2 constraint



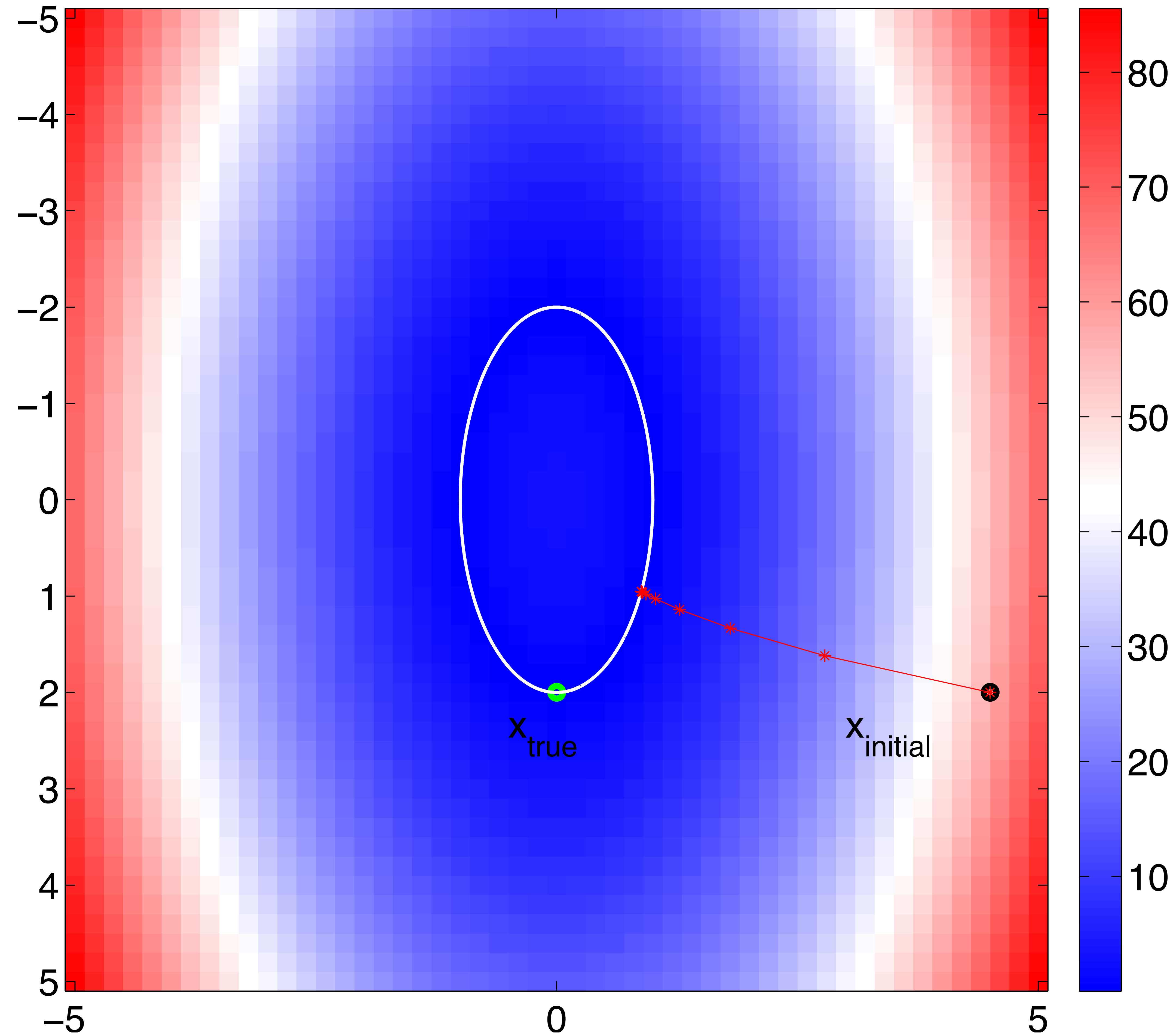
Nonlinear problem with multiple solutions

$$\|\mathbf{d} - \mathbf{m}^T \mathbf{A}^T \mathbf{A} \mathbf{m}\|_2^2$$

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{d} = -4$$

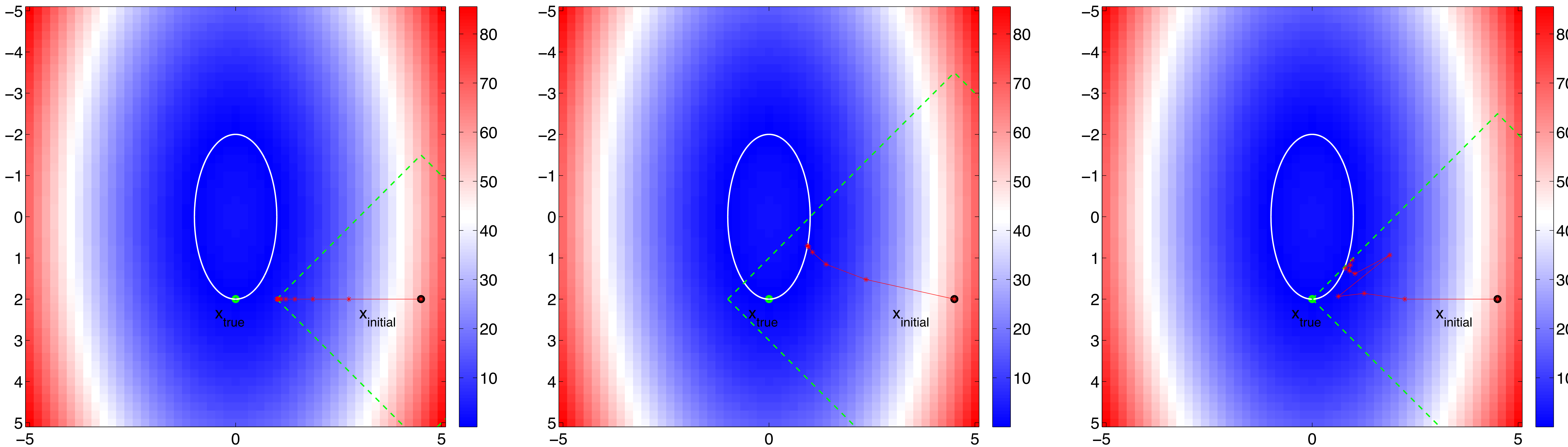
$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$



GN with unconstrained objective

GN with sparse constrained objective function

$$\min_{\mathbf{x}} \Phi(\mathbf{x}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{S}^{\mathrm{H}}\mathbf{x}]\|_2^2 \right\} \quad \text{subject to} \quad \|\mathbf{x} - \mathbf{x}_0\|_{\ell_1} \leq \tau$$



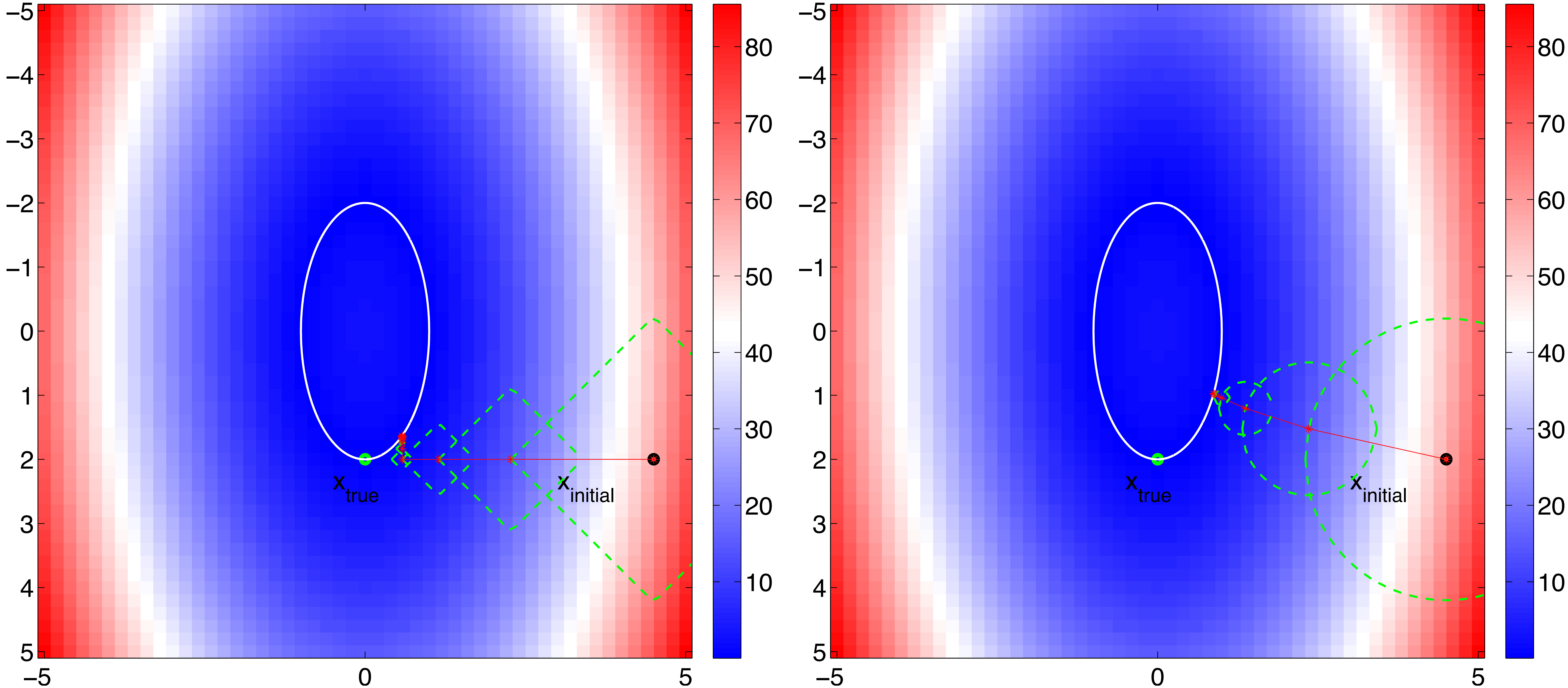
with wrong constraint $\tau < \tau_{true}$

with wrong constraint $\tau > \tau_{true}$

with right constraint $\tau = \tau_{true}$

The modified Gauss-Newton

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := \left\{ \frac{1}{2} \|\mathbf{d} - \boldsymbol{\mathcal{F}}[\mathbf{m}]\|_2^2 \right\} \qquad \delta\mathbf{m} = \mathbf{S}^{\mathrm{H}} \arg\min_{\delta\mathbf{x}} \|\delta\mathbf{d} - \mathbf{J}[\mathbf{m}_k]\mathbf{S}^{\mathrm{H}}\delta\mathbf{x}\|_2^2 \quad \text{subject to} \quad \|\delta\mathbf{x}\|_{\ell_1} \le \tau$$



Observations

- For convex problems with unique solution, the modified Gauss-Newton method will find the solution as other methods with unconstrained objective.

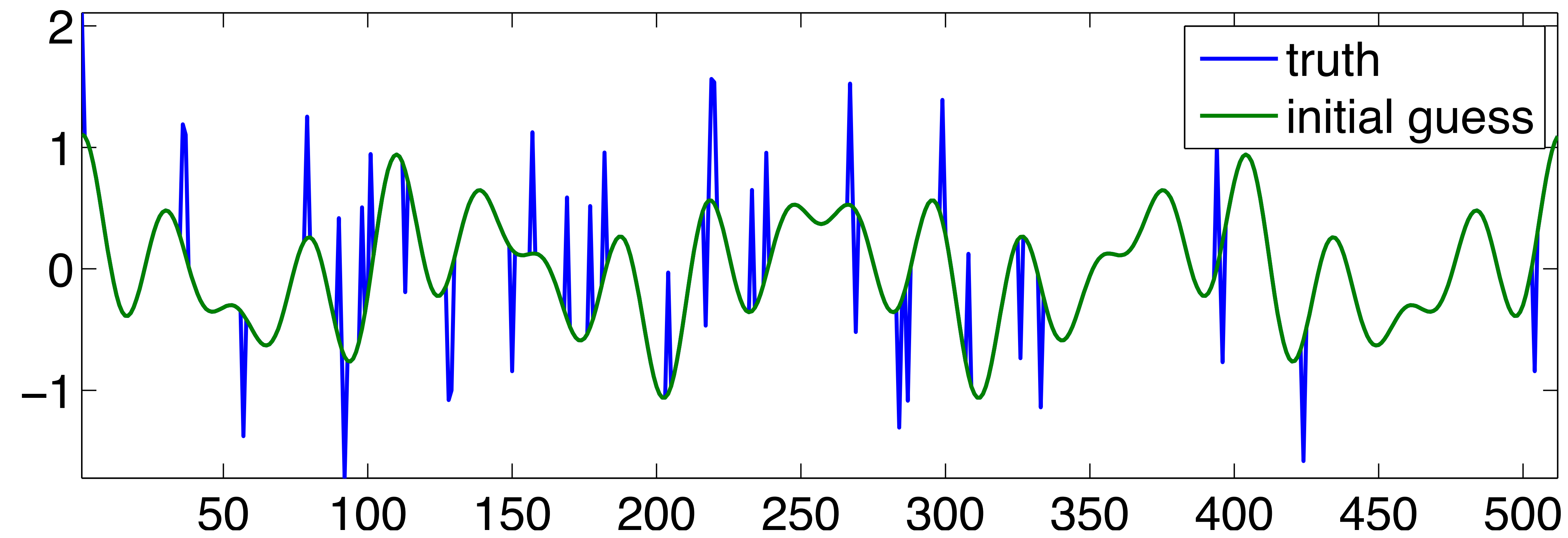
- For problems with multiple solutions, the modified Gauss-Newton method can find a solution with sparse perturbation of the initial guess, if updates share the same support.

The phase retrieval problem

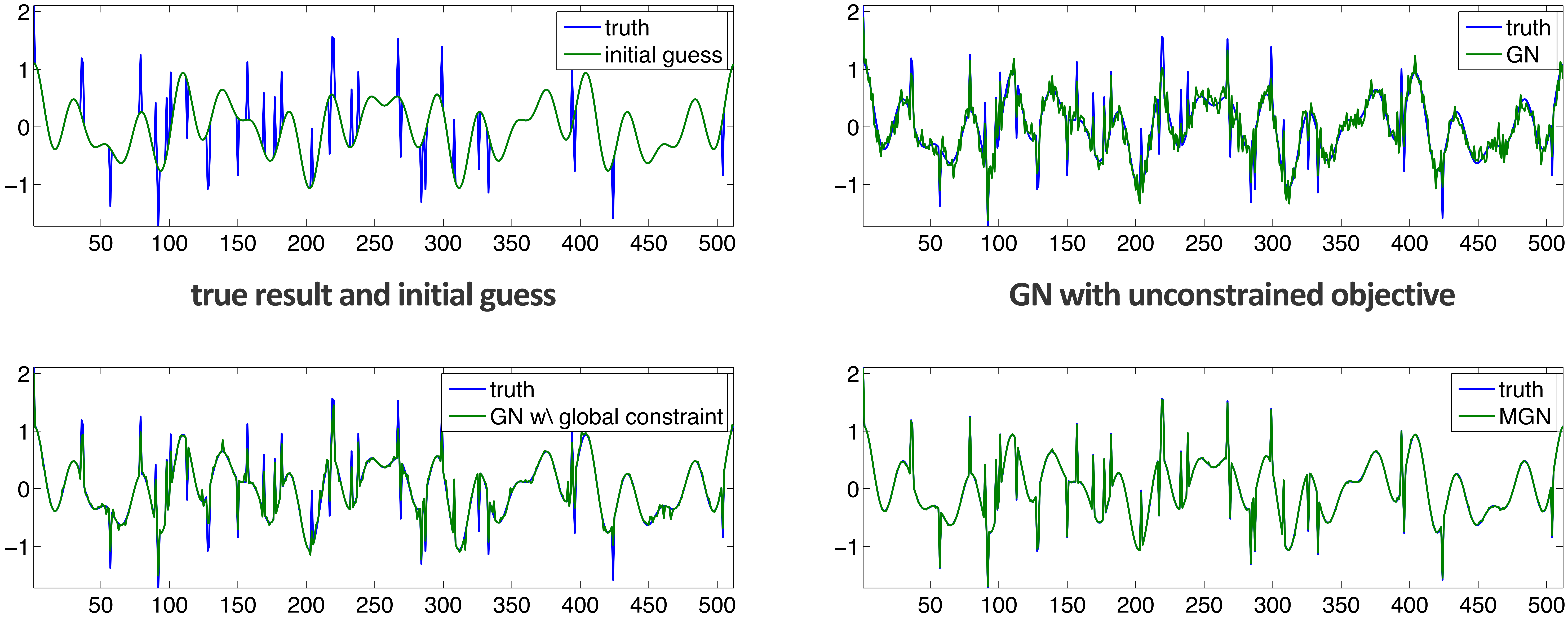
objective function:

$$\Phi(\mathbf{x}) := \left\{ \frac{1}{2} \| \mathbf{d} - \mathrm{diag}(\mathbf{Ax})(\mathbf{Ax}) \|_2^2 \right\}$$

$$\begin{aligned} \mathbf{A} &: 400 \times 512 \text{ matrix} \\ \mathbf{d} &: 400 \times 1 \text{ vector} \\ \mathbf{x} &: 512 \times 1 \text{ unknown vector} \end{aligned}$$



Results



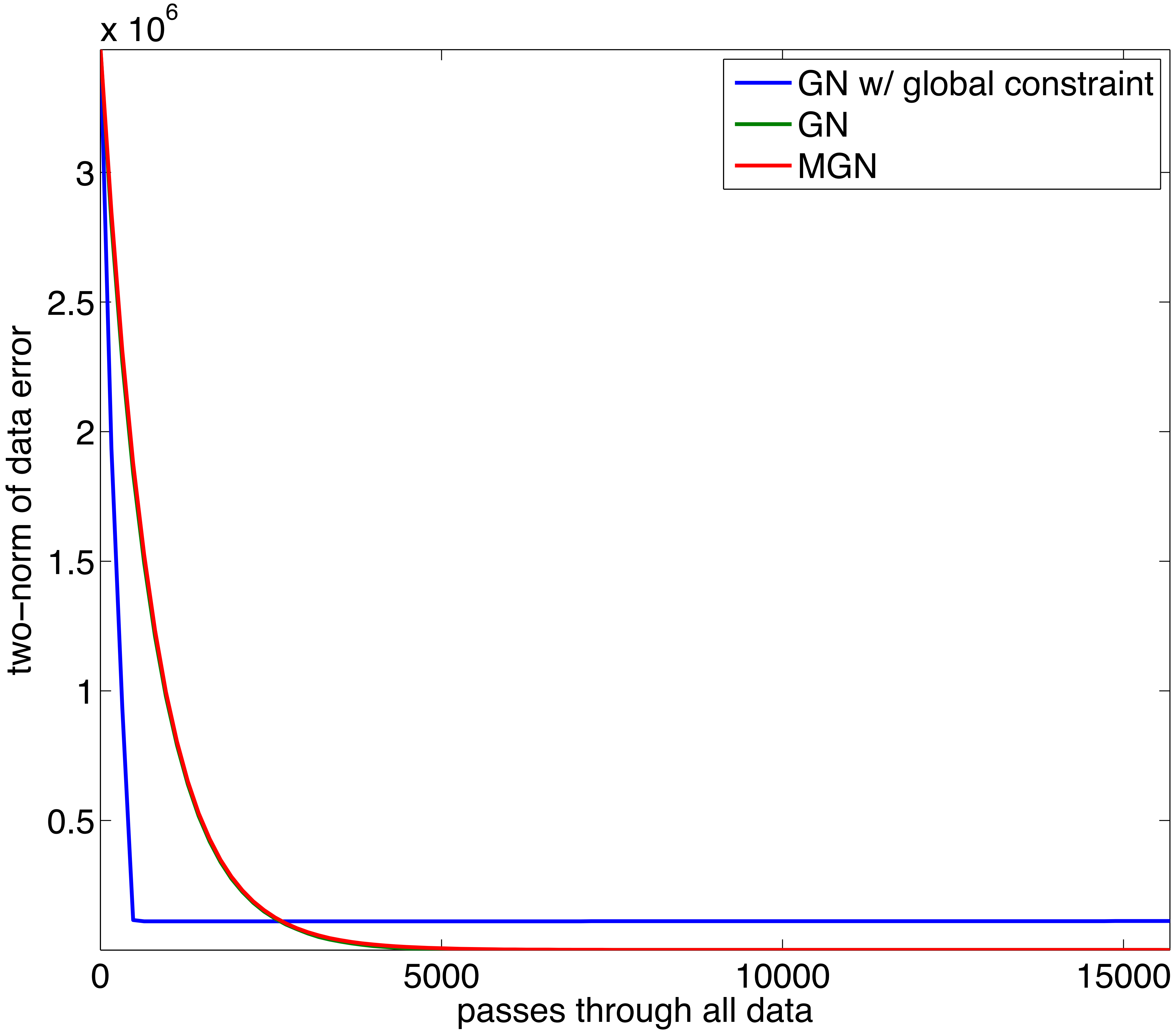
true result and initial guess

GN with unconstrained objective

GN with ℓ_1 constrained objective

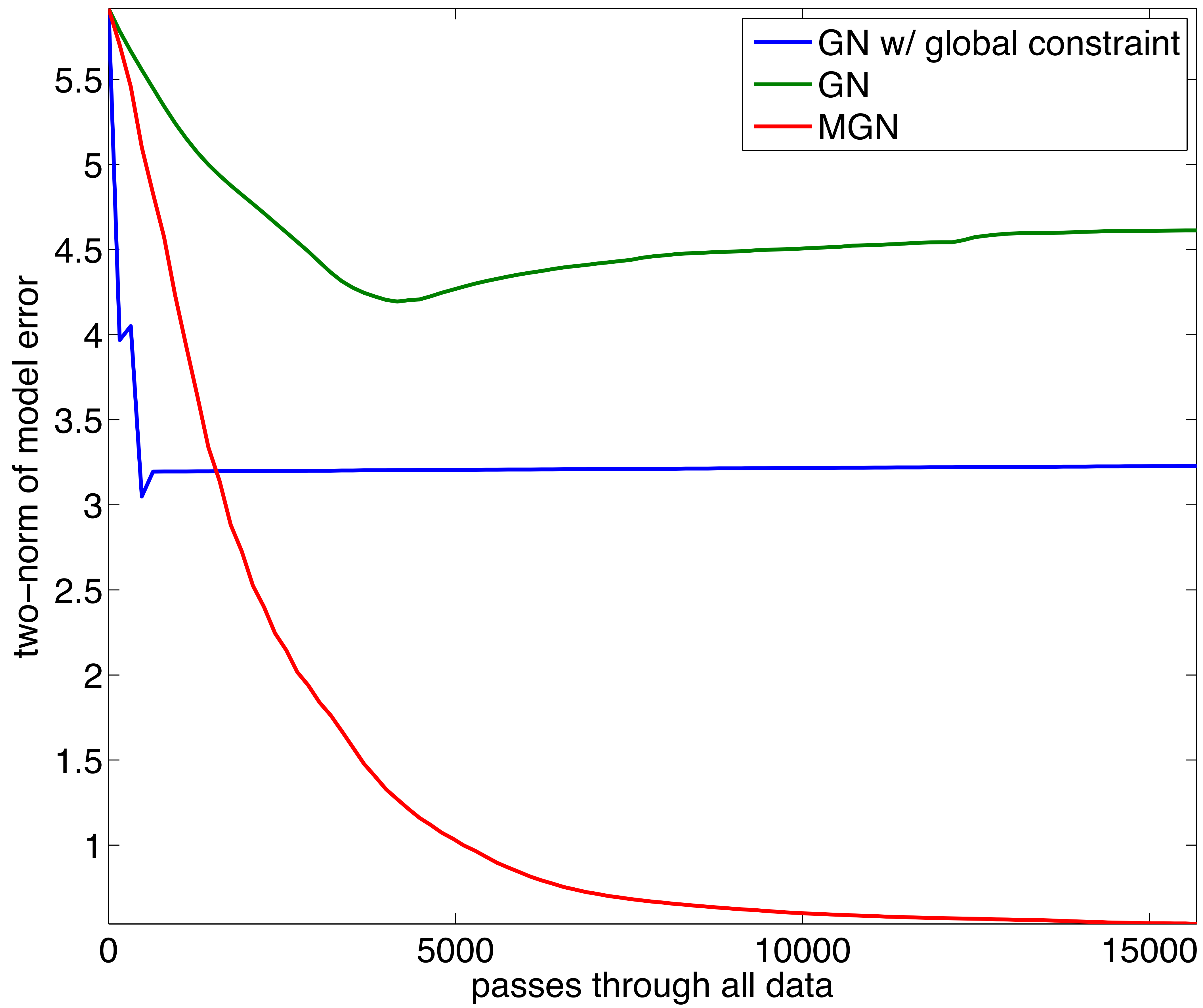
modified Gauss-Newton

Convergency



objective

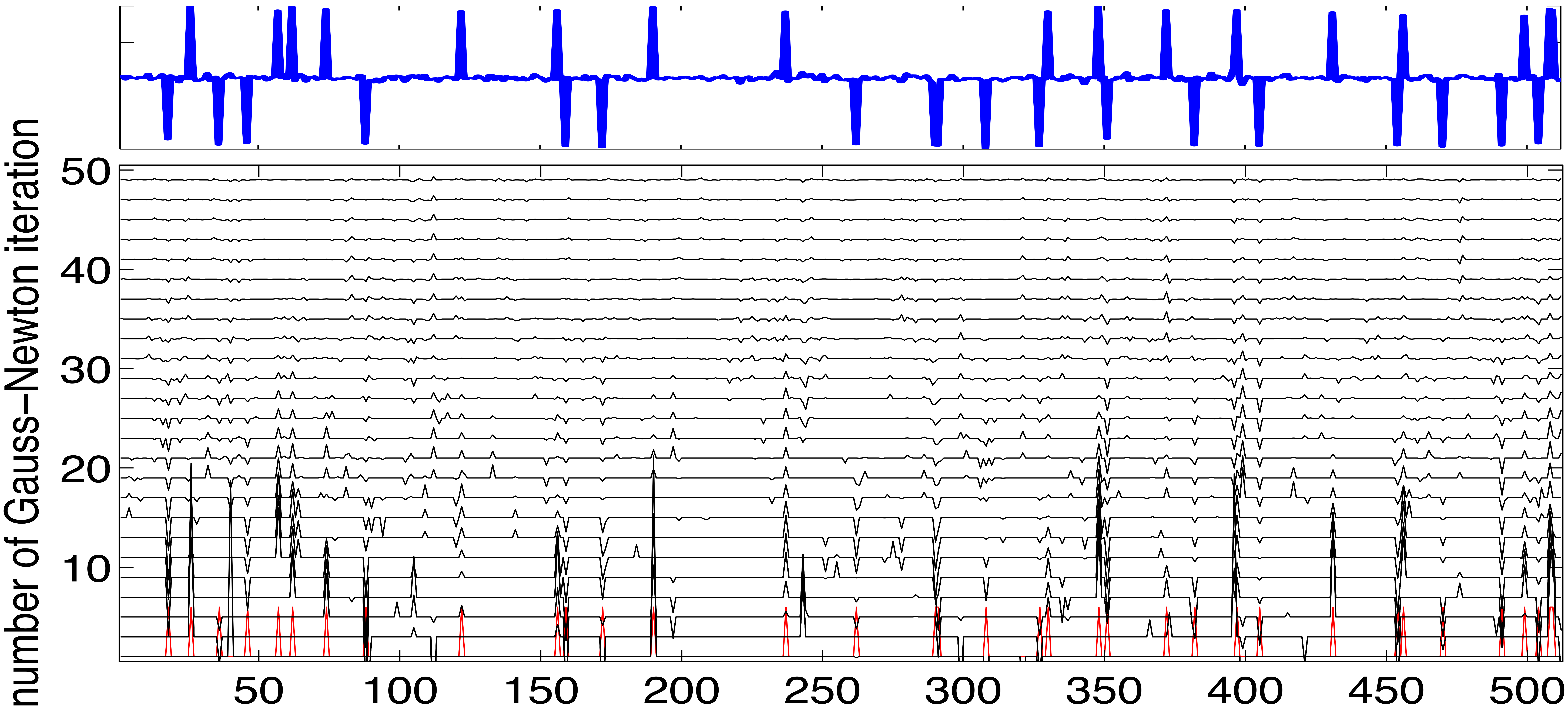
$$\frac{1}{2}\|\mathbf{d} - \mathrm{diag}(\mathbf{A}\mathbf{x}_k)(\mathbf{A}\mathbf{x}_k)\|_2^2$$



relative model error

$$\|\mathbf{x}_k - \mathbf{x}_{true}\|_2$$

Modified Gauss-Newton updates

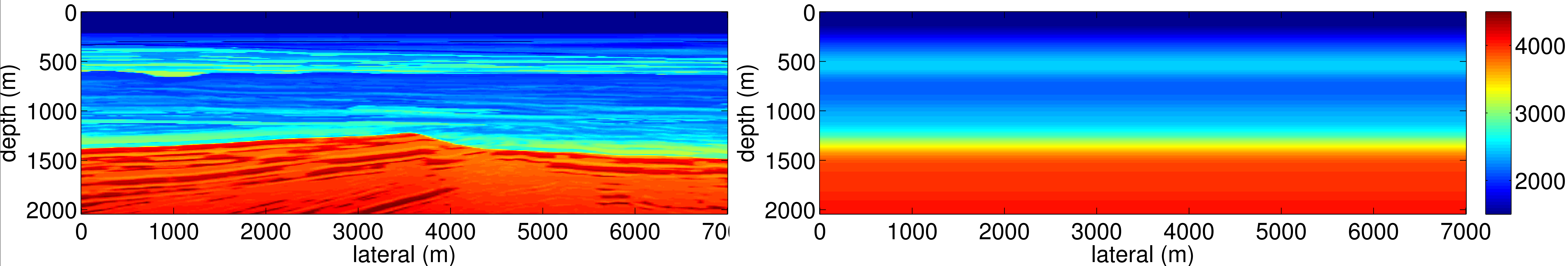


red is the position of sparse support

BG model example

BG Compass model

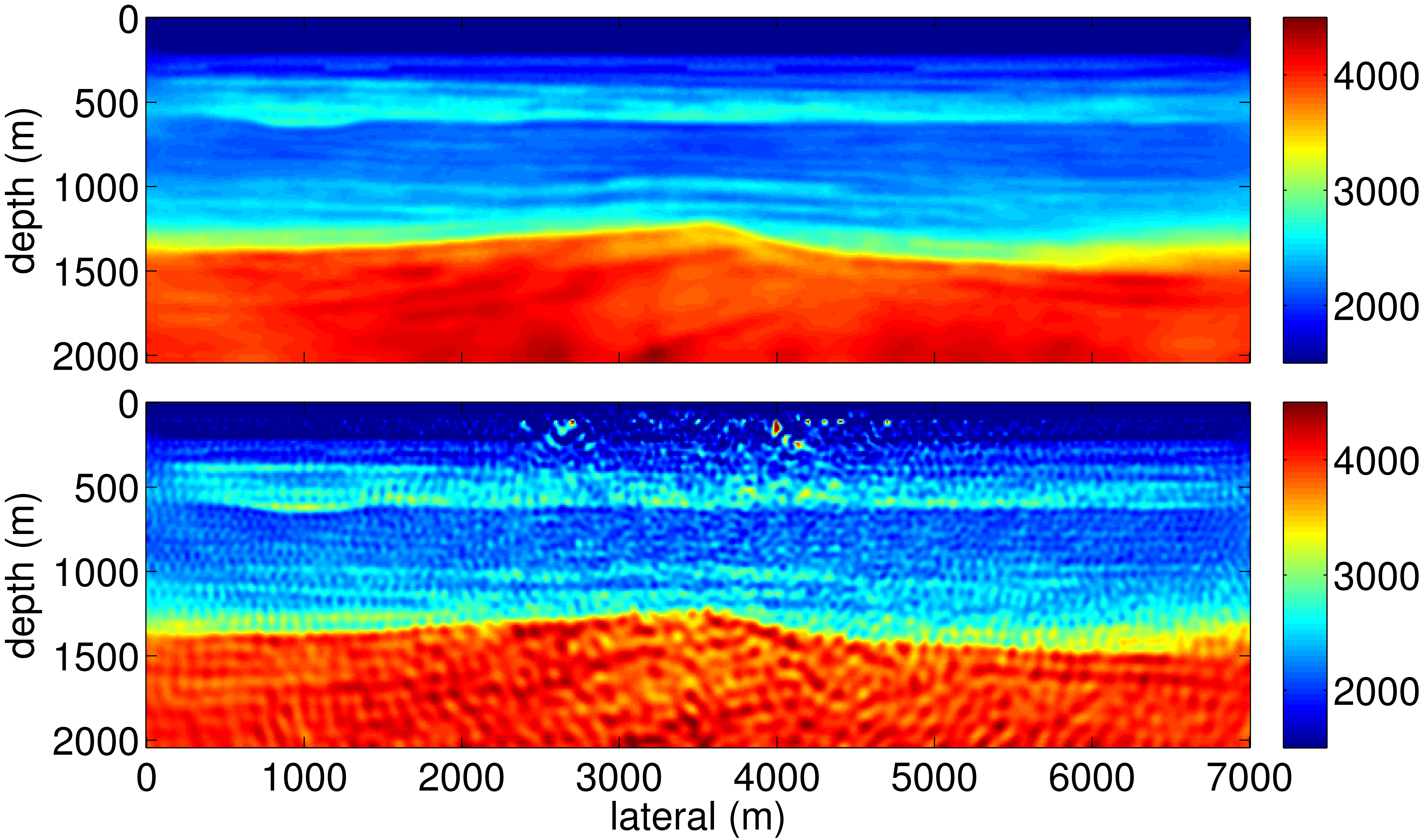
- 2 x 7 km
- 350 shot positions, 700 fixed receivers
- 3-15Hz, 10 frequency bands
- 5 GN updates for each band
- observed data is from time domain finite difference



true model

initial model

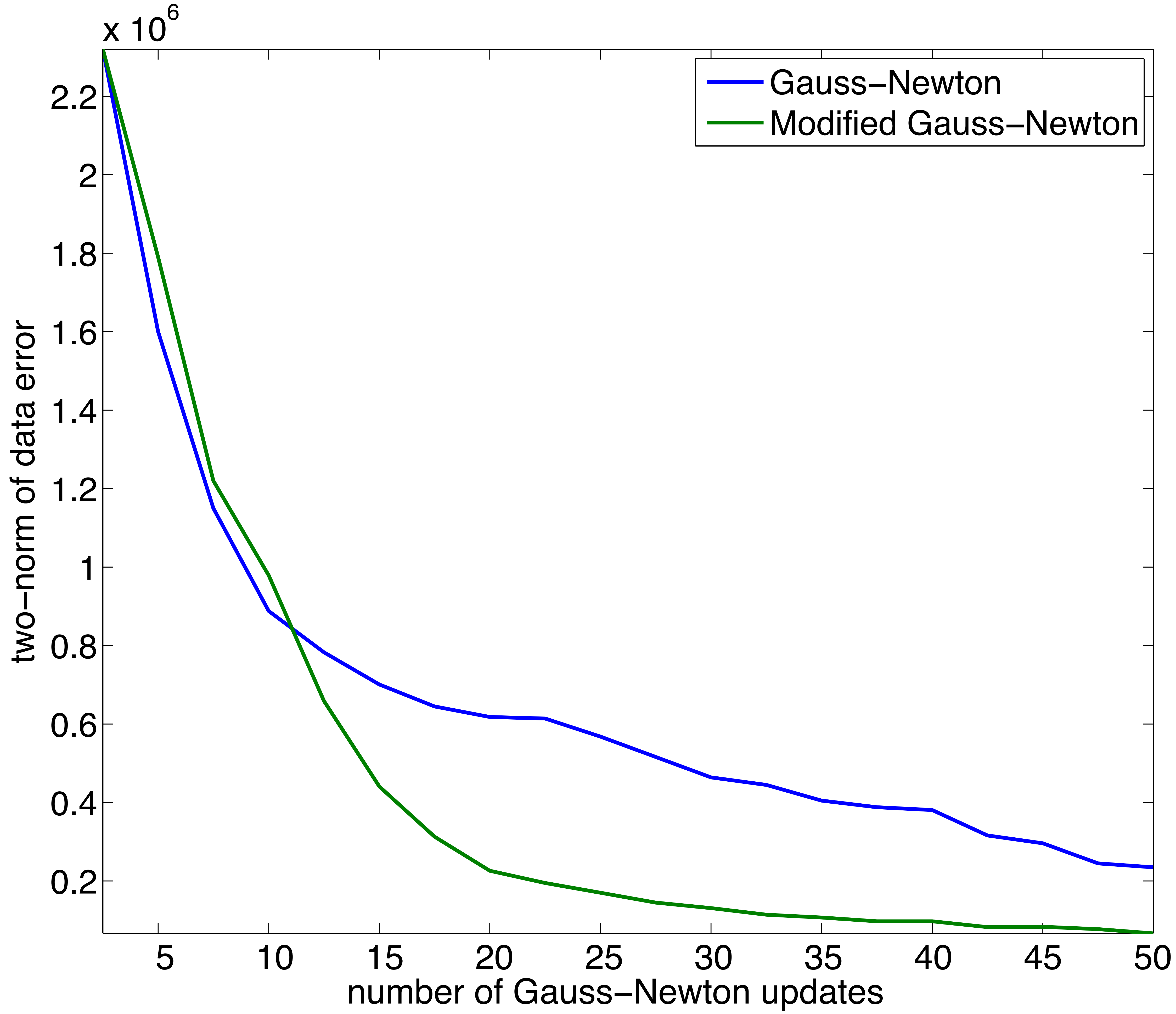
Inversion results



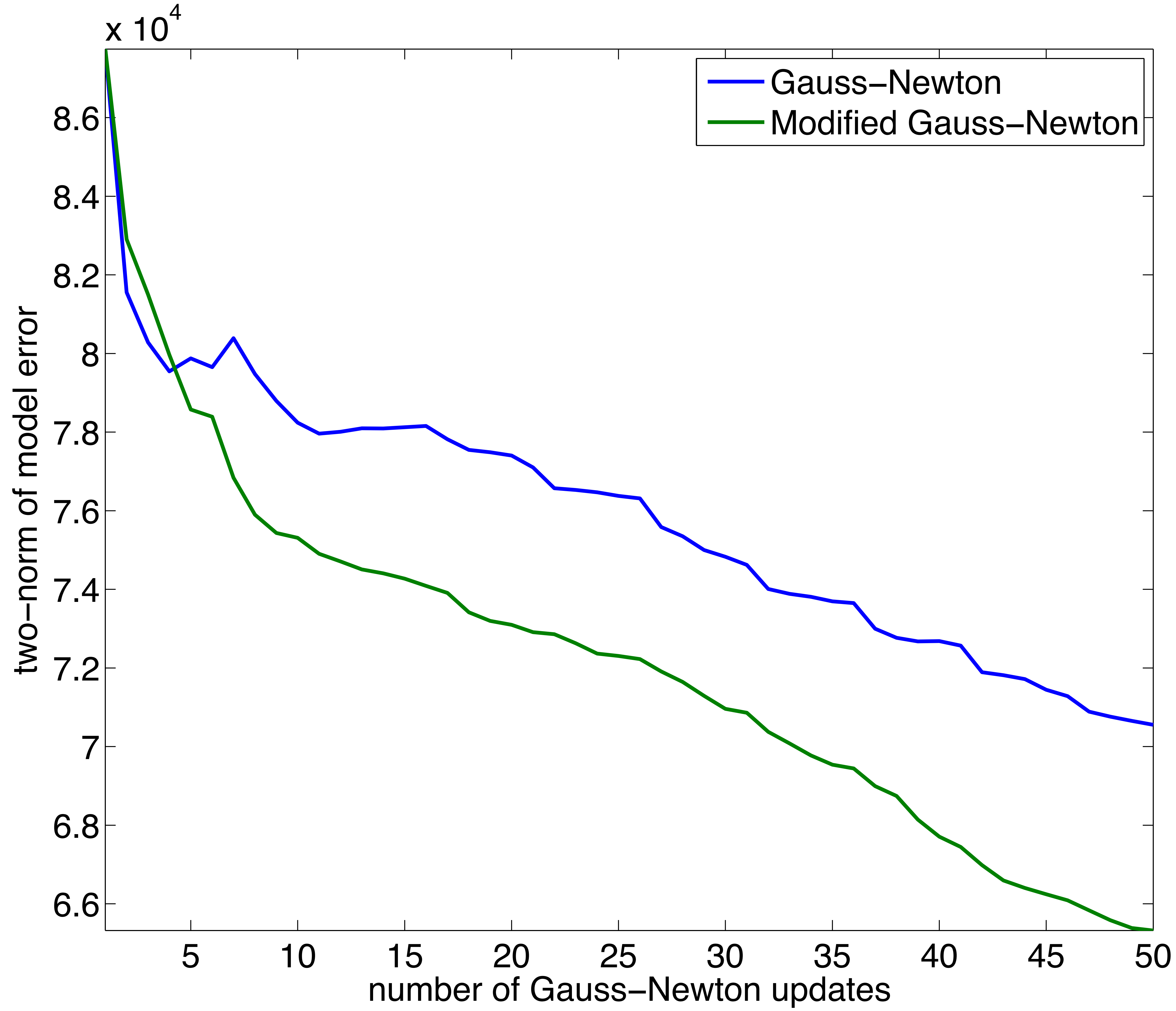
the modified Gauss-Newton method with L1 constraint

the modified Gauss-Newton method with L2 constraint

Convergence

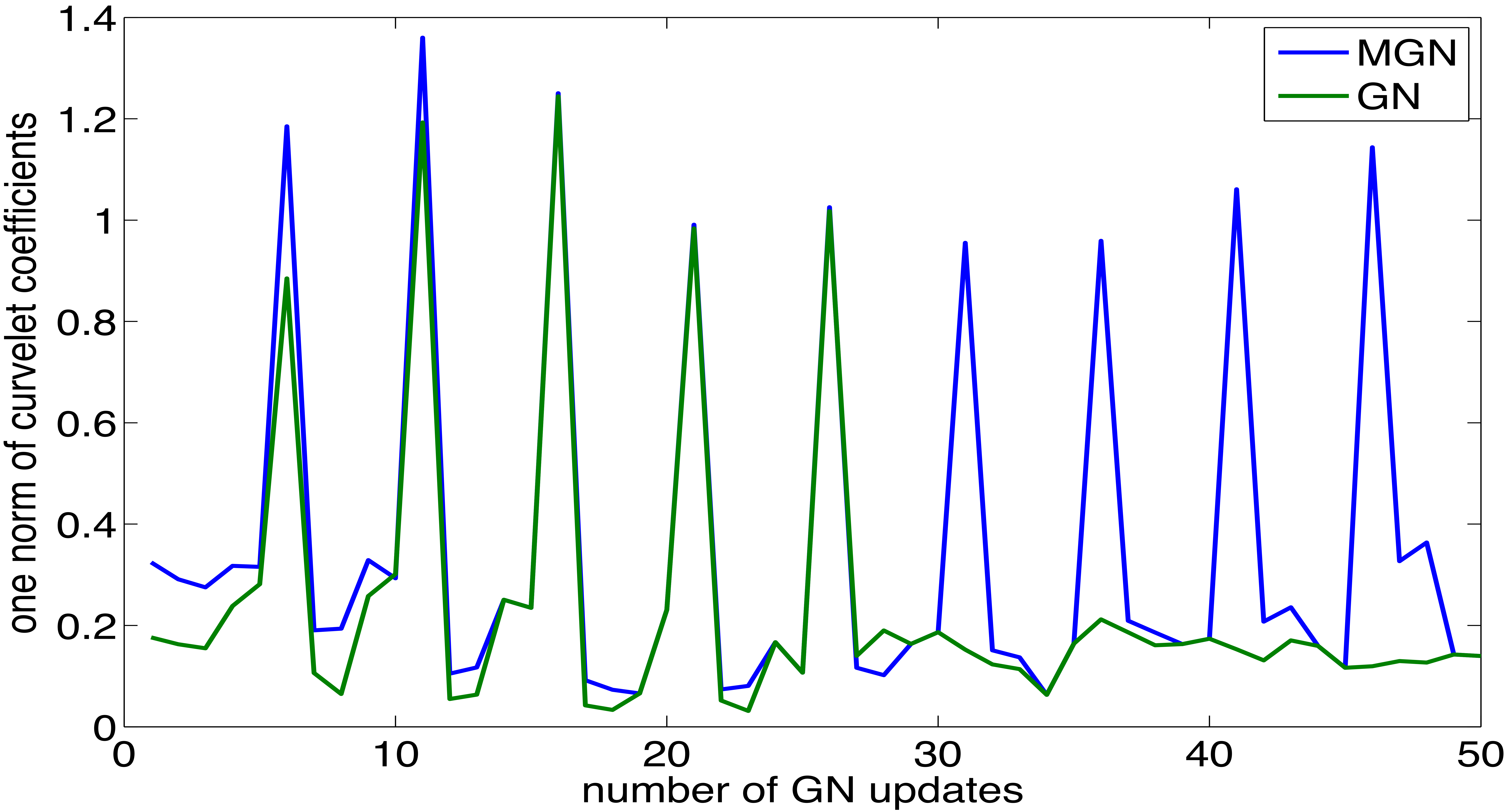


FWI objective



model error

Curvelet coefficients of updates



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Conclusion

- introduce sparsity constraint by changing the objective function does not necessarily generating the solution with sparse perturbation of the initial guess
- the modified Gauss-Newton method can provides us the solution with sparse perturbation, if all updates share the same sparsity support
- the modified Gauss-Newton works for linear problems (seismic imaging)
- the modified Gauss-Newton works for FWI, if a reasonable initial model is provided

Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

SINBAD

NSERC CRSNG

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