Wavefield Reconstruction Inversion (WRI) — a new take on wave-equation based inversion

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Wavefield Reconstruction Inversion (WRI) — a new take on wave-equation based inversion

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Motivation

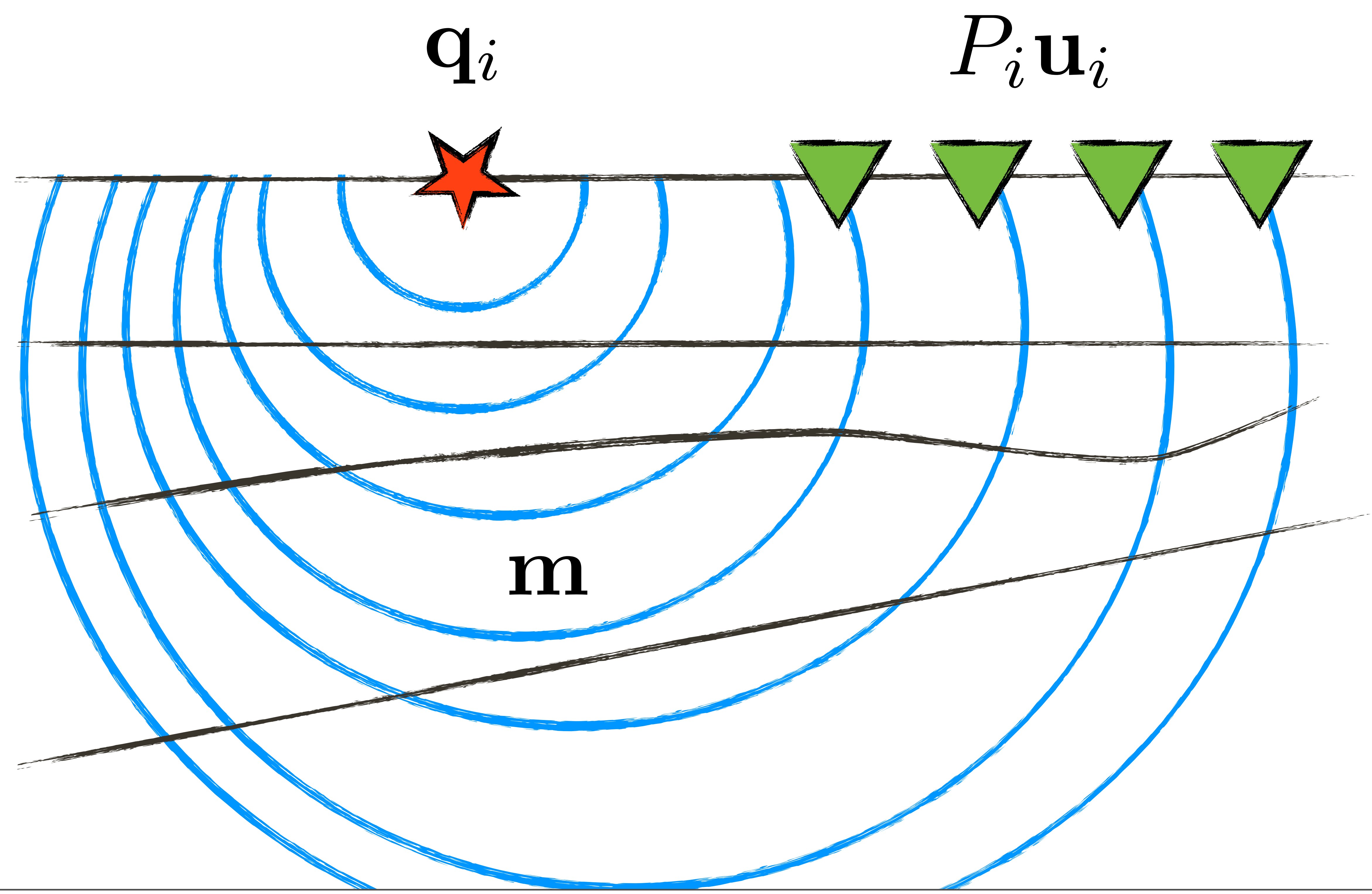
Full-waveform inversion is plagued with local minima

Derive an alternative formulation

- less prone to local minima
- computationally feasible
- relaxes the physics while staying solidly grounded

Waveform inversion

Retrieve the medium parameters from partial measurements of the solution of the wave-equation: $A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$



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Waveform inversion

Adjoint-state/reduced-space methods:

- Optimize over earth models to minimize the misfit between observed and simulated data while solving the wave equation exactly for each earth model.

Full-space or all-at-once methods:

- Optimize over earth models & wavefields jointly to minimize the misfit between observed and simulated data subject to wavefields that satisfy the wave equation.

*Wave*form inversion

Both approaches assume *flawless* wave physics–i.e.,

“known” physics “known” source

$$A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

“unknown” wavefield

- holds *exactly* for each source *i*
- *differ* on *insisting* wave equations to *hold* for *each* iteration
- *different* unknowns: **m** $\longleftrightarrow$ **m** & **u**

[Richter, ’81]

Equation error approach

If we “know” the wavefields everywhere, we solve for $\mathbf{m}$ from

$$A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

via

$$\min_{\mathbf{m}} \|A(\mathbf{m})P_i^{-1}\mathbf{d}_i - \mathbf{q}_i\|_2^2 \qquad \left(\text{cf. } \min_{\mathbf{m}} \|P_i A(\mathbf{m})^{-1}\mathbf{q}_i - \mathbf{d}_i\|_2^2\right)$$

The challenge is to reconstruct wavefields from partial measurements...

[van Leeuwen & FJH, 2013]

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WRI – Wavefield Reconstruction Inversion

For $\mathbf{m}$ fixed, reconstruct wavefields by jointly fitting observed shots

$$P\mathbf{u}_i \approx \mathbf{d}_i$$

and wave-equations

$$A(\mathbf{m})\mathbf{u}_i \approx \mathbf{q}_i$$

via least-squares solutions of the data-augmented wave-equation

$$\min_{\mathbf{u}_i} \left\| \begin{pmatrix} P_i \\ A(\mathbf{m}) \end{pmatrix} \mathbf{u}_i - \begin{pmatrix} \mathbf{d}_i \\ \mathbf{q}_i \end{pmatrix} \right\|_2^2$$

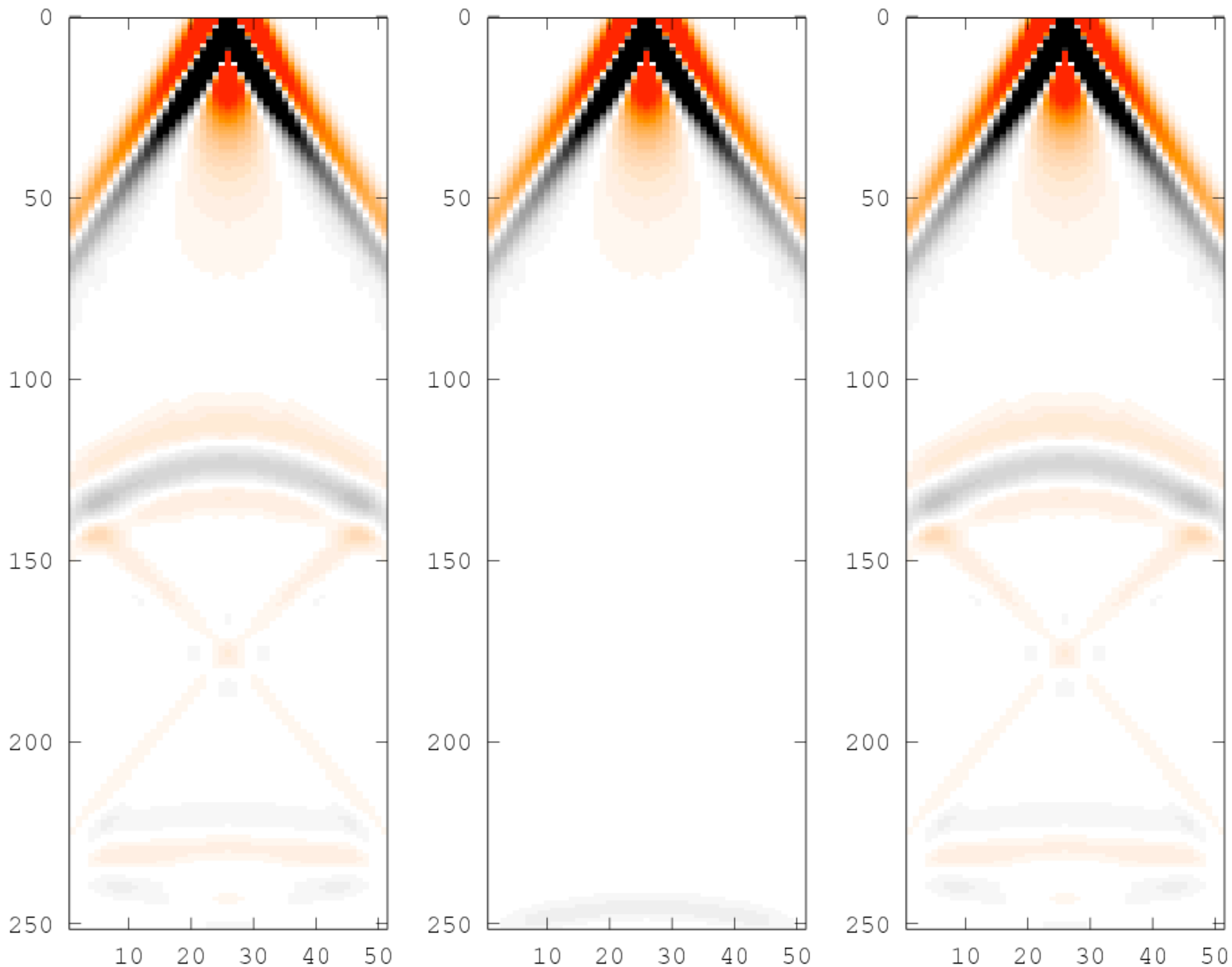
followed by fixing $\mathbf{u}_i$ and solving

$$\min_{\mathbf{m}} \|A(\mathbf{m})\mathbf{u}_i - \mathbf{q}_i\|_2^2$$

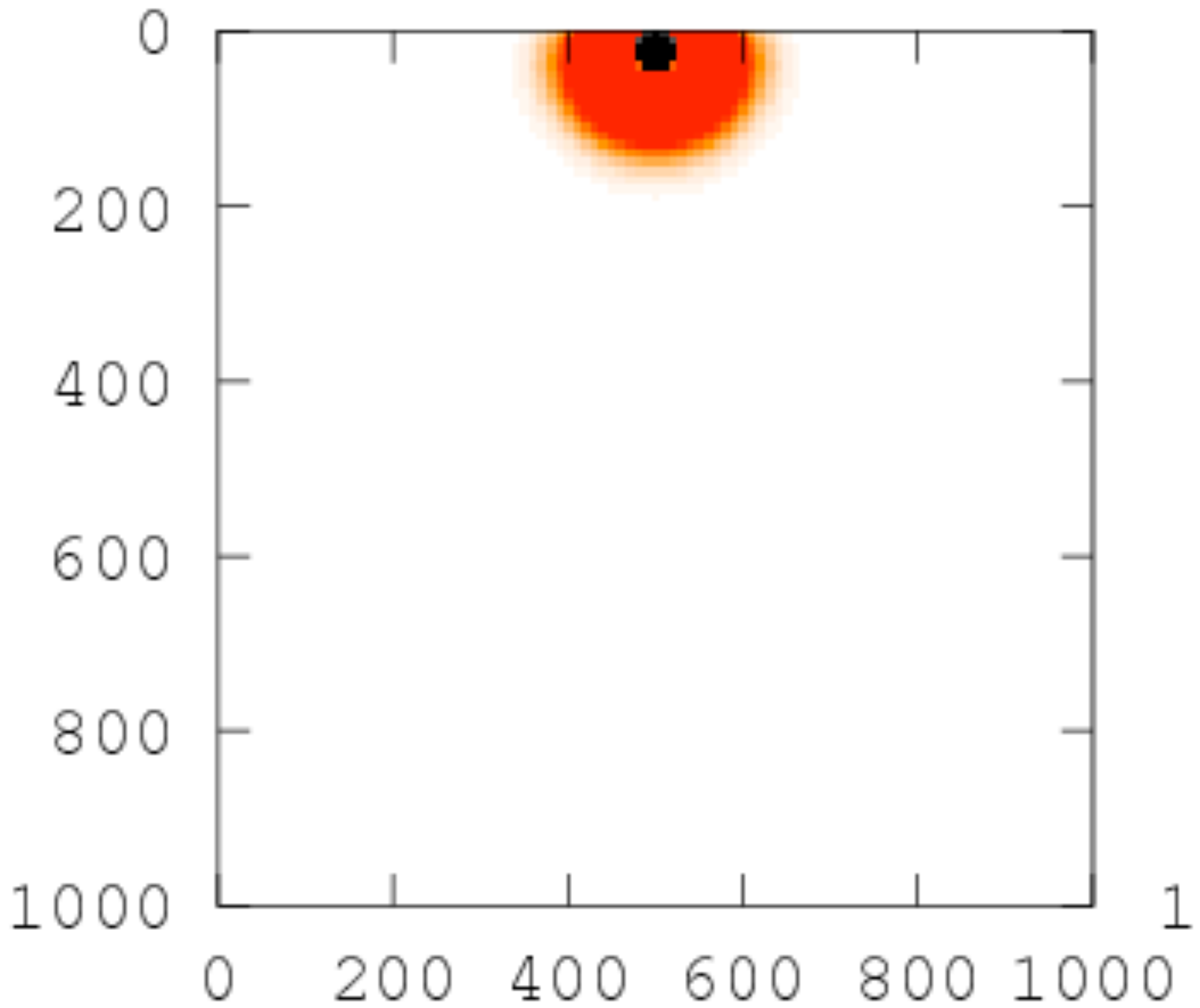
wave-equation X **wavefield** = **source**

versus

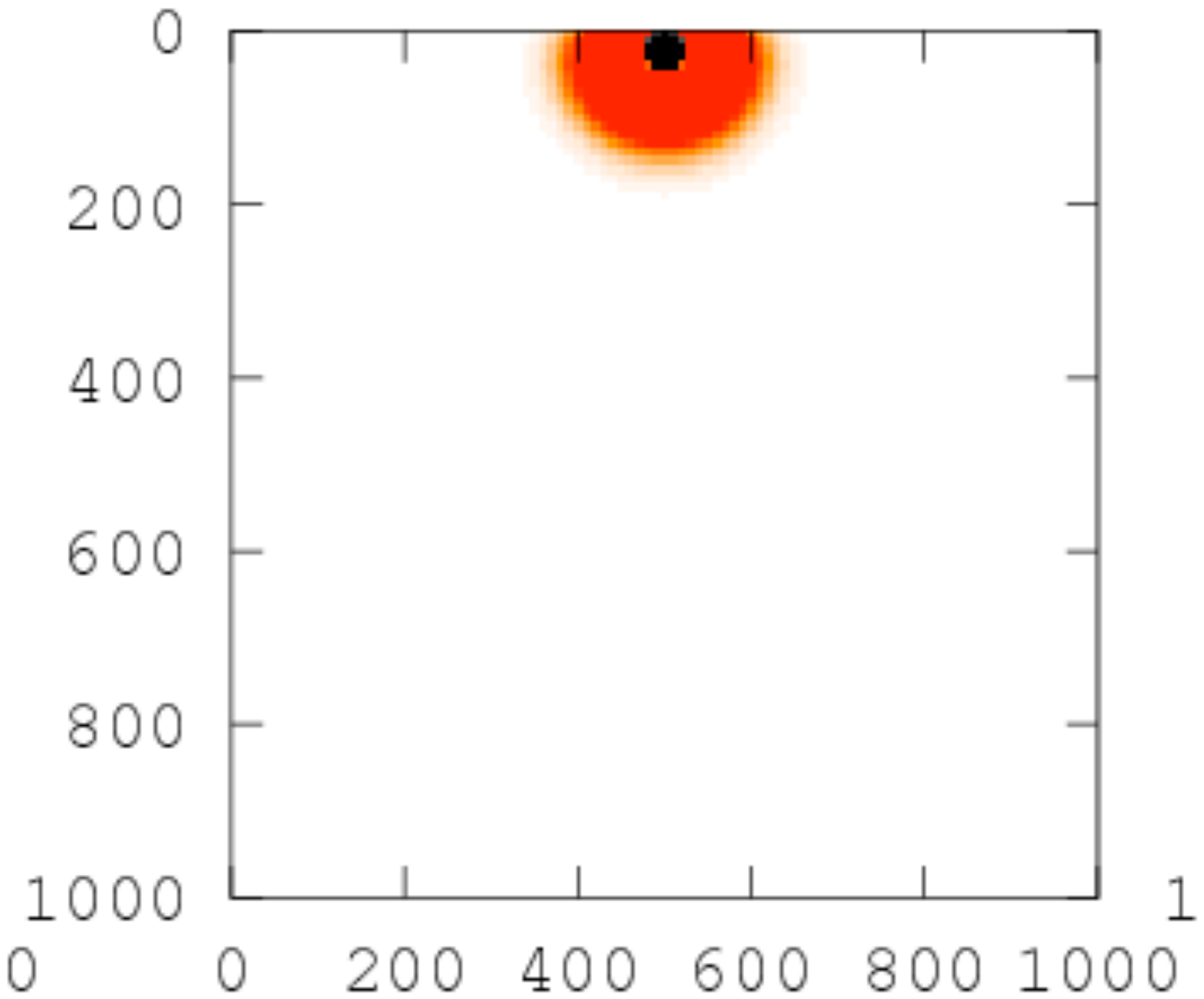
(**wave-equation** / **sampling operator**) X **wavefield** = (**source** / **data**) observed data   initial data   data-augmented solution



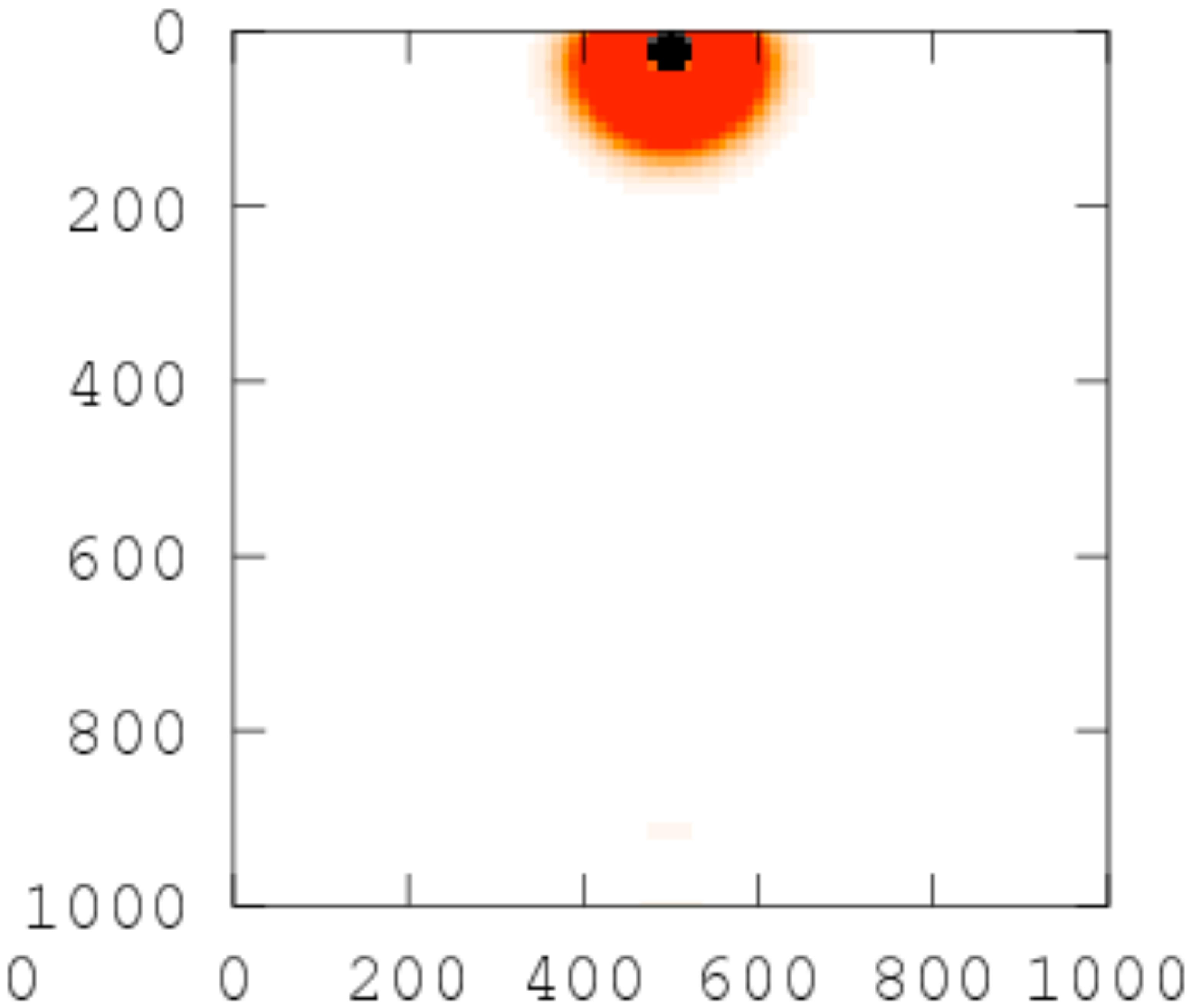
wavefield in *true* model

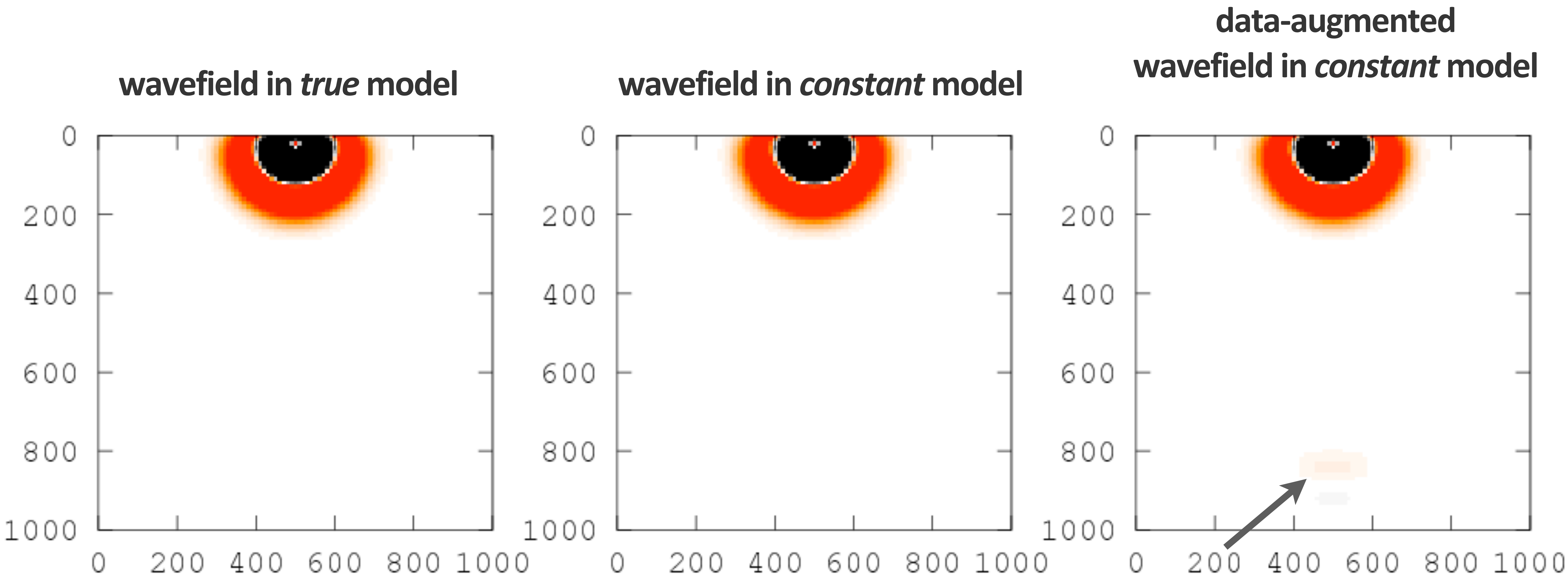


wavefield in *constant* model

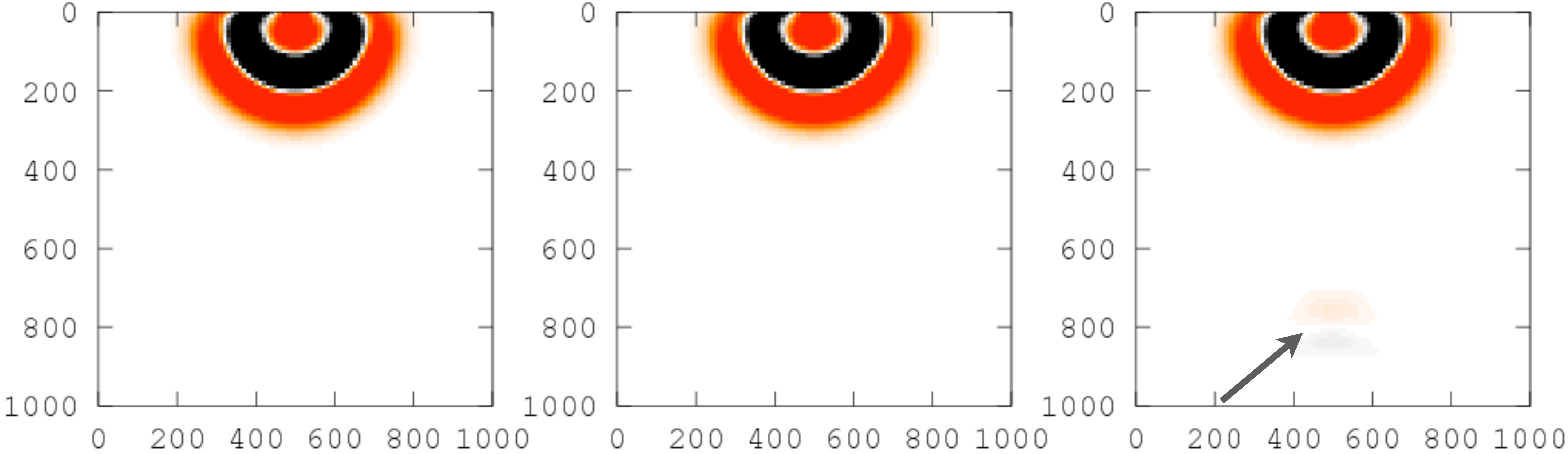


data-augmented wavefield in *constant* model

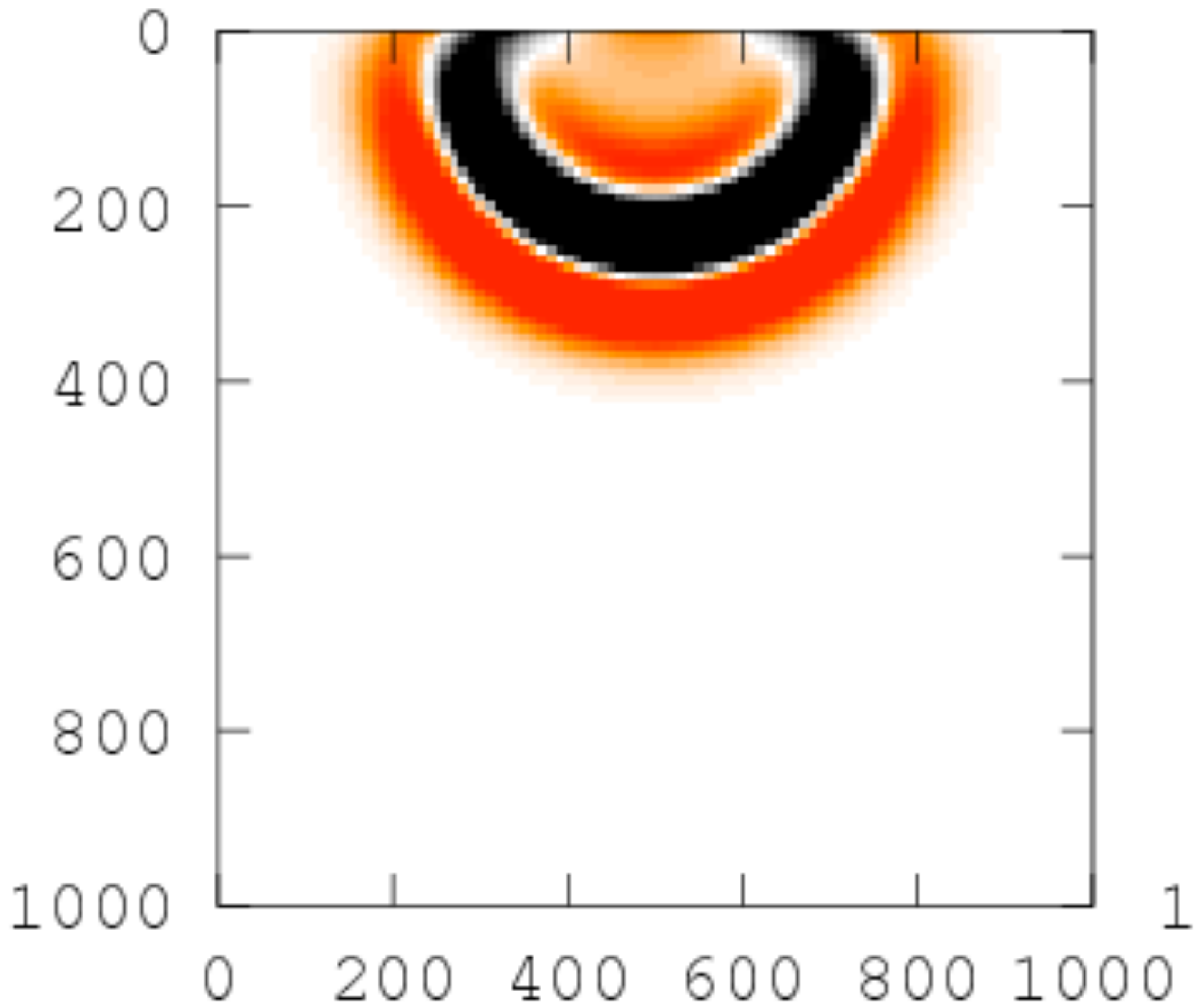



wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
0
200
400
600
800
1000

wavefield in *true* model | wavefield in *constant* model | data-augmented wavefield in *constant* model



wavefield in *true* model



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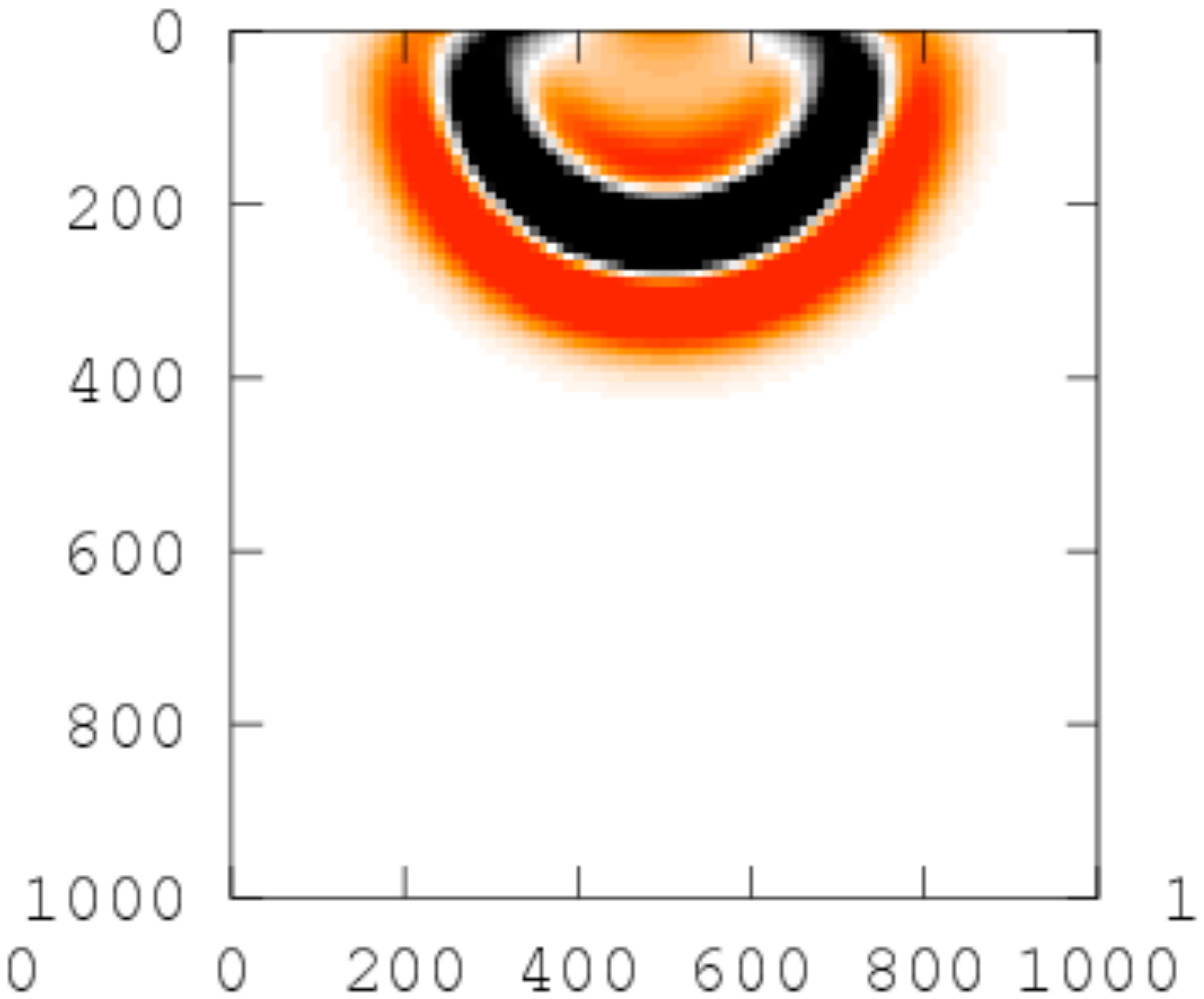

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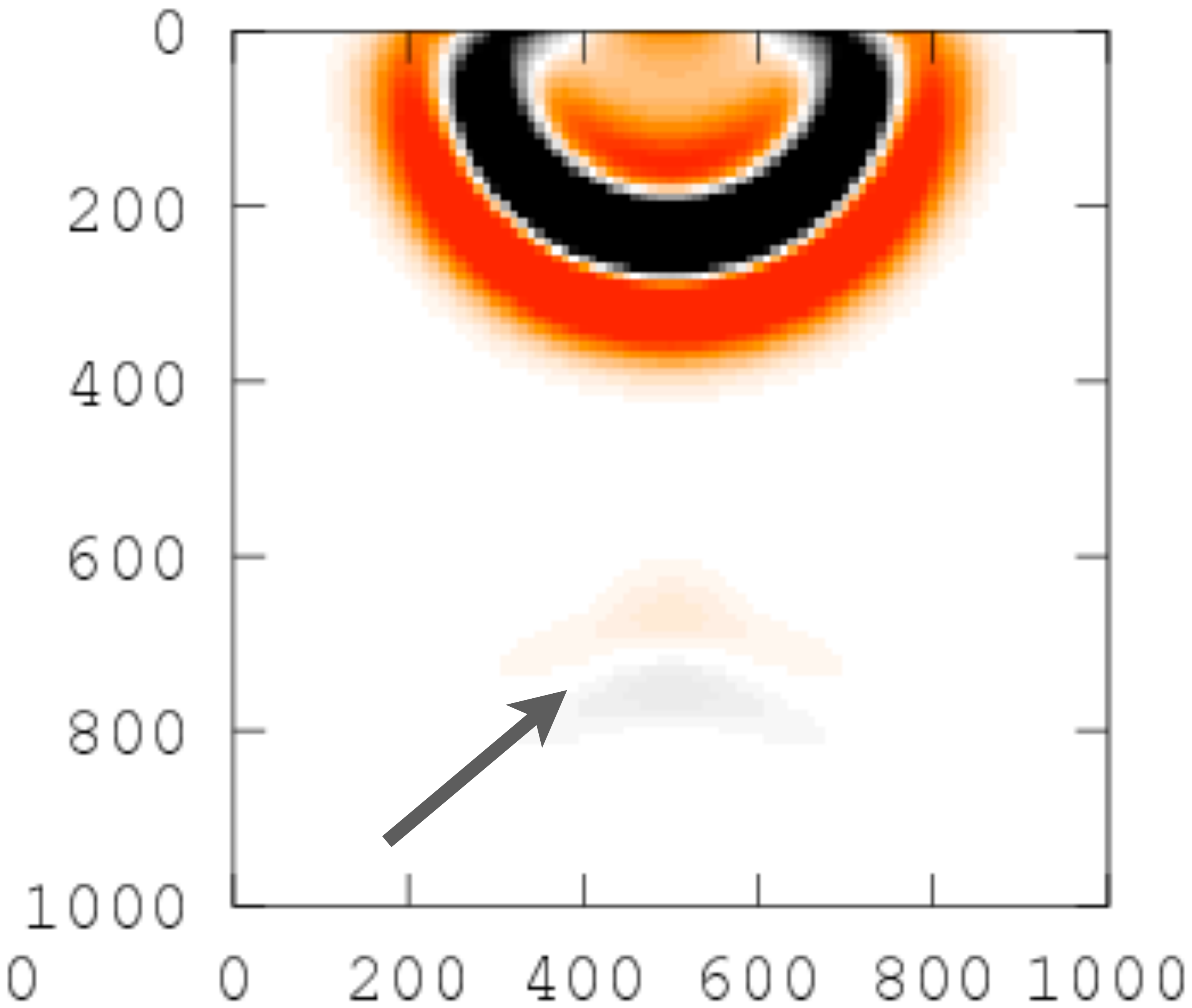
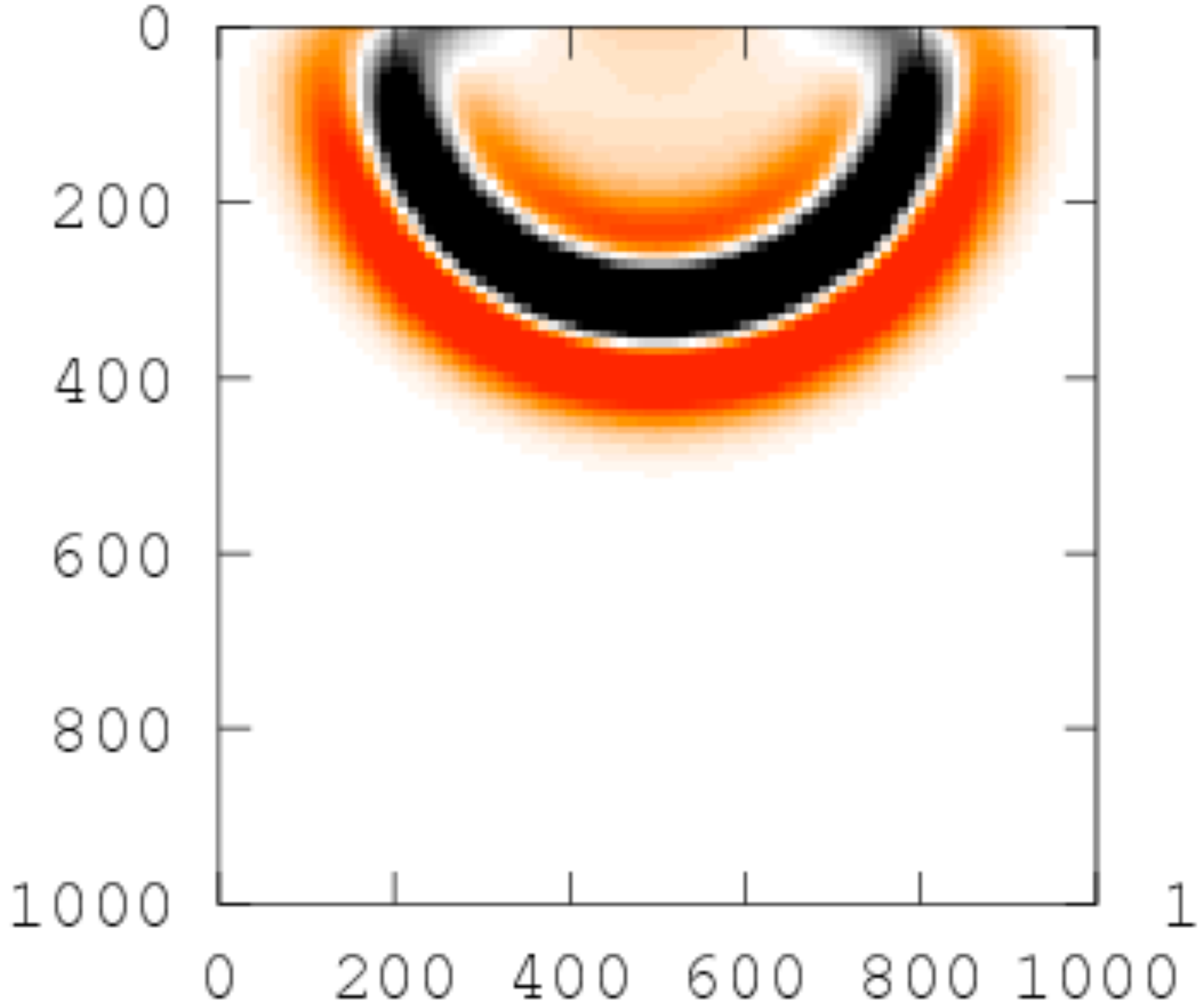
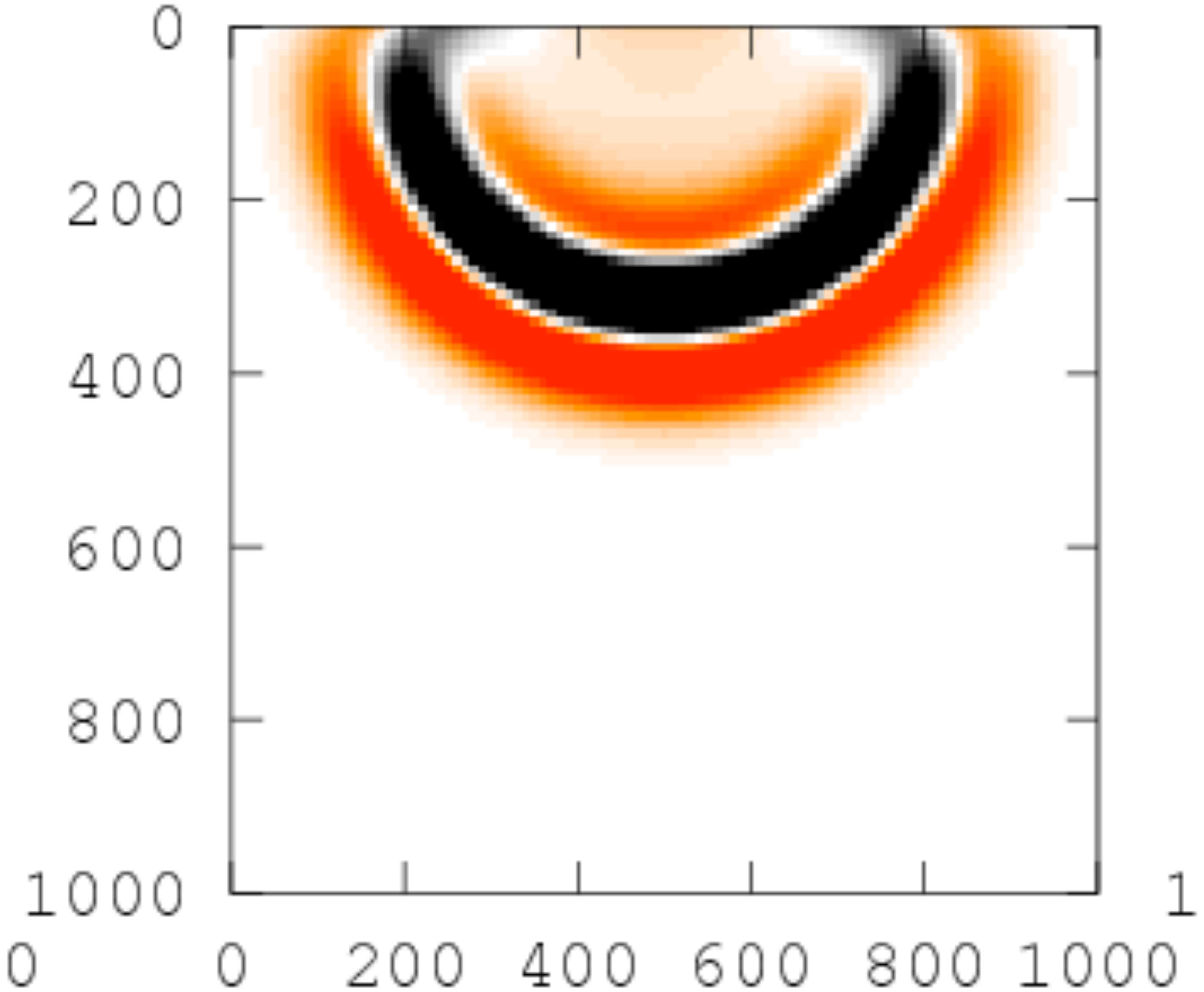


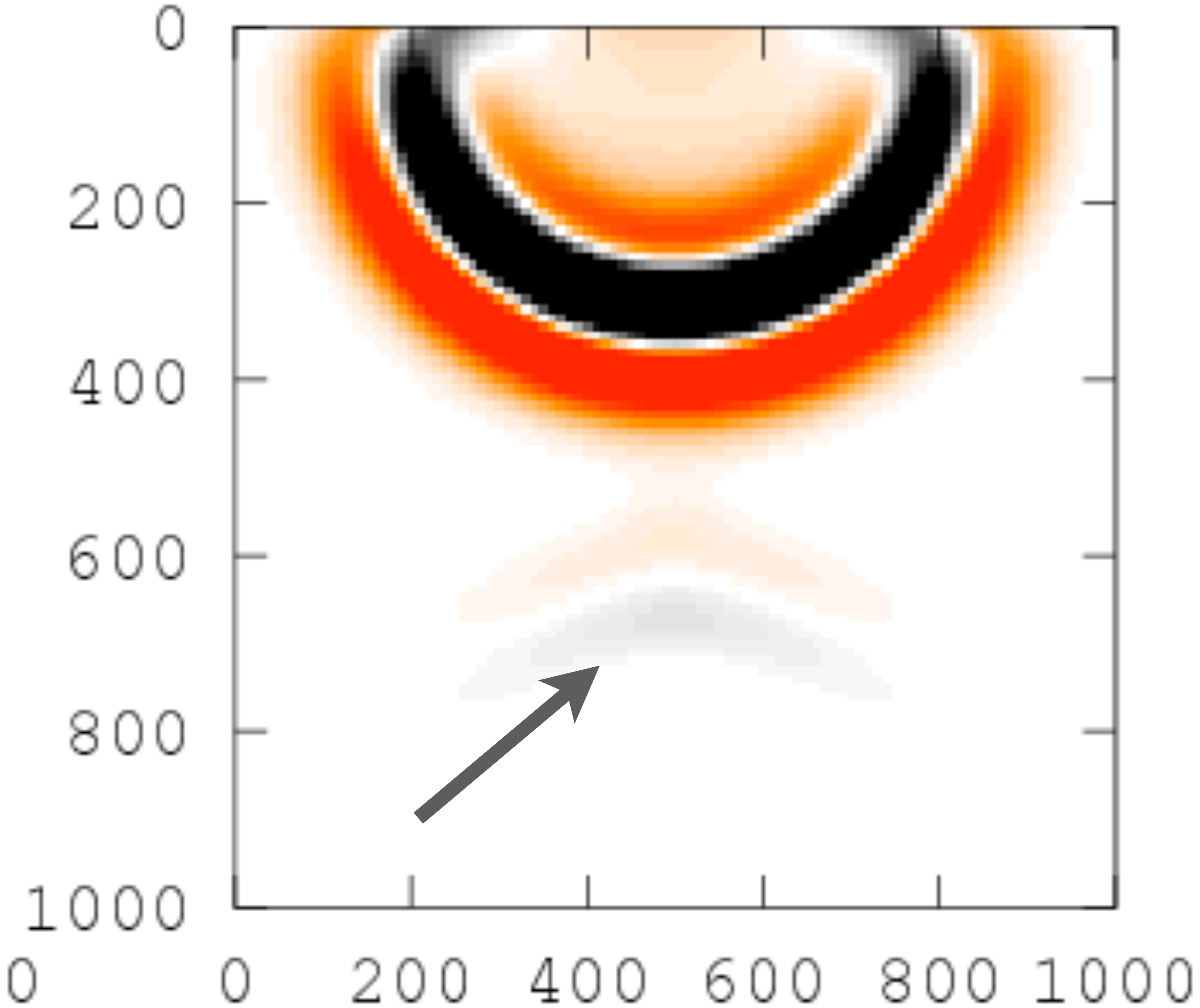
wavefield in *true* model



wavefield in *constant* model



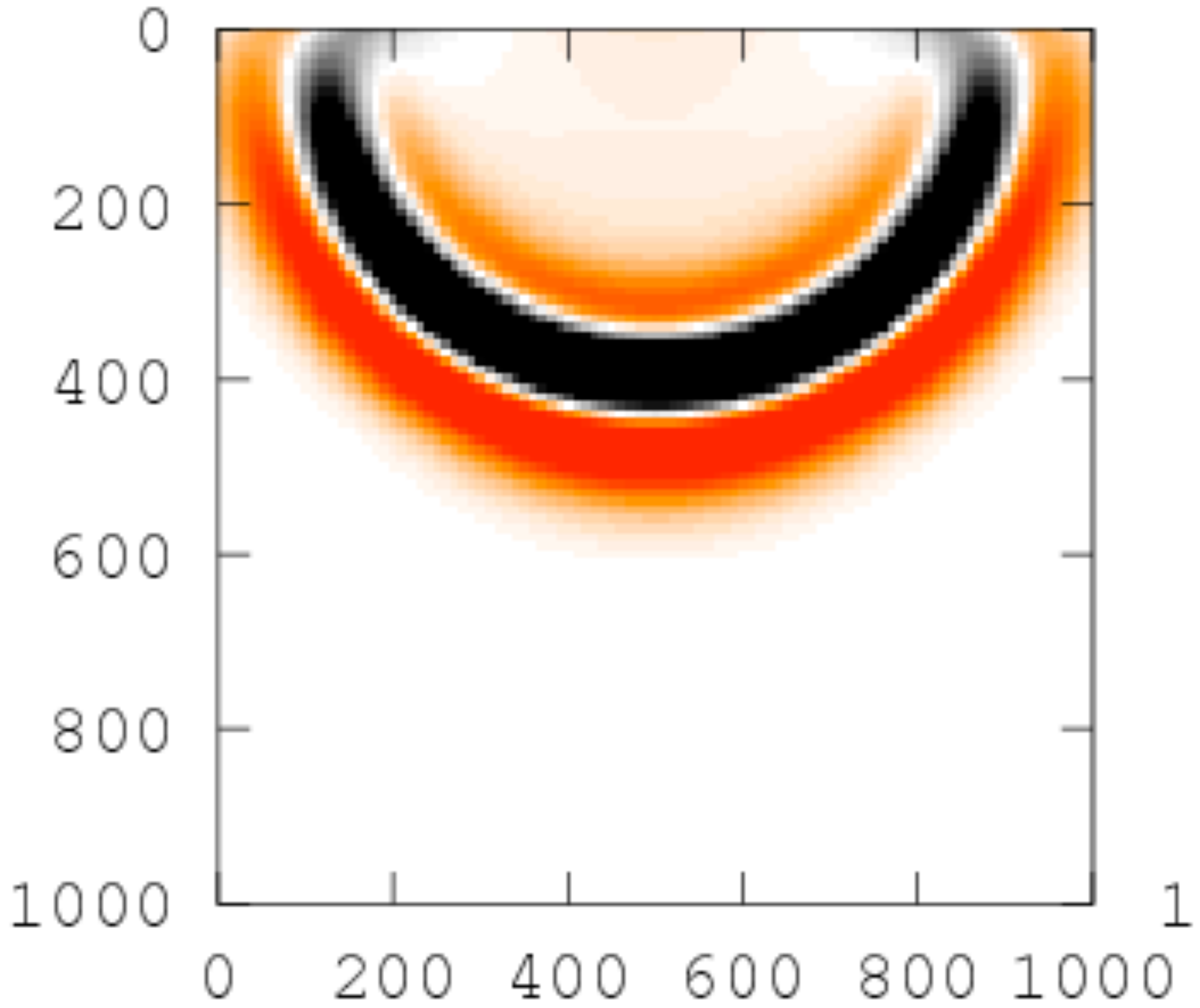
data-augmented
wavefield in *constant* model

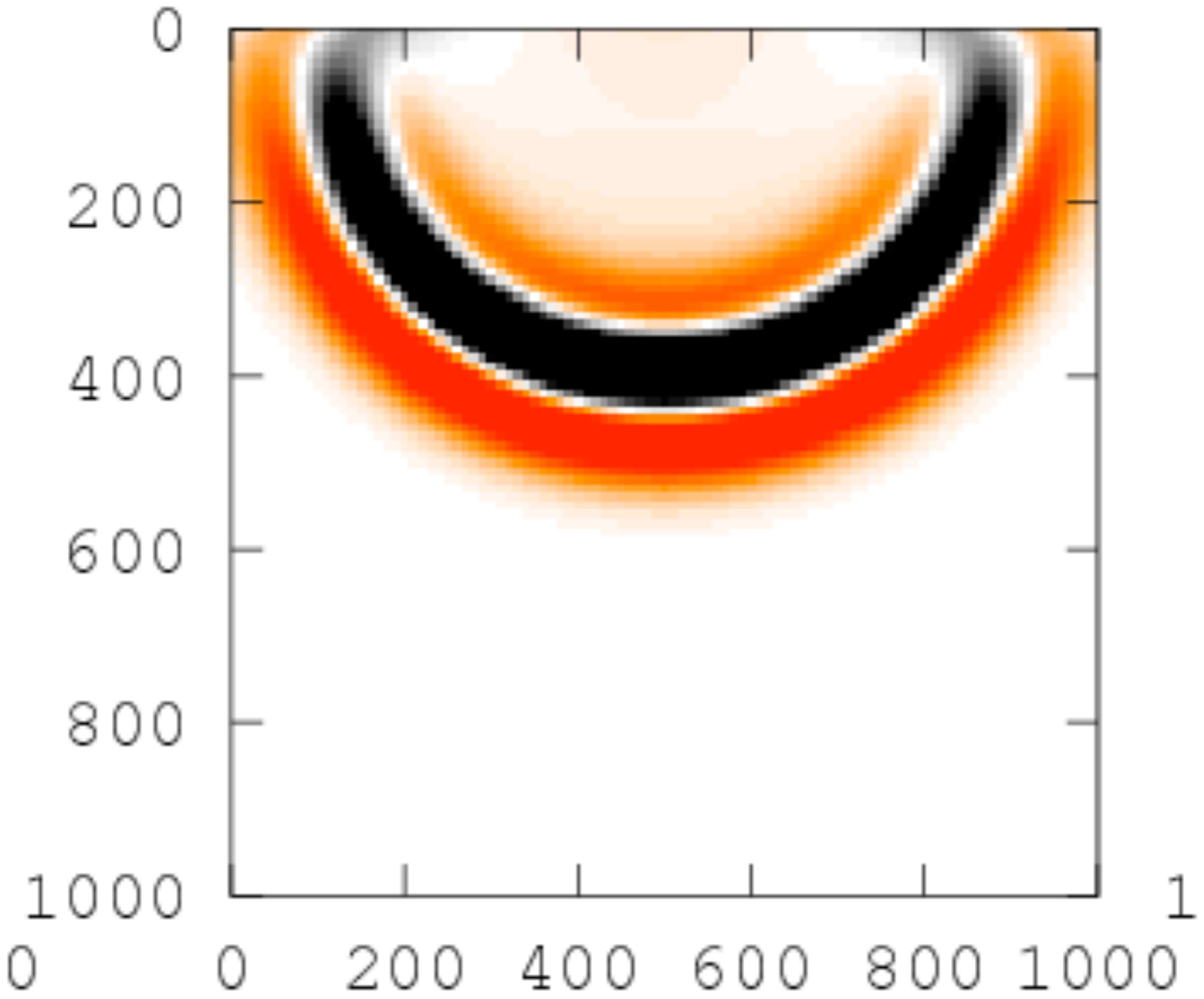


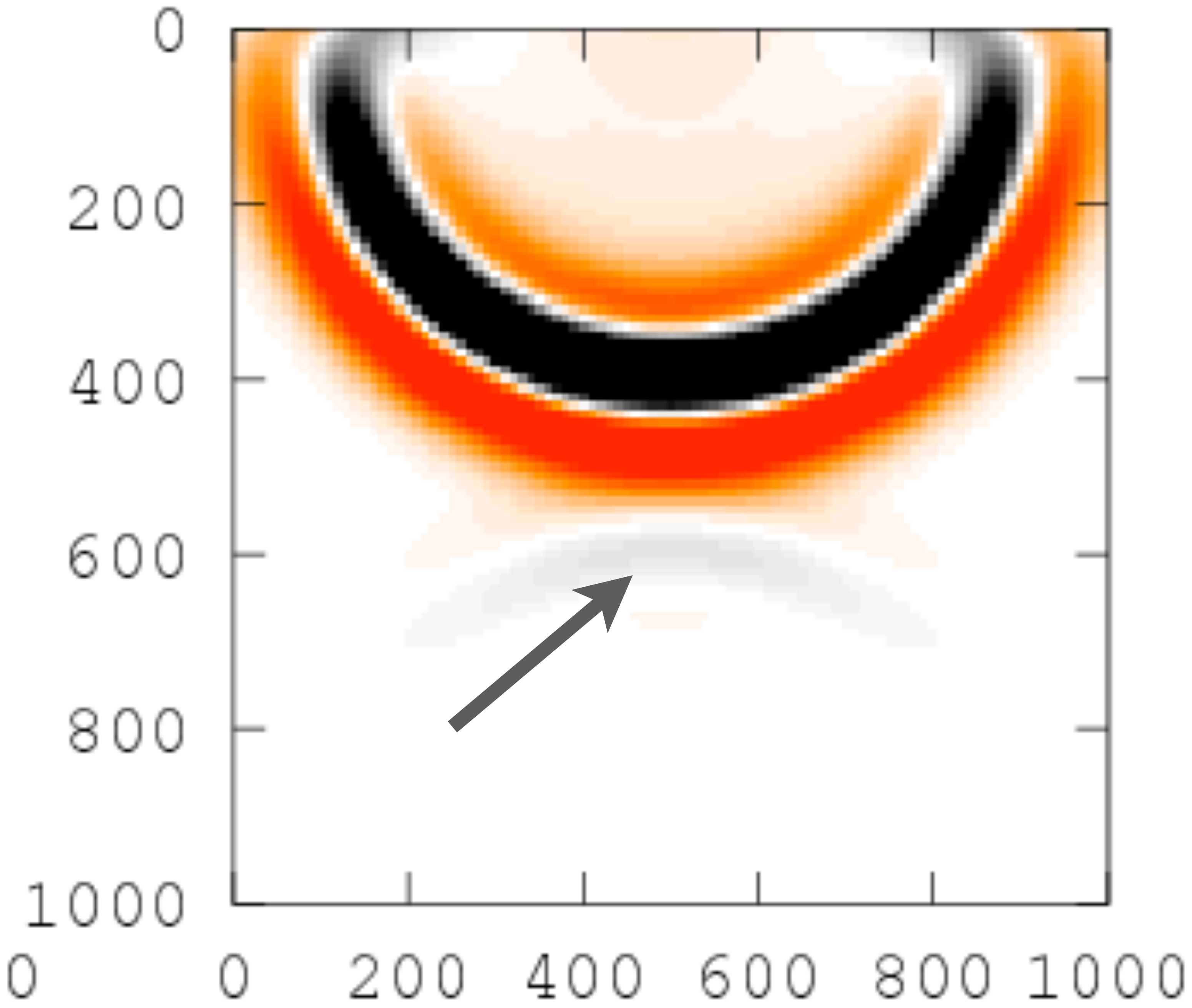
wavefield in *true* model

wavefield in *constant* model

data-augmented wavefield in *constant* model

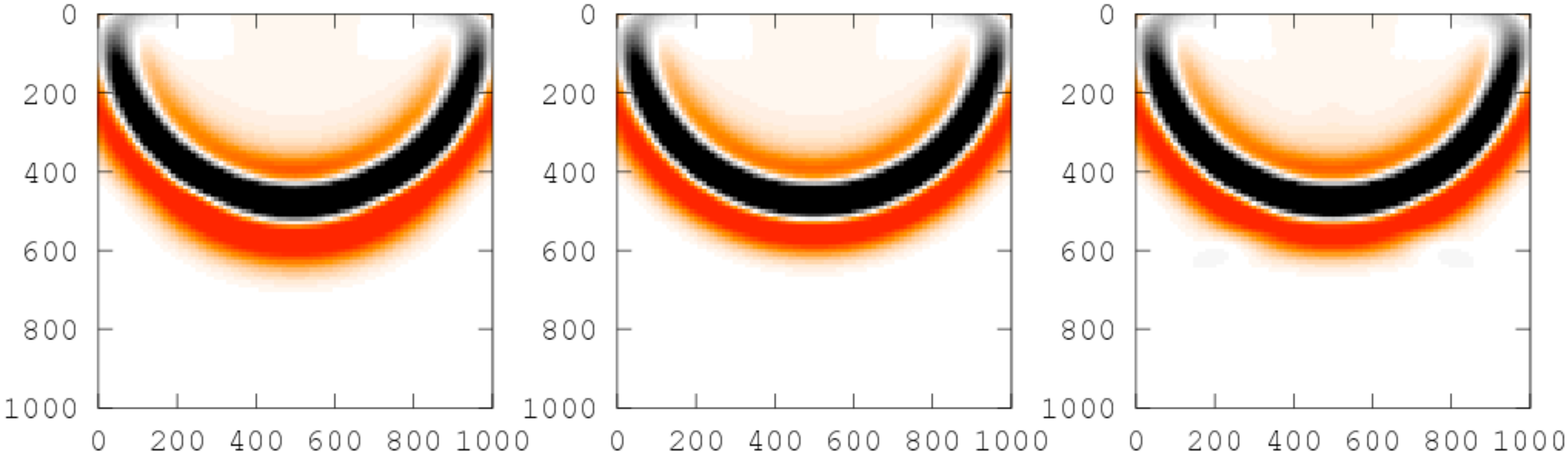




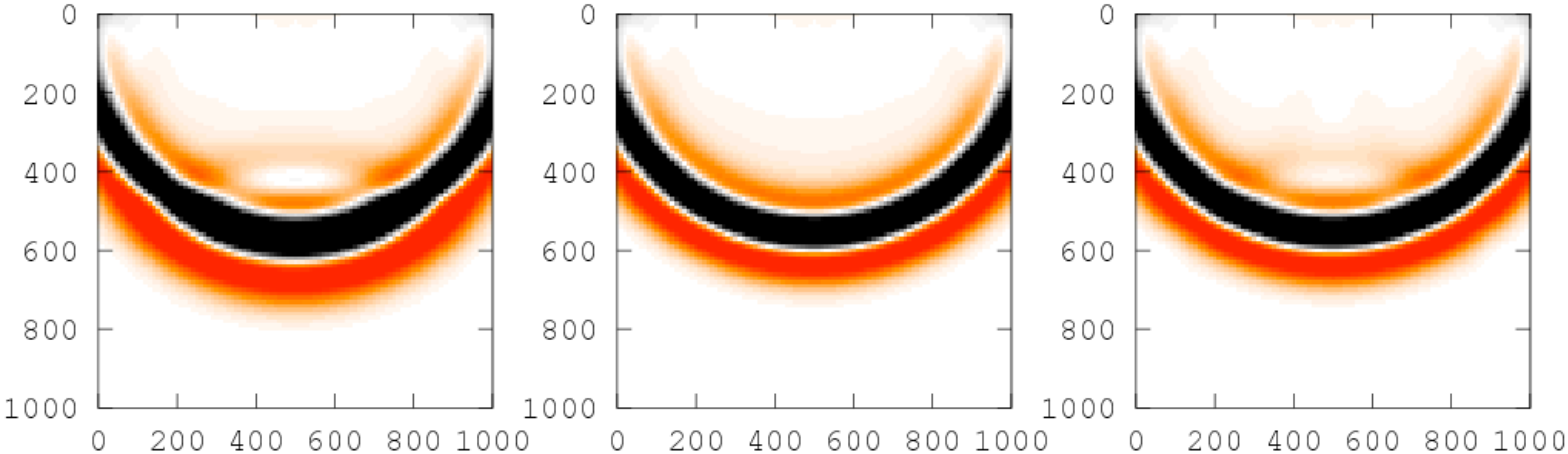


data-augmented wavefield in *constant* model

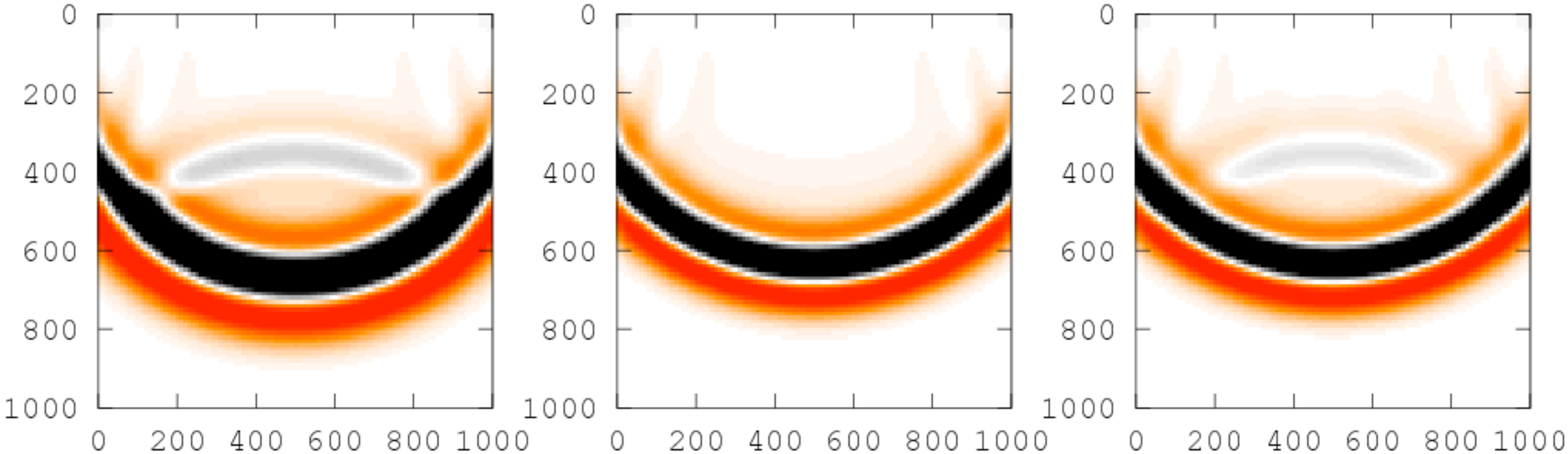
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



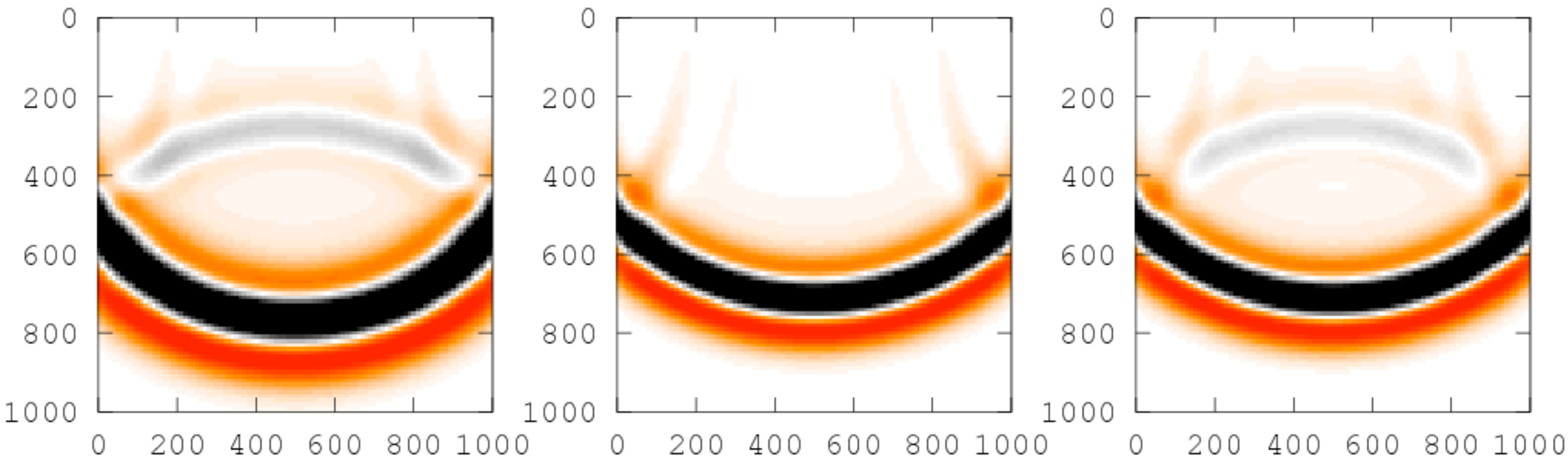
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



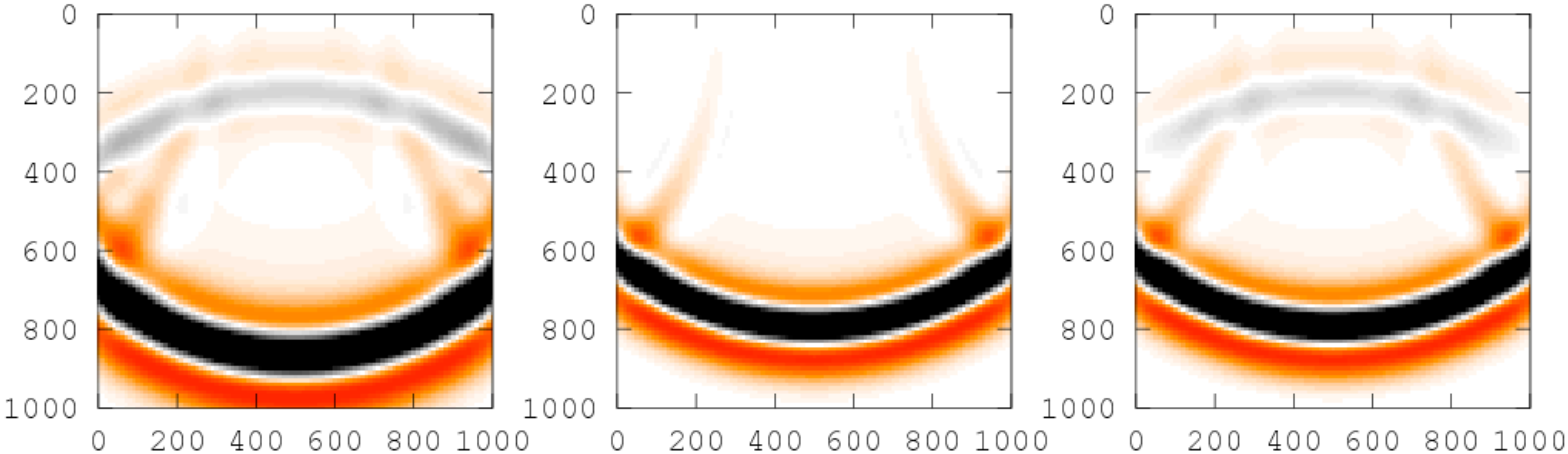
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



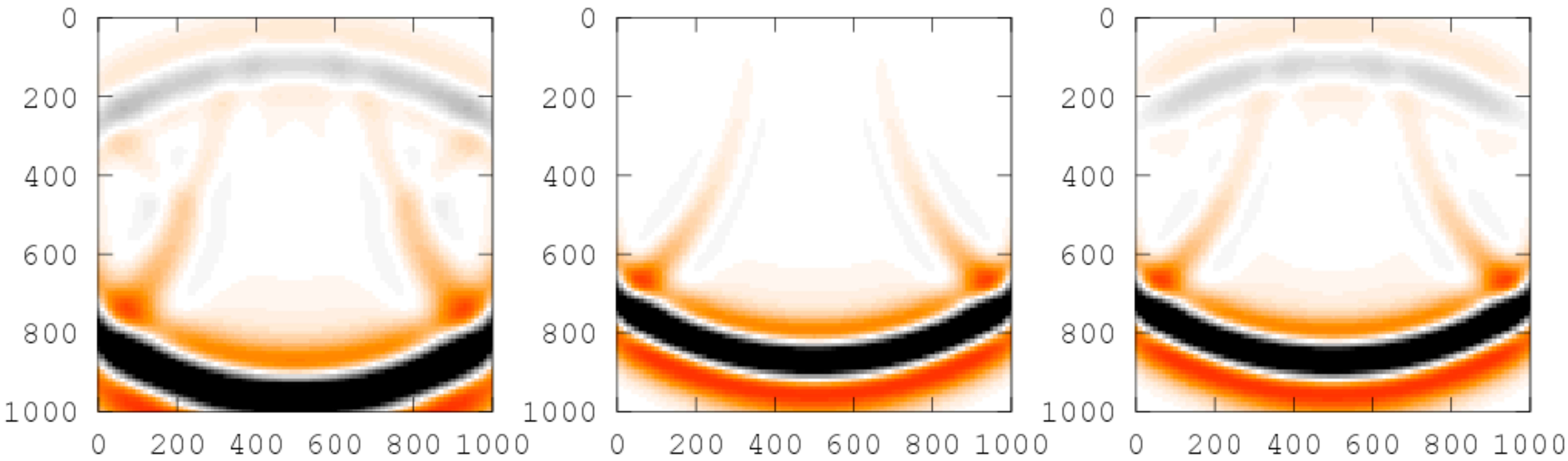
wavefield in *true* model

wavefield in *constant* model

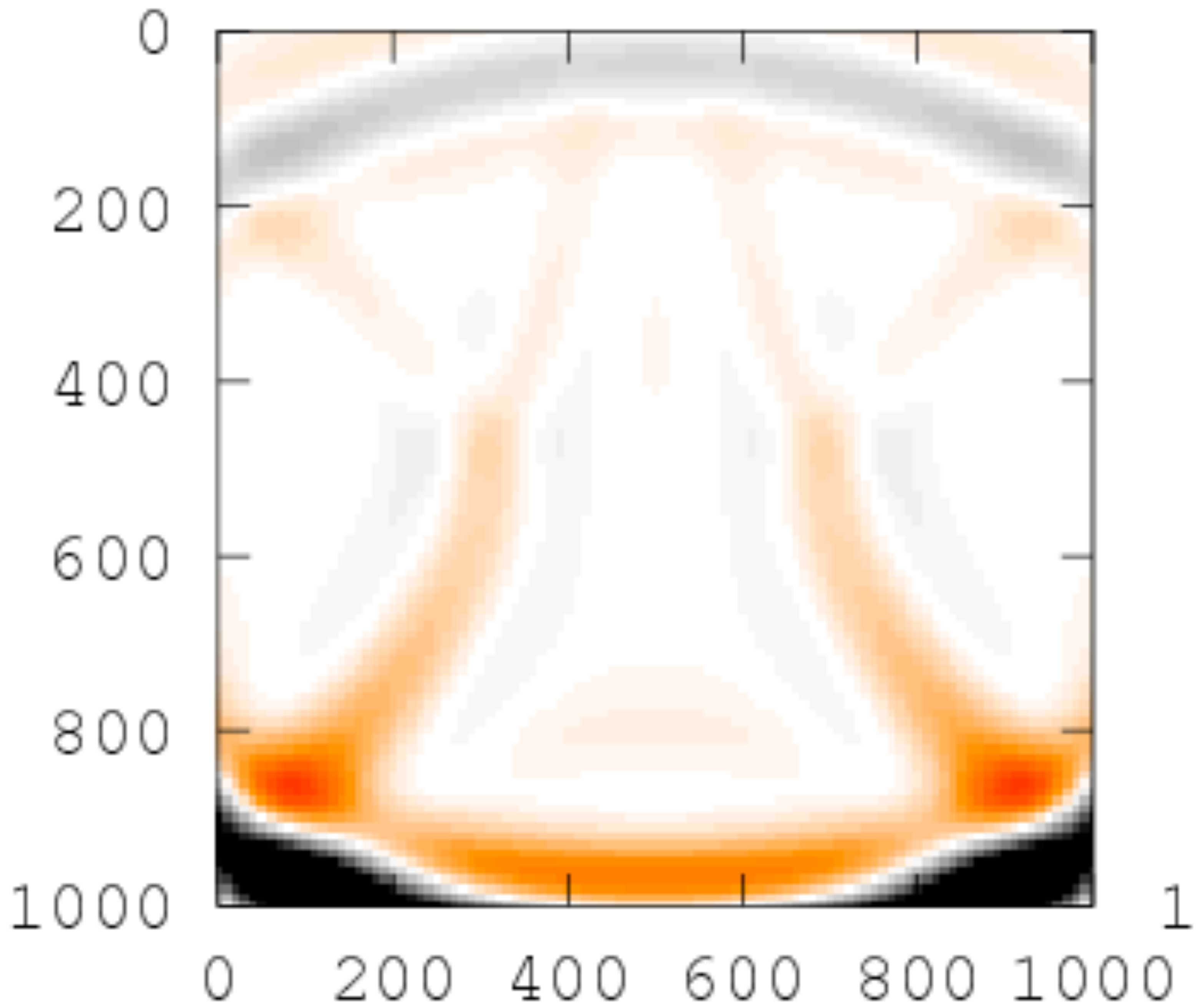
data-augmented wavefield in *constant* model



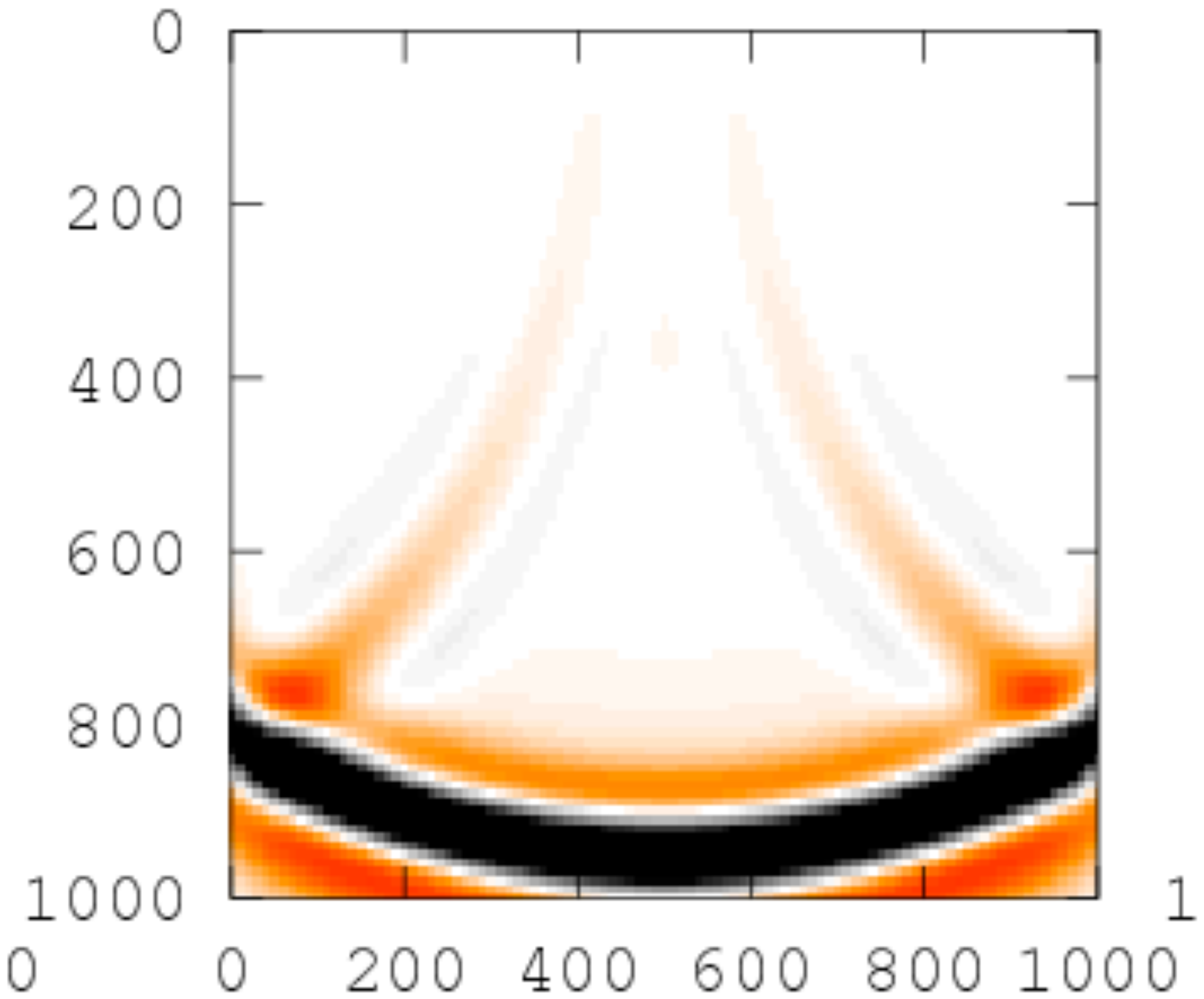
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



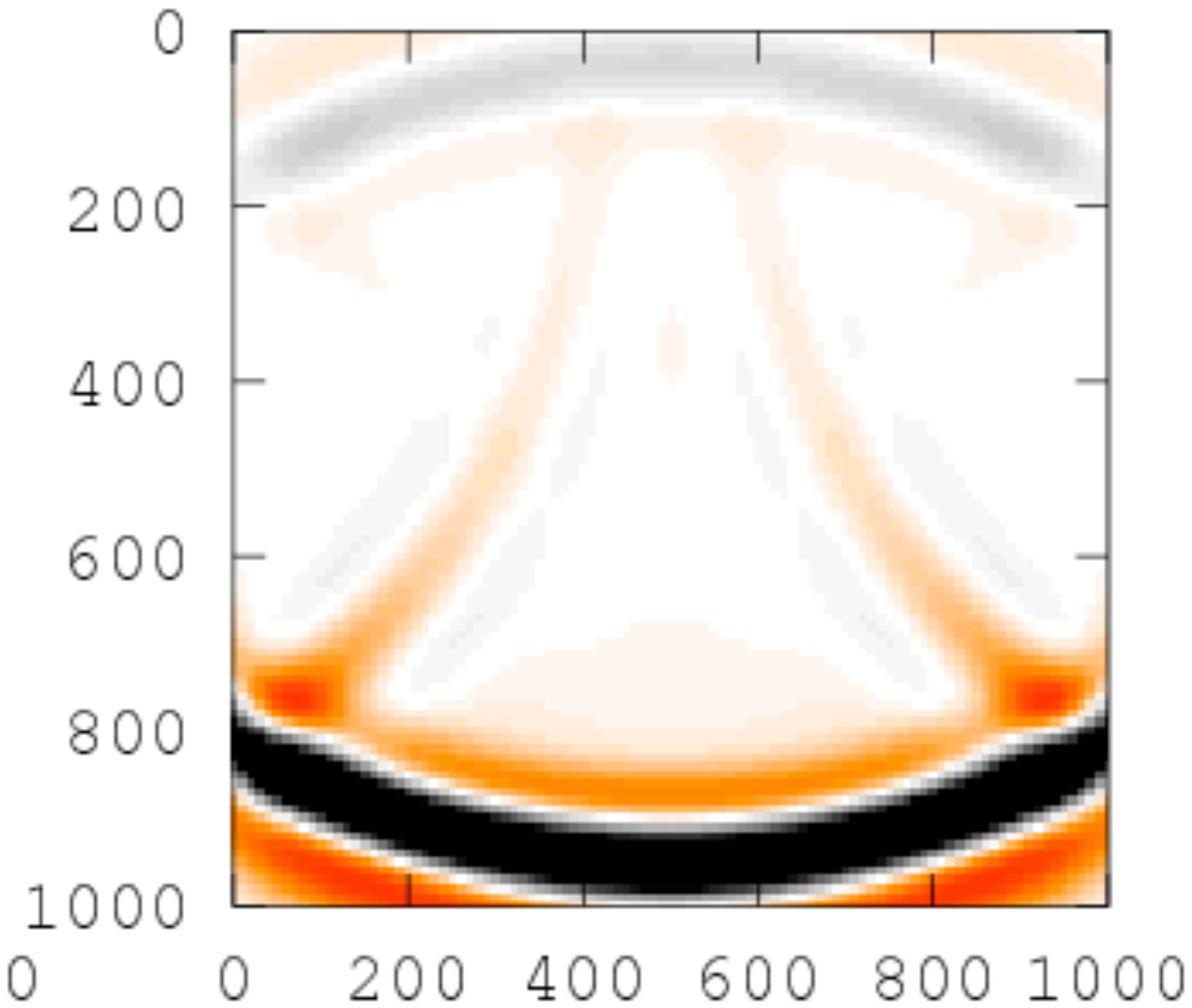
wavefield in *true* model



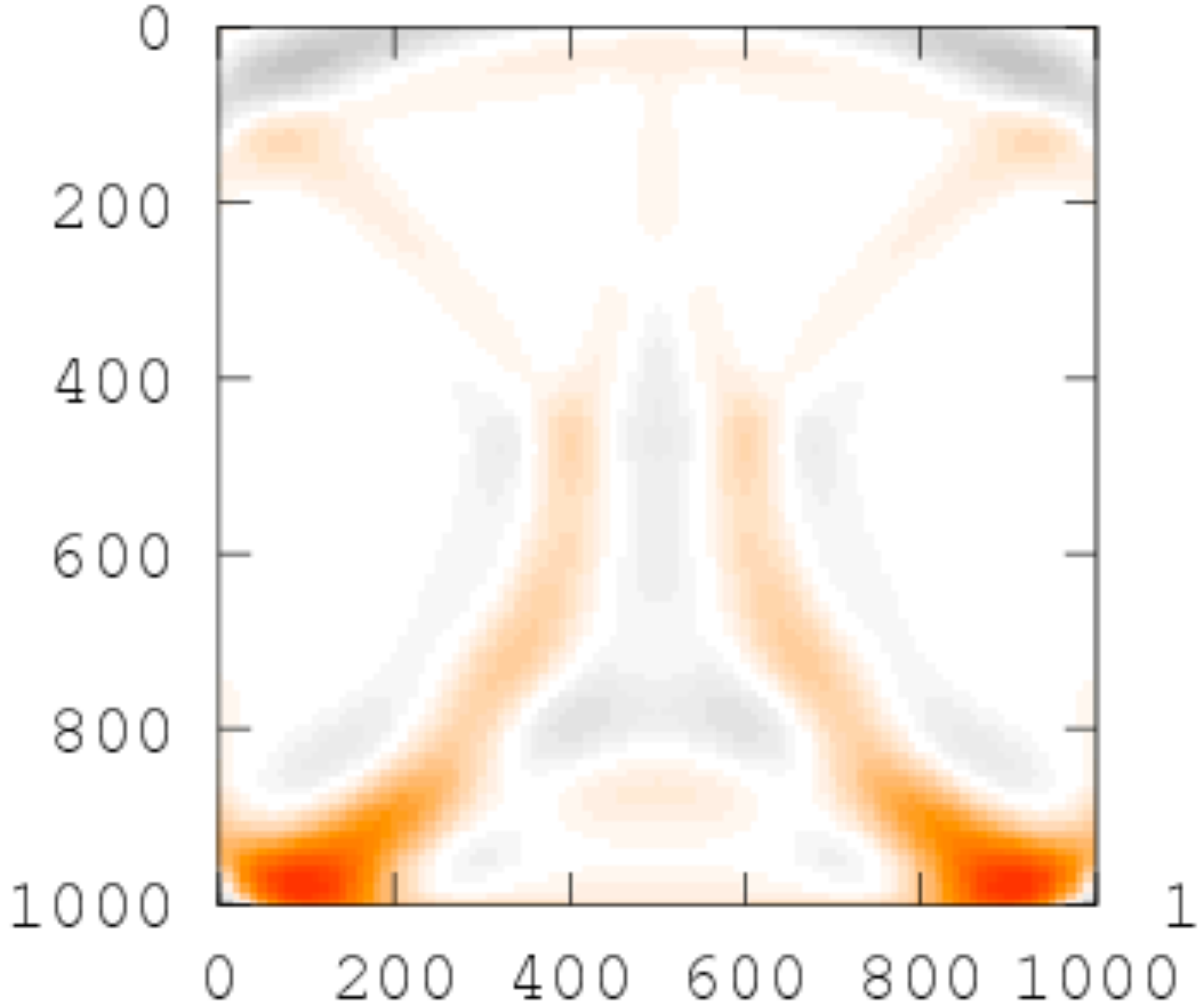
wavefield in *constant* model



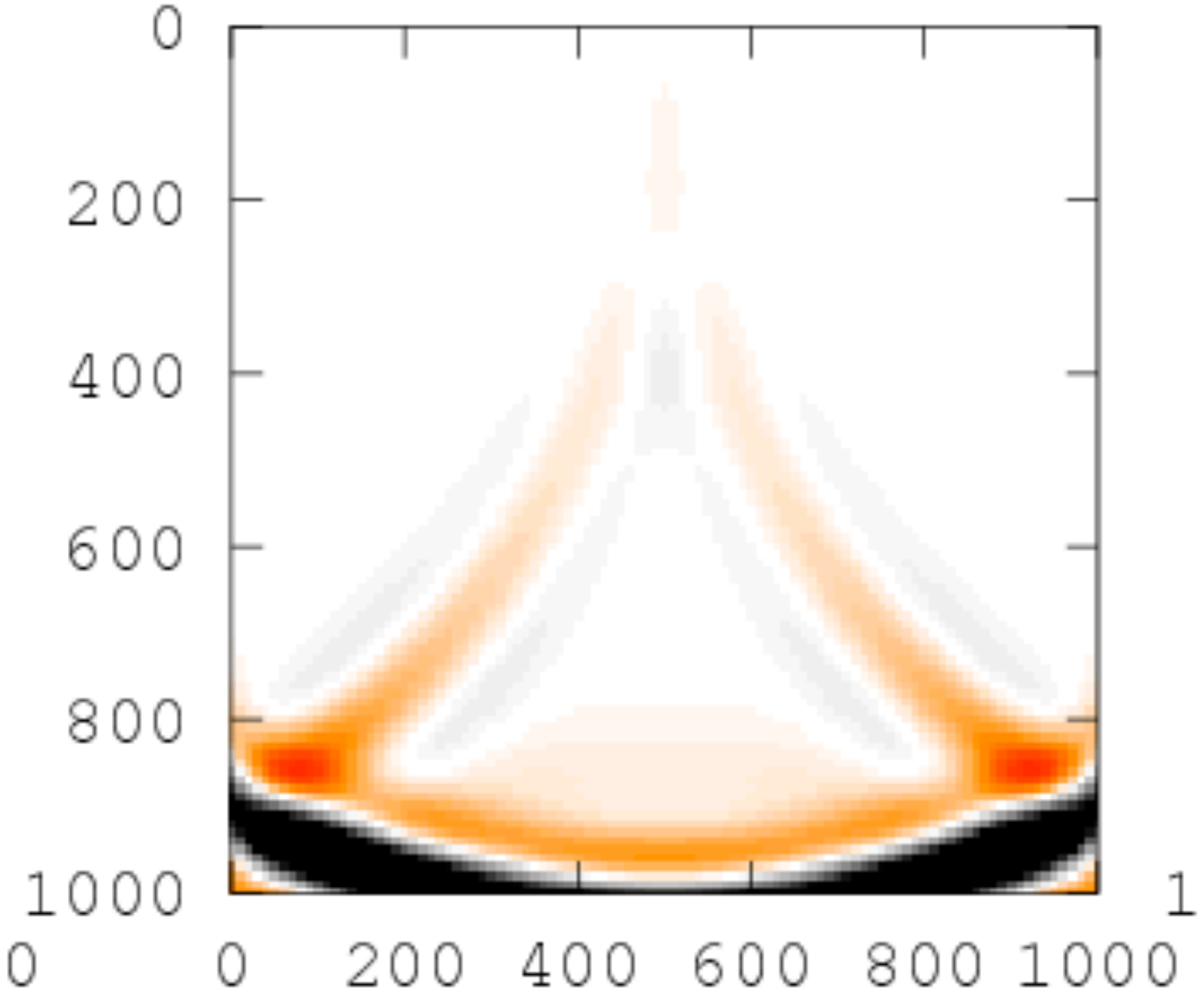
data-augmented wavefield in *constant* model



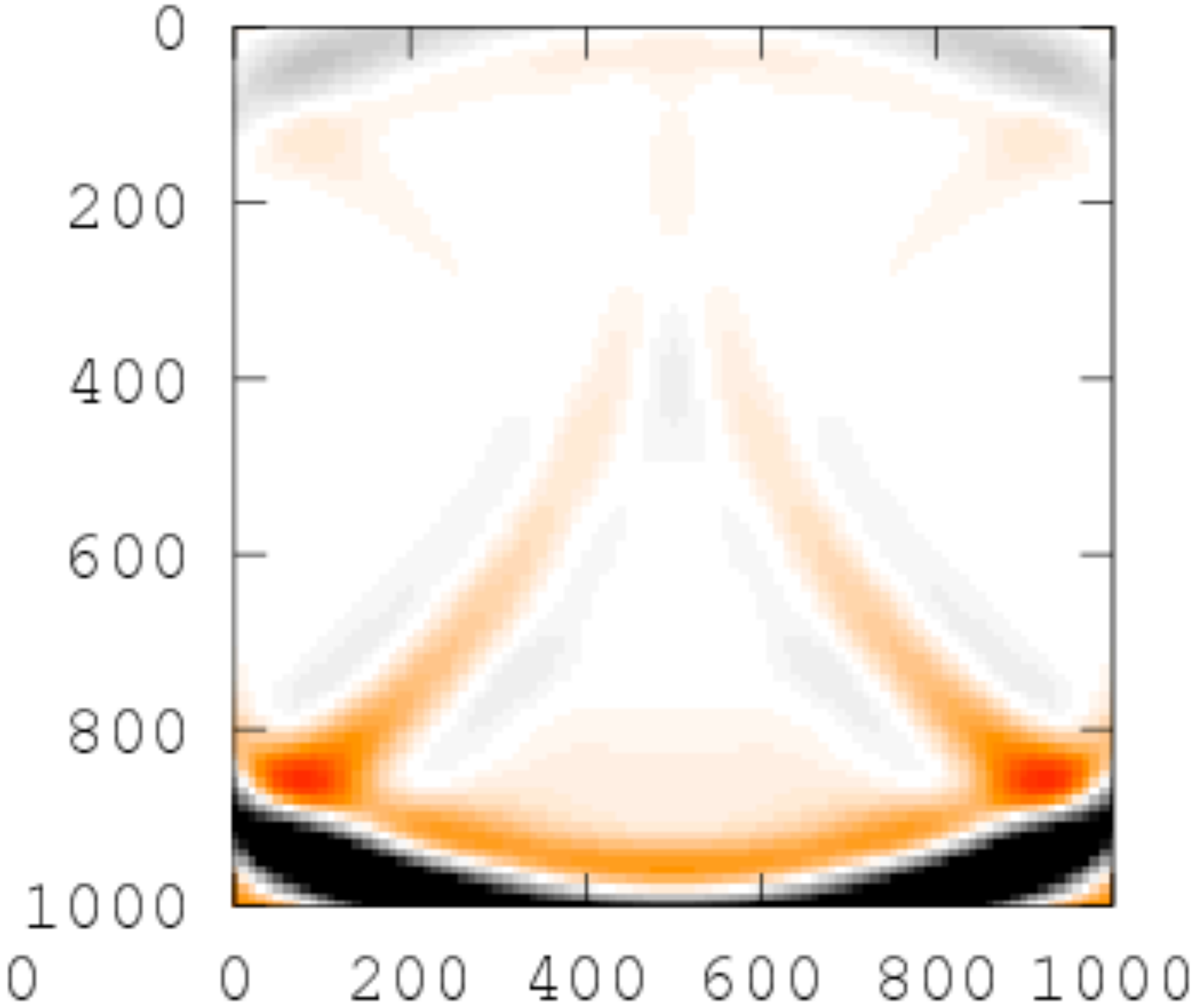
wavefield in *true* model



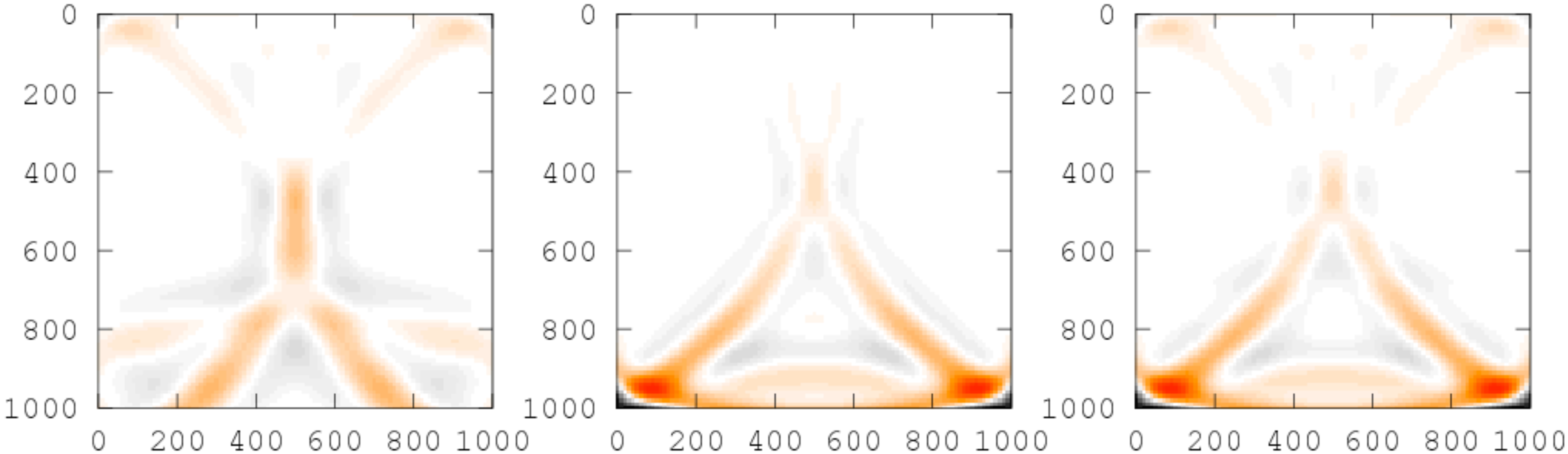
wavefield in *constant* model



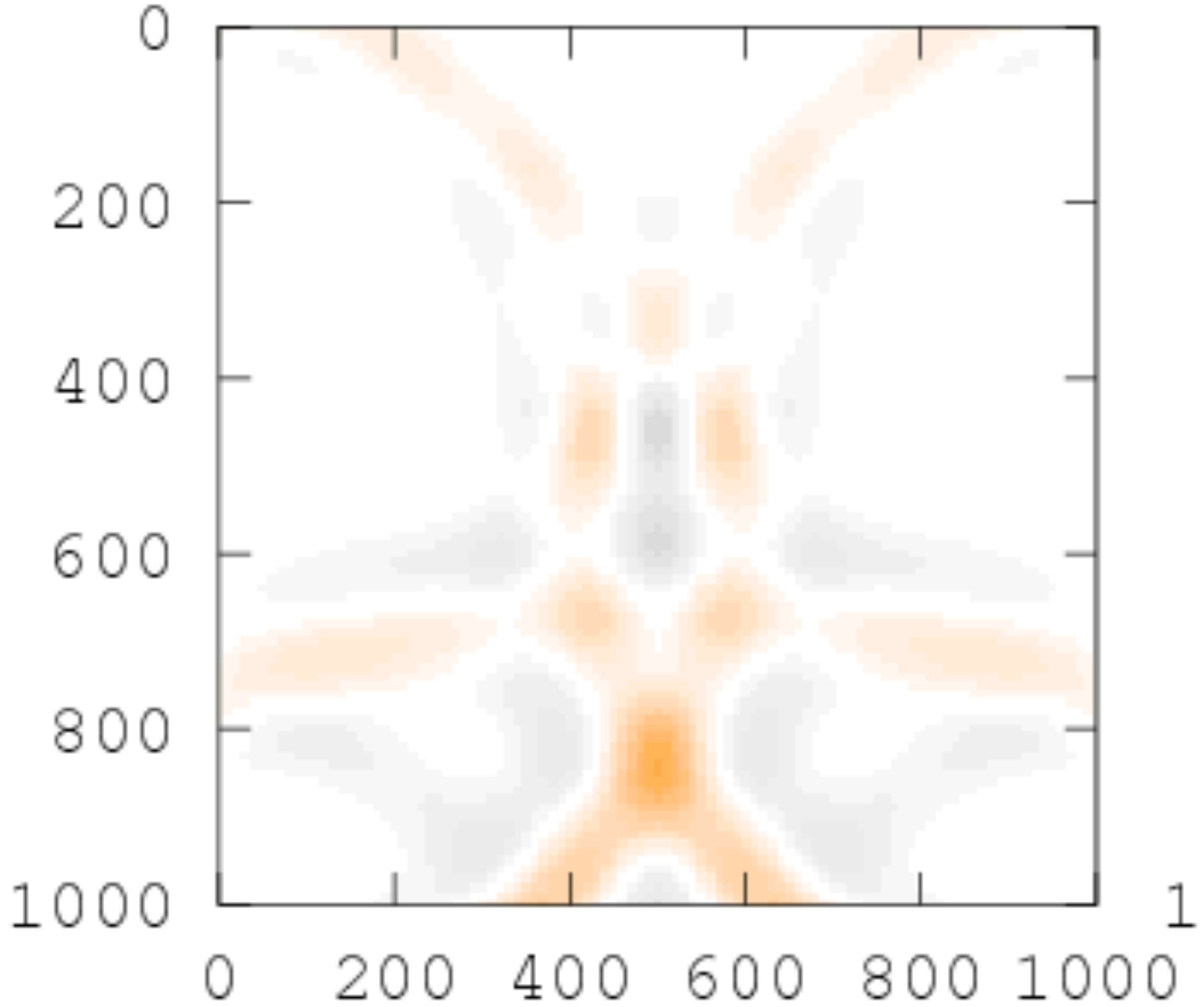
data-augmented wavefield in *constant* model



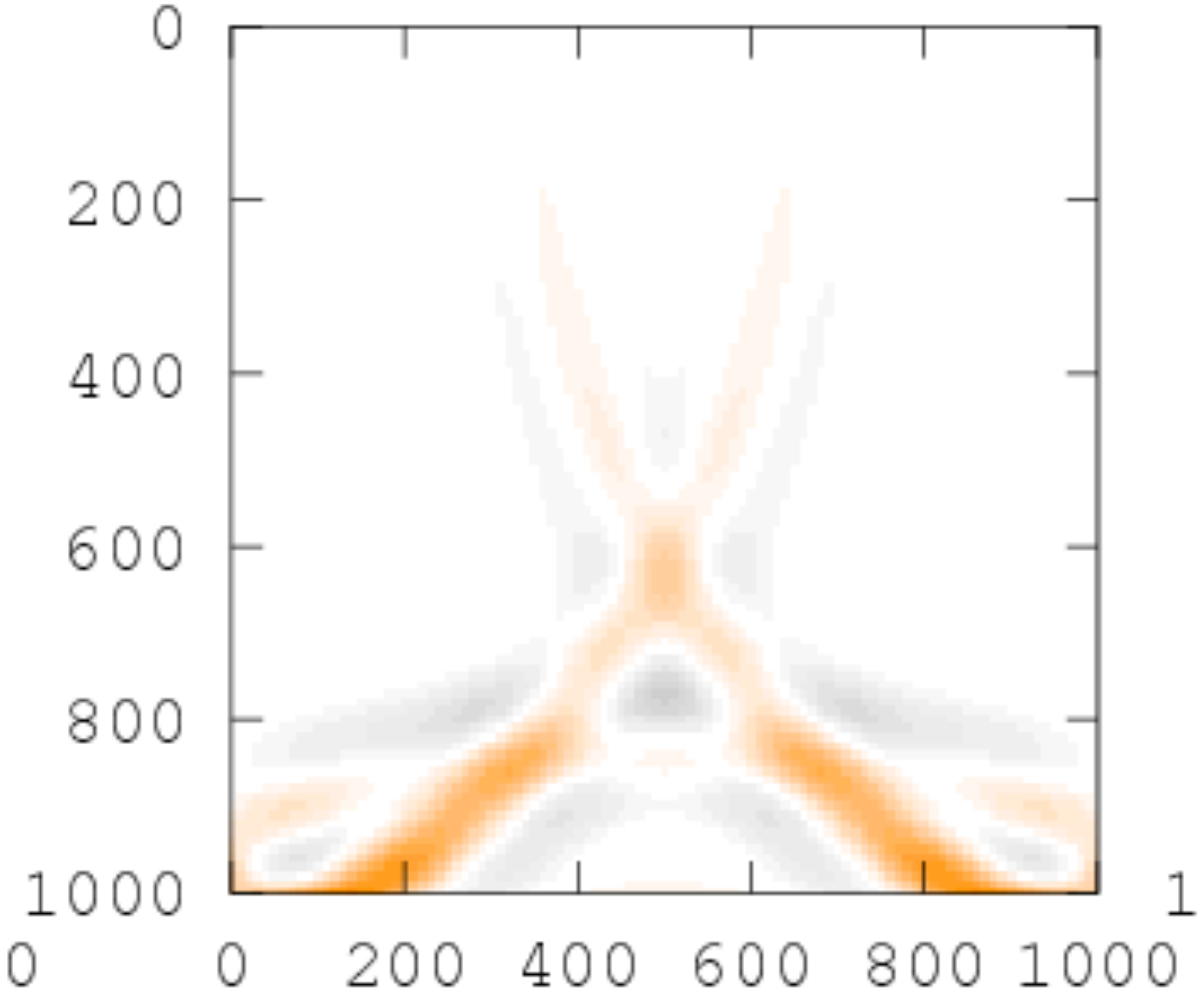
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



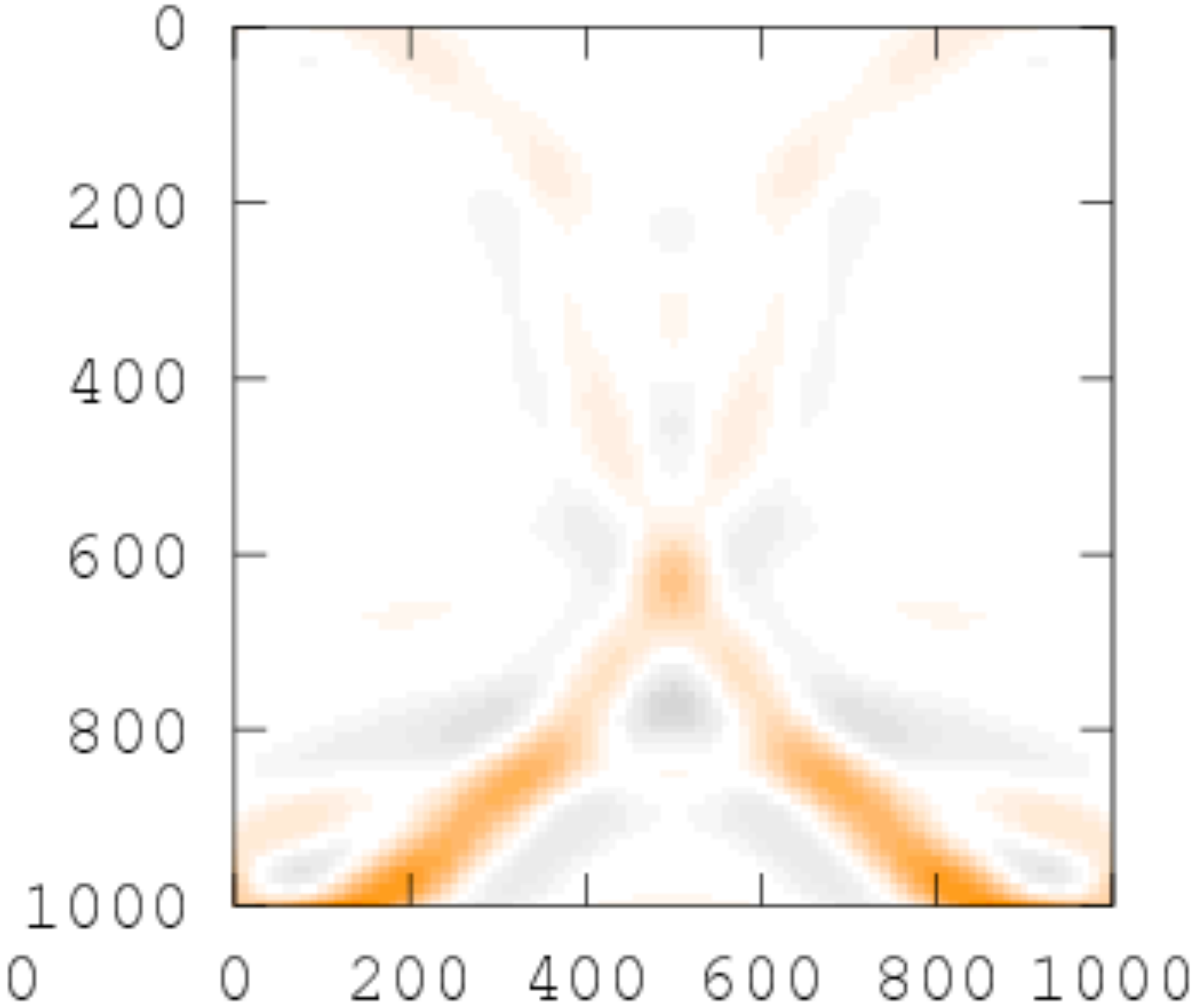
wavefield in *true* model



wavefield in *constant* model



**data-augmented
wavefield in *constant* model**



Data-*augmented* wave equation

Solutions *jointly* fit *wave* equation & *observed* data

$$\begin{pmatrix} P_i \\ A(\mathbf{m}) \end{pmatrix} \mathbf{u}_i \approx \begin{pmatrix} \mathbf{d}_i \\ \mathbf{q}_i \end{pmatrix}$$

by creating additional “reflection-like” events to explain the *data*.

Contains *information* to *update* the earth *model.*

[Heinkenschloss, ’98 , Haber, ’00]

PDE-constrained optimization
all-at-once full-space approach

simulated data

simulated wavefield

$$\min_{\mathbf{m},\mathbf{u}} \sum_{i=1}^{M} \|P_i \mathbf{u}_i - \mathbf{d}_i\|_2^2 \quad \text{s.t.} \quad A_i(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

observed data

source

Helmholtz equation

- avoids having to solve the PDE explicitly
- sparse (GN) Hessian
- requires storing all variables (**m,u**)
- does **not** scale to industry-scale seismic problems

PDE-constrained optimization

Reformulate as *unconstrained* problem via the *Lagrangian*

$$\mathcal{L}(\mathbf{m},\mathbf{u},\mathbf{v}) = ||P\mathbf{u}-\mathbf{d}||_2^2 + \mathbf{v}^*(A(\mathbf{m})\mathbf{u}-\mathbf{q})$$

Leads to *large*-scale *root*-finding problem

- *avoids* having to solve the PDE *explicitly*
- *sparse* (GN) Hessian
- *all-at-once* approach based on KKT system requires storing *all* variables
- does ***not*** scale to *industry*-scale seismic problems

[Heinkenschloss, '98 , Haber, '00]

Adjoint-state/reduced-space formulation

Elimination of the constraint leads for all sources to

$$\min_{\mathbf{m}} \phi_{\text{red}}(\mathbf{m}) = \sum_{i=1}^{M} \|P_i A_i(\mathbf{m})^{-1}\mathbf{q}_i - \mathbf{d}_i\|_2^2$$

- no need to store all wavefields (block-elimination)
- suitable for black-box optimization (e.g., l-BFGS)
- need to solve forward & adjoint PDEs
- very non-linear in earth model (m)
- dense (GN) Hessian, involves additional PDE solves

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WRI – penalty formulation

Instead of eliminating, we add constraints as penalties–i.e.,

$$\min_{\mathbf{m},\mathbf{u}} \phi_\lambda(\mathbf{m},\mathbf{u}) = \sum_{i=1}^{M} \|P\mathbf{u}_i - \mathbf{d}_i\|_2^2 + \lambda^2 \|A_i(\mathbf{m})\mathbf{u}_i - \mathbf{q}_i\|_2^2$$

coincides with original problem when $\lambda \uparrow \infty$

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Variable projection

Solve data-augmented wave equation for each source

$$\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$$

Define reduced objective

$$\phi_\lambda(\mathbf{m}) = \phi_\lambda(\mathbf{m}, \bar{\mathbf{u}}_\lambda) = \|P\bar{\mathbf{u}}_\lambda - \mathbf{d}\|_2^2 + \lambda^2 \|A(\mathbf{m})\bar{\mathbf{u}}_\lambda - \mathbf{q}\|_2^2$$

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Gradient

Wavefield eliminated–i.e., $\nabla_{\bar{\mathbf{u}}}\phi_\lambda(\mathbf{m},\bar{\mathbf{u}}) = 0$ by solving

$$\left(\lambda^2 A^*(\mathbf{m})A(\mathbf{m}) + P^*P\right)\mathbf{u} = \lambda^2 A^*(\mathbf{m})\mathbf{q} + P^*\mathbf{d}$$

yielding the gradient

Jacobian of $A(\mathbf{m})\bar{\mathbf{u}}_\lambda$

$$\nabla\phi_\lambda(\mathbf{m}) = G(\mathbf{m},\bar{\mathbf{u}}_\lambda)^*\bar{\mathbf{v}}_\lambda$$

with

$$\bar{\mathbf{v}}_\lambda = \lambda^2(A(\mathbf{m})\bar{\mathbf{u}}_\lambda - \mathbf{q}) \quad \text{(PDE residual)}$$

[van Leeuwen & FJH, '13]

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Wavefield Reconstruction Inversion

WRI method

for each source i

solve $\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u}_{\lambda,i} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$

$\mathbf{g} = \mathbf{g} + \lambda^2 \omega^2 \mathrm{diag}(\bar{\mathbf{u}}_{i,\lambda})^* (A(\mathbf{m})\bar{\mathbf{u}}_{i,\lambda} - \mathbf{q}_i)$

$\mathbf{m} = \mathbf{m} - \alpha \mathbf{g}$

end

correlation proxy wavefield & PDE residual

Conventional method

for each source i

solve $A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$

solve $A(\mathbf{m})^* \mathbf{v}_i = P_i^* (P_i \mathbf{u}_i - \mathbf{d}_i)$

$\mathbf{g} = \mathbf{g} + \omega^2 \mathrm{diag}(\mathbf{u}_i)^* \mathbf{v}_i$

$\mathbf{m} = \mathbf{m} - \alpha \mathbf{g}$

end

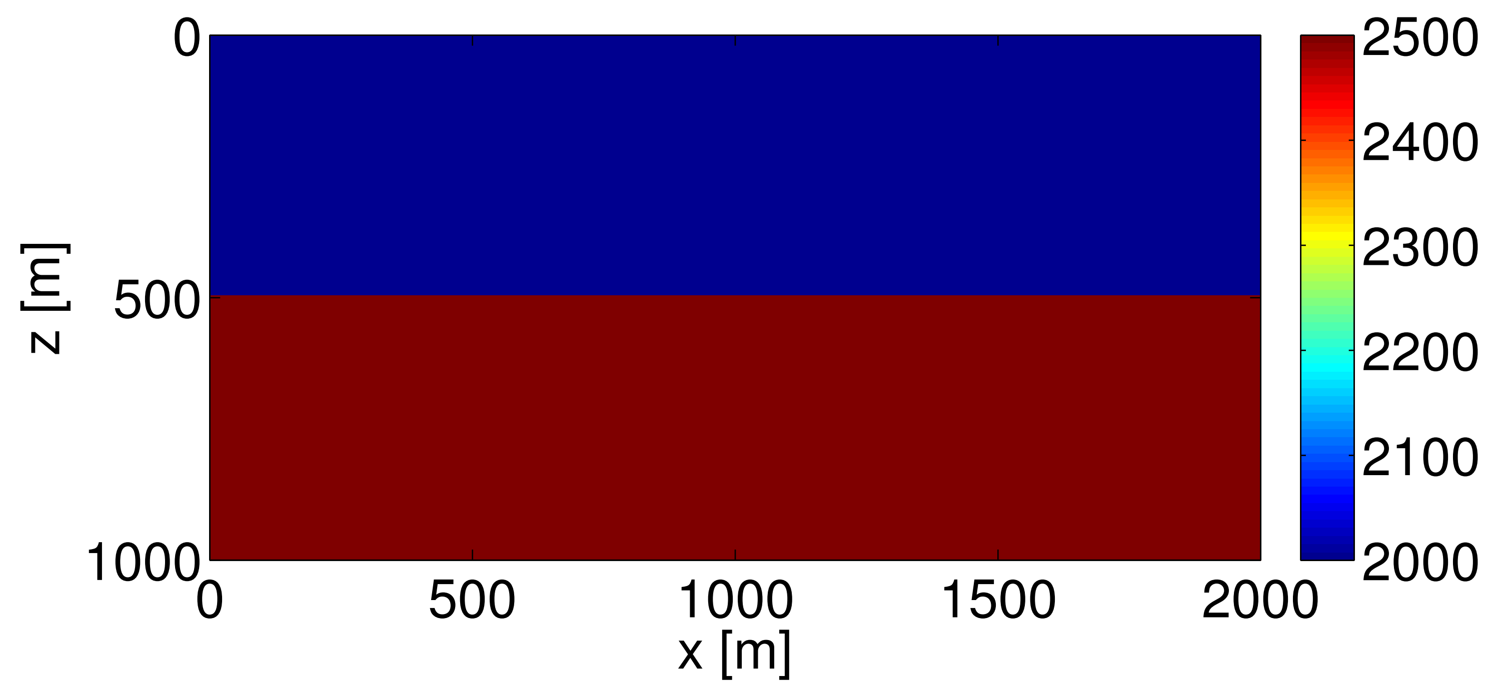
correlation wavefield & data residual

Wavefield Reconstruction Inversion

- **no** need to store all the fields (**u**)
- **no** adjoint solves
- sparse approximation of GN Hessian for small
- less non-linear in **m**
- need to solve overdetermined PDE
- not clear how to pick λ
- ...

One reflector example

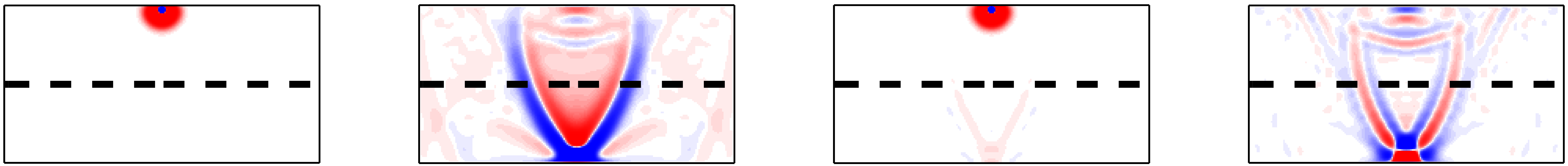
true model



Wavefields in *homogeneous* background

FWI | **WRI**

forward | adjoint | reconstructed wavefield | PDE residual

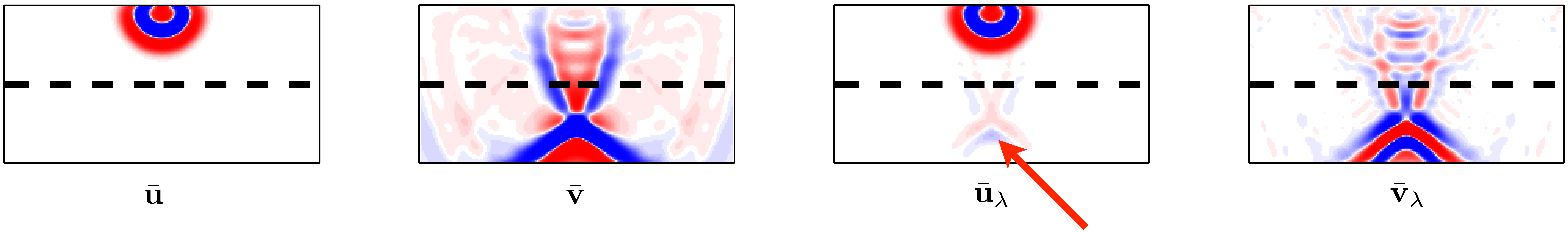


$\bar{\mathbf{u}}$ | $\bar{\mathbf{v}}$ | $\bar{\mathbf{u}}_\lambda$ | $\bar{\mathbf{v}}_\lambda$

Wavefields in *homogeneous* background

FWI | **WRI**

forward | adjoint | reconstructed wavefield | PDE residual

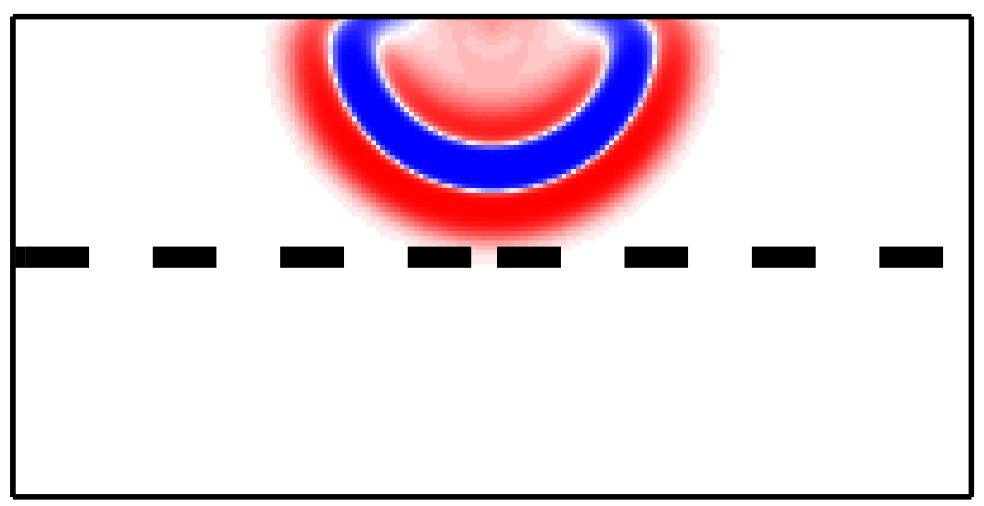


$\bar{\mathbf{u}}$ | $\bar{\mathbf{v}}$ | $\bar{\mathbf{u}}_\lambda$ | $\bar{\mathbf{v}}_\lambda$

Wavefields in *homogeneous* background

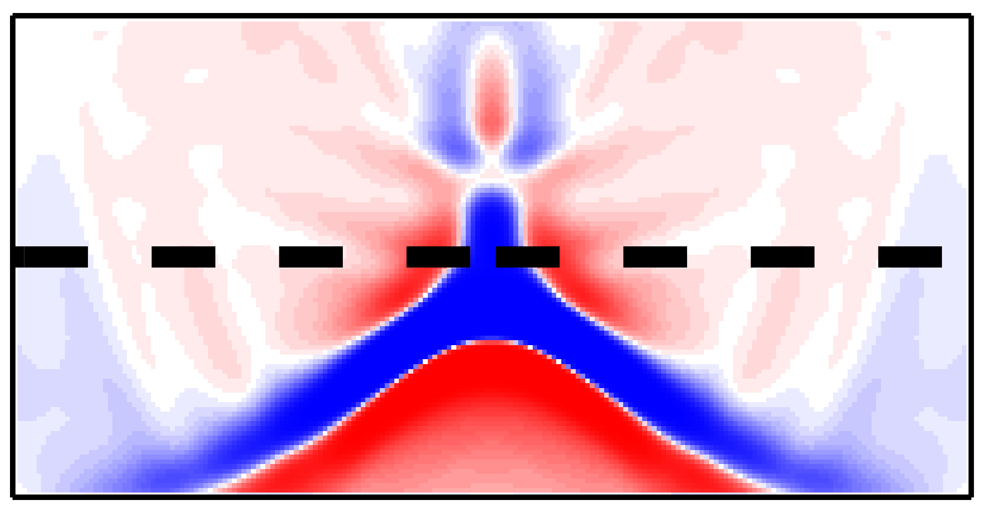
FWI

forward



$\bar{\mathbf{u}}$

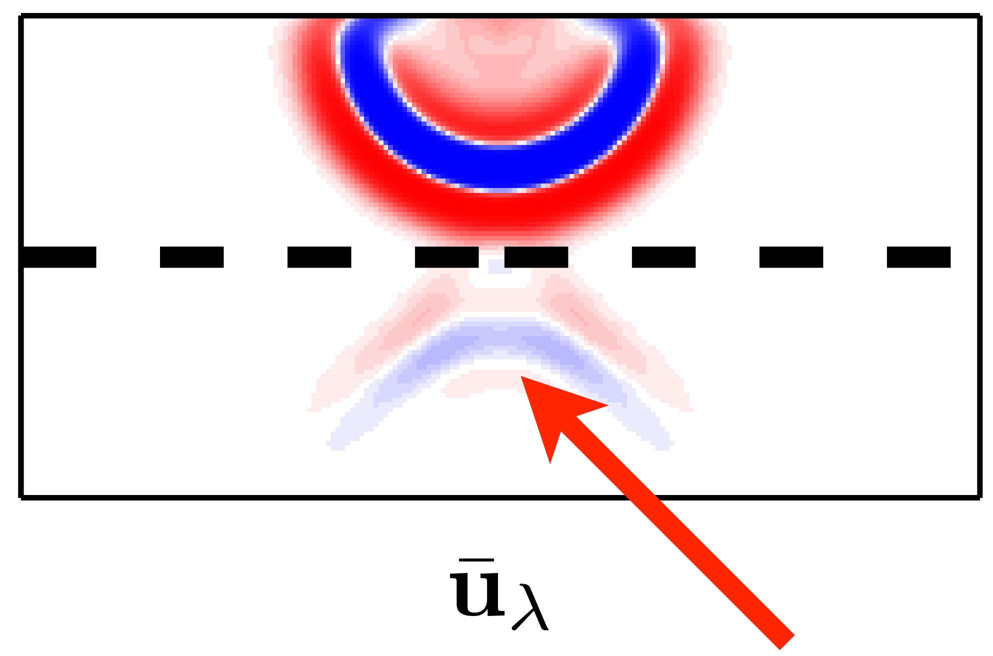
adjoint



$\bar{\mathbf{v}}$

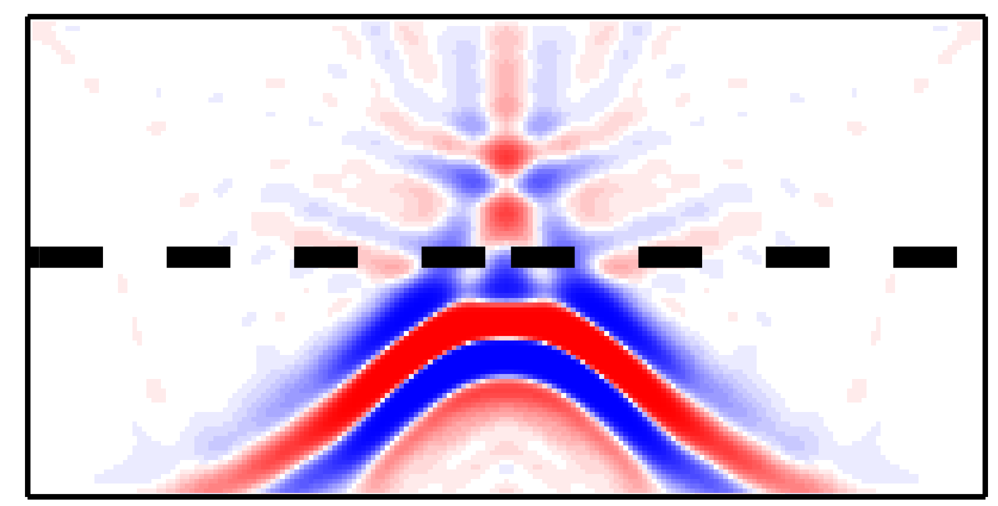
WRI

reconstructed wavefield



$\bar{\mathbf{u}}_{\lambda}$

PDE residual



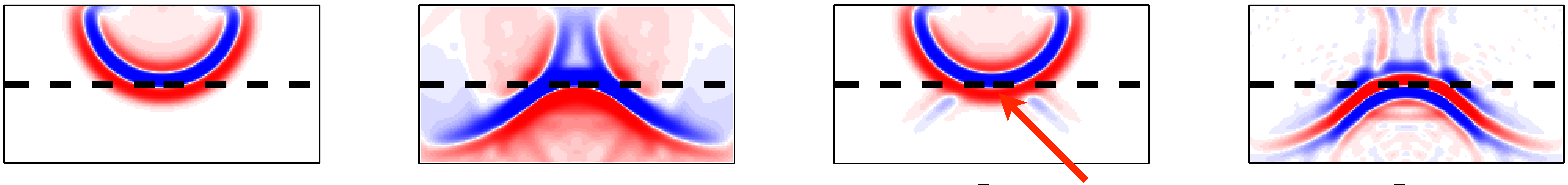
$\bar{\mathbf{v}}_{\lambda}$

Wavefields in *homogeneous* background

FWI

WRI

forward | adjoint | reconstructed wavefield | PDE residual



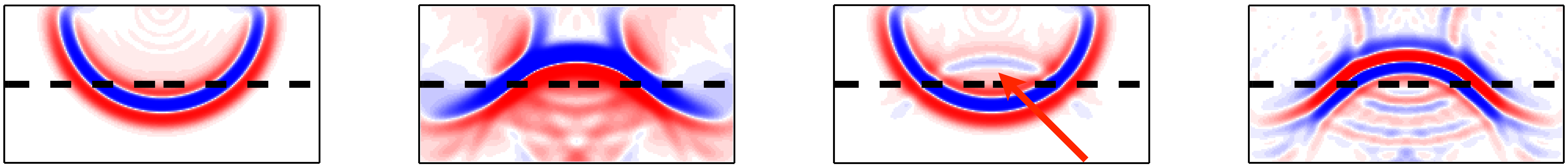
$\bar{\mathbf{u}}$ | $\bar{\mathbf{v}}$ | $\bar{\mathbf{u}}_\lambda$ | $\bar{\mathbf{v}}_\lambda$

Wavefields in *homogeneous* background

FWI

WRI

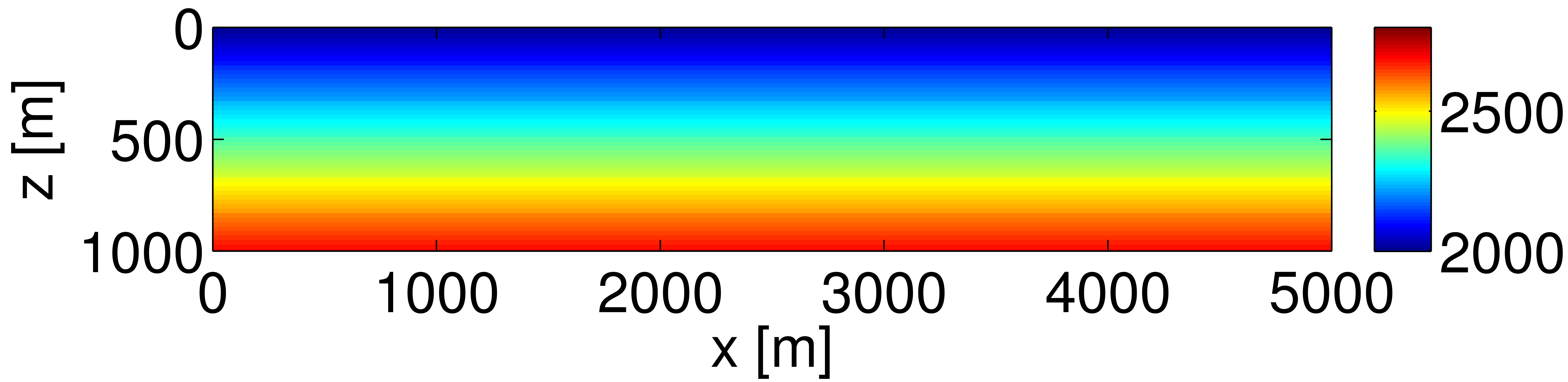
forward | adjoint | reconstructed wavefield | PDE residual

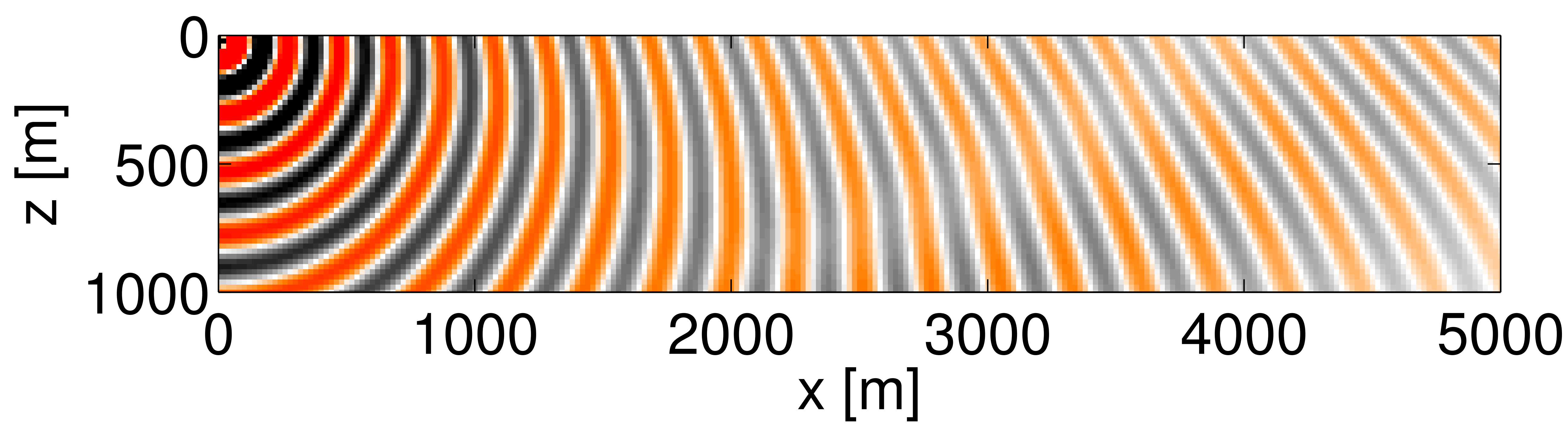


$\bar{\mathbf{u}}$ | $\bar{\mathbf{v}}$ | $\bar{\mathbf{u}}_\lambda$ | $\bar{\mathbf{v}}_\lambda$

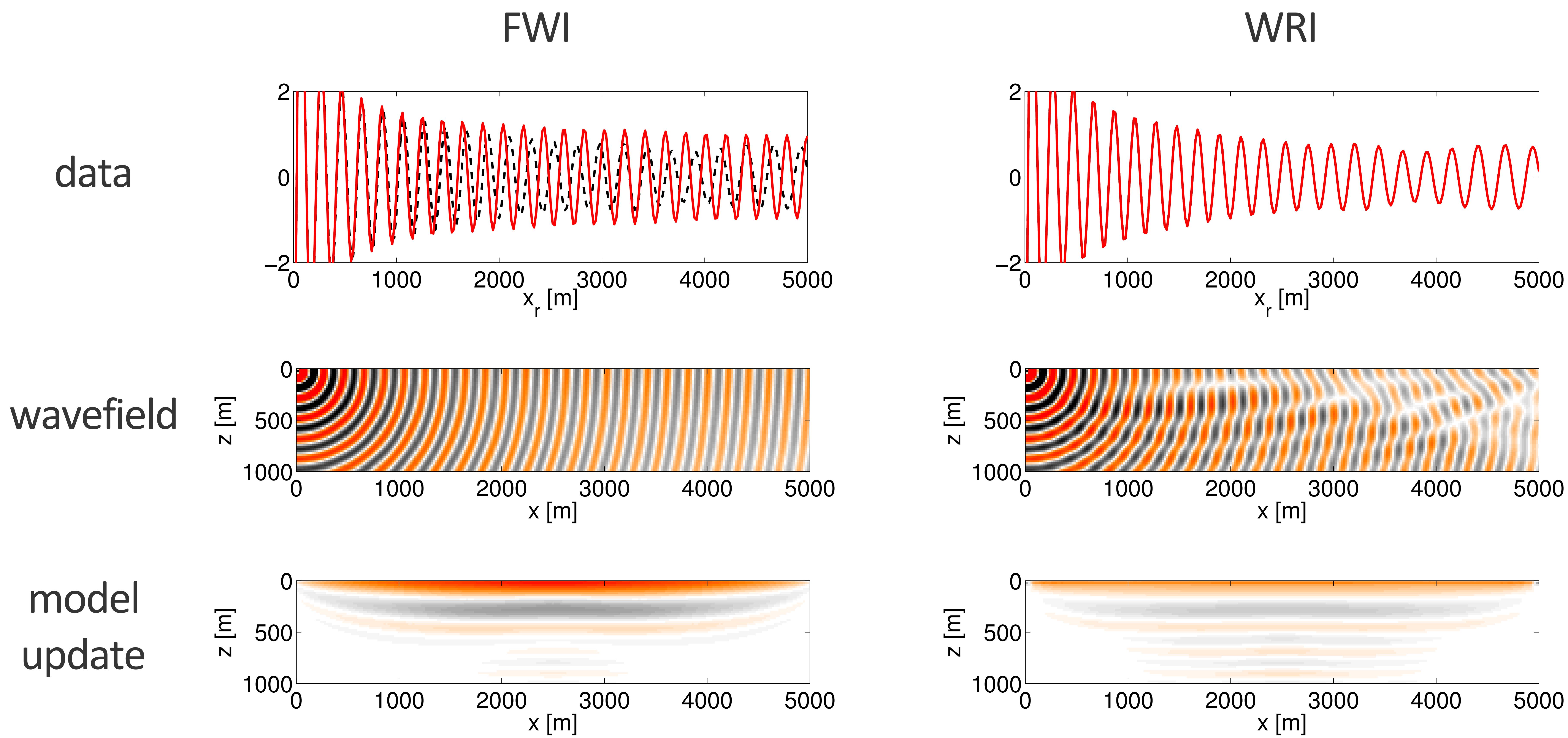
Diving wave example

true model and wavefield





Wavefields in *homogeneous* background



Connections

Related work

Contrast-source formulation

- combined objective is similar
- but does not eliminate wavefields via variable projection
- **requires storage of wavefields for all sources**

Extended modelling

The penalty formulation

$$\min_{\mathbf{m},\mathbf{u}} ||P\mathbf{u} - \mathbf{d}||_2^2 + \lambda^2 ||A(\mathbf{m})\mathbf{u} - \mathbf{q}||_2^2$$

can be interpreted as

$$\min_{\tilde{\mathbf{m}}} \text{misfit}(\tilde{\mathbf{m}}) + \text{annihilator}(\tilde{\mathbf{m}})$$

with

$$\tilde{\mathbf{m}} = (\mathbf{m}, \mathbf{u})$$

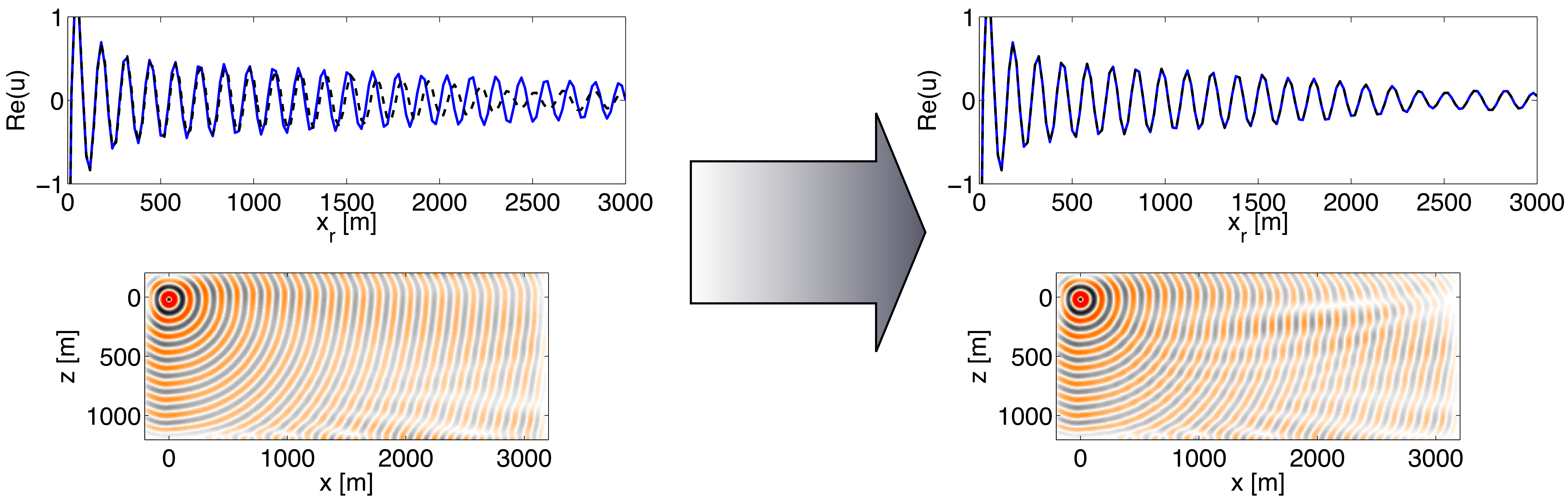
For a physically plausible model we have

$$\text{annihilator}(\tilde{\mathbf{m}}) = 0$$

[Symes, personal communication]

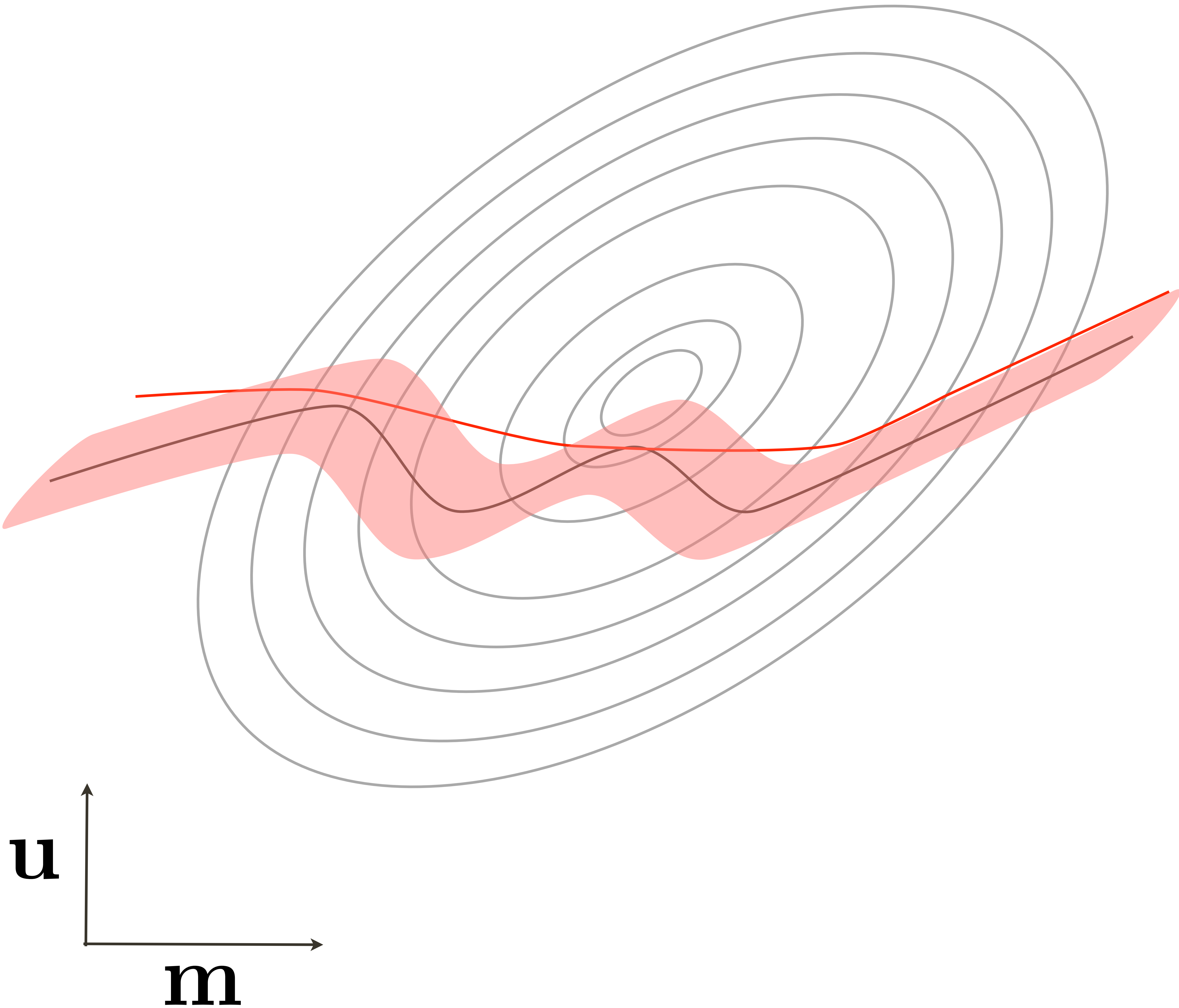
Warping

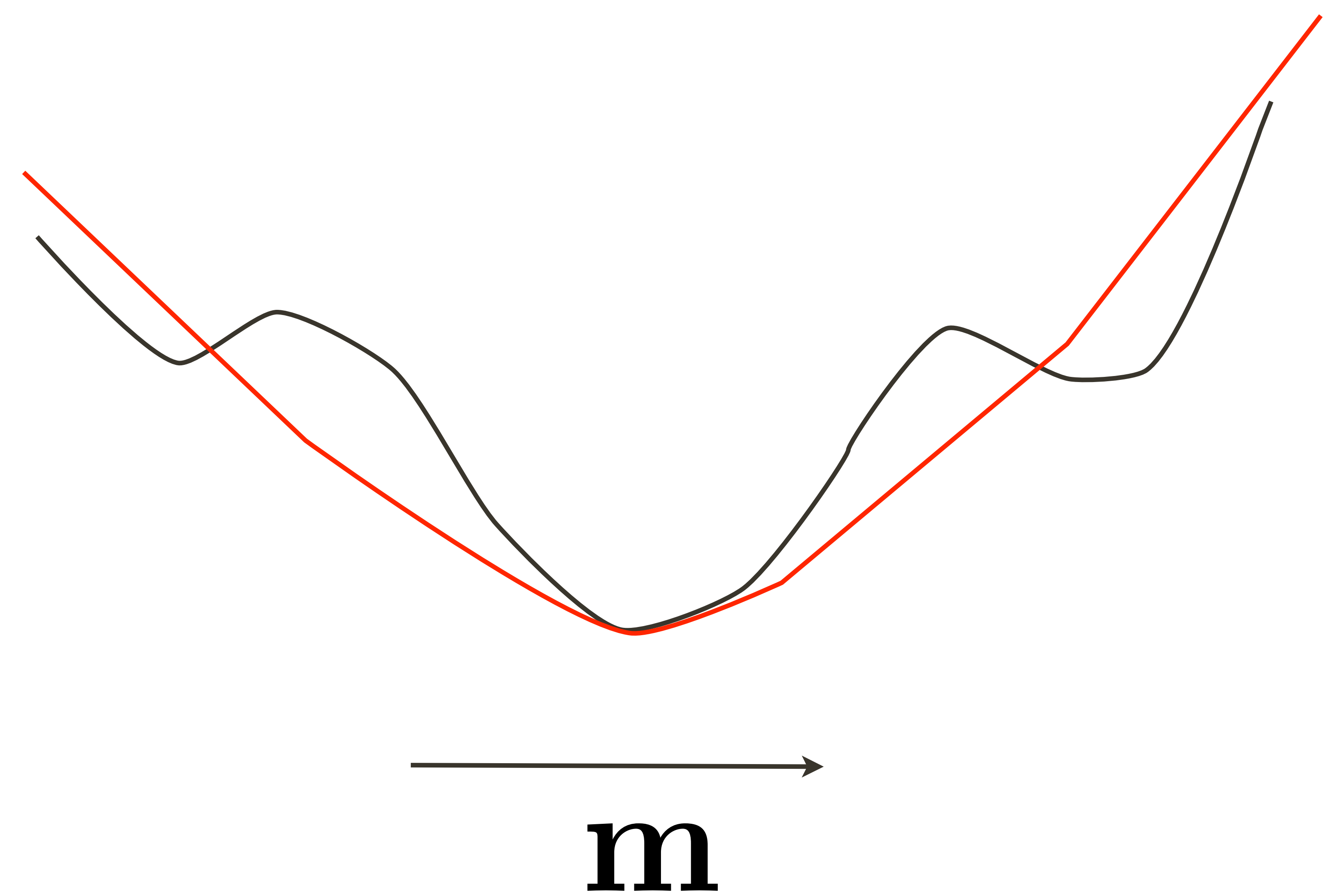
The overdetermined WE is a way of warping



[Baek ’13, Ma ’13]

WRI vs. FWI





[van Leeuwen & FJH, ’13]

SLIM

Wavefield Reconstruction Inversion

WRI method

for each source i

solve $\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u}_{\lambda,i} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$

$\mathbf{g} = \mathbf{g} + \lambda^2\omega^2 \text{diag}(\bar{\mathbf{u}}_{i,\lambda})^* (A(\mathbf{m})\bar{\mathbf{u}}_{i,\lambda} - \mathbf{q}_i)$

$H_{GN} = H_{GN} + \lambda^2\omega^4 \text{diag}(\mathbf{u}_i)^* \text{diag}(\mathbf{u}_i)$

$\mathbf{m} = \mathbf{m} - \alpha H_{GN}^{-1}\mathbf{g}$

end

diagonal = pseudo Hessian

Conventional method

for each source i

solve $A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$

solve $A(\mathbf{m})^*\mathbf{v}_i = P_i^*(P_i\mathbf{u}_i - \mathbf{d}_i)$

$\mathbf{g} = \mathbf{g} + \omega^2 \text{diag}(\mathbf{u}_i)^* \mathbf{v}_i$

$\mathbf{m} = \mathbf{m} - \alpha\mathbf{g}$

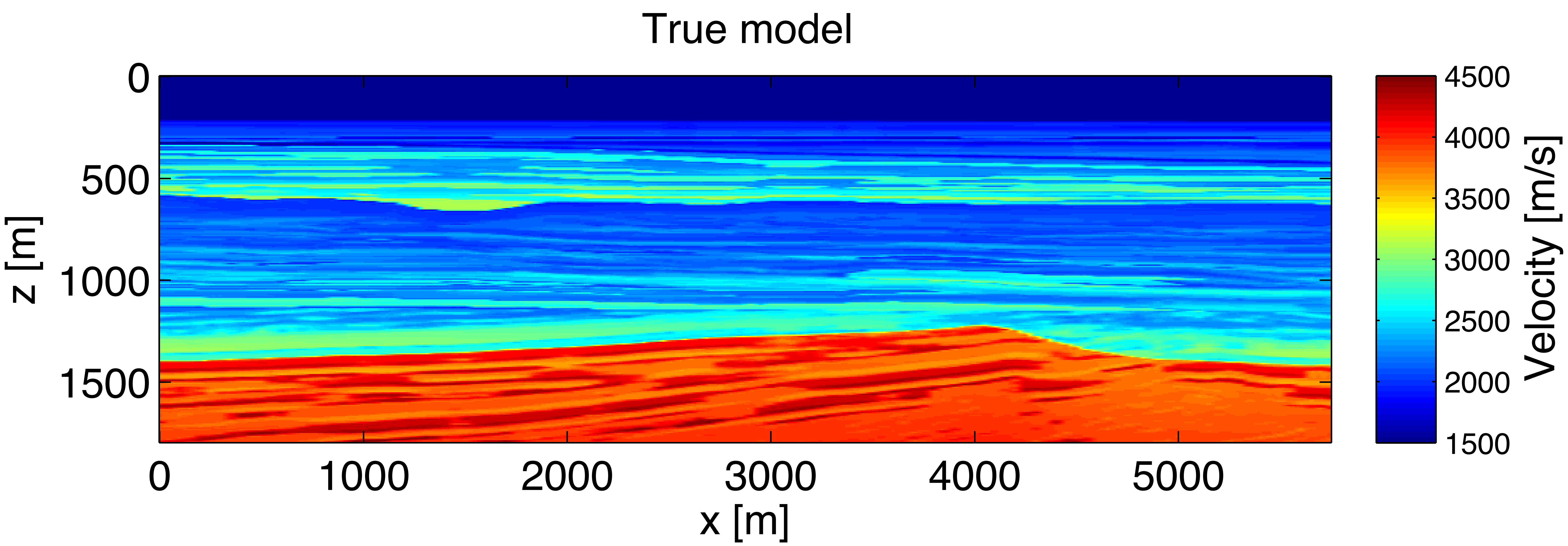
end

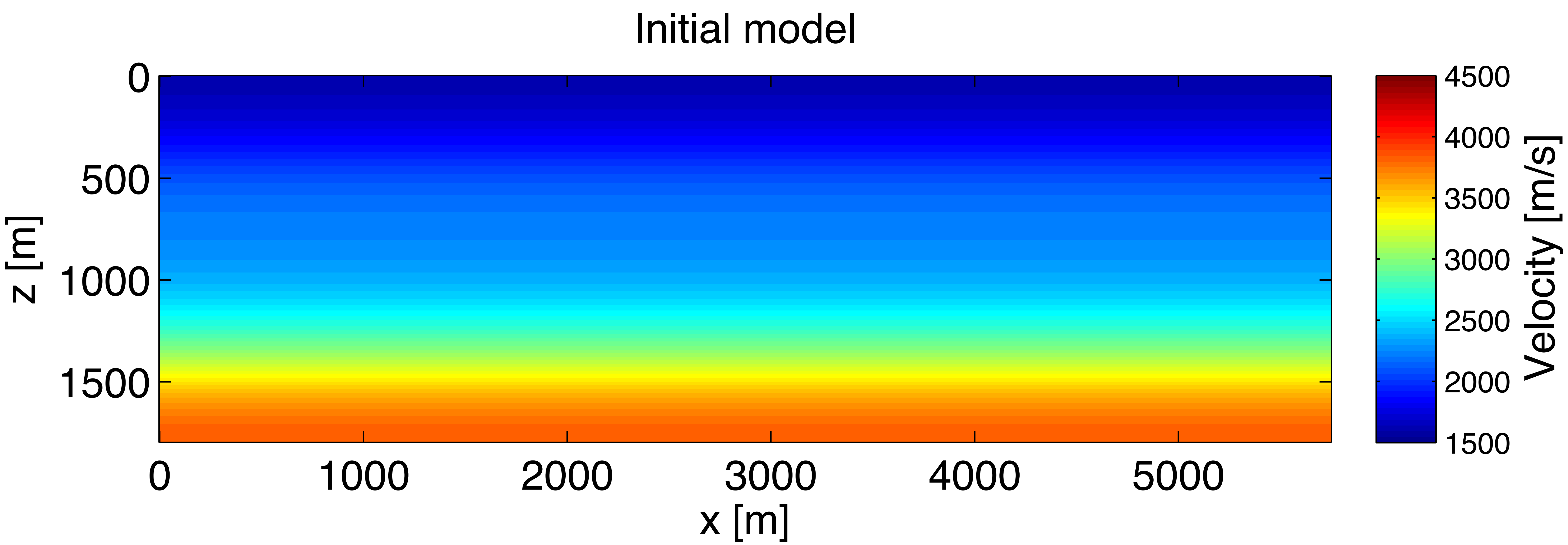
dense & too expsensive

Example – BG Compass model

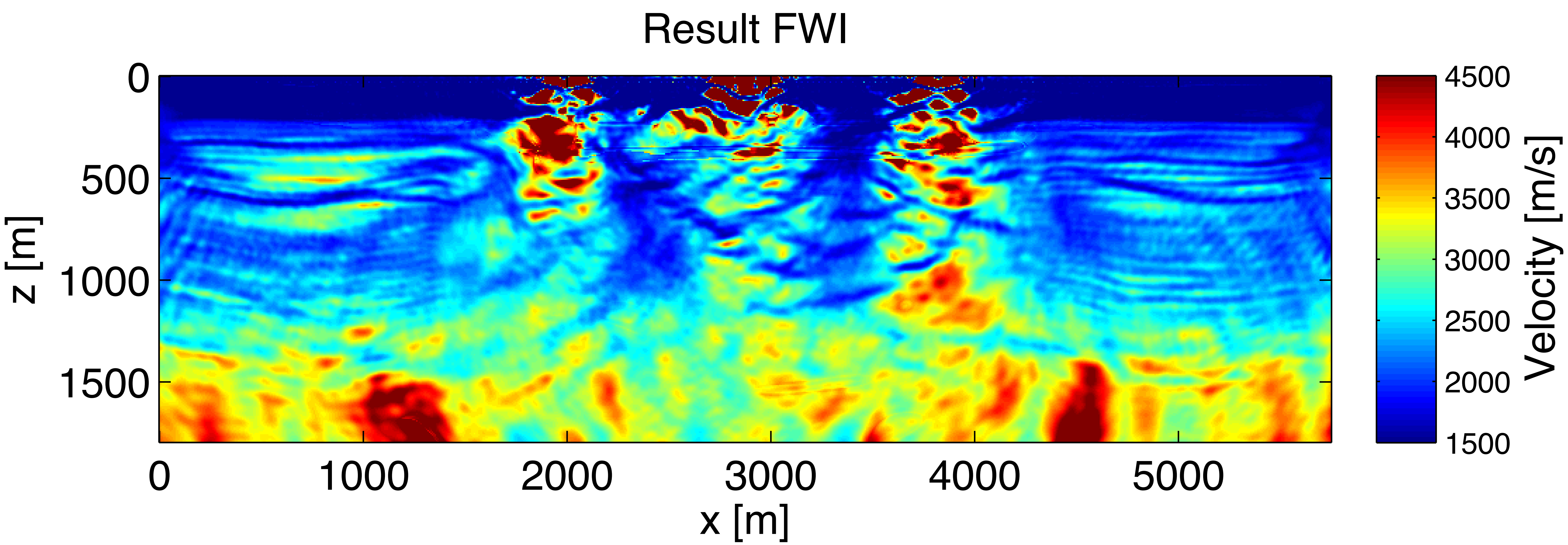
- *Low* frequencies missing, 24 frequency batches (15 iterations each) {5 6} ,{6 7},... ,{28 29} Hertz. Each interval contains 5 frequencies.
- 103 sources/receivers w/ 55m sample interval
- Inaccurate *initial* model

True & initial model

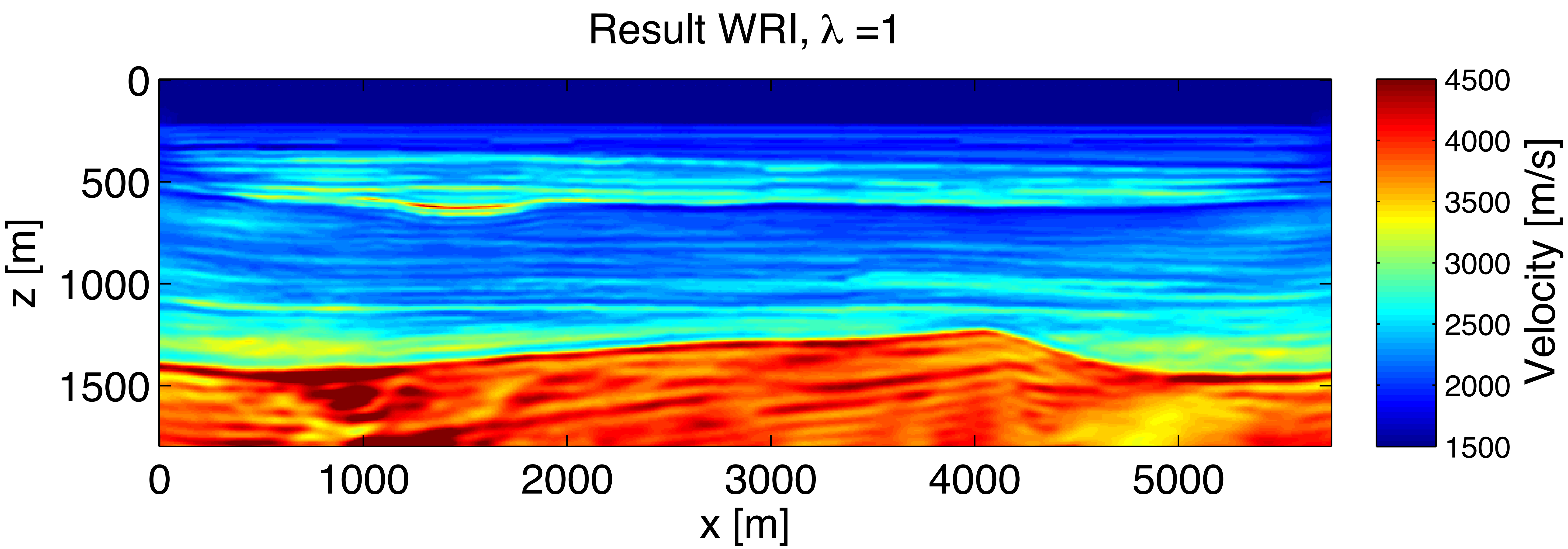




FWI vs WRI

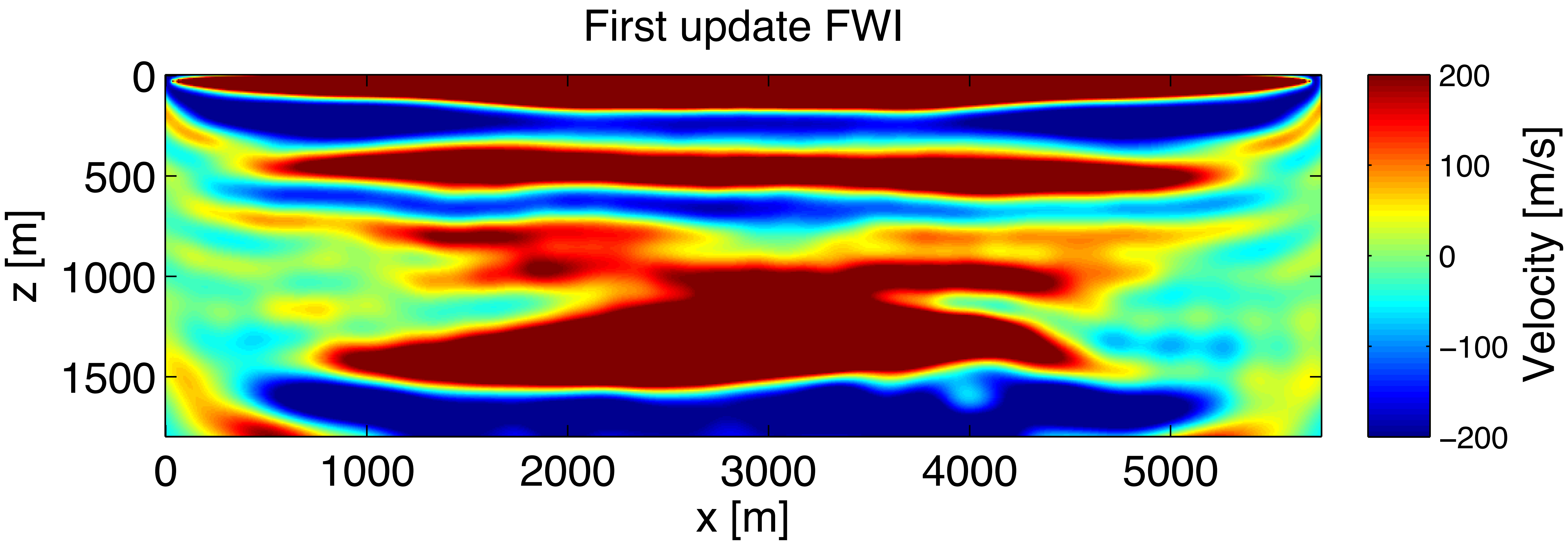


Result FWI
0
500
1000
1500
z [m]
0
1000
2000
3000
4000
5000
x [m]
4500
4000
3500
3000
2500
2000
1500
Velocity [m/s]

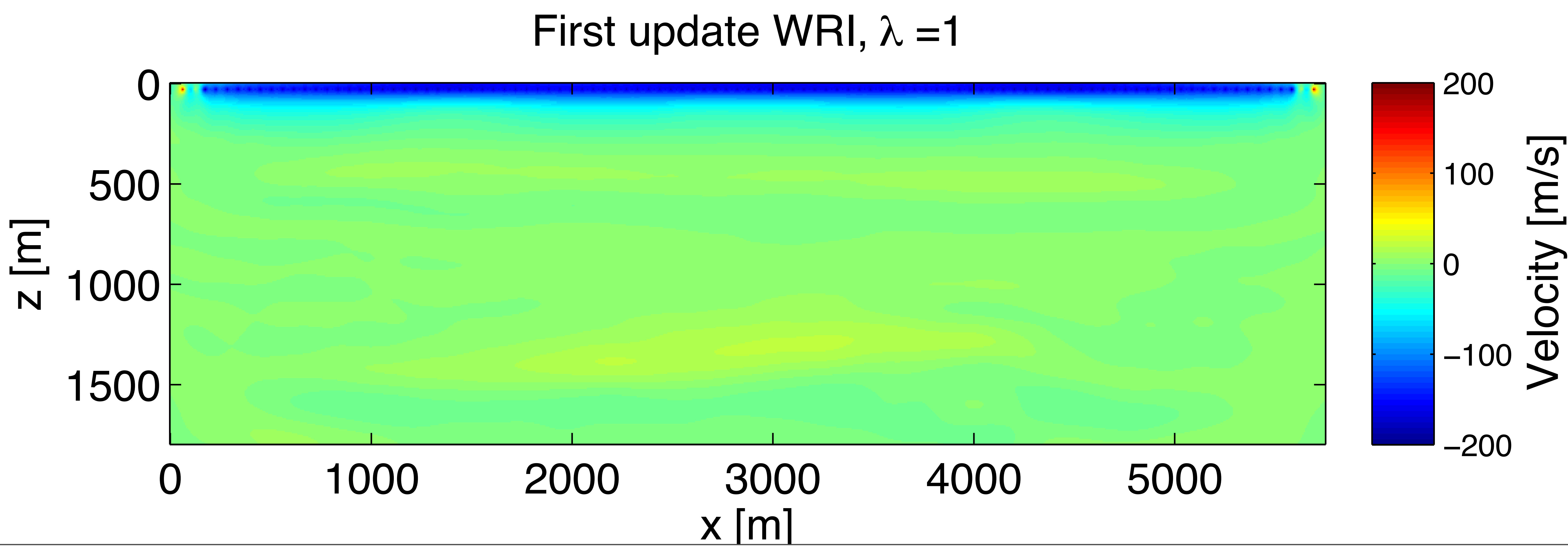


Result WRI, λ =1
0
500
1000
1500
z [m]
0
1000
2000
3000
4000
5000
x [m]
4500
4000
3500
3000
2500
2000
1500
Velocity [m/s]

Gradients

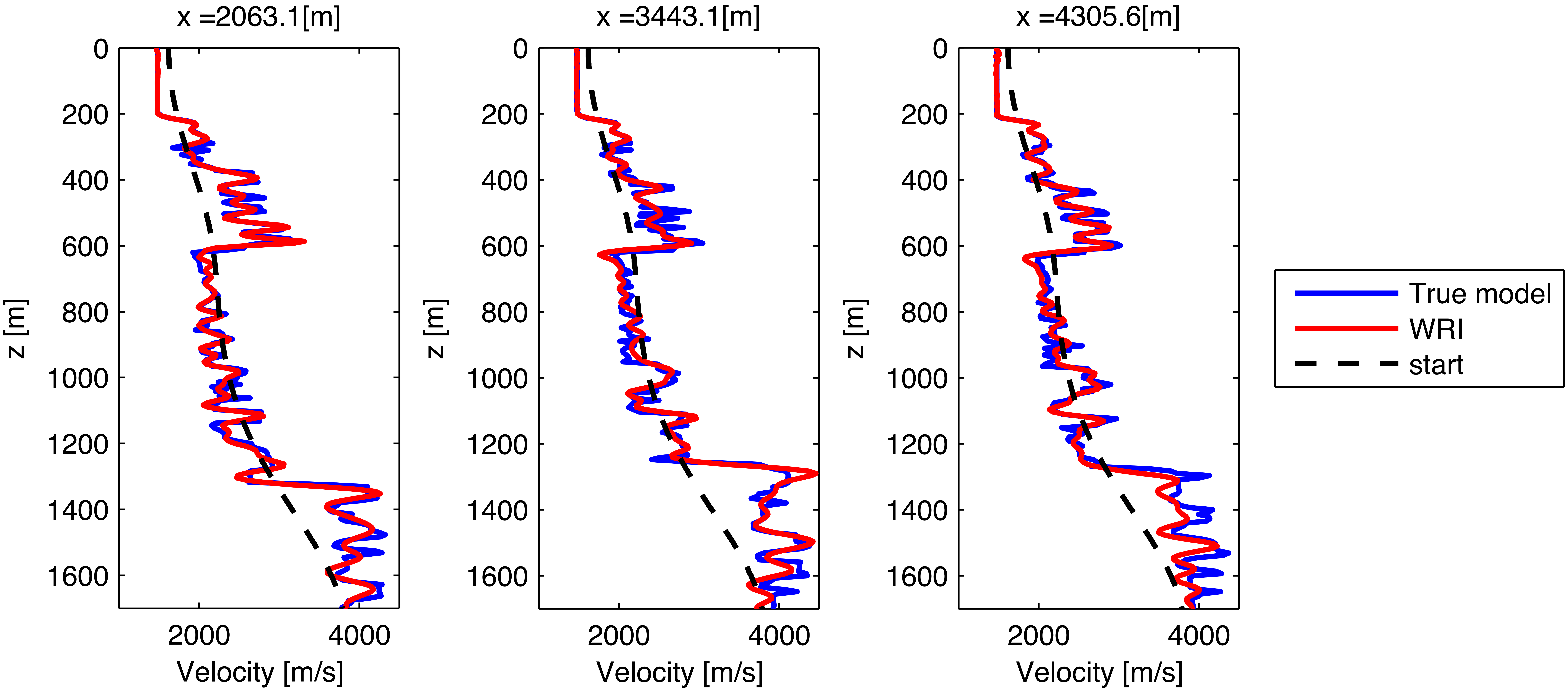


First update FWI
z [m]
x [m]
Velocity [m/s]

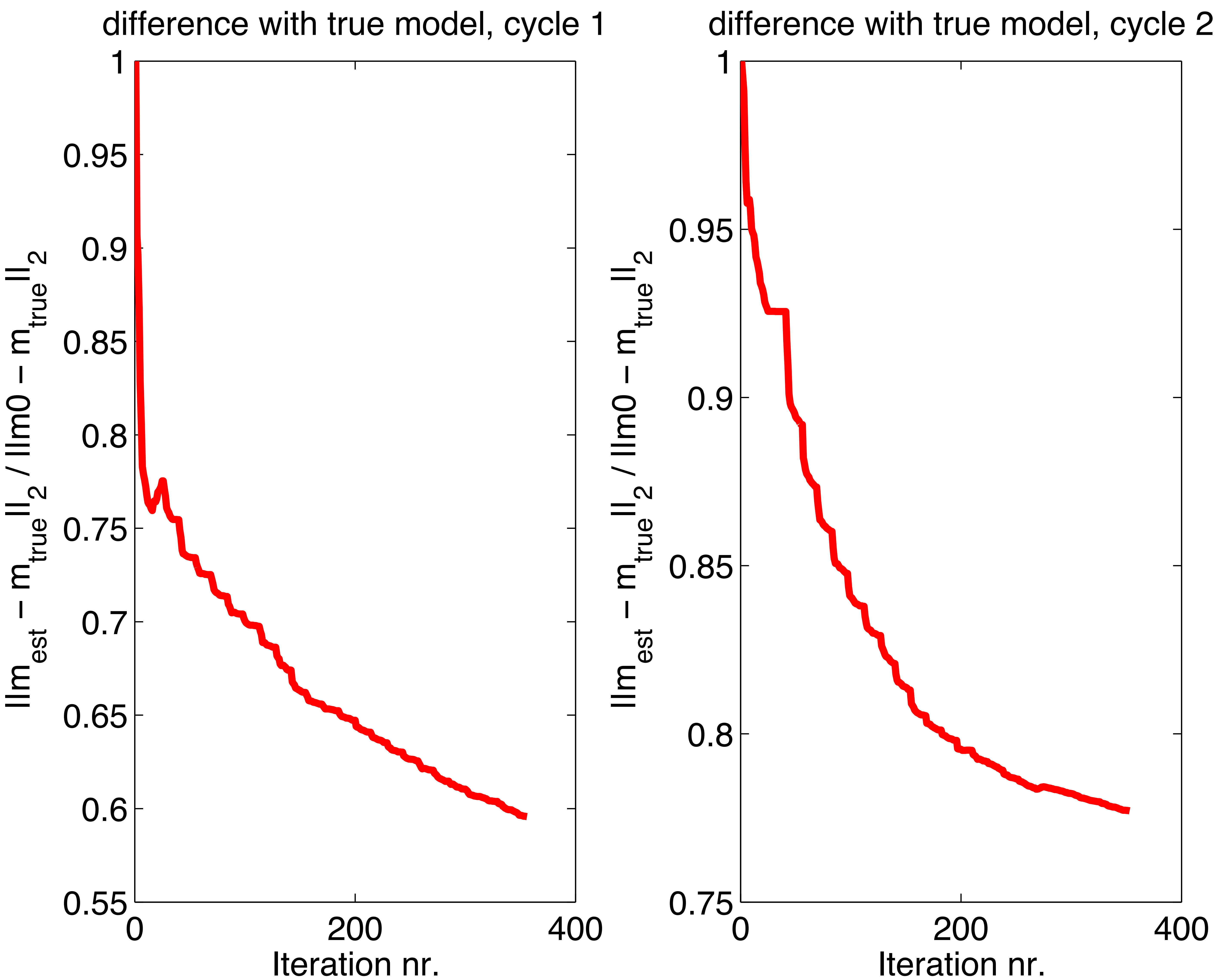


First update WRI, λ =1
z [m]
x [m]
Velocity [m/s]

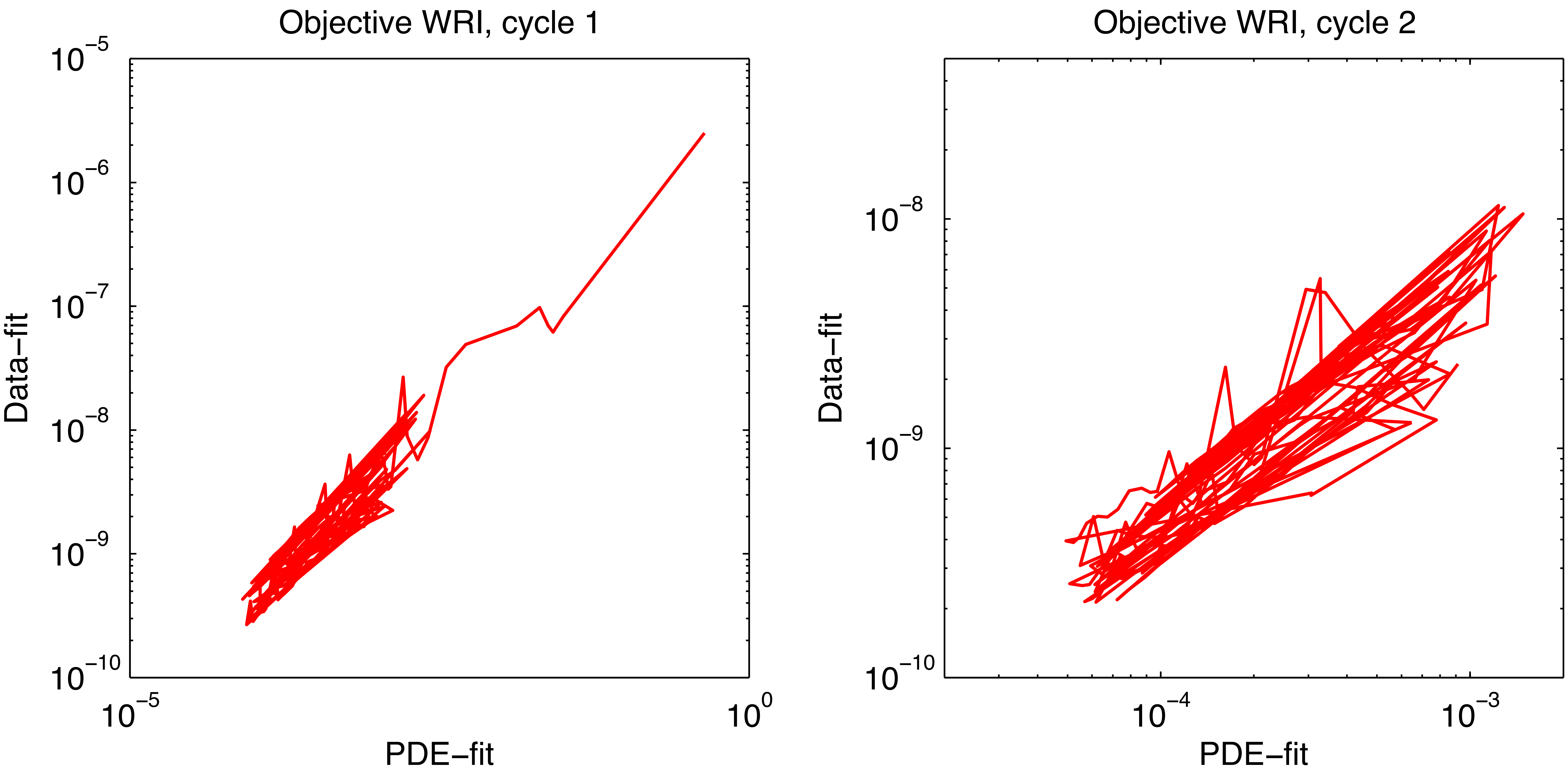
Cross sections



Relative model errors

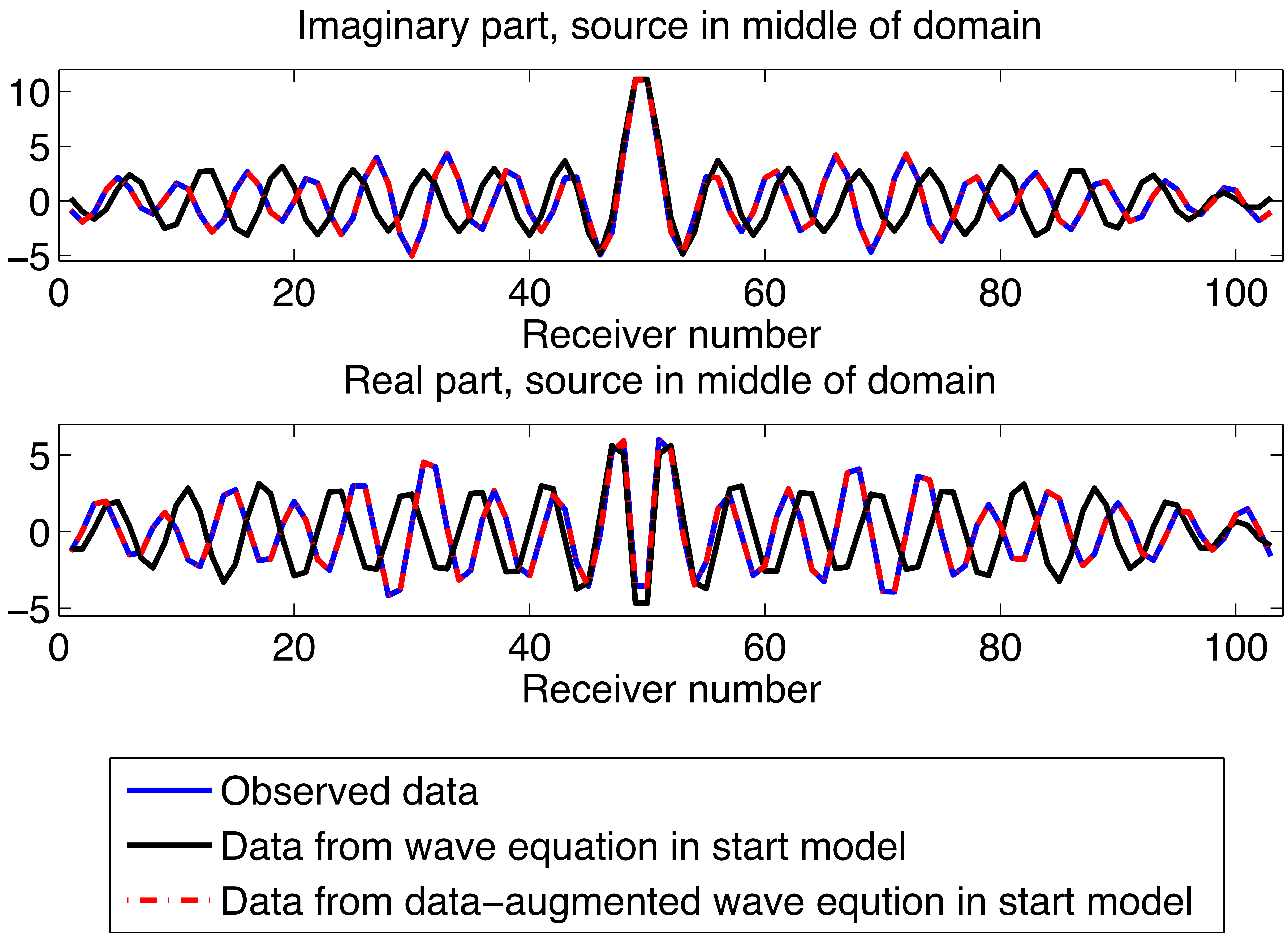


Objective function value



Data fit increases at some iterates

Data fit



Imaginary part, source in middle of domain
10
5
0
−5
0
20
40
60
80
100
Receiver number
Real part, source in middle of domain
5
0
−5
0
20
40
60
80
100
Receiver number
Observed data
Data from wave equation in start model
Data from data−augmented wave eqution in start model

Perspectives

Gauss-Newton Hessian is sparse – fast evaluation of inverse

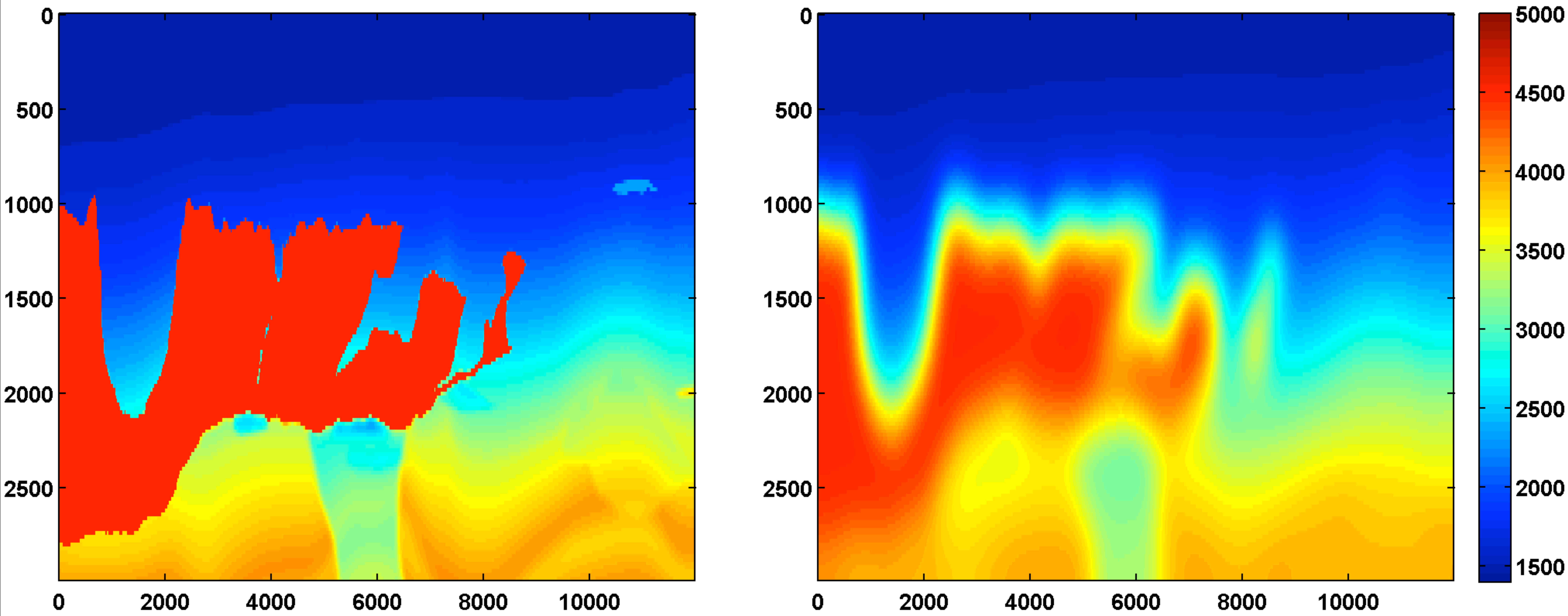
- good for UQ (Zhilong's talk)
- possibility to include TV or one-norm minimization (Ernie's talk)
- possibility to exploit for multi-parameter (WRI) (Bas's talk)

Data-augmented system can be solved for large problems (Bas's talk)

Data fit also possible w/ data-constrained formulation (Rong's talk)

Penalty formulation offers possibility for joint formulation (Bas's talk)

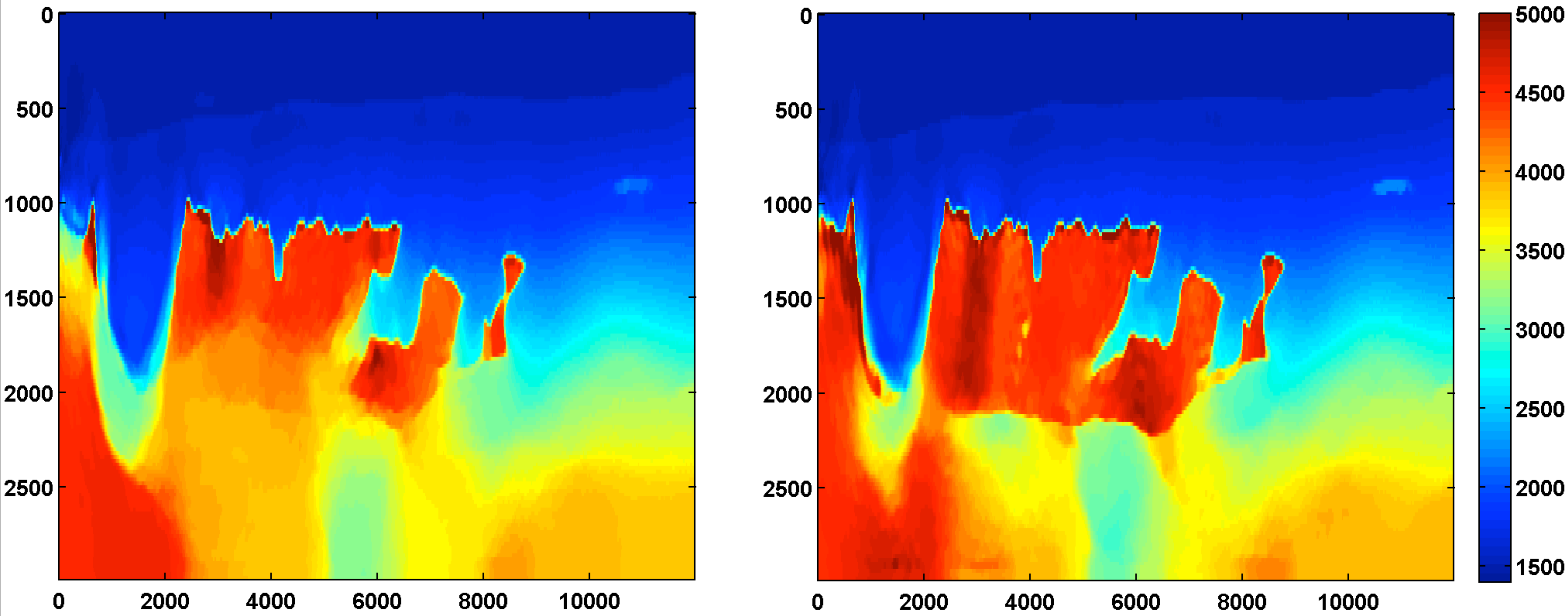
True and initial velocity



Results w/ TV

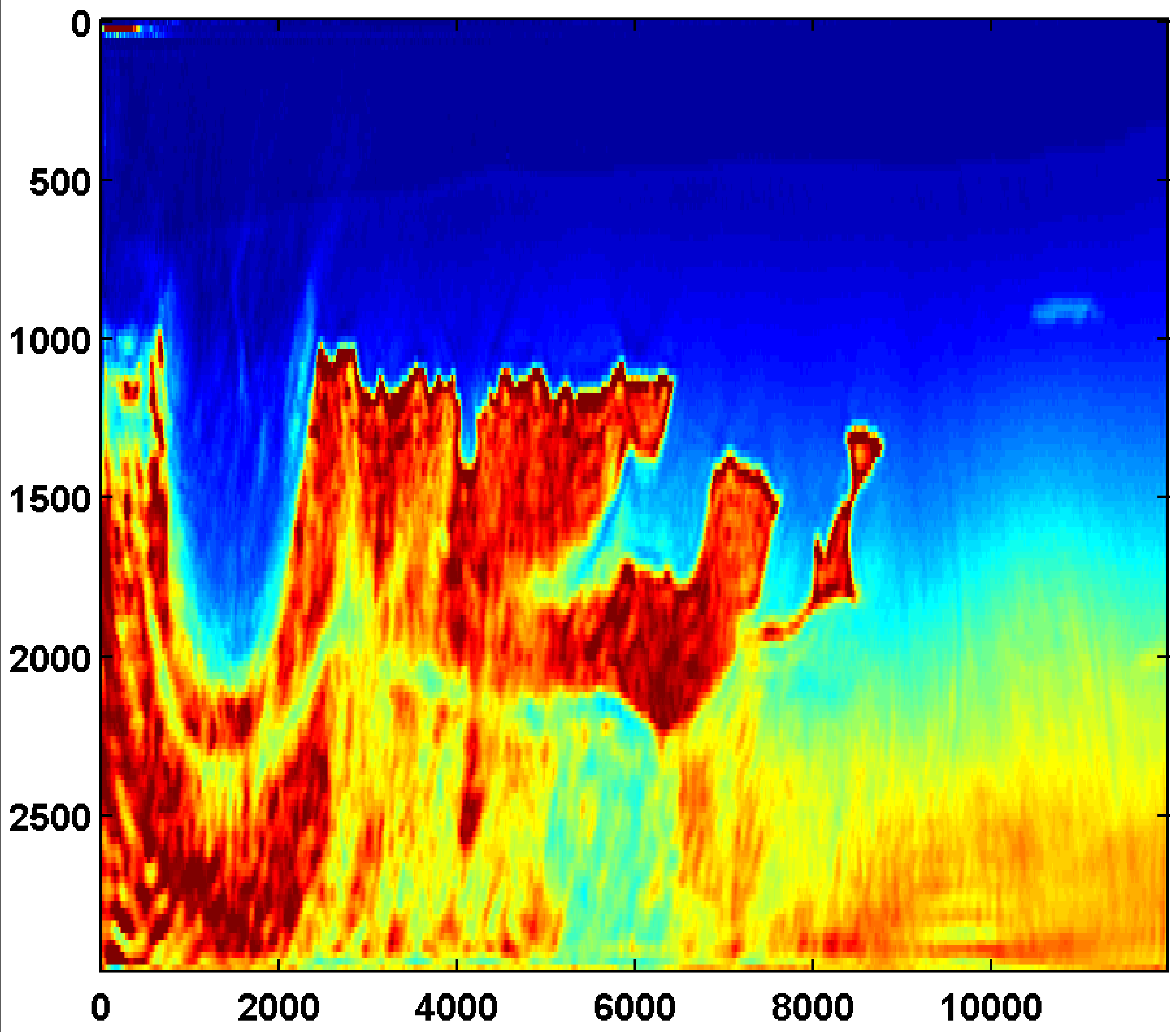
After one cycle through the frequencies

After two cycles through the frequencies

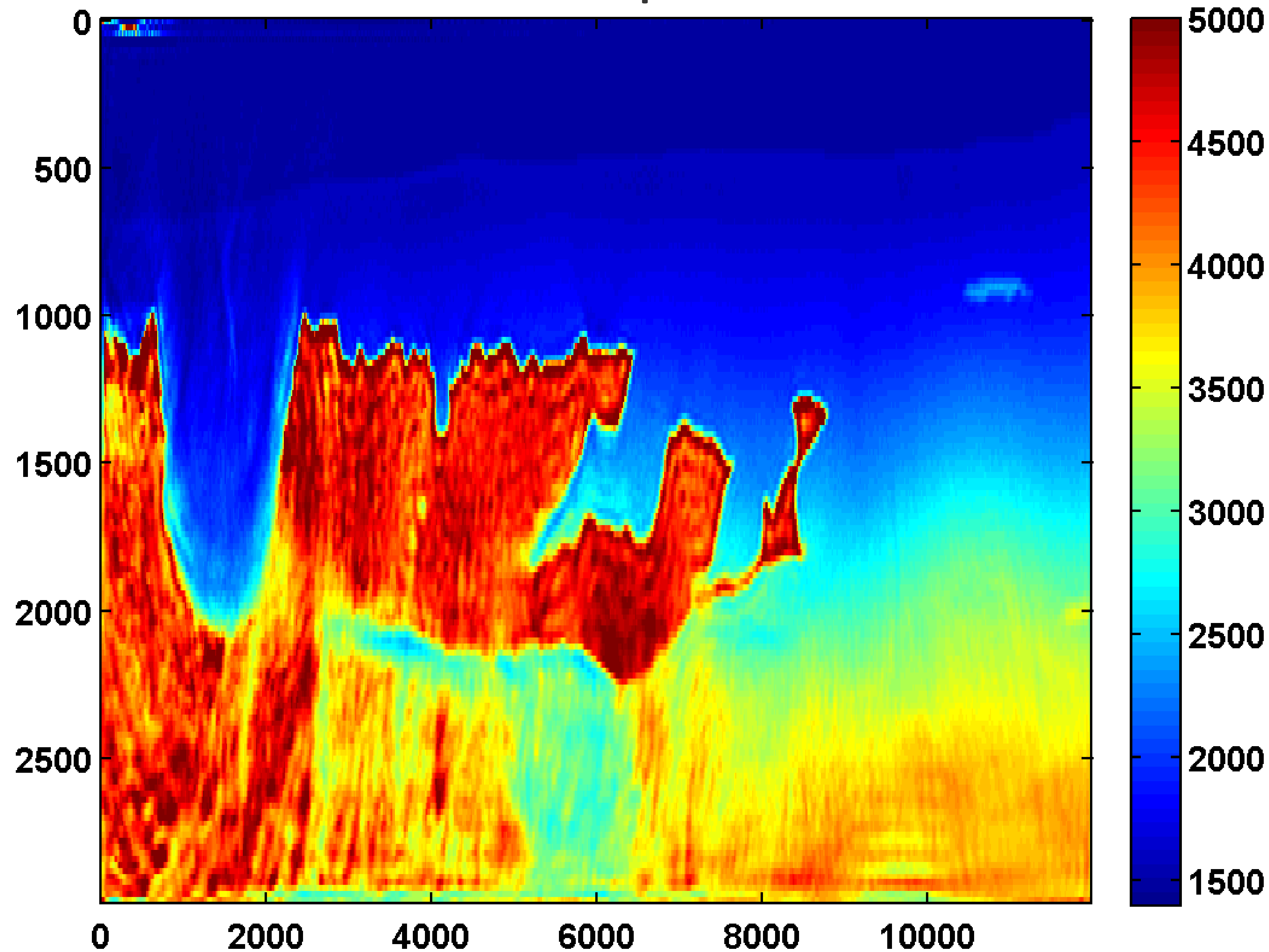


Results w/o TV

After one cycle through the frequencies



After two cycles through the frequencies



Conclusions

New alternating method for wave-equation based inversion:

- same extended search space as in all-at-once but with memory & CPU requirements as in adjoint-state approach
- no adjoints & sparse GN-Hessian approximation
- less susceptible to local minima due to data fit
- sparse GN Hessians
- bilinear

Challenge: Stationary points are not necessary global minima

Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

This work was in part financially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant (22R81254) and the Collaborative Research and Development Grant DNOISE II (375142-08). This research was carried out as part of the SINBAD II project with support from the following organizations: BG Group, BGP, CGG, Chevron, ConocoPhillips, ION, Petrobras, PGS, Statoil, Total SA, WesternGeco, and Woodside.