Total Variation Constrained WRI with Continuation

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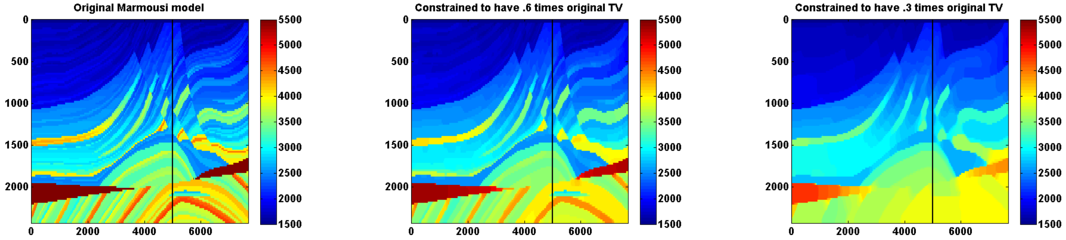
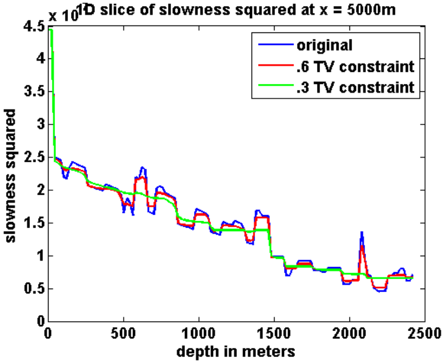
Projection onto Total Variation Constraint

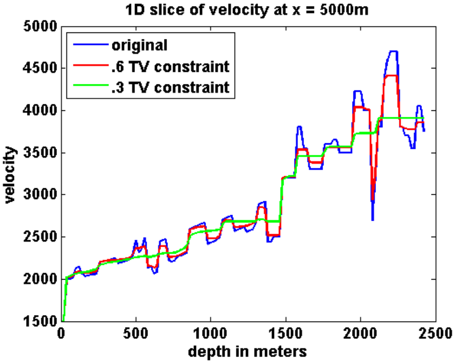
Consider projecting the Marmousi model, denoted m_0 onto the intersection of box and TV constraints

$$\Pi_C(m_0) = \arg\min_{m} \frac{1}{2}\|m - m_0\|^2$$
$$\text{s.t. } m_i \in [b_i, B_i] \text{ and } \|m\|_{TV} \leq \tau \ .$$

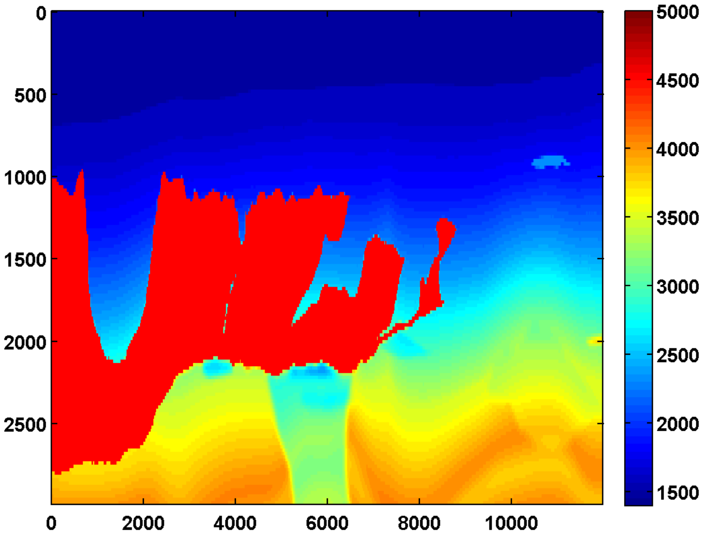


Comparison of Slices from TV Projections

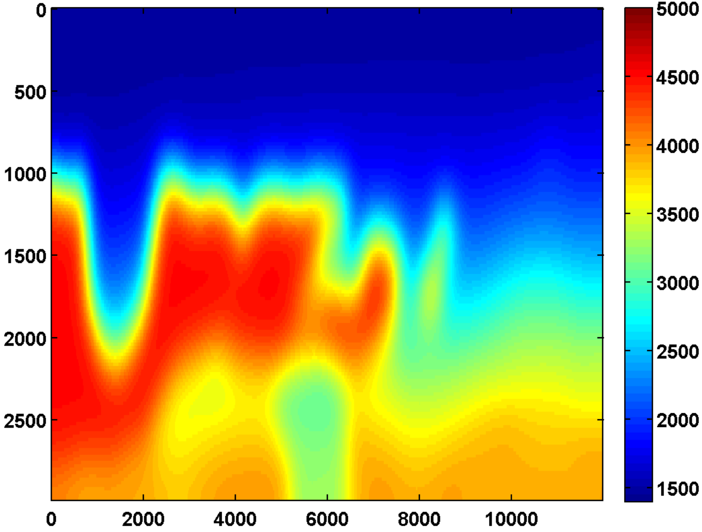




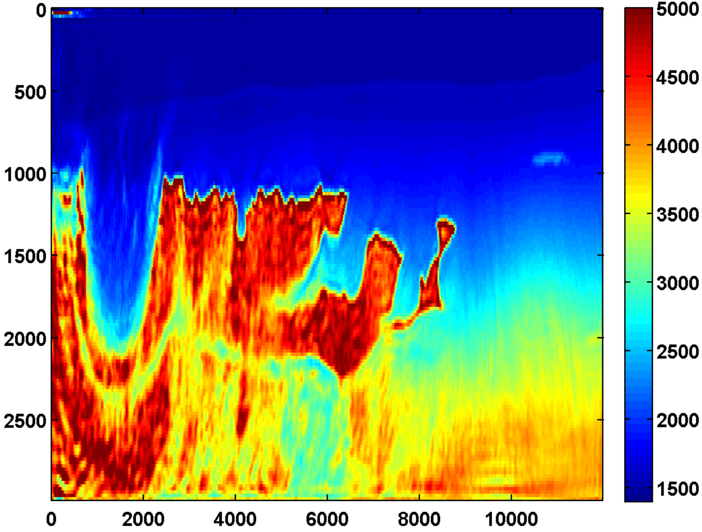
Top Left Portion of BP 2004 Velocity Benchmark



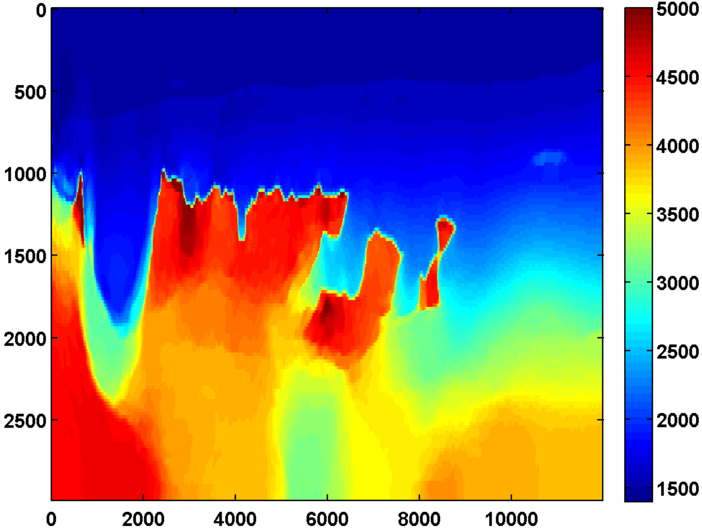
Smooth Initial Velocity Model



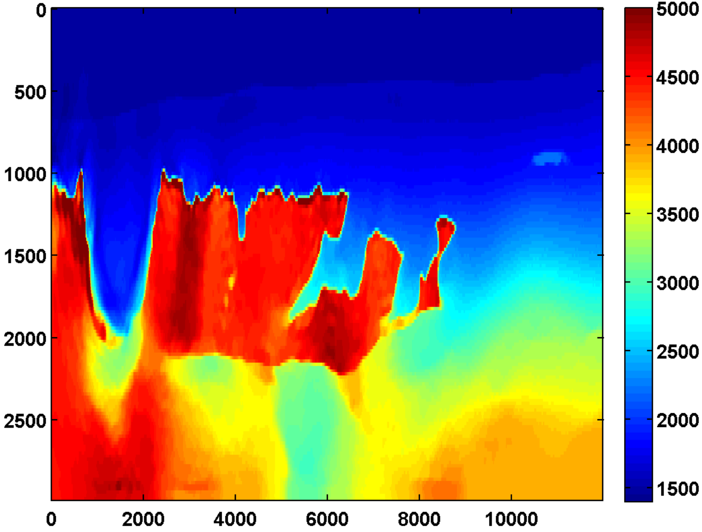
WRI without TV



TV Constrained WRI



Second Pass of TV Constrained WRI with Relaxed Constraint



Acoustic Full Waveform Inversion in Frequency Domain

$$\min_{m,u} \sum_{sv} \frac{1}{2} \|P u_{sv} - d_{sv}\|^2 \quad \text{s.t.} \quad A_v(m) u_{sv} = q_{sv}$$ [Tarantola 1984, Virieux and Operto 2009]

where $A_v(m)u_{sv} = q_{sv}$ is the discrete Helmholtz equation

$$A_v(m) = \omega_v^2 \operatorname{diag}(m) + L \ .$$

- ω_v is angular frequency and L is a discrete Laplacian
- $s = 1, ..., N_s$ is the source index and $v = 1, ..., N_v$ is the frequency index
- m is the model, the reciprocal of velocity squared
- N is the number of points in the spatial discretization
- $u_{sv} \in \mathbb{C}^N$ denotes the wavefield for source s and frequency v
- $q_{sv} \in \mathbb{C}^N$ denotes the sources
- $d_{sv} \in \mathbb{C}^{N_r}$ denotes the observed data
- P projects the wavefields onto the N_r receiver locations

Wavefield Reconstruction Inversion

Relax the PDE constraint to a quadratic penalty and solve

$$\min_{m,u} \sum_{sv} \frac{1}{2}\|Pu_{sv} - d_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(m)u_{sv} - q_{sv}\|^2$$

[van Leeuwen and Herrmann 2013]

Variable Projection and Gauss Newton

Solve for u as a function of m:

$$\bar{u}_{sv}(m^n) = \arg\min_{u_{sv}} \frac{1}{2}\|Pu_{sv} - d_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(m^n)u_{sv} - q_{sv}\|^2 \quad \text{for all } s, v$$

Let $F(m) = \sum_{sv} F_{sv}(m)$, where

$$F_{sv}(m) = \frac{1}{2}\|P\bar{u}_{sv}(m) - d_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(m)\bar{u}_{sv}(m) - q_{sv}\|^2 \ ,$$

and use Gauss Newton to minimize $F(m)$

Computing the Gradient and Gauss Newton Hessian

Using a variable projection argument [Aravkin and van Leeuwen 2012],

$$\nabla F(m^n) = \sum_{sv} \mathrm{Re}\left(\lambda^2 \omega_v^2 \operatorname{diag}(\bar{u}_{sv}(m^n))^* (\omega_v^2 \operatorname{diag}(\bar{u}_{sv}(m^n)) m^n + L\bar{u}_{sv}(m^n) - q_{sv})\right)$$

The Gauss Newton approximation to the Hessian of F at m^n is **diagonal**, given by

$$H^n = \sum_{sv} \mathrm{Re}(\lambda^2 \omega_v^4 \operatorname{diag}(\bar{u}_{sv}(m^n))^* \operatorname{diag}(\bar{u}_{sv}(m^n)))$$ [van Leeuwen and Herrmann 2013]

Scaled Gradient Descent Framework

A scaled gradient descent approach [Bertsekas 1999] for minimizing F can be written as

$$\Delta m = \arg \min_{\Delta m \in R^N} \sum_{sv} \Delta m^T \nabla G_{sv}(m^n) + \frac{1}{2} \Delta m^T H_{sv}^n \Delta m + c_n \Delta m^T \Delta m$$

$$m^{n+1} = m^n + \Delta m \ .$$

In addition to Gauss Newton, this general framework includes gradient descent and Newton's method.

Including Convex Constraints

Add the constraint $m \in C$, where C is a convex set and iterate

$$\Delta m = \arg \min_{\Delta m \in R^N} \sum_{sv} \Delta m^T \nabla F_{sv}(m^n) + \frac{1}{2} \Delta m^T H_{sv}^n \Delta m + c_n \Delta m^T \Delta m$$
$$\text{s.t. } m^n + \Delta m \in C$$

$$m^{n+1} = m^n + \Delta m \ .$$

Bound Constraints without Increasing Computational Cost

Bound constraint example: $C = \{m \in \mathbb{R}^N : m_j \in [B_1, B_2]\}$.

When H is diagonal and positive, the update for Δm is simple.

$$\begin{aligned}
\Delta m = \arg \min_{\Delta m \in R^N} & \Delta m^T \nabla F(m^n) + \frac{1}{2} \Delta m^T H \Delta m \\
\text{s.t. } & m^n + \Delta m \in C \\
= \max & \left((B_1 - m^n), \min \left((B_2 - m^n), -H^{-1} \nabla F(m^n) \right) \right)
\end{aligned}$$

Total Variation Regularization

If we represent m as a N_1 by N_2 image, we can define

$$
\begin{aligned}
\|m\|_{TV} &= \frac{1}{h} \sum_{ij} \sqrt{(m_{i+1,j} - m_{i,j})^2 + (m_{i,j+1} - m_{i,j})^2} \\
&= \sum_{ij} \frac{1}{h} \left\| \begin{bmatrix} (m_{i,j+1} - m_{i,j}) \\ (m_{i+1,j} - m_{i,j}) \end{bmatrix} \right\| \\
&= \|Dm\|_{1,2} \ ,
\end{aligned}
$$

where D is a discrete gradient operator applied to a vectorized m.

Proposed Model and Algorithm

Solve

$$\min_{m} F(m) \qquad \text{s.t.} \qquad m \in [B_1, B_2] \text{ and } \|m\|_{TV} \leq \tau$$

by iterating

$$\Delta m = \arg\min_{\Delta m} \Delta m^T \nabla F(m^n) + \frac{1}{2} \Delta m^T H^n \Delta m + c_n \Delta m^T \Delta m$$
$$\text{s.t.} \qquad m^n + \Delta m \in [B_1, B_2] \;\text{ and }\; \|m^n + \Delta m\|_{TV} \leq \tau$$

$$m^{n+1} = m^n + \Delta m \ .$$

Solving the Convex Subproblem

There are many effective primal dual methods for solving the convex subproblem for Δm based on finding a saddle point of the Lagrangian

$$\mathcal{L}(\Delta m, p) = \Delta m^T \nabla F(m^n) + \frac{1}{2}\Delta m^T (H^n + 2c_n \mathrm{I})\Delta m + p^T D(m^n + \Delta m) - \tau \|p\|_{\infty,2}$$
$$\text{for} \quad m^n + \Delta m \in [B_1, B_2] \; ,$$

which can be related to the primal problem by noting that

$$\sup_p p^T Dm - \tau \|p\|_{\infty,2} = \begin{cases} 0 & \text{if } \|Dm\|_{1,2} \leq \tau \\ \infty & \text{otherwise.} \end{cases}$$

Modified PDHG Iterations

The modified PDHG method [Zhu and Chan 2008, Chambolle and Pock 2011, Esser, Zhang and Chan 2010, He and Yuan 2012] finds a saddle point by iterating

$$
\begin{aligned}
p^{k+1} &= \arg\min_{p} \tau\|p\|_{\infty,2} - p^T D(m^n + \Delta m^k) + \frac{1}{2\delta}\|p - p^k\|^2 \\
\Delta m^{k+1} &= \arg\min_{\Delta m} \Delta m^T \nabla F(m^n) + \frac{1}{2}\Delta m^T (H^n + 2c_n \mathrm{I})\Delta m \\
&\quad + \Delta m^T D^T (2p^{k+1} - p^k) + \frac{1}{2\alpha}\|\Delta m - \Delta m^k\|^2 \\
&\text{s.t.} \qquad m^n + \Delta m \in [B_1, B_2]
\end{aligned}
$$

The p^{k+1} update involves a projection that can be efficiently computed, and

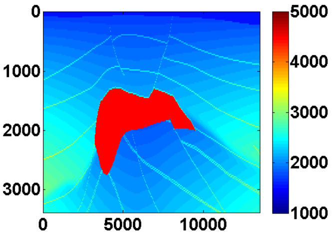
$$
\begin{aligned}
\Delta m^{k+1} = (H^n + \xi_n \mathrm{I})^{-1} \max\big((H^n + \xi_n \mathrm{I})(B_1 - m^n), \\
\min\left((H^n + \xi_n \mathrm{I})(B_2 - m^n), -\nabla F(m^n) + \frac{\Delta m^k}{\alpha} - D^T(2p^{k+1} - p^k)\right)\big)
\end{aligned}
$$

where $\xi_n = 2c_n + \frac{1}{\alpha}$

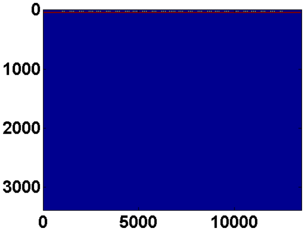
Numerical Experiment

SEG/EAGE salt model, sources and receivers near the surface, two simultaneous shots, and a very good initial guess

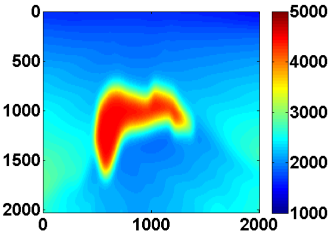
$$\bar{q}_{jv} = \sum_{s=1}^{N_s} w_{js} q_{sv} \quad j = 1,2 \qquad w_{js} \in \mathcal{N}(0,1) \qquad \bar{d}_{jv} = P A_v^{-1}(m) \bar{q}_{jv}$$



True velocity



Source and receiver locations

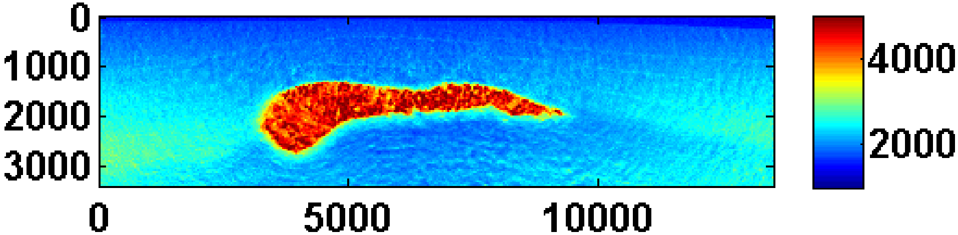


Initial velocity

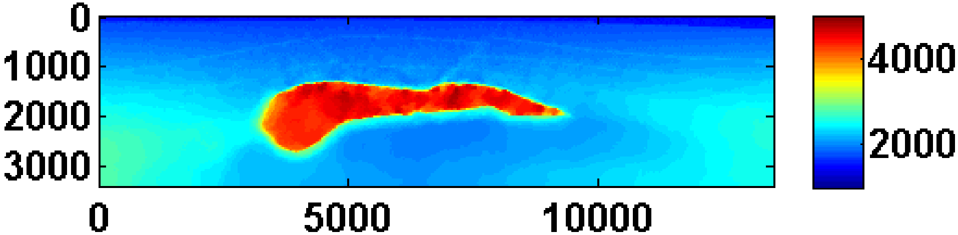
Modeling Details

- model size: 170 by 676
- mesh size: 20m
- number of sources: 116
- number of receivers: 673
- frequency range: 3-33Hz in overlapping batches of 2
- maximum number of outer iterations per frequency batch: 25
- maximum number of inner iterations for convex subproblems: 2000
- known Ricker wavelet sources with 30Hz peak frequency
- two simultaneous shots with Gaussian weights, without redraws
- no added noise

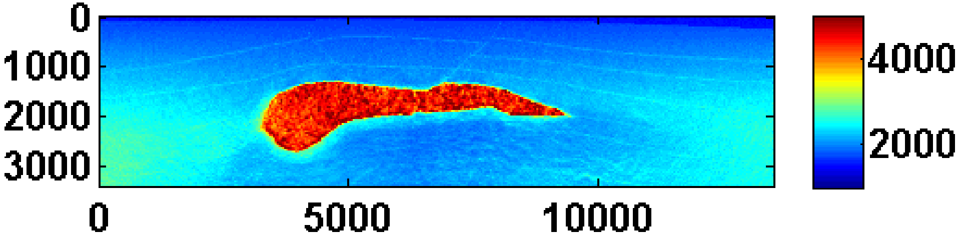
No TV Constraint, Smooth Initial Model



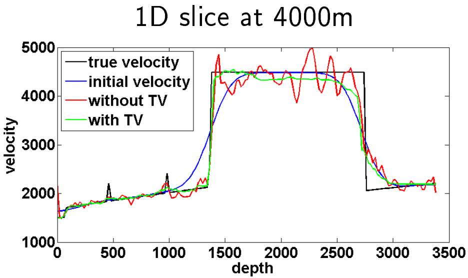
With TV Constraint, Smooth Initial Model

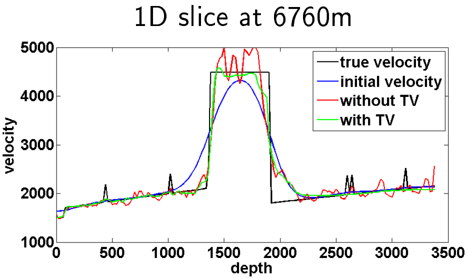


No TV Constraint, Using TV Result as Initial Model



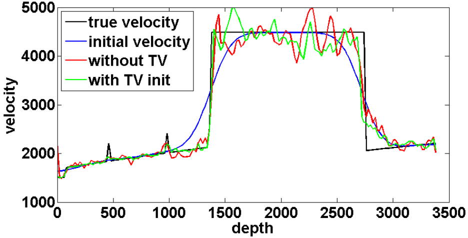
1D Slices from Salt Inversion using TV



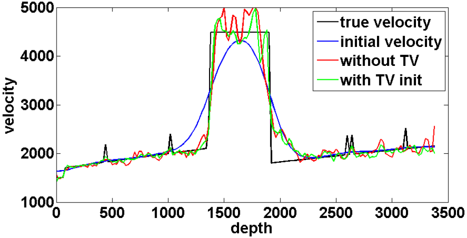


1D Slices without TV but using TV Initialization

1D slice at 4000m

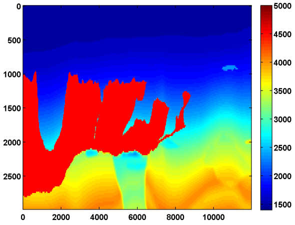


1D slice at 6760m

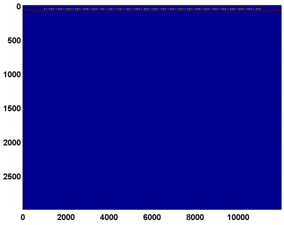


Simultaneous Shot with Redraws Experiment

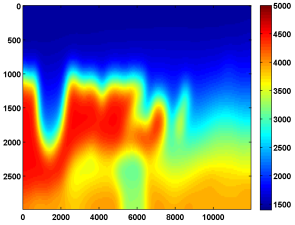
Top left portion of BP 2004 velocity benchmark, sources and receivers near the surface, two simultaneous shots, and good smooth initial model



True velocity



Source and receiver locations

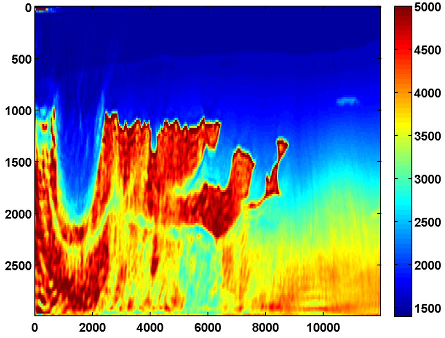


Initial velocity

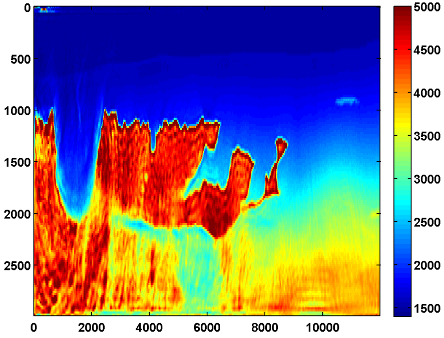
Modeling Details

- model size: 150 by 600
- mesh size: 20m
- number of sources: 126 (starting 1000m in from boundary)
- number of receivers: 299
- frequency range: 3-20Hz in overlapping batches of 2
- maximum number of outer iterations per frequency batch: 25
- maximum number of inner iterations for convex subproblems: 2000
- known Ricker wavelet sources with 15Hz peak frequency
- two simultaneous shots with Gaussian weights, WITH redraws
- no added noise

Results without TV Regularization

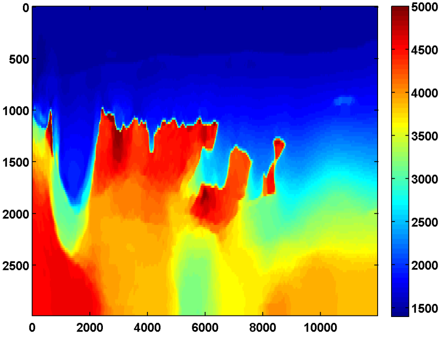


First pass

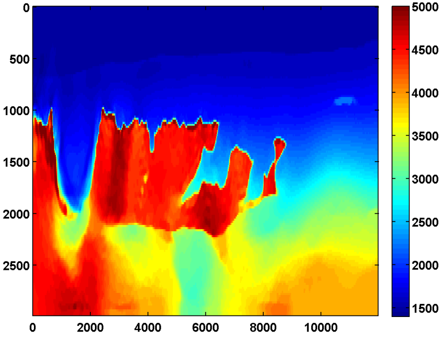


Second pass

Results with TV Regularization

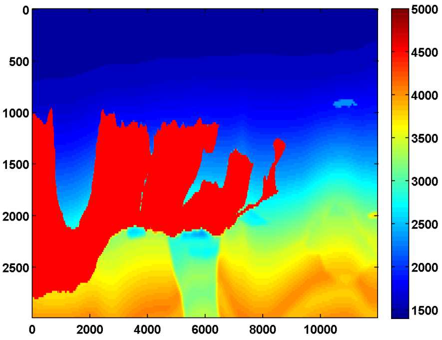


First pass

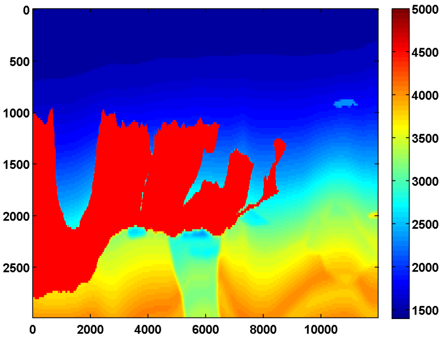


Second pass

True Velocity Model



True velocity



True velocity

Conclusions and Future Work

- Repeatedly solving TV constrained WRI problems while relaxing the TV constraint appears to maintain some benefits of the regularization while still allowing fine details into the solution.
- Determining effective automatic continuation strategies for the TV constraint is ongoing work.
- We aim to update the TVWRI software release with an automatic strategy for selecting a sequence of regularization parameters.