Cosparse seismic data interpolation

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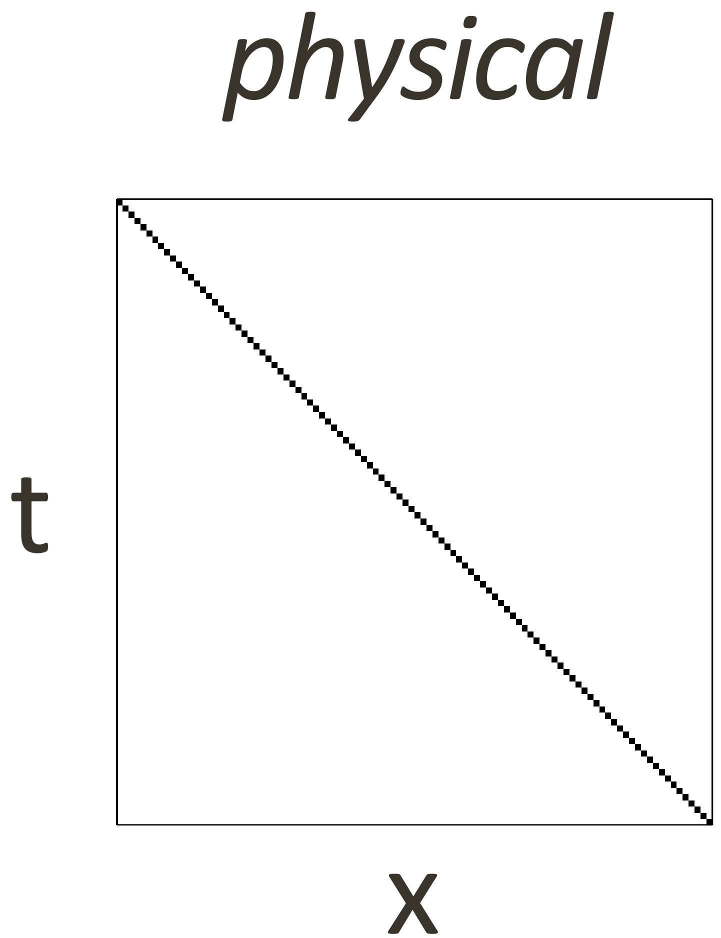
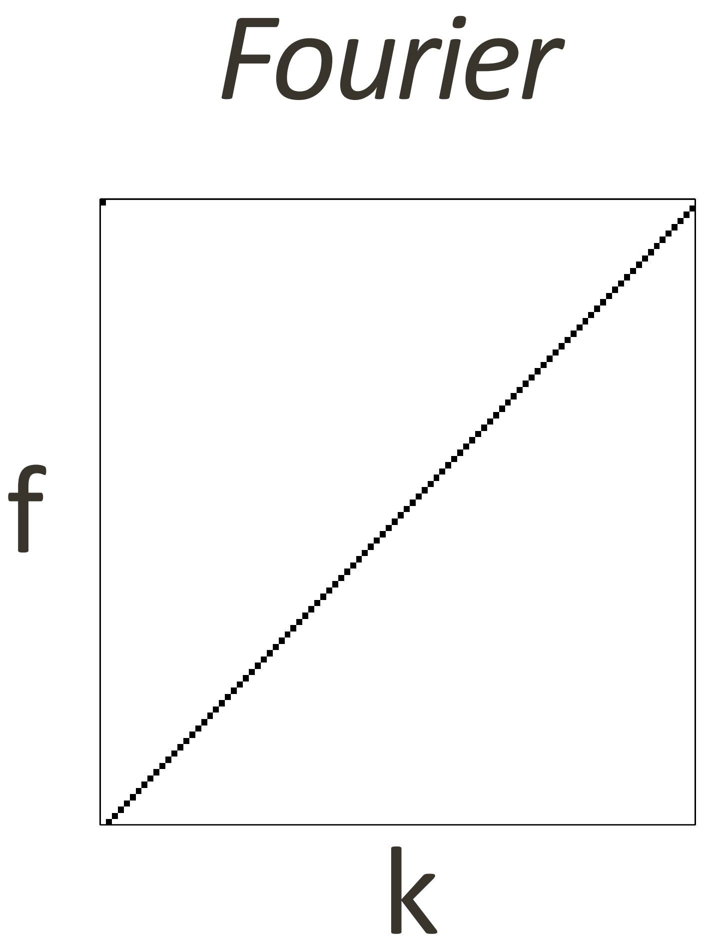
What is meant by “sparsity”?

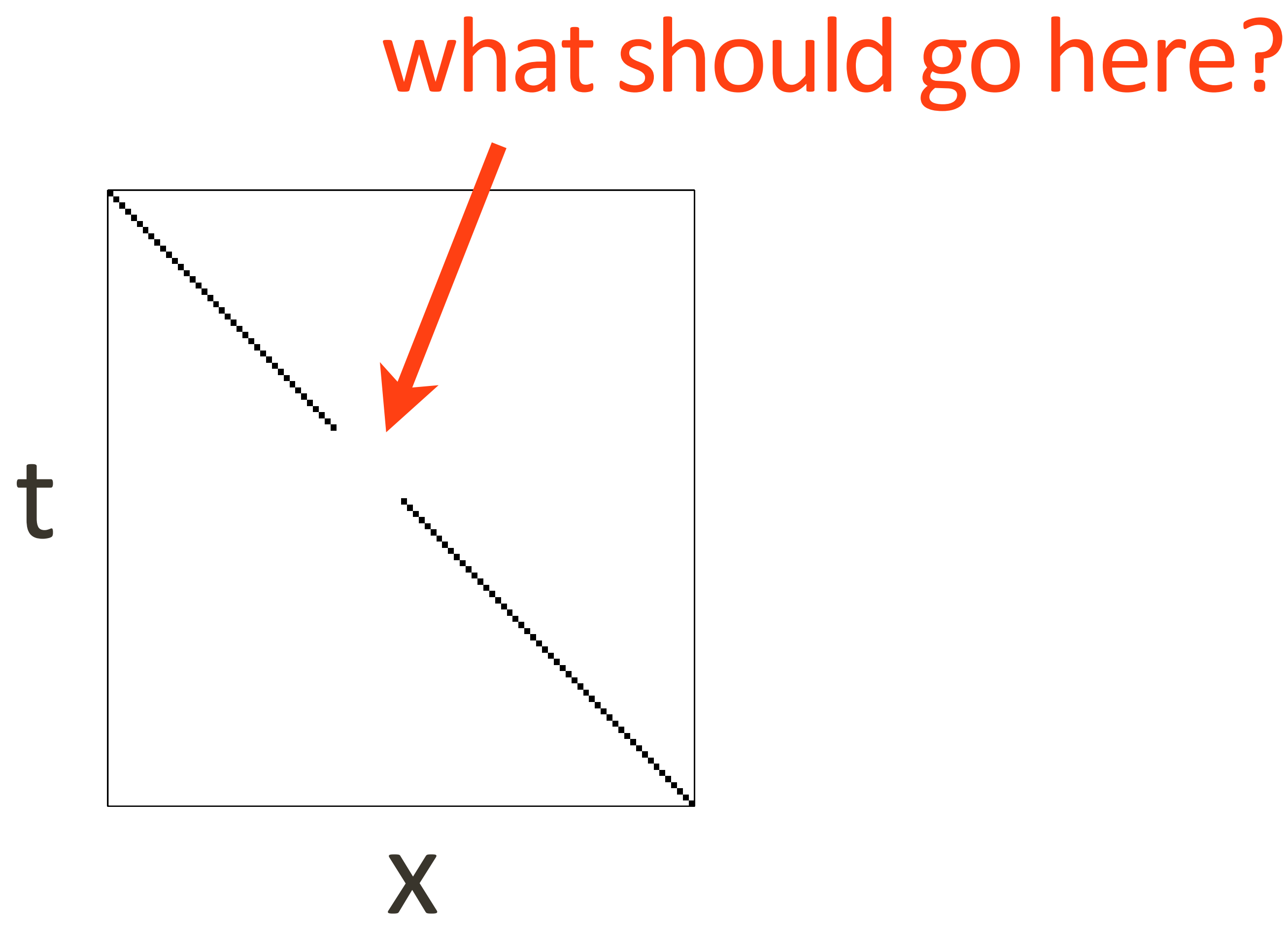
sparsity infers structure

Sparsity infers structure under transforms





Sparsity/structure resolves ambiguity



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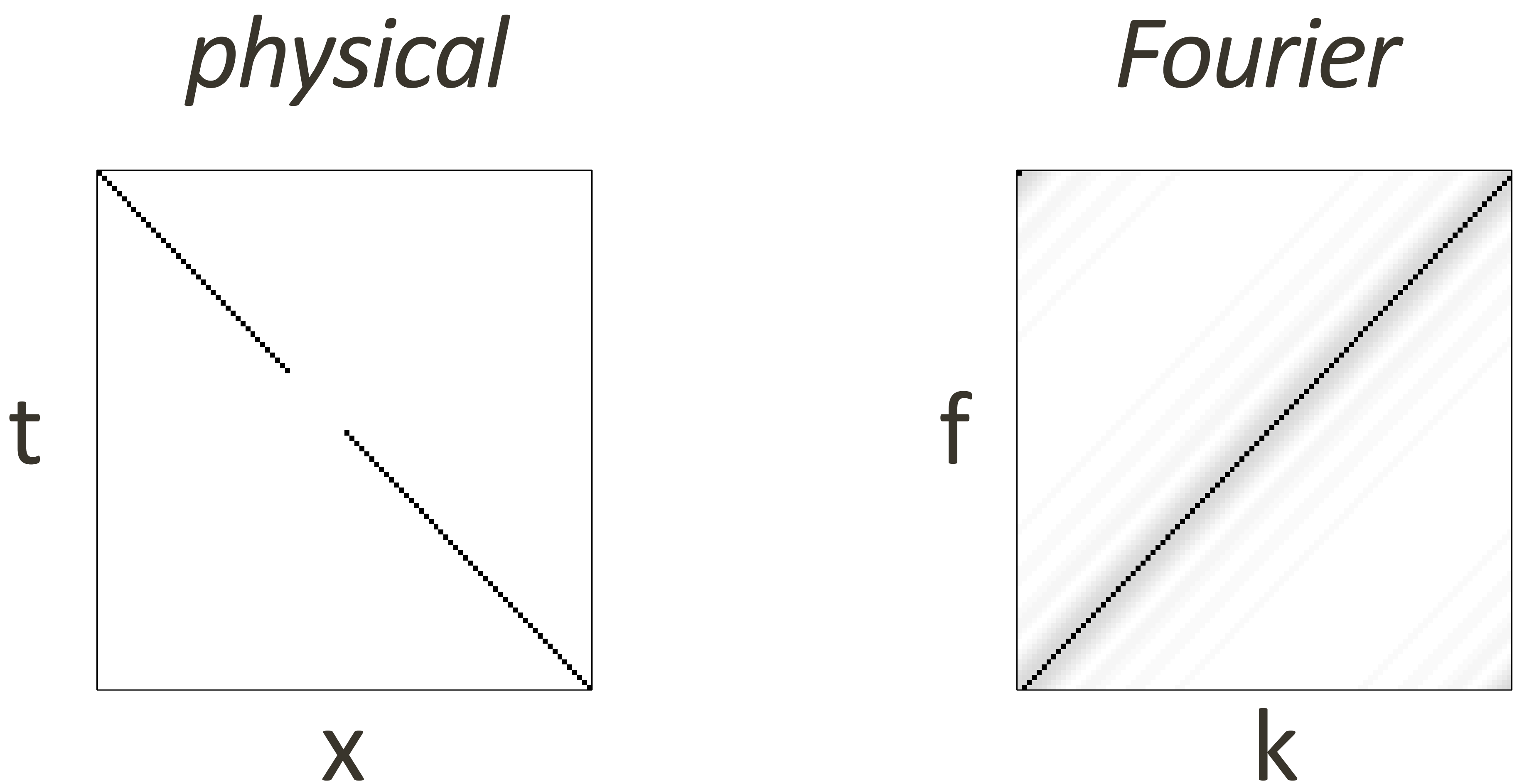
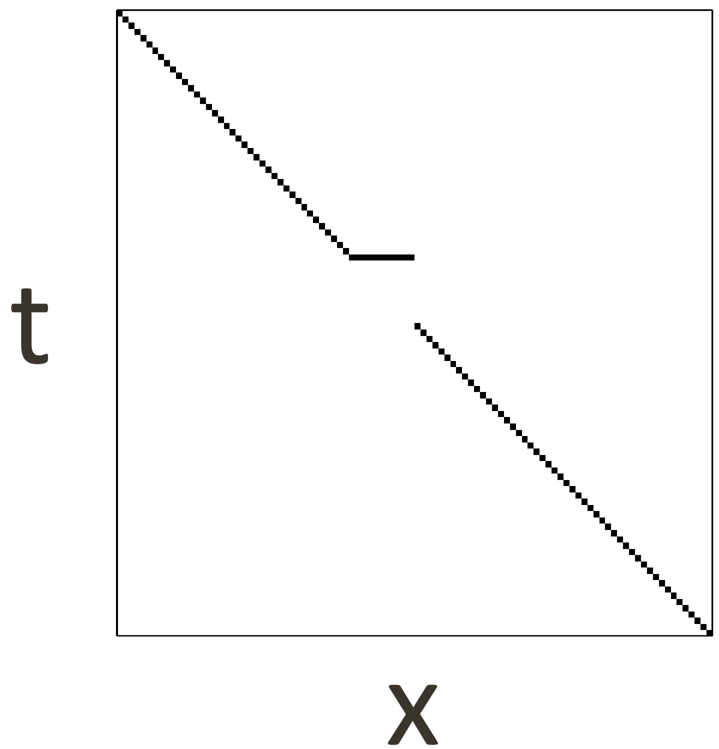
Sparsity infers structure under transforms

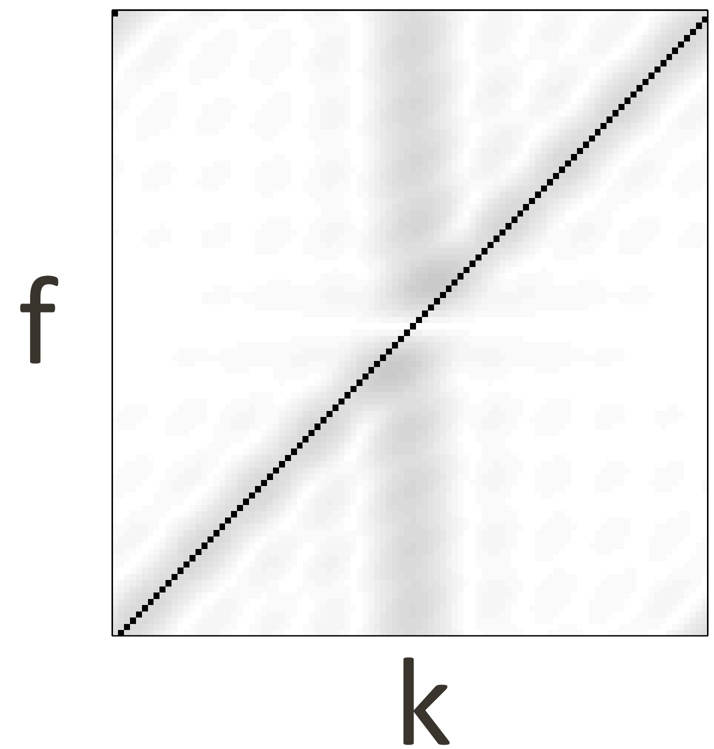


Sparsity infers structure under transforms

physical

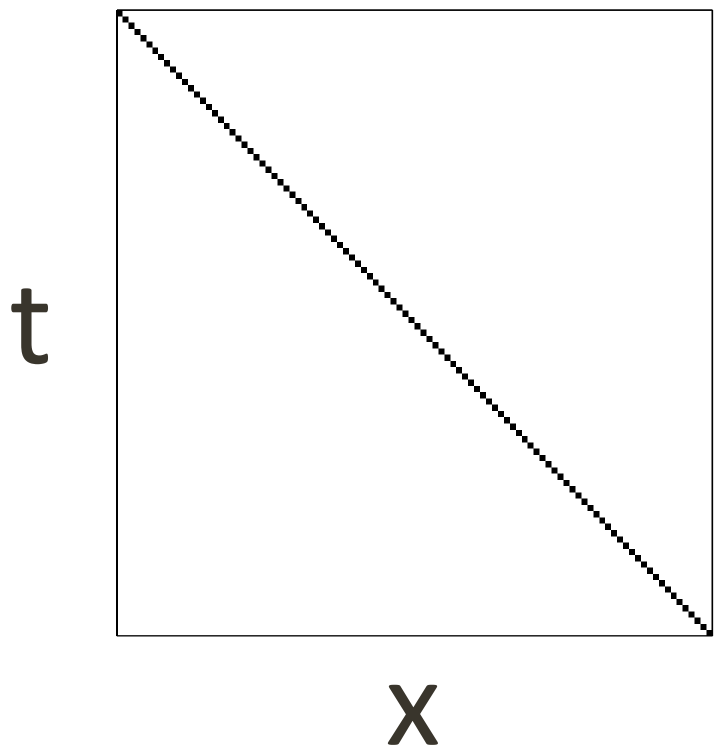


Fourier

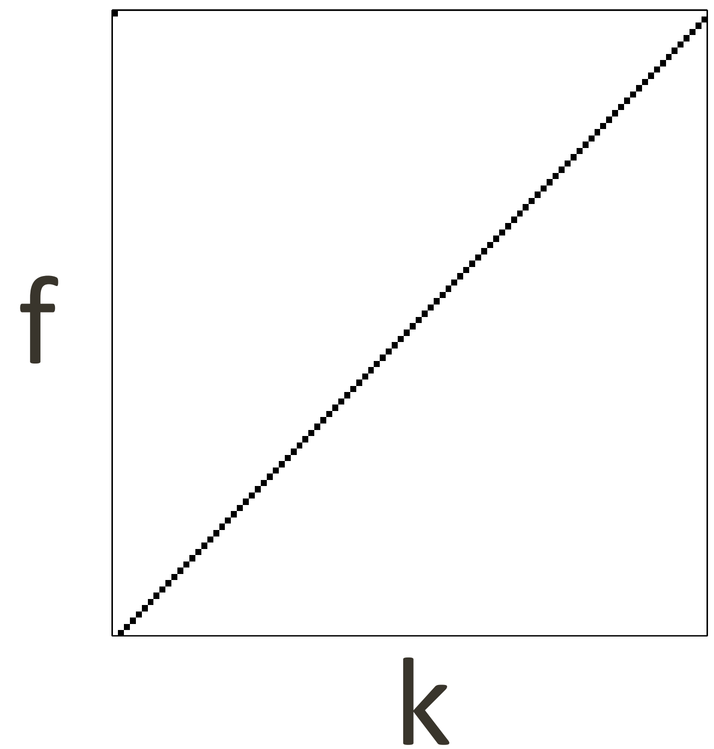


Sparsity infers structure under transforms

physical



Fourier



Curvelets

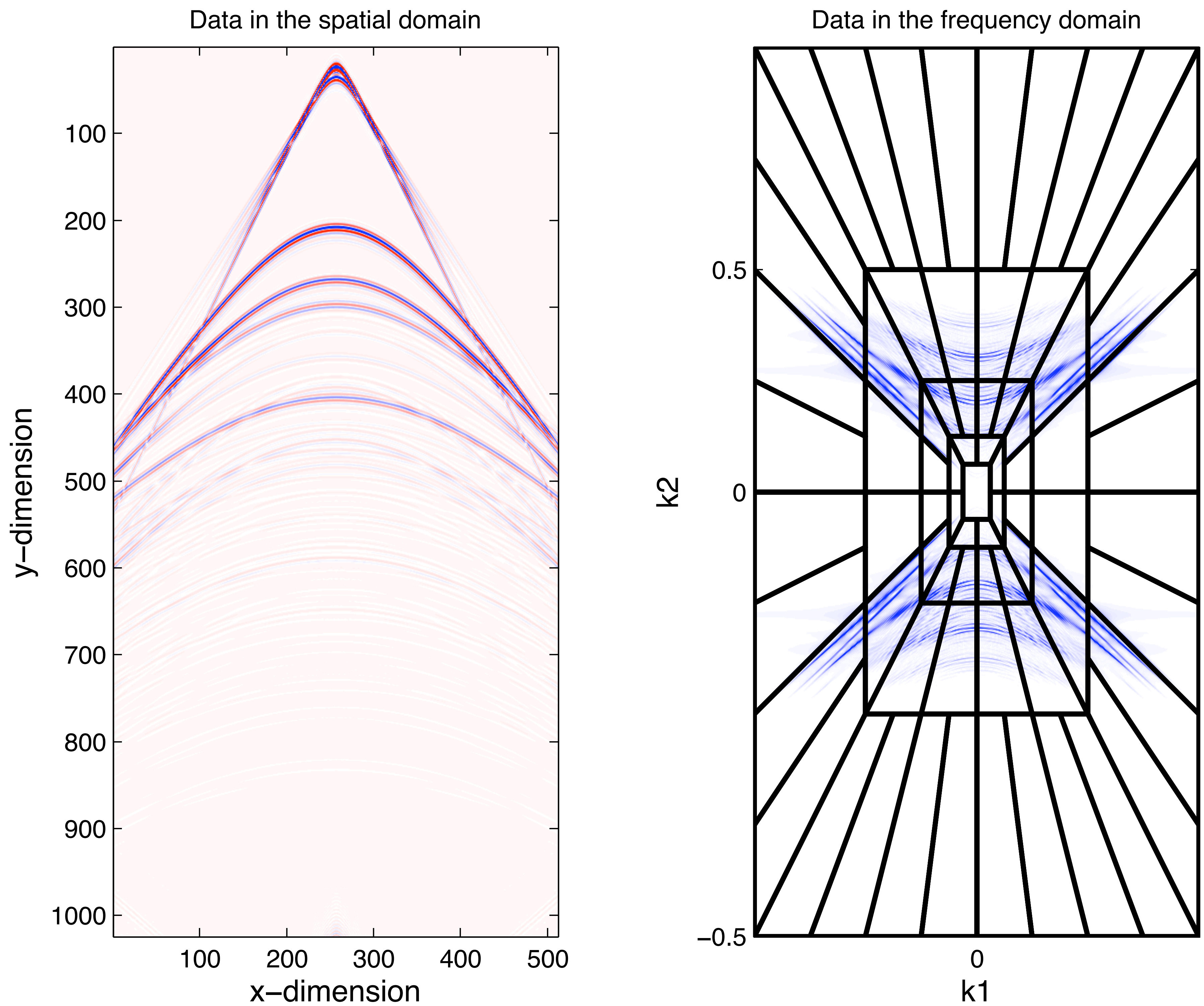
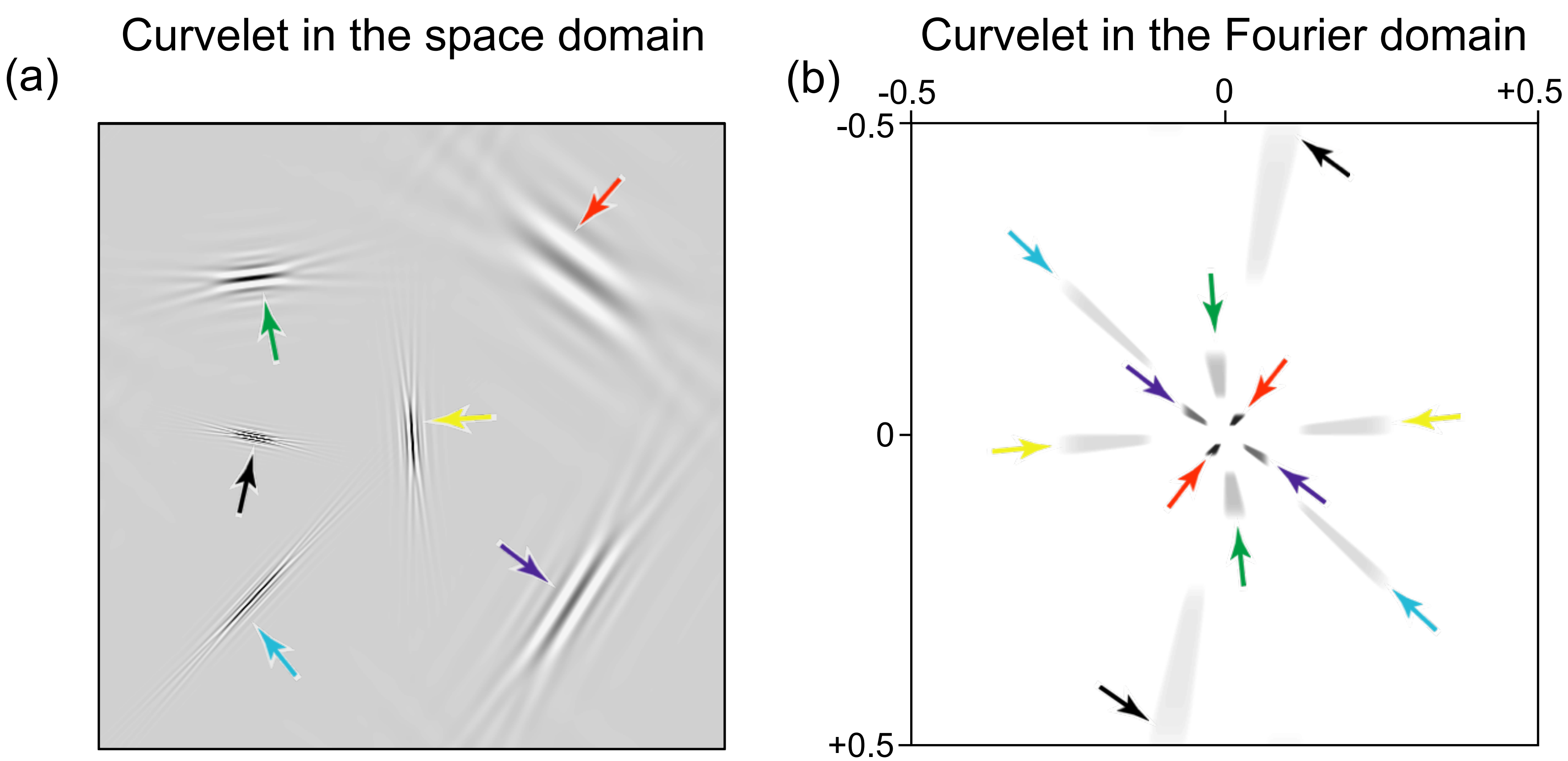


figure from Hennenfent 06

Curvelets



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Curvelets

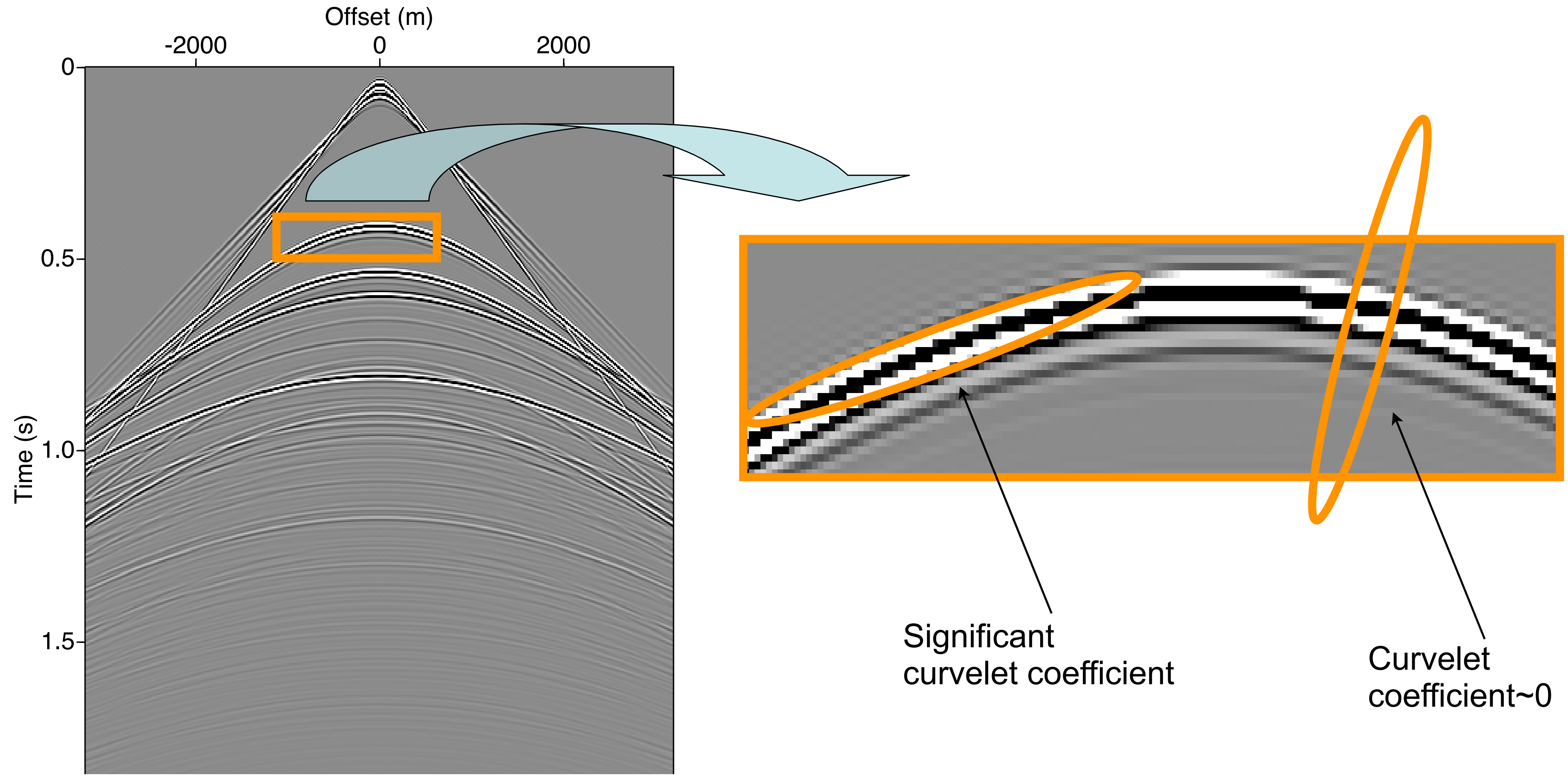


figure from Hennenfent 06

Curvelet synthesis

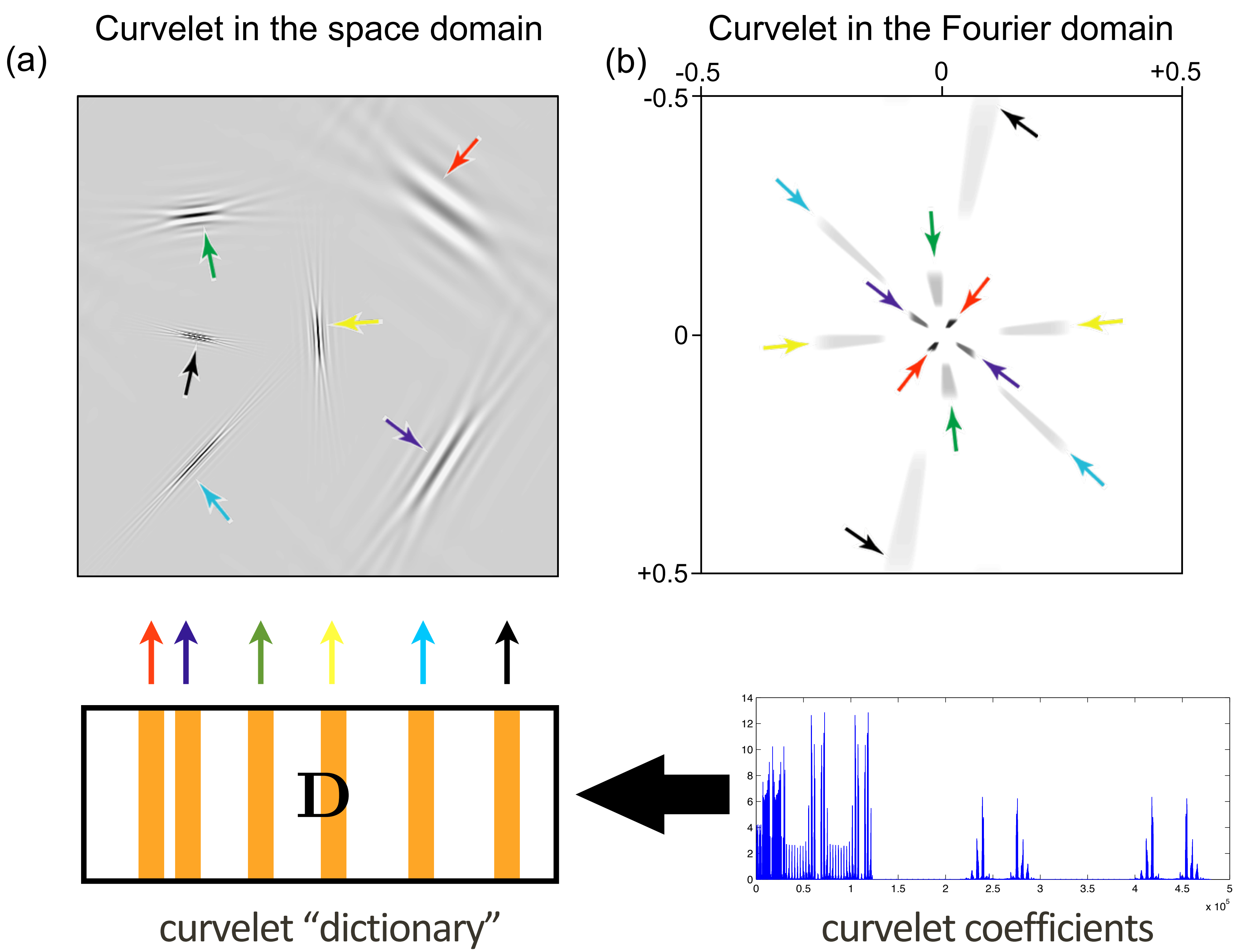
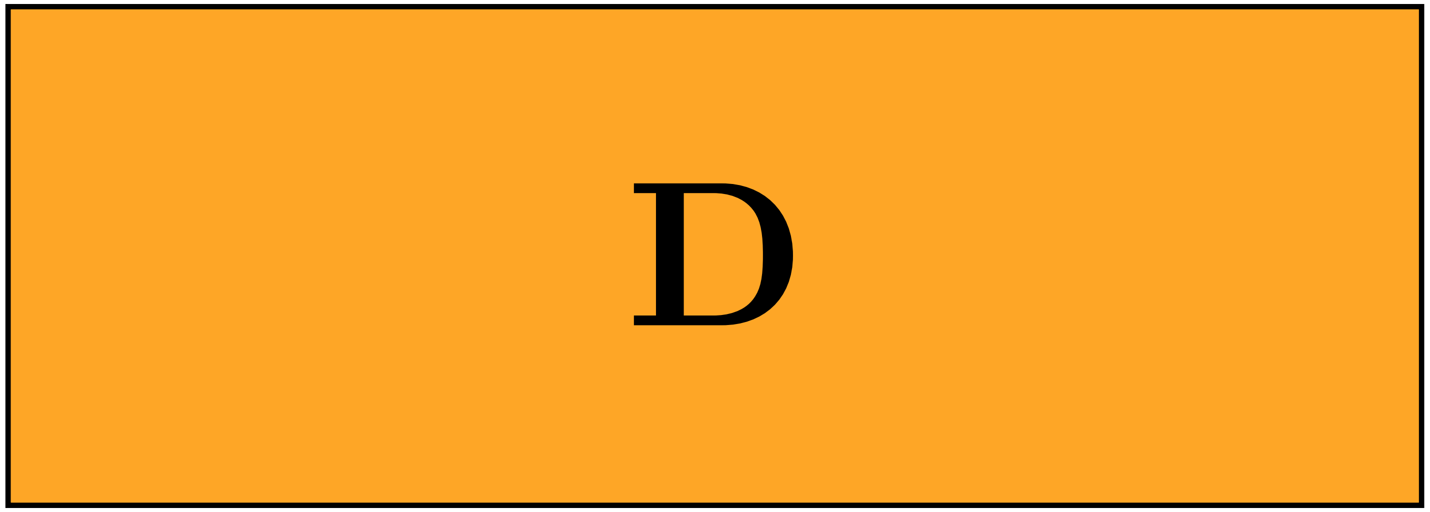


figure from Hennenfent 06

Curvelet dictionary is redundant

m < n

Size of physical image *m*



Number of coefficients *n*

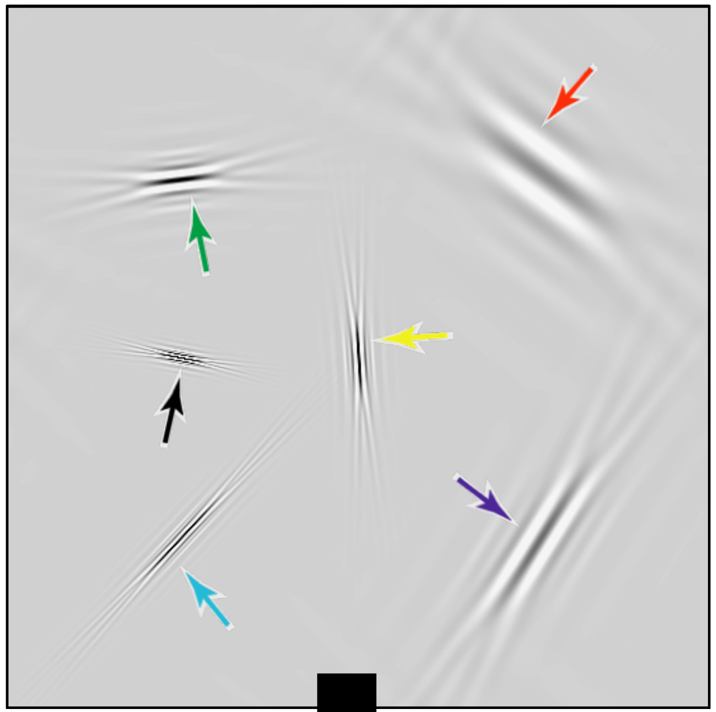
Curvelet dictionary is redundant

Many useful ones also:

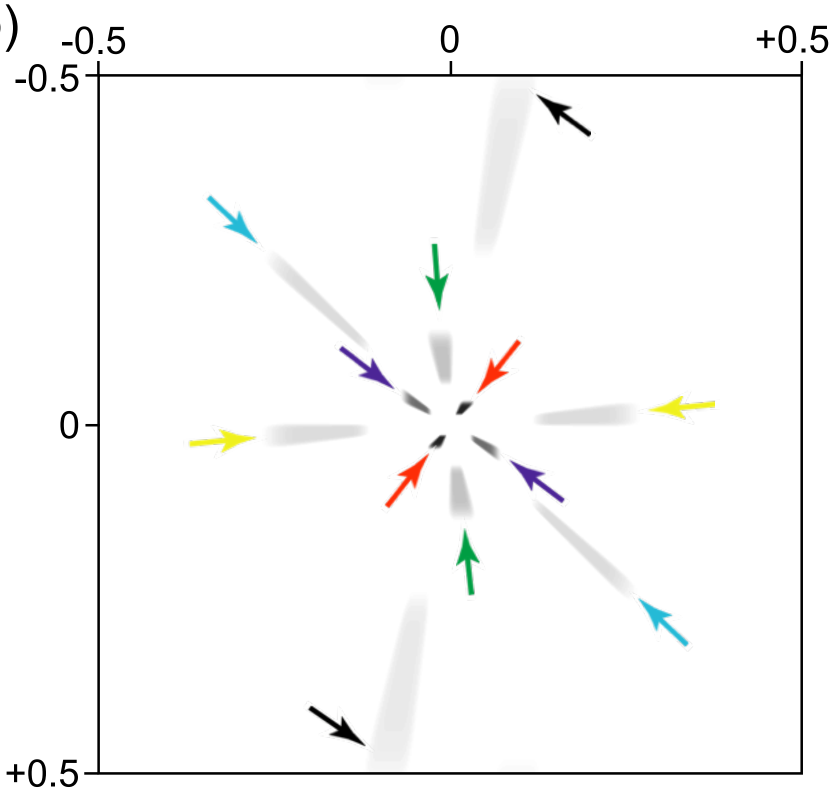
- Stationary wavelet
- Windowed Fourier/Cosine
- Ridgelets
- Wave atoms
- Radon
- *many more...*

Curvelet analysis

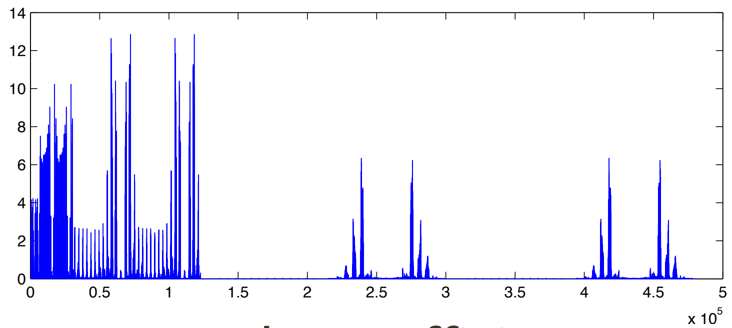
(a) Curvelet in the space domain



(b) Curvelet in the Fourier domain



?



curvelet coefficients

figure from Hennenfent 06

Curvelet analysis

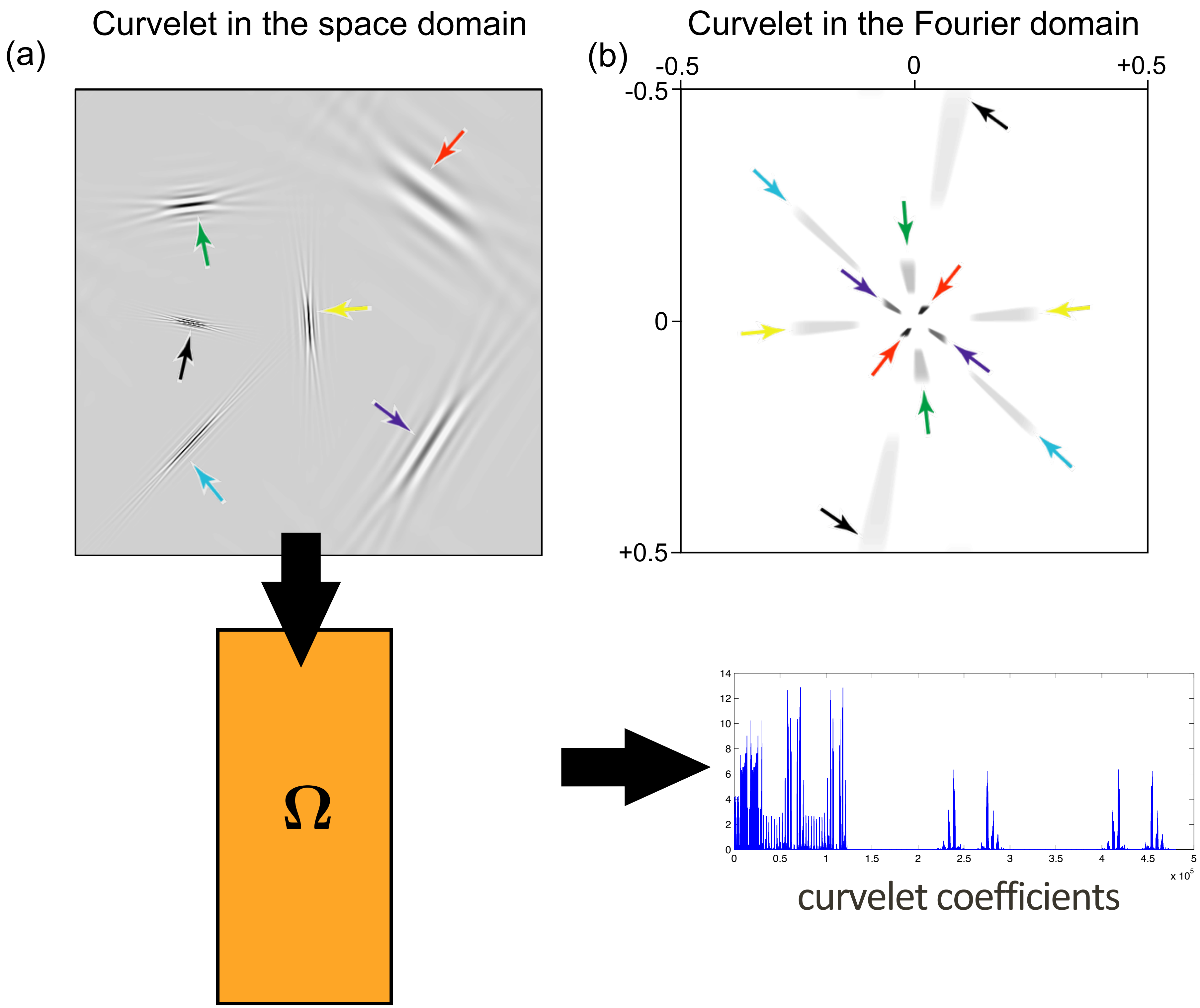
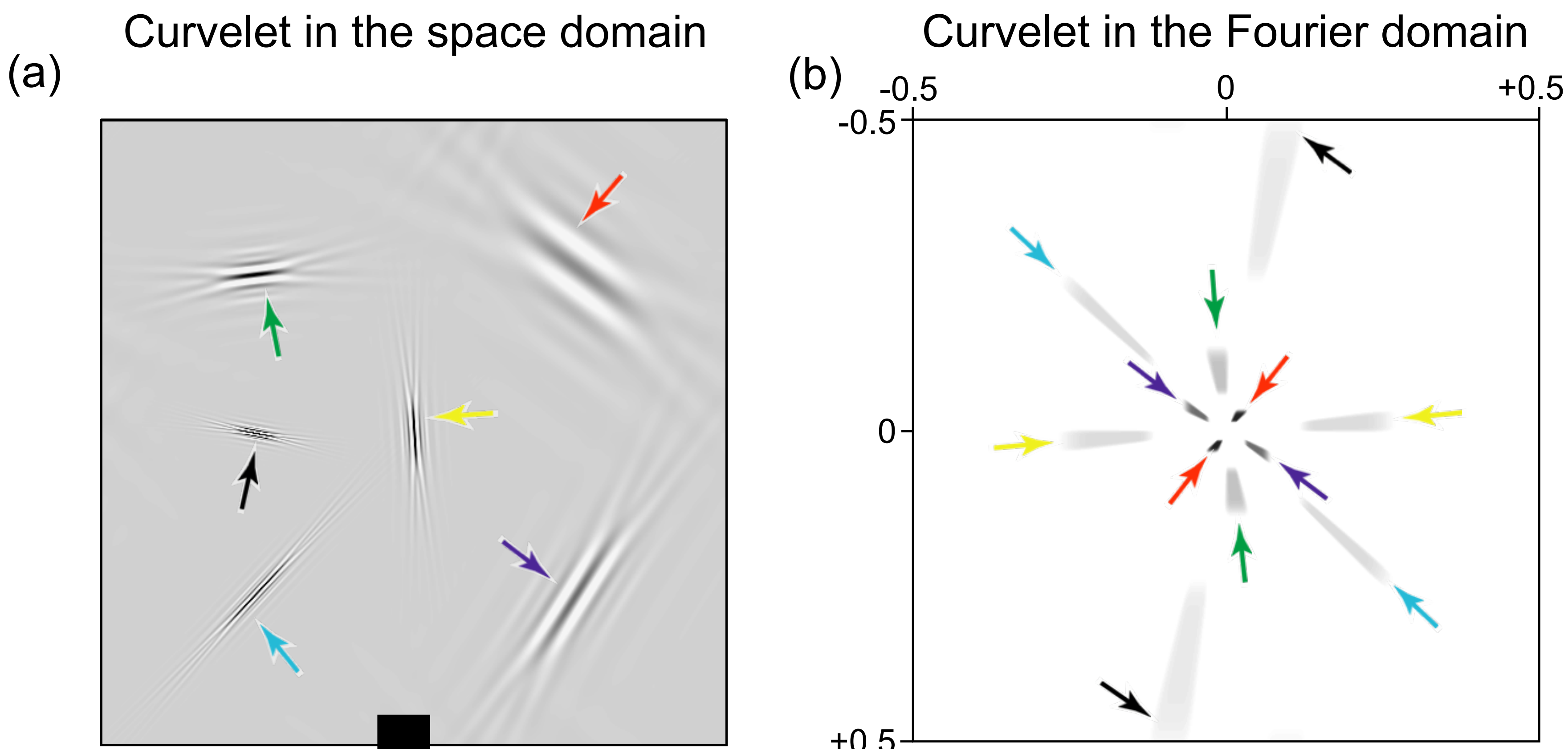


figure from Hennenfent 06

Curvelet analysis

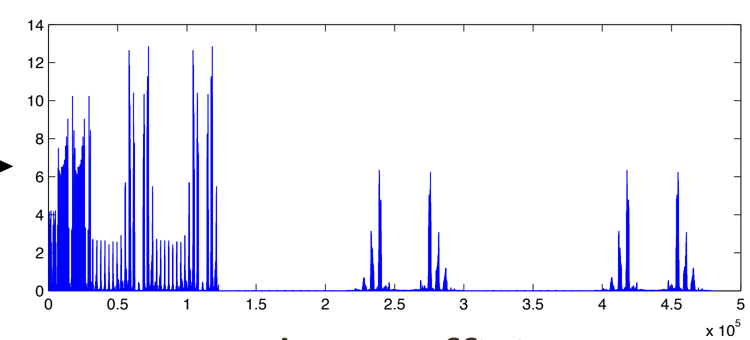


Generally,

$$\mathbf{\Omega} = \mathbf{D}^{\dagger}$$

but for curvelet,

$$\mathbf{\Omega} = \mathbf{D}^{H}$$



curvelet coefficients

figure from Hennenfent 06

Trace interpolation via sparsity

$\mathbf{x} = \mathbf{D}\mathbf{z}$ (assume **x** is not sparse, but **z** is)

$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_0 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

- 0-norm measure sparsity (# of non-zero coefficients)
- $\mathbf{y}$ is data with missing traces
- $\mathbf{A}$ is trace mask (match data at observed trace positions)
- $\mathbf{x}$ is estimate interpolated gather
- $\mathbf{z}$ is *a* choice of curvelet coefficients for $\mathbf{x}$

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Trace interpolation via sparsity

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$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

convexify

- 0-norm measure sparsity (# of non-zero coefficients)
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- $\mathbf{A}$ is trace mask (match data at observed trace positions)
- $\mathbf{x}$ is estimate interpolated gather
- $\mathbf{z}$ is *a* choice of curvelet coefficients for $\mathbf{x}$

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Constructing signals with…

Sparsity

Analysis vs Synthesis

$\mathbf{x} = \mathbf{D}\mathbf{z}$ (assume **x** is not sparse, but **z** is)

$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} ||\mathbf{z}||_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

"Synthesis"-based sparse signal reconstruction

Analysis vs Synthesis

(assume **x** is not sparse, but $\mathbf{D}^\dagger \mathbf{x}$ is)

$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \, ||\mathbf{z}||_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

"Synthesis"-based sparse signal reconstruction

SLIM

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"Analysis"-based sparse signal reconstruction

Equivalence?

Synthesis $\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$

"Synthesizes" the signal using sparse sets of columns of **D**

Analysis $\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^{\dagger}\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$

"Analyses" the sparsity of the signal under an operator

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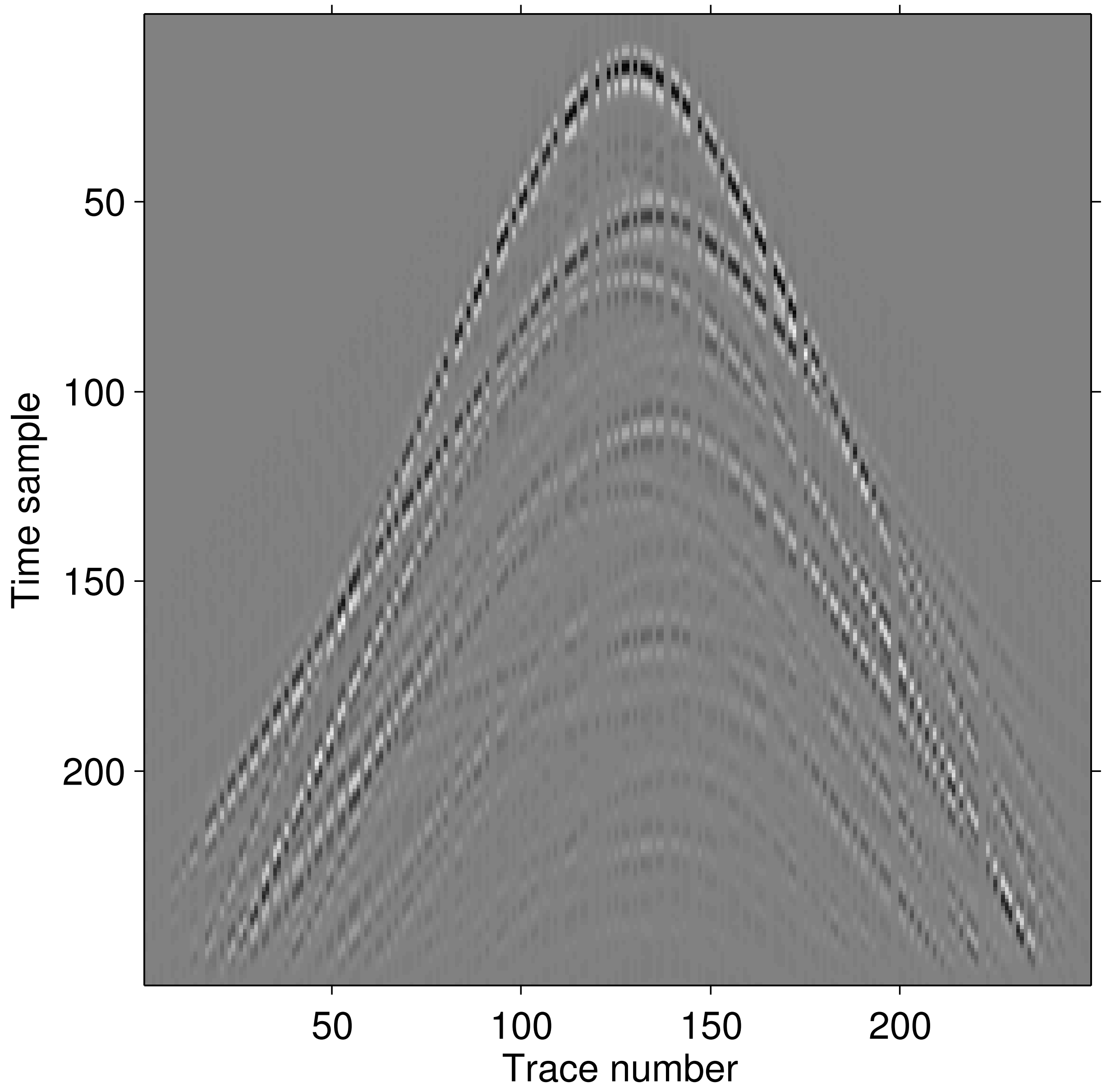
"Analyses" the sparsity of the signal under an operator

Equivalence?

40% random missing traces

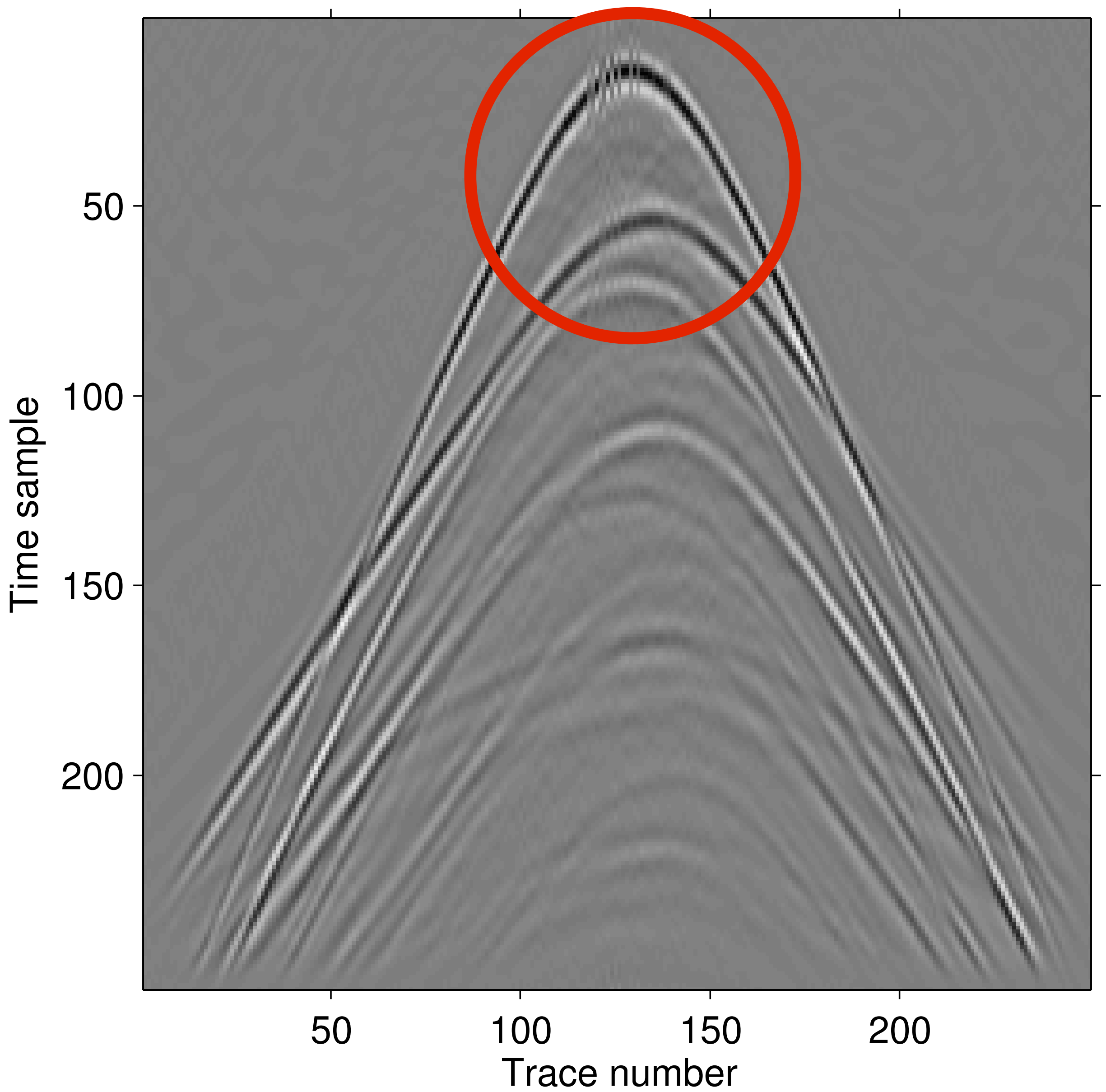


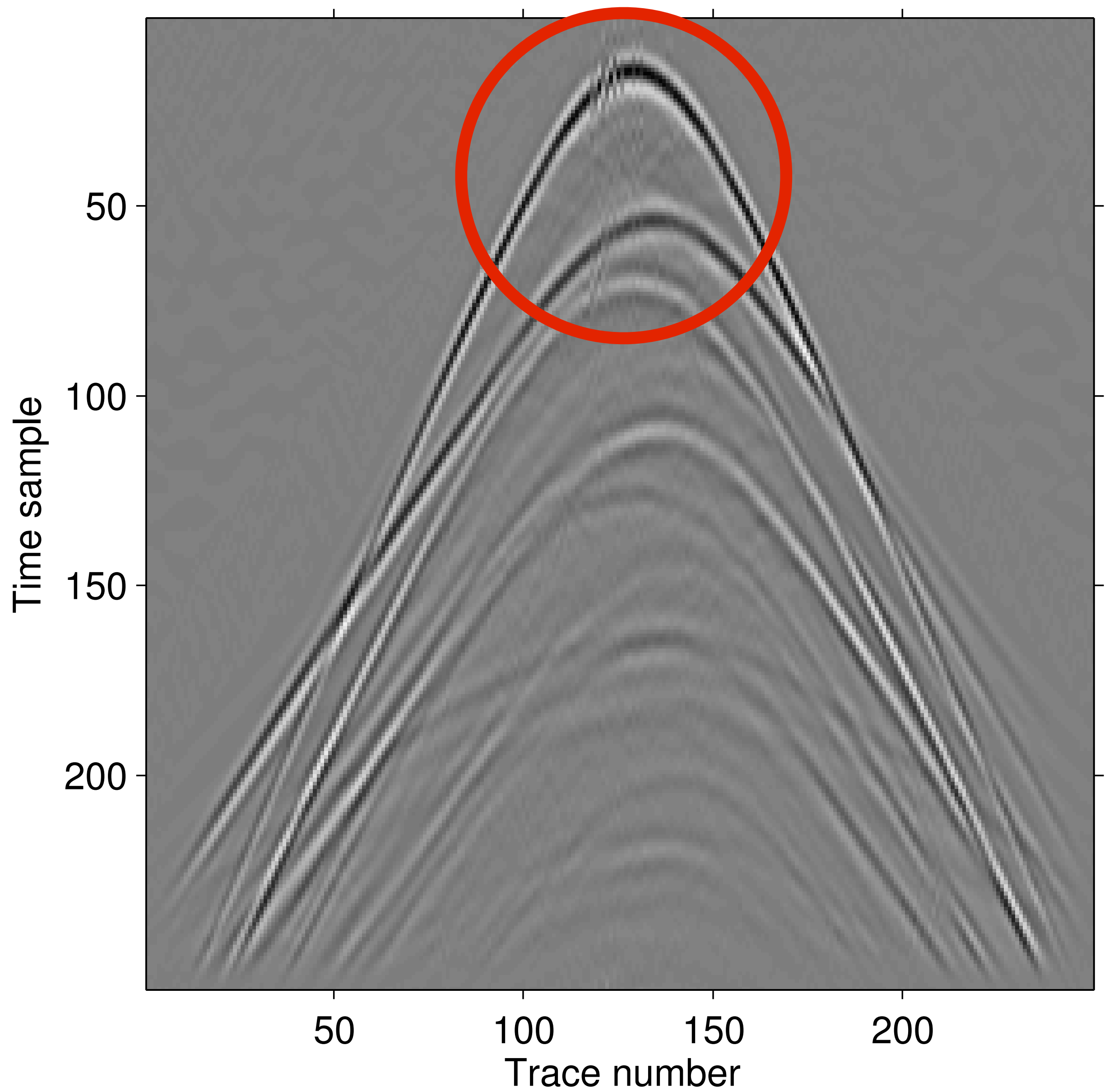
observed data

Recover complete shot-record using synthesis/analysis problem

Analysis $\underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^{\dagger}\mathbf{x}\|_1$

Synthesis $\underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1, \ \mathbf{x} = \mathbf{Dz}$



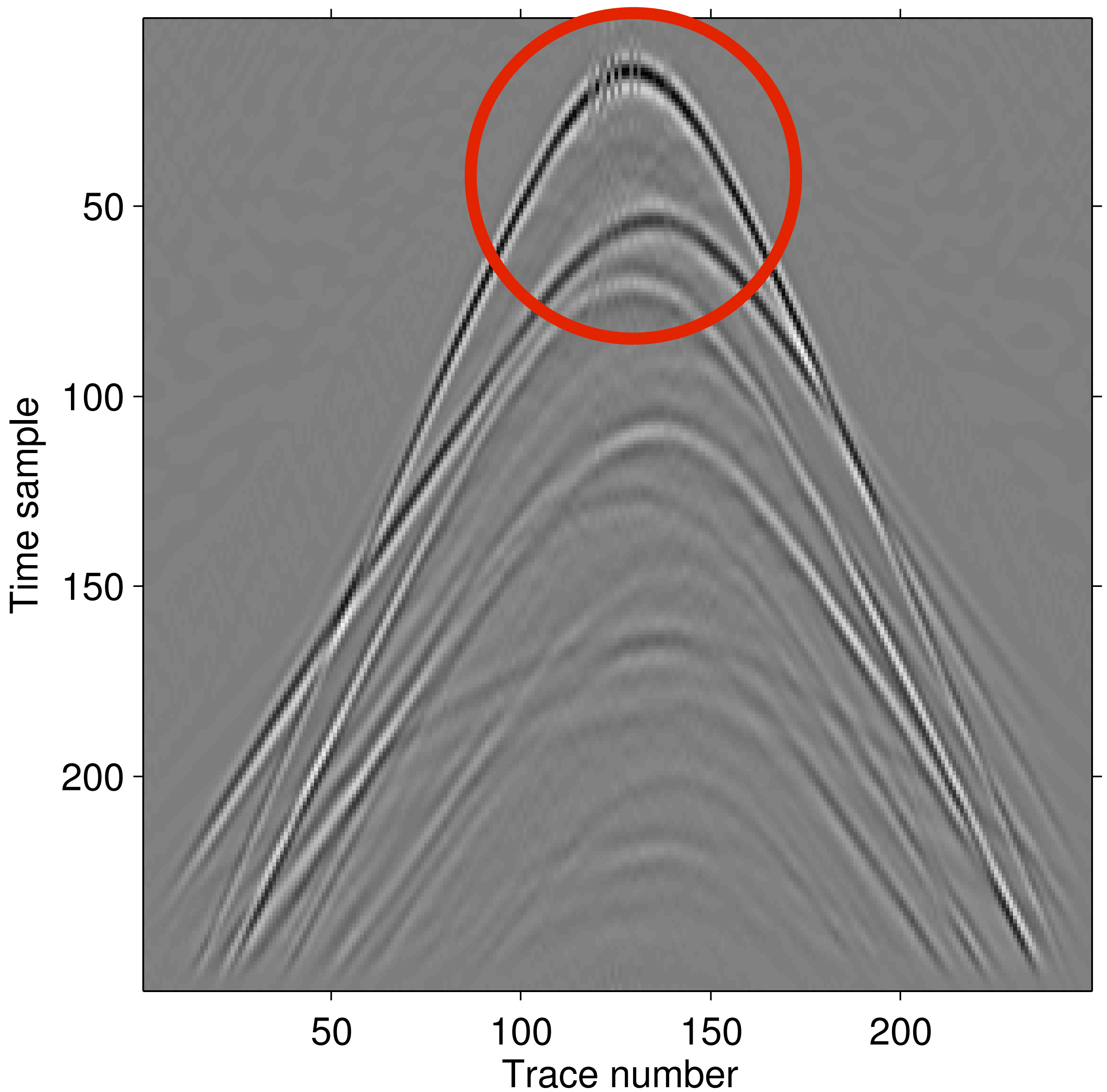


Rel error: 1.7E-01

Rel error: 1.6E-01

Analysis $\underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^{\dagger}\mathbf{x}\|_1$

Synthesis $\underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1, \ \mathbf{x} = \mathbf{Dz}$



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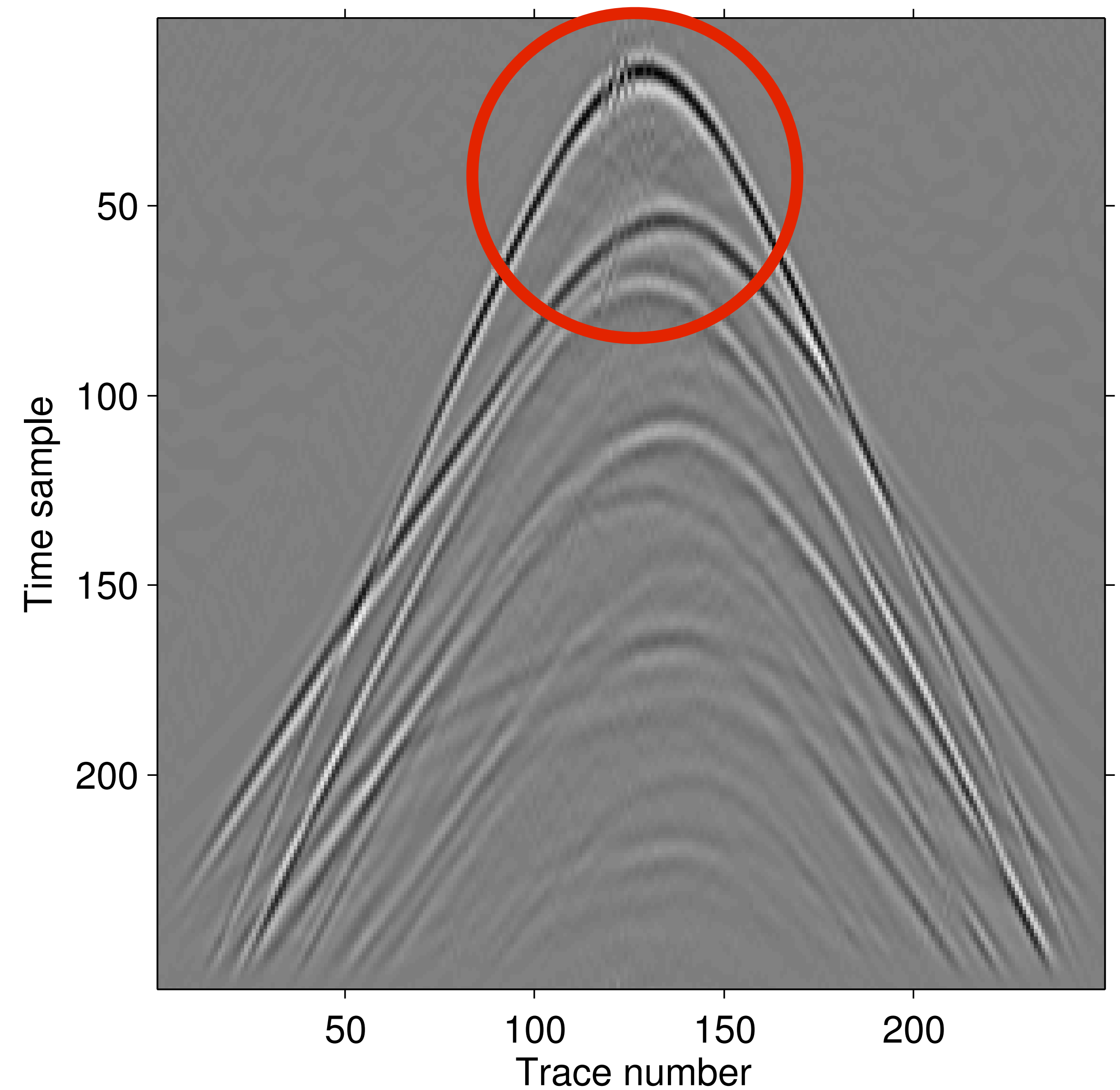
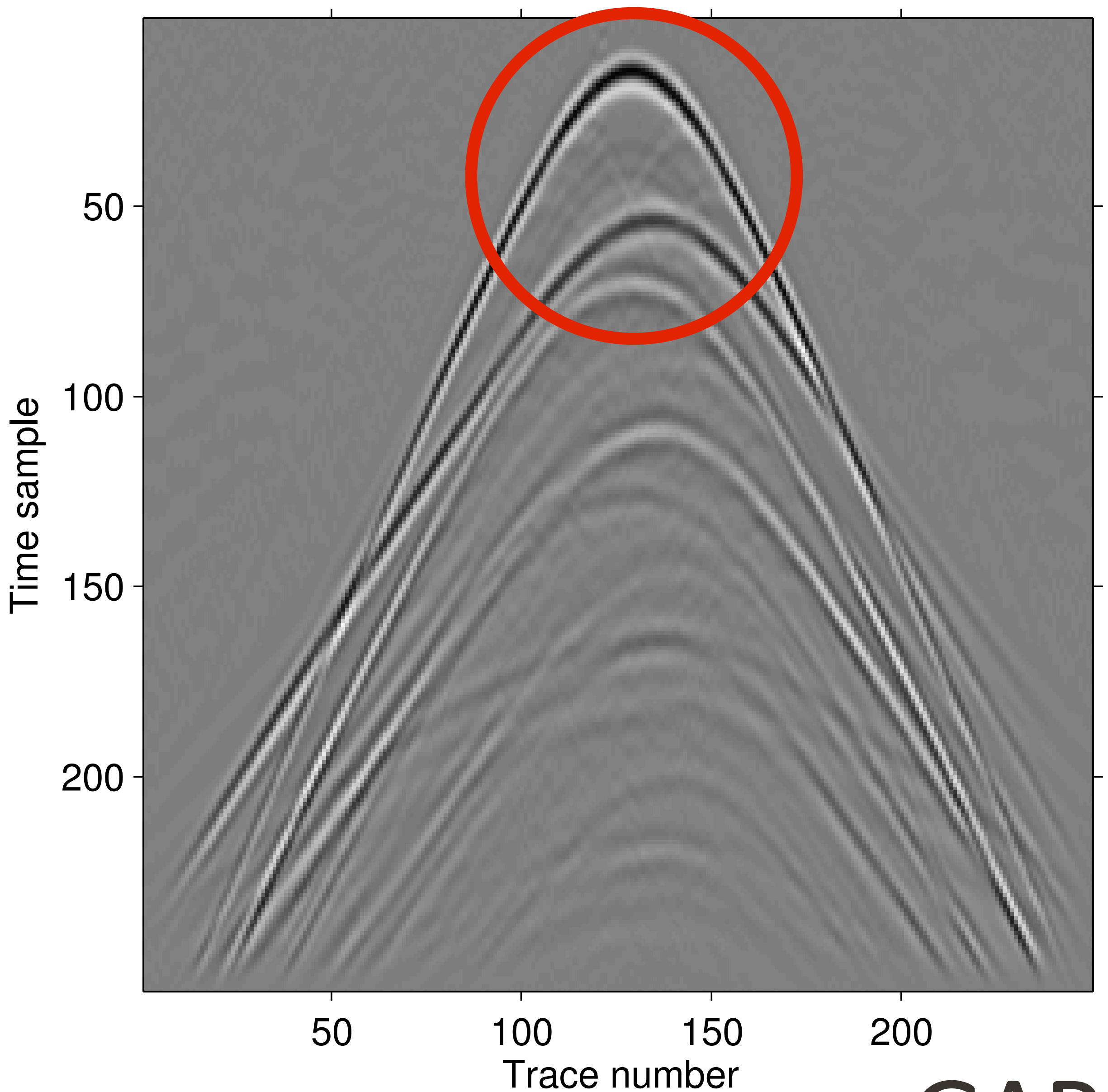


Rel error: 1.6E-01

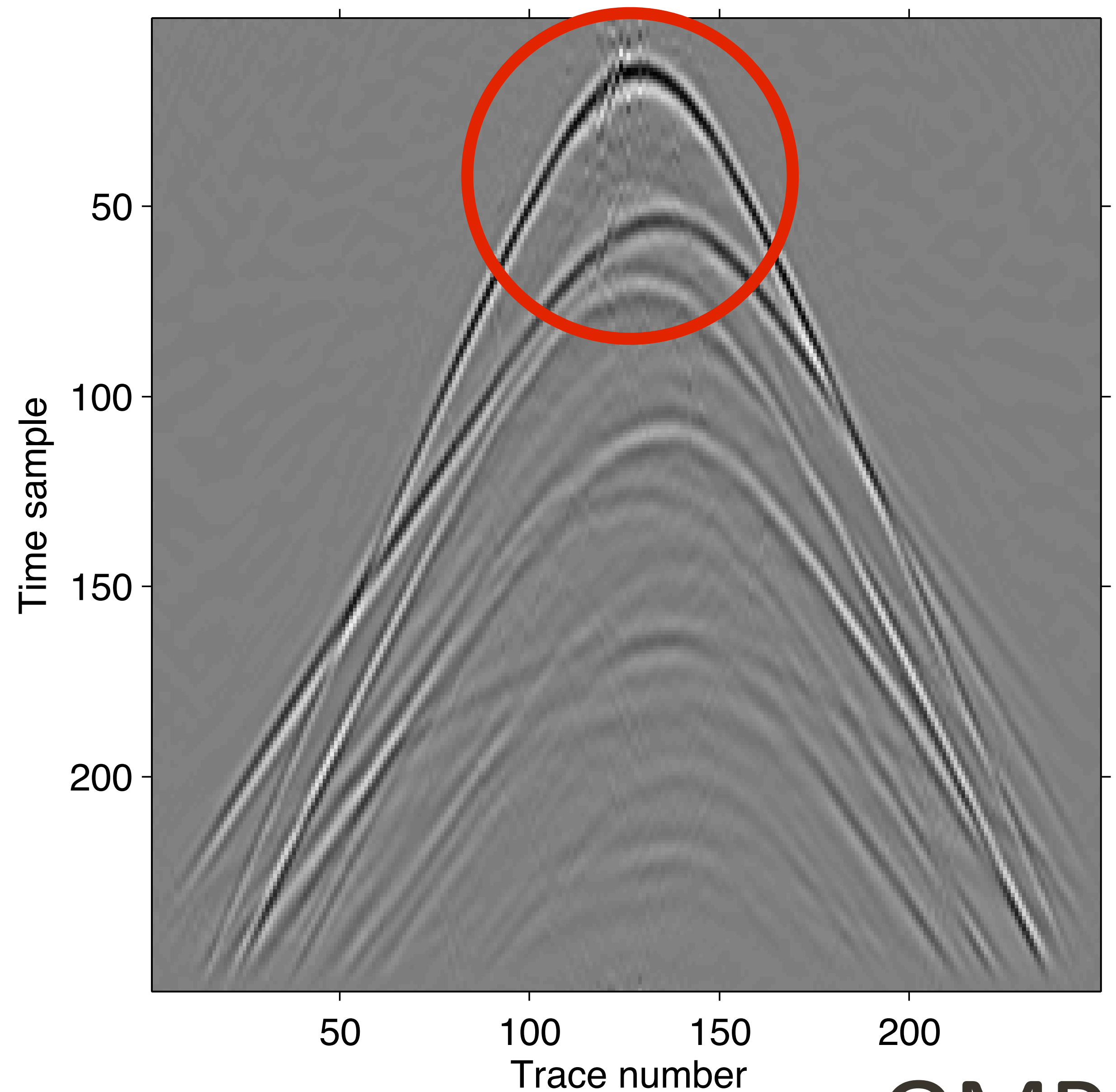
Analysis $\underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{x}\|_0$



GAP

Rel error: 5.4E-02

Synthesis $\underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_0, \ \mathbf{x} = \mathbf{D}\mathbf{z}$



OMP

Rel error: 3.0E-01

Equivalence?

$\mathbf{D}$ $\mathbf{D}^{-1}$

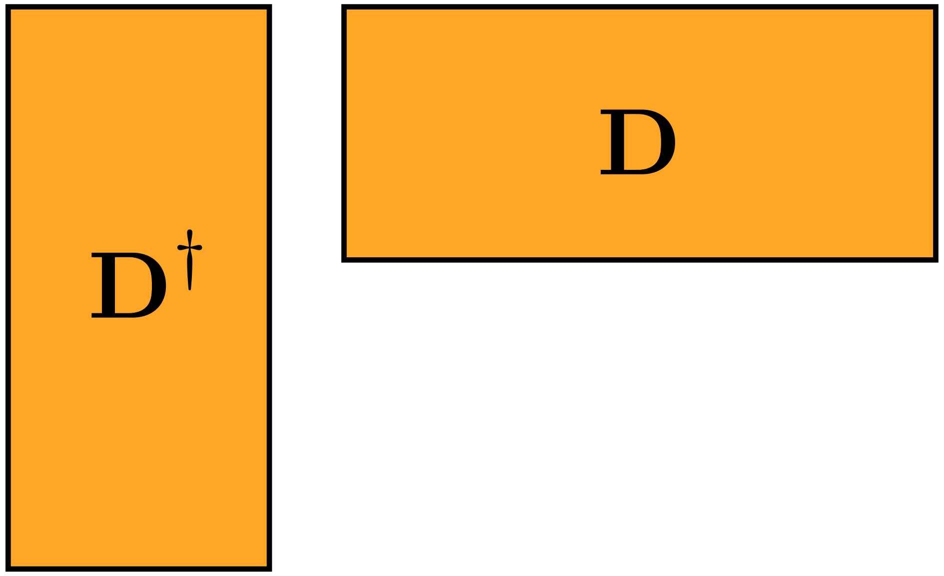
If **D** is square and invertible, then synthesis = analysis

Equivalence?

If **D** is “flat” and redundant, then *not equal*

Equivalence?

Not identity



If **D** is "flat" and redundant, then *not equal*

Equivalence?

Synthesis $$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

Analysis $$\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Equivalence

Synthesis* $\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1$ subject to $\mathbf{y} = \mathbf{ADz}$

$\mathbf{z} = \mathbf{D}^\dagger \mathbf{Dz}$

⇕

Analysis $\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{x}\|_1$ subject to $\mathbf{y} = \mathbf{Ax}$

Analysis-sparsity is a **stronger** condition than Synthesis-sparsity

Equivalence?

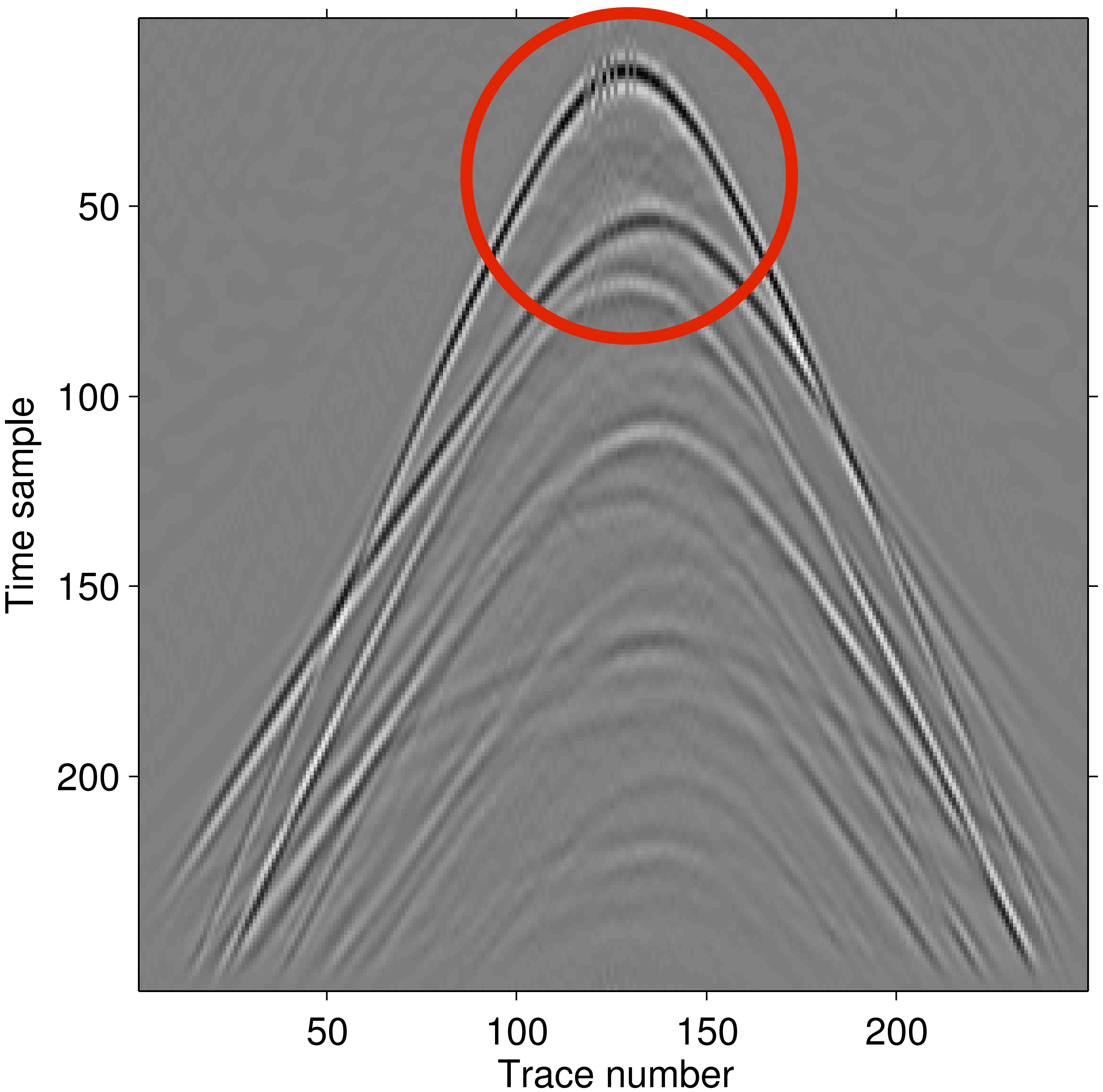
- Many ways to choose

$$\mathbf{z} \text{ s.t. } \mathbf{x} = \mathbf{D}\mathbf{z}$$

- But there is only one

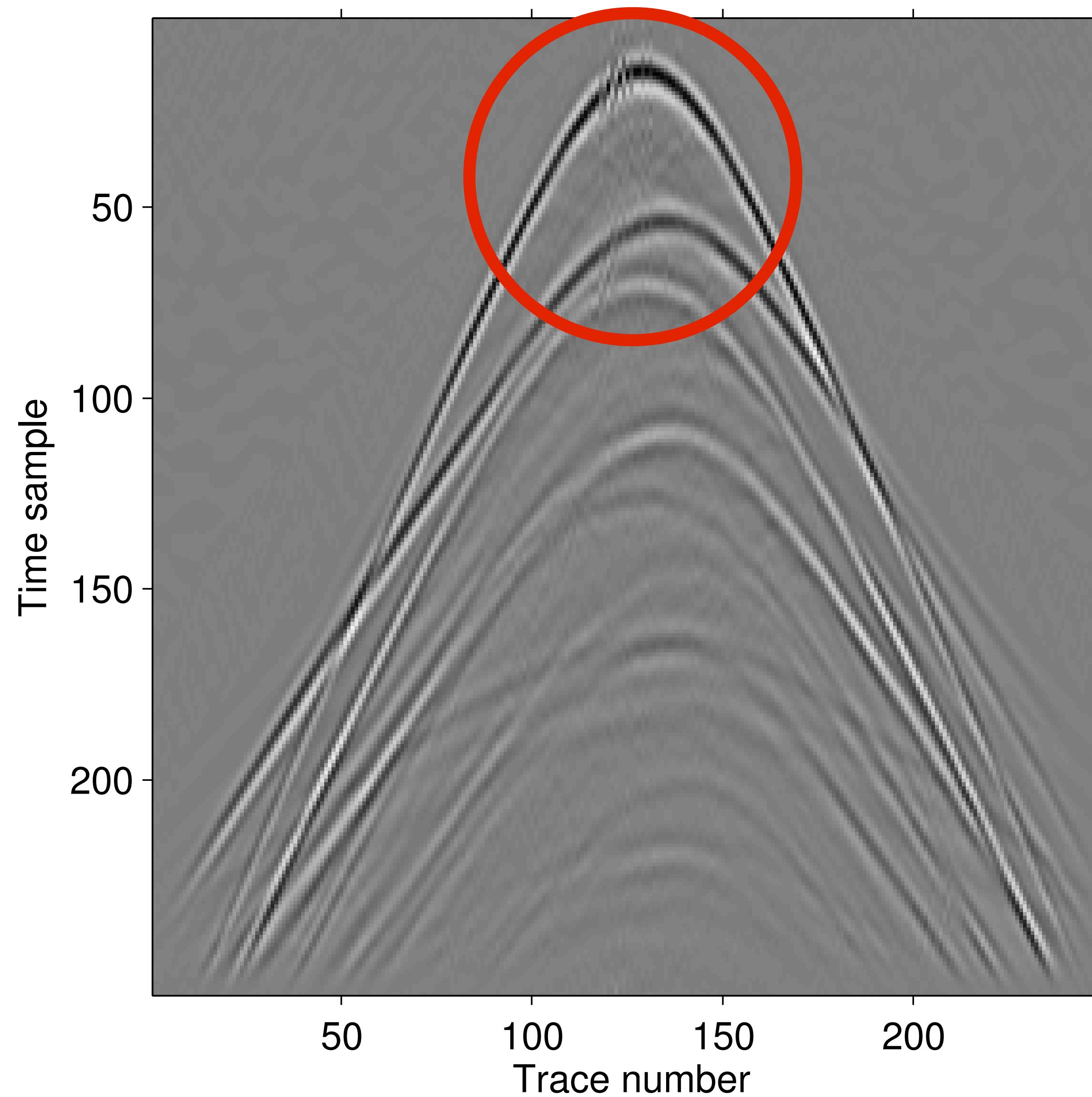
$$\mathbf{D}^{\dagger}\mathbf{x}$$

Analysis $\underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^{\dagger}\mathbf{x}\|_1$



Rel error: 1.7E-01

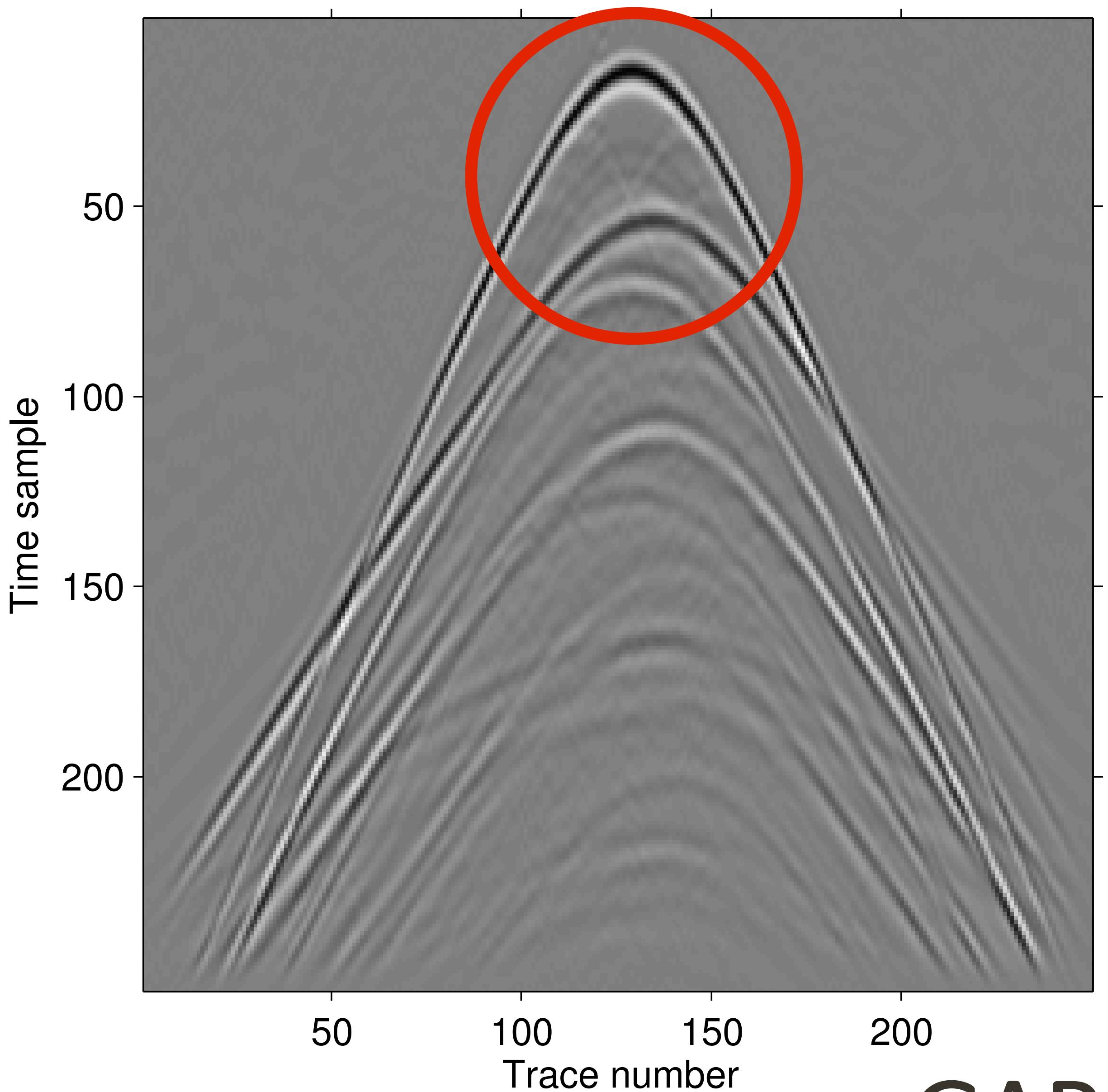
Synthesis $\underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1, \ \mathbf{x} = \mathbf{D}\mathbf{z}$



Rel error: 1.6E-01

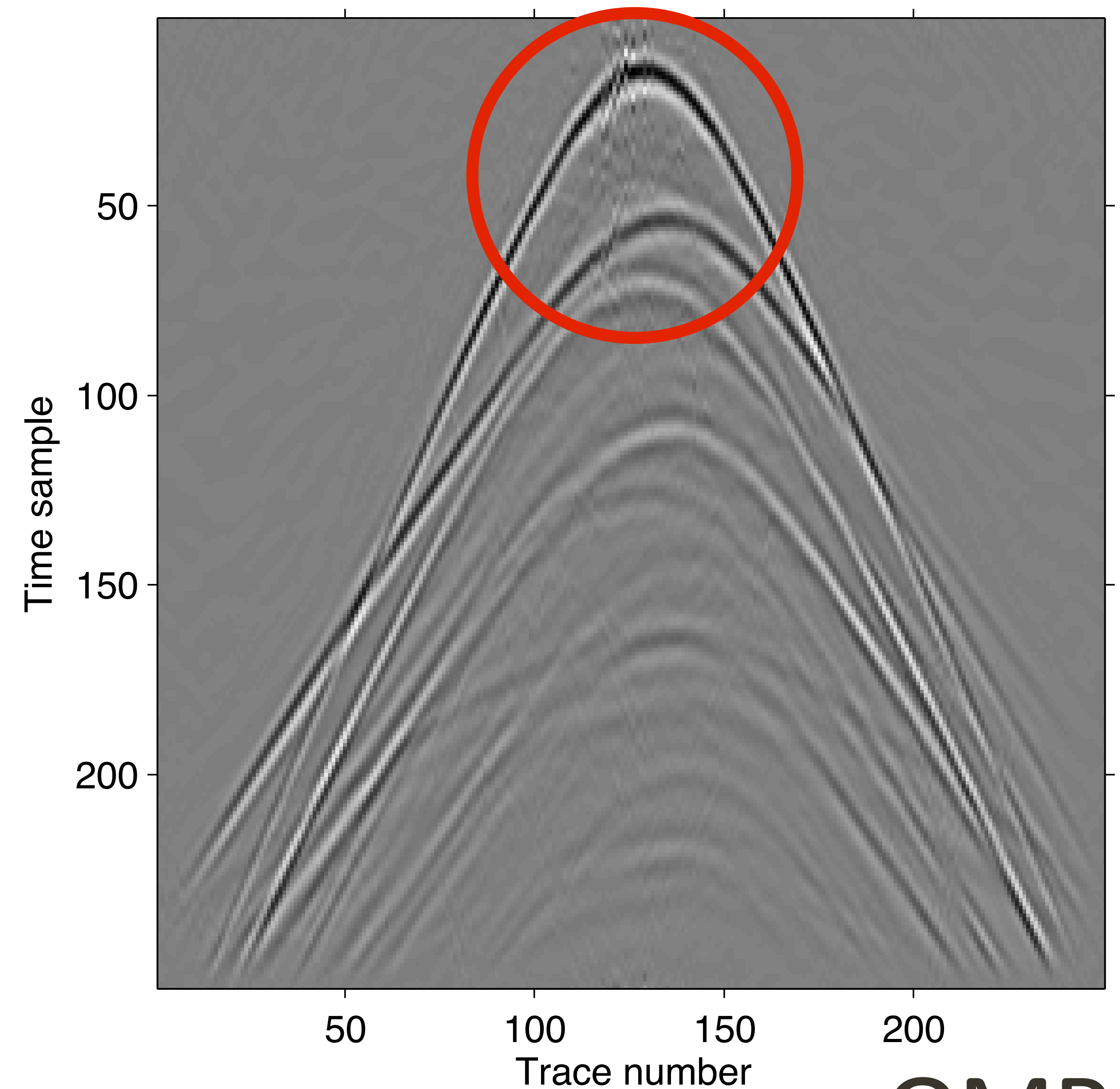
Analysis $\underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^{\dagger}\mathbf{x}\|_0$

Synthesis $\underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_0, \ \mathbf{x} = \mathbf{D}\mathbf{z}$



GAP

Rel error: 5.4E-02

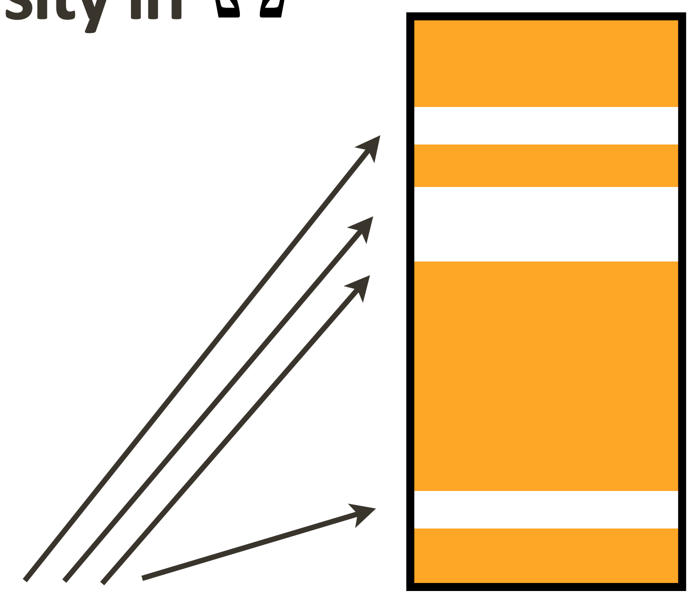


OMP

Rel error: 3.0E-01

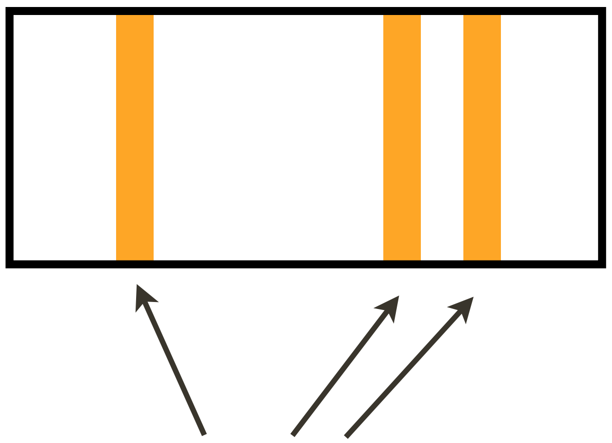
Introducing Cosparsity

Cosparsity in Ω



Constrains signal to be orthogonal to some number of rows

Sparsity in $\mathbf{D}$

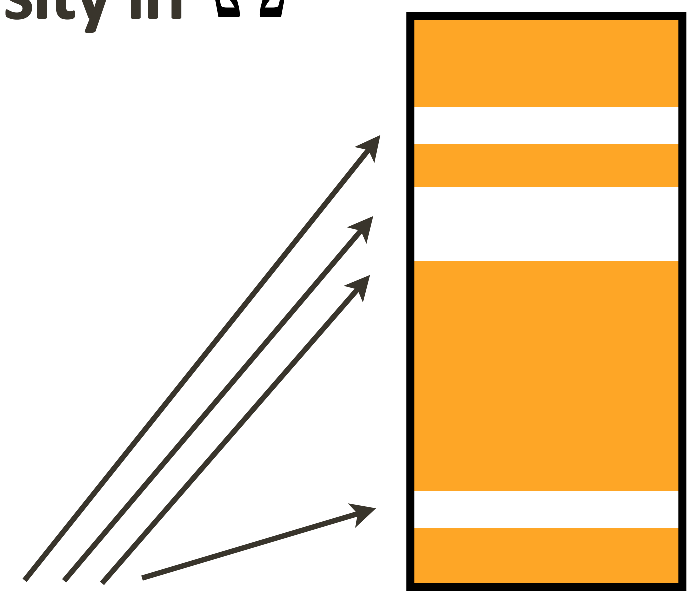


Constrains signal to lie on the support of a few number of columns

S. Nam et al., The cosparse analysis model and algorithms, 2012

Introducing Cosparsity

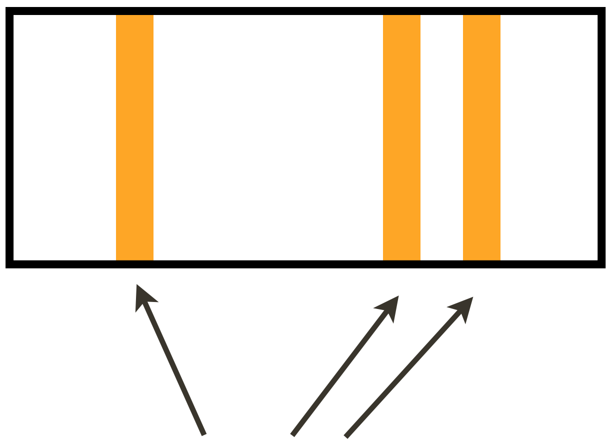
Cosparsity in Ω



Constrains signal to be orthogonal to some number of rows

cosparsity ℓ

Sparsity in $\mathbf{D}$



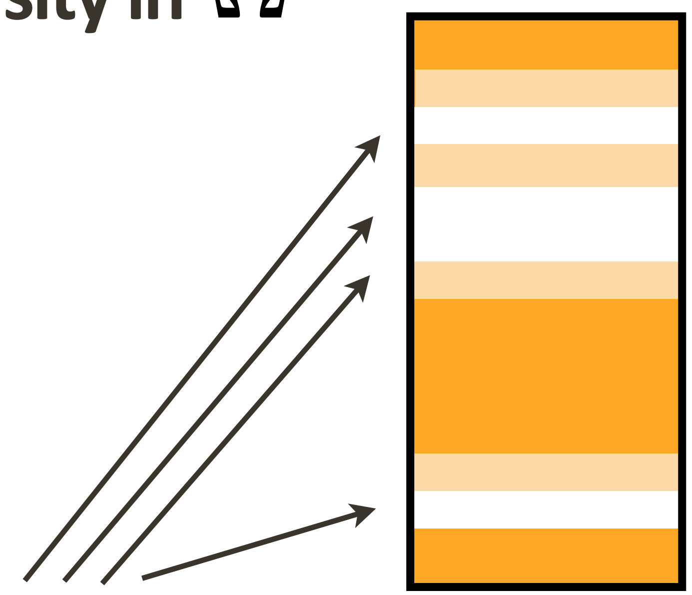
Constrains signal to lie on the support of a few number of columns

sparsity k

S. Nam et al., The cosparse analysis model and algorithms, 2012

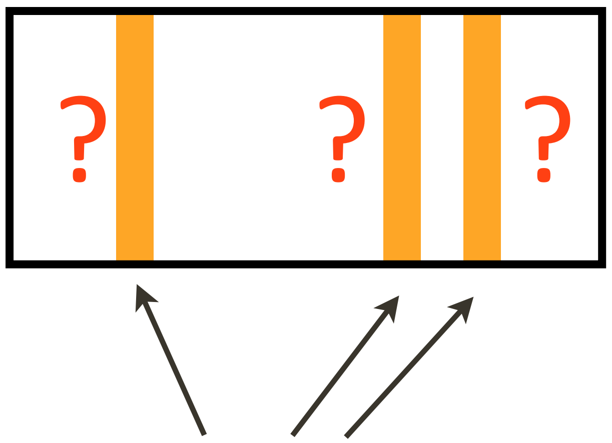
Introducing Cosparsity

Cosparsity in $\boldsymbol{\Omega}$



Constrains signal to also be nearly-orthogonal to rows that are lin. dep. with the zero rows

Sparsity in $\mathbf{D}$



Constrains signal to lie on the support of a few number of columns

sparsity k

S. Nam et al., The cosparse analysis model and algorithms, 2012

Example: PDE solving

Monochromatic Helmholtz system

$$\min \|s - \mathbf{H}_{x \in x_s} \mathbf{u}\|_2 \quad \text{subject to} \quad \mathbf{H}_{x \notin x_s} \mathbf{u} = 0$$

enforcing non-source position to be zero

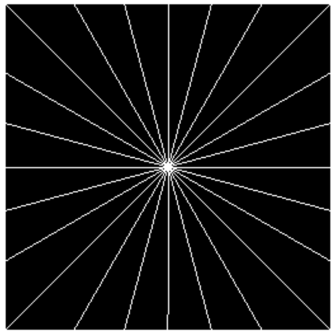
Cosparsity results

- uniqueness of solution when recovering from undersampled cosparse signals
- sufficient condition (“ERC-like”) for success of L1-Analysis and GAP in reconstructing the above

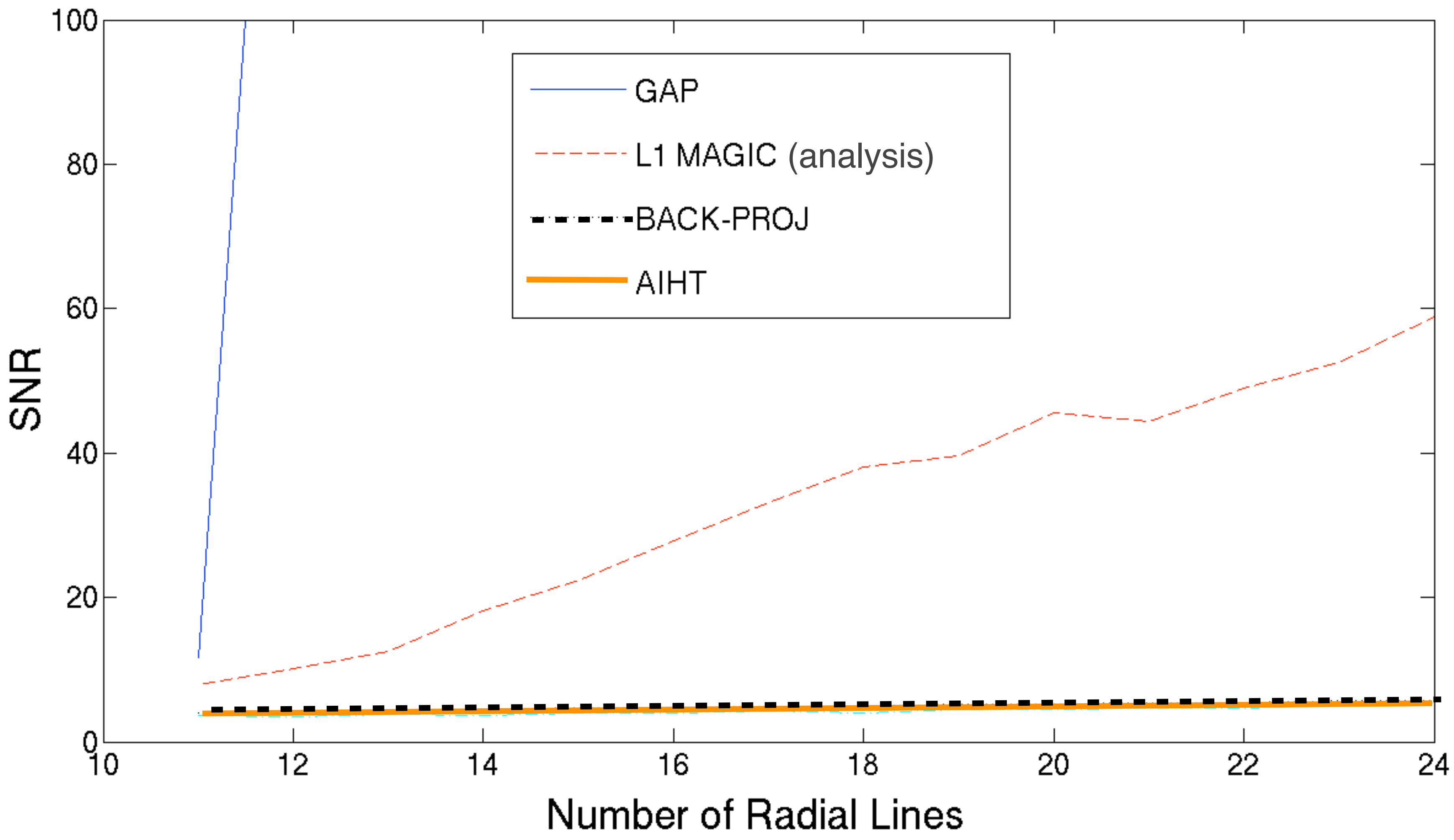
S. Nam et al., The cosparse analysis model and algorithms, 2012

Cosparsity in MRI

Observe radial lines in spatial frequency domain



Recover piecewise-constant image using "TV"



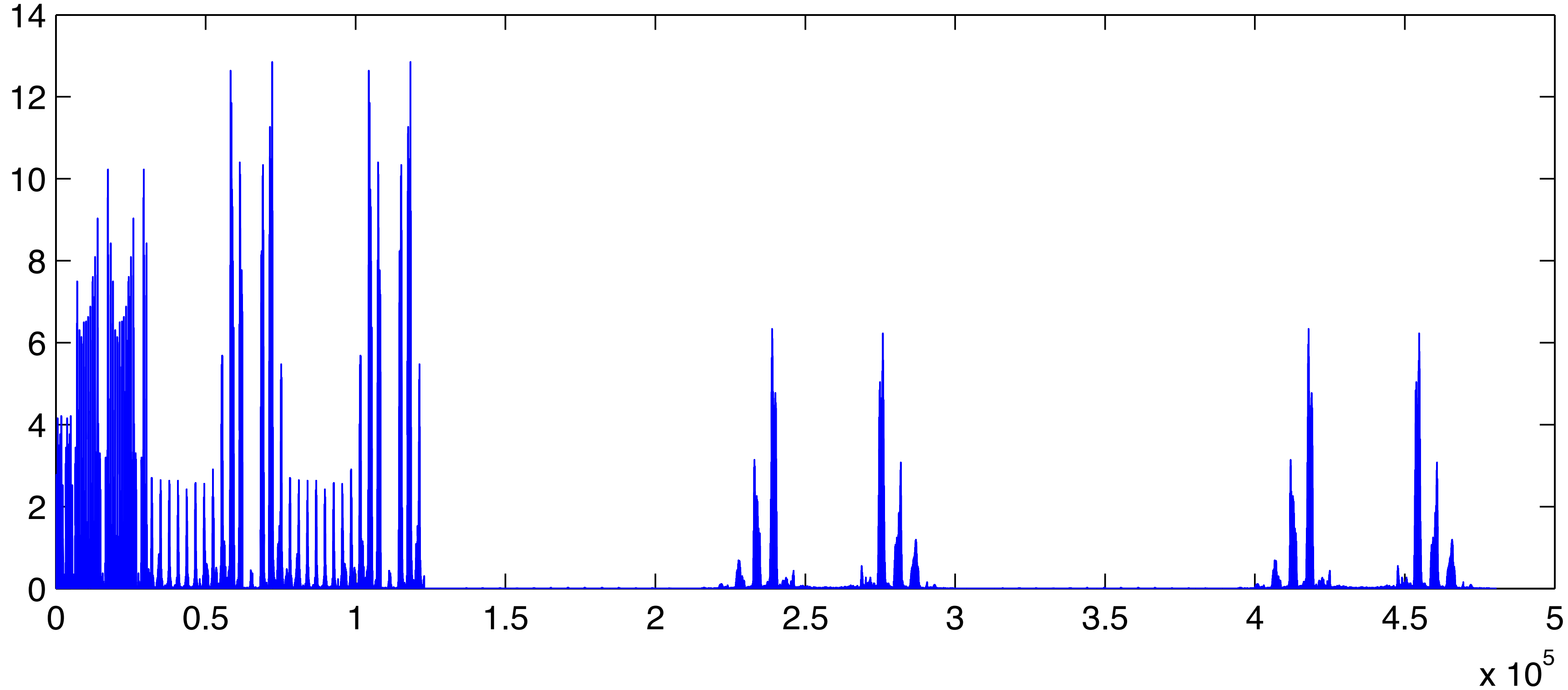
from S. Nam et al., The cosparse analysis model and algorithms, 2012

GAP does not detect "support"

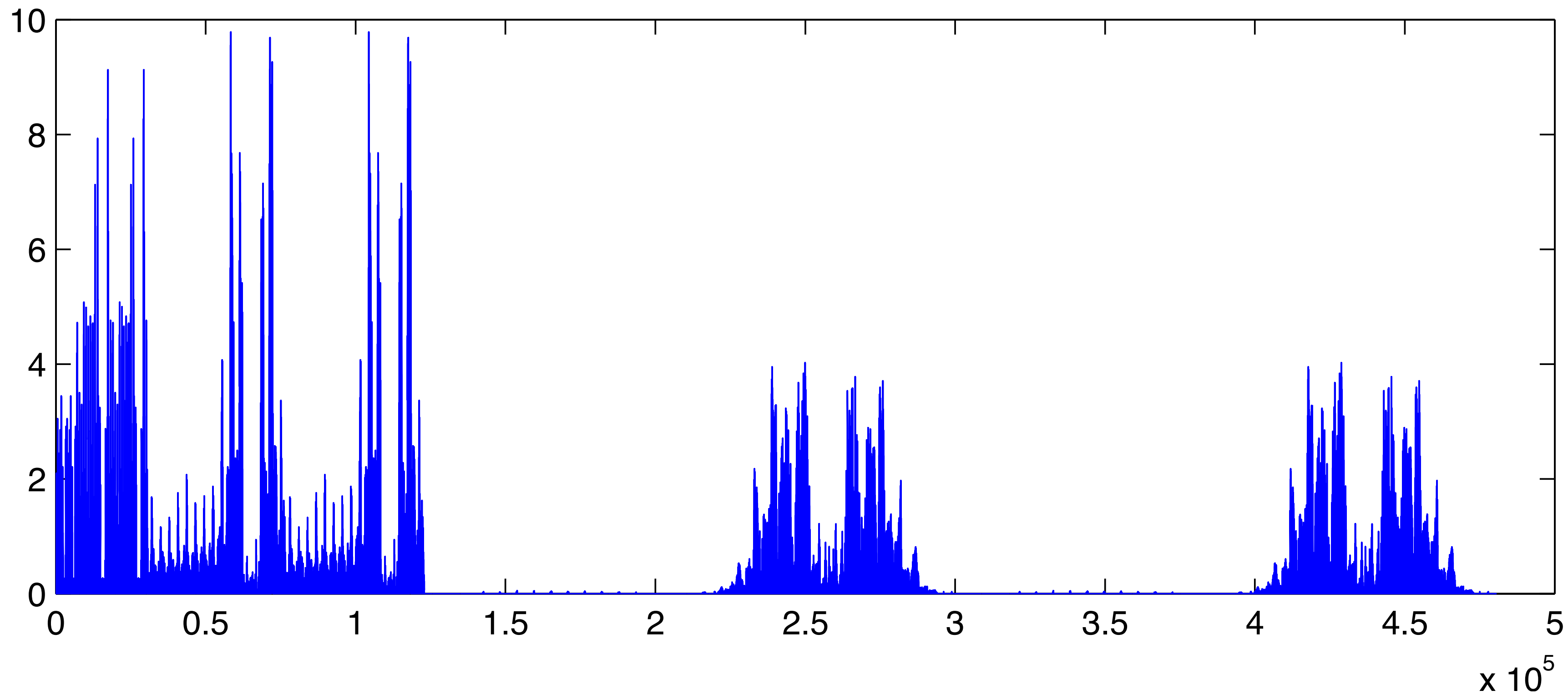
Out of 480617 coefficients:

GAP kept 150951 **(31.4%)**
L1-Synthesis kept 49519 **(10.3%)**

(signal size > 0.5% of largest)

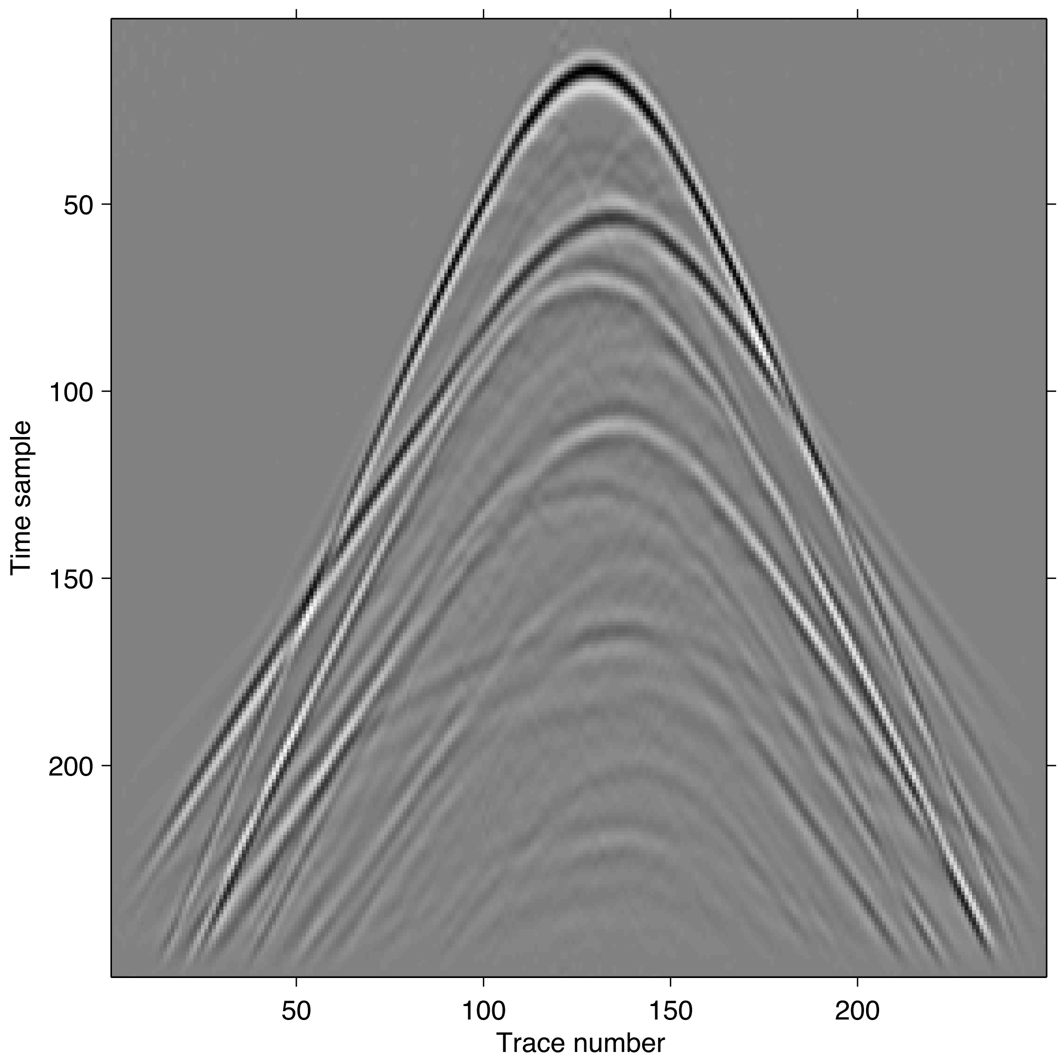


$\min_{\mathbf{x}} \|\mathbf{C}_\Lambda \mathbf{x}\|_2$ subject to $\mathbf{y} = \mathbf{A}\mathbf{x}$ GAP solution

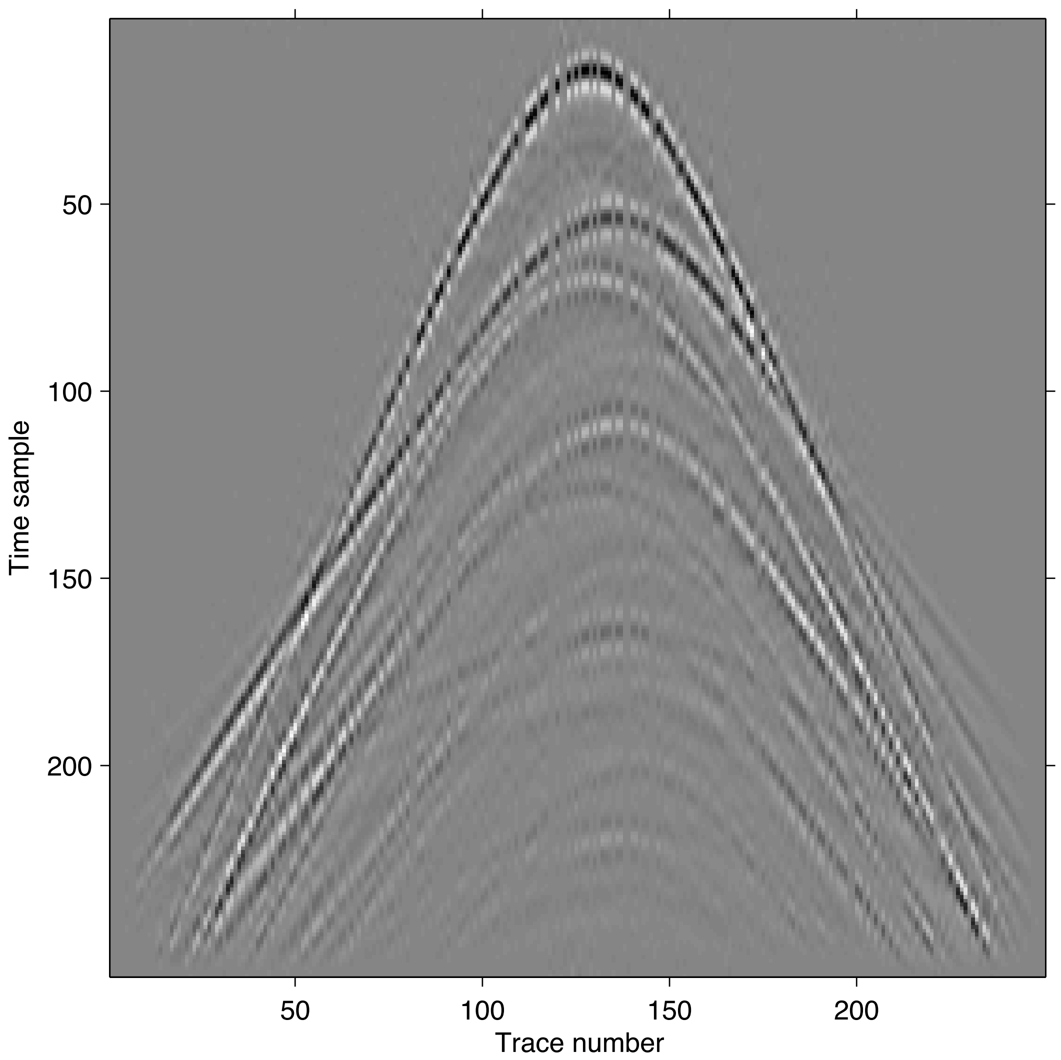


$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{C}^H_{\Lambda^c}\mathbf{x}\|_2$ treating GAP solution as “support”

GAP does not detect “support”

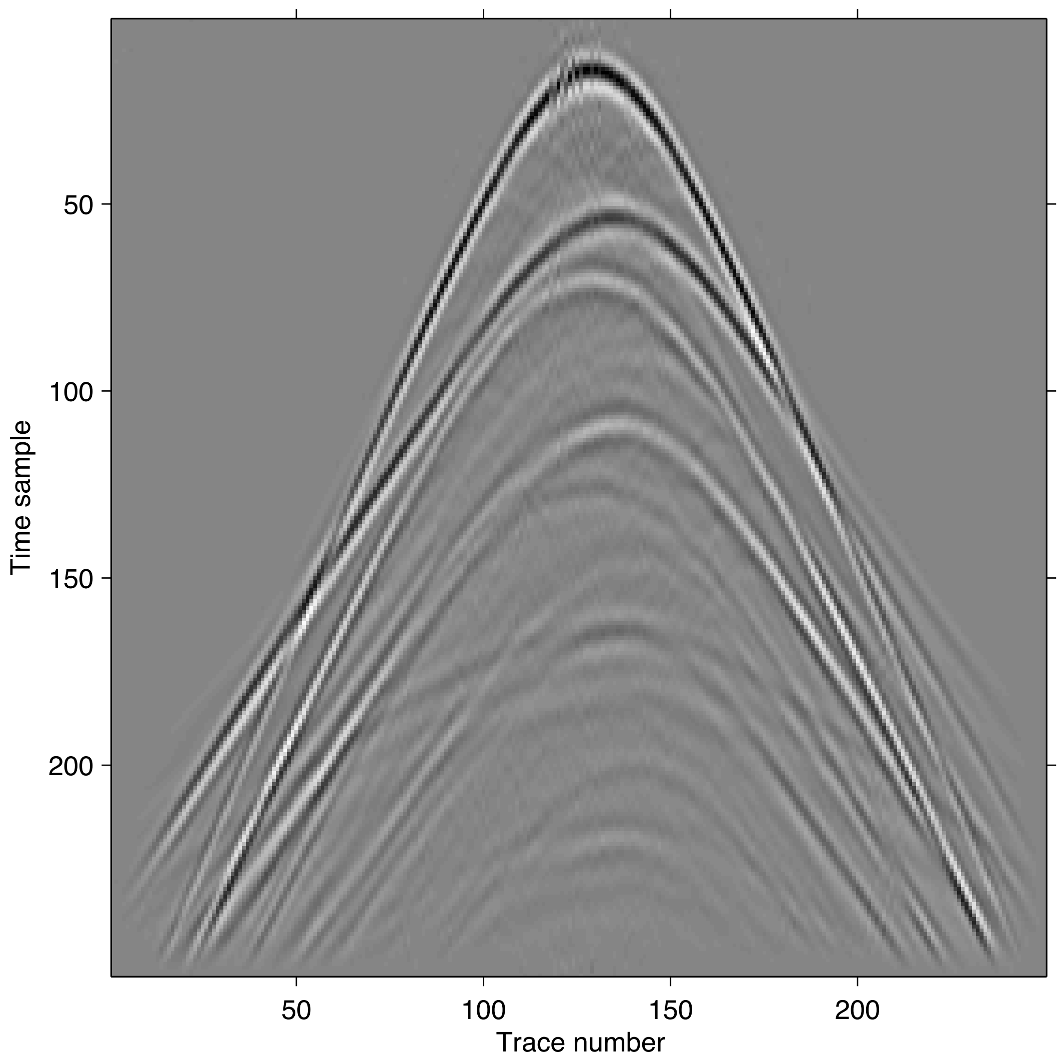


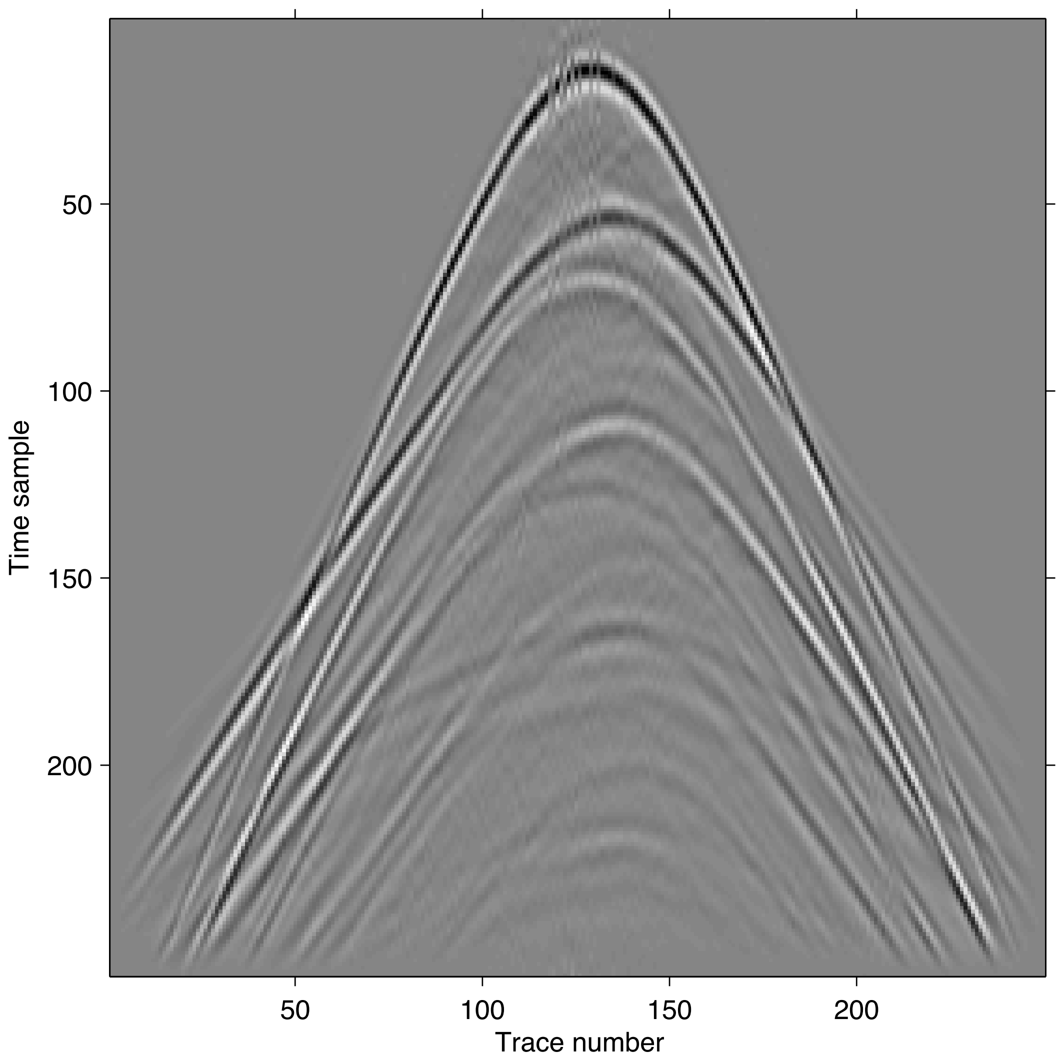
GAP solution



treating GAP solution as “support”

GAP does not detect “support”





L1-Synth solution

L1-Synth “debiased” solution

Cosparse algorithm

Goal is to pick out the rows of $\boldsymbol{\Omega}$ that should be orthogonal to the solution (as many as possible)

Greedy Analysis Pursuit (GAP)

S. Nam et al., The cosparse analysis model and algorithms, 2012

GAP basic outline

Start with full index set of rows of $\Omega \in \mathbb{C}^{n \times d}$

$$\Lambda = \{1, 2, 3, \ldots, n\}$$

1. **Projection:** compute $\mathbf{z} = \Omega \mathbf{x}_k$
2. **Find largest** element(s) of $\mathbf{z}$
3. **Remove** the corresponding row(s) from Λ
4. **Update solution** estimate

$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

GAP basic outline

$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

GAP basic outline

SLIM

obtained by solving least-squares problem

$$\min_{\mathbf{x}} \left\| \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix} - \begin{pmatrix} \mathbf{A} \\ \lambda \boldsymbol{\Omega} \end{pmatrix} \mathbf{x} \right\|_2$$

(adjust λ dynamically based on residual and expected noise level in the data)

$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

GAP basic outline

Start with full index set of rows of $\Omega \in \mathbb{C}^{n \times d}$

$$\Lambda = \{1, 2, 3, \ldots, n\}$$

$$\tilde{\mathbf{x}}_0 = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

1. **Projection:** compute $\mathbf{z} = \Omega \mathbf{x}_k$
2. **Find largest** element(s) of $\mathbf{z}$
3. **Remove** the corresponding row(s) from Λ
4. **Update solution** estimate

$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

GAP basic outline

Start with full index set of rows of $\Omega \in \mathbb{C}^{n \times d}$

$$\Lambda = \{1, 2, 3, \ldots, n\}$$

$$\tilde{\mathbf{x}}_0 = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

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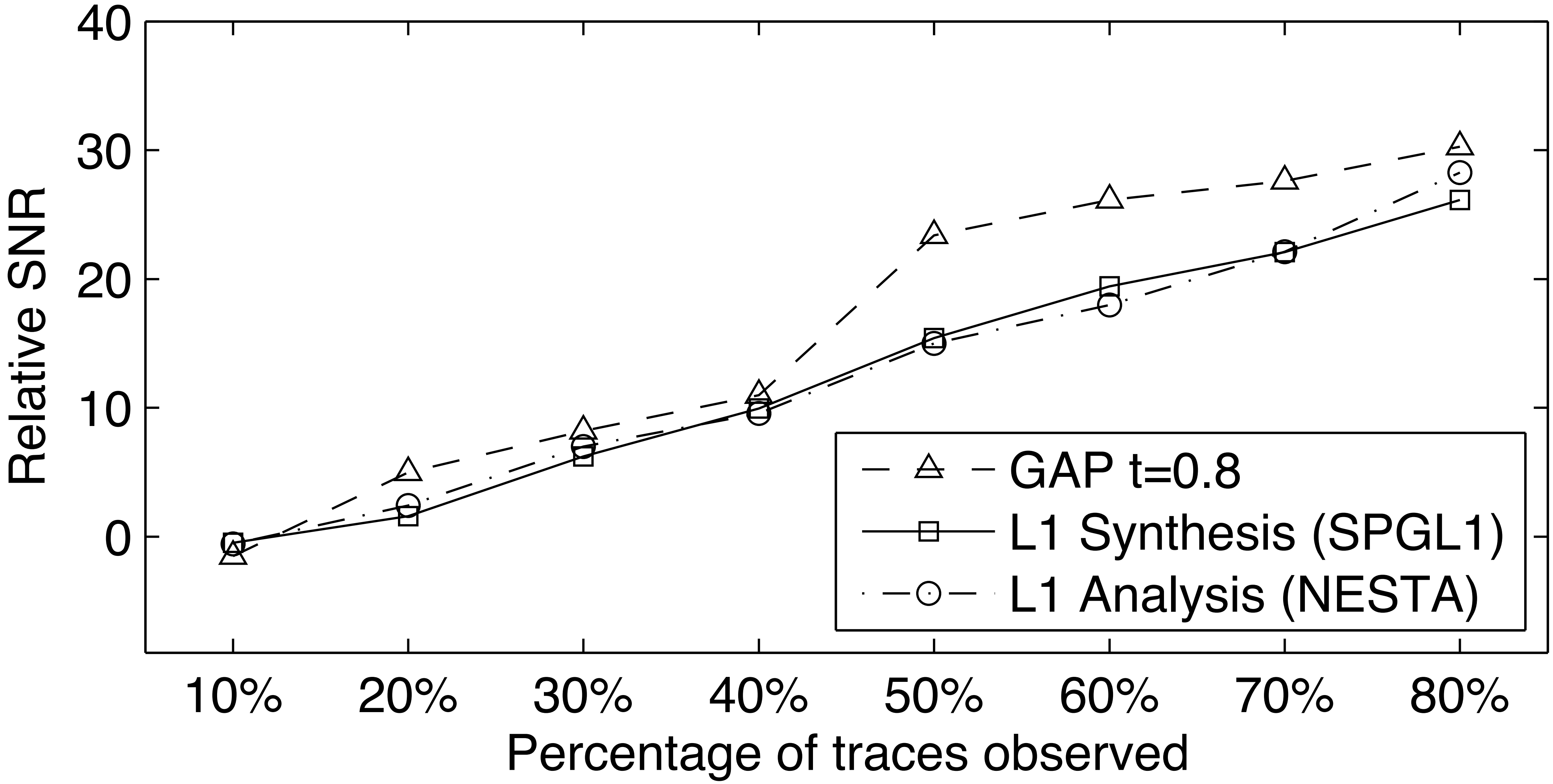
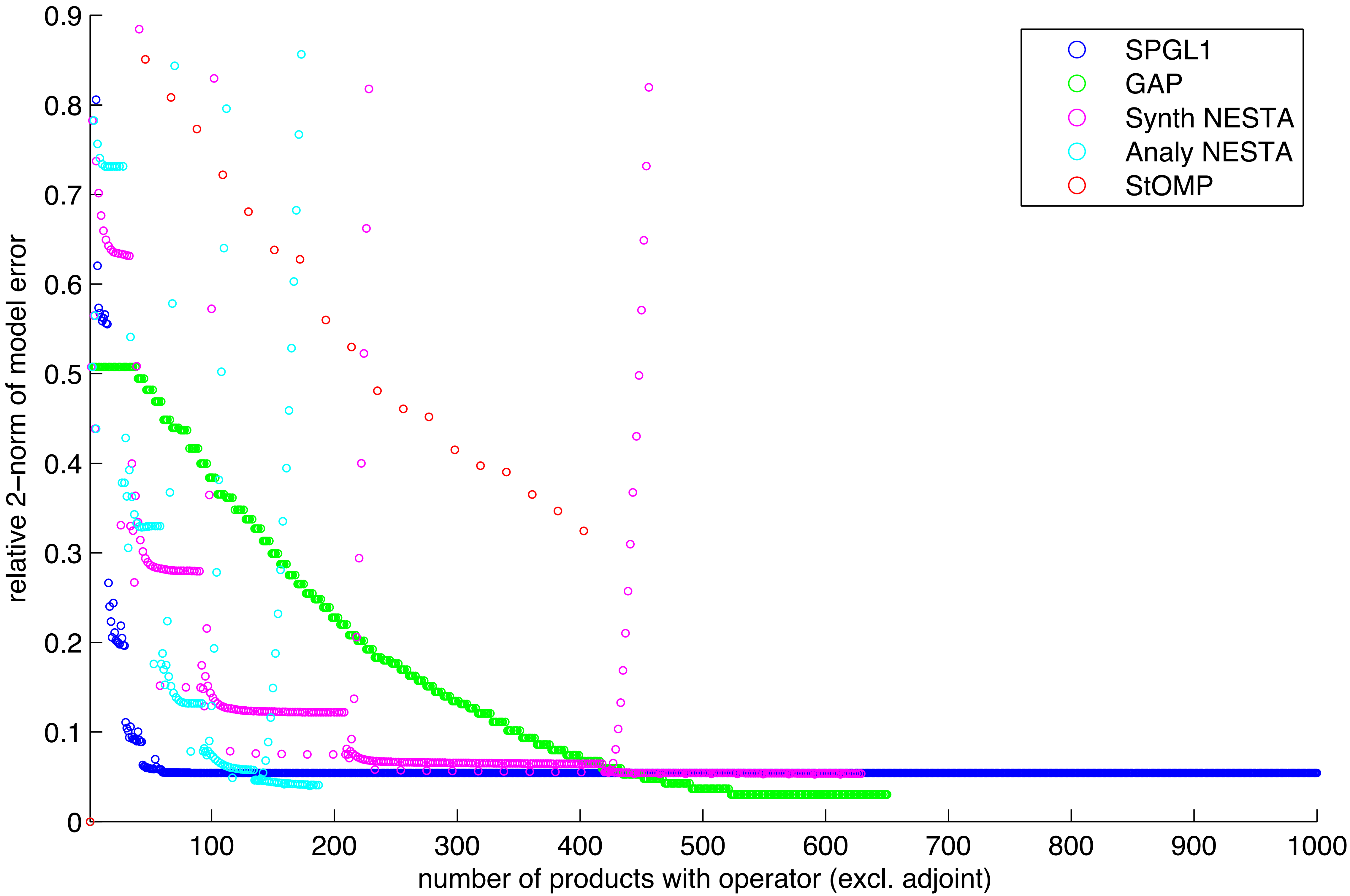
$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Stop at convergence of $\Delta \mathbf{x}$, or small $\|\Omega_\Lambda \mathbf{x}\|_\infty$

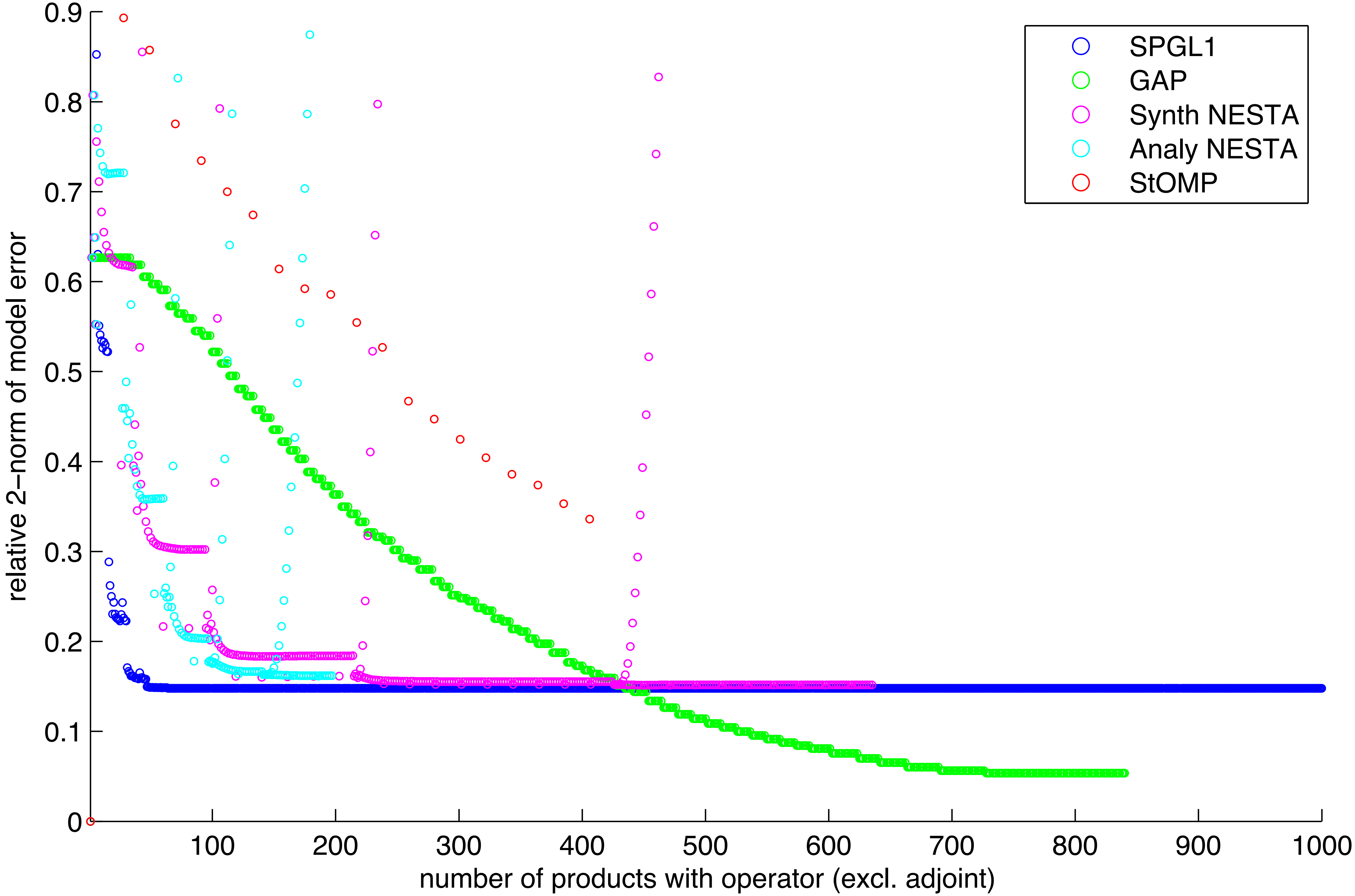
Recovery performance



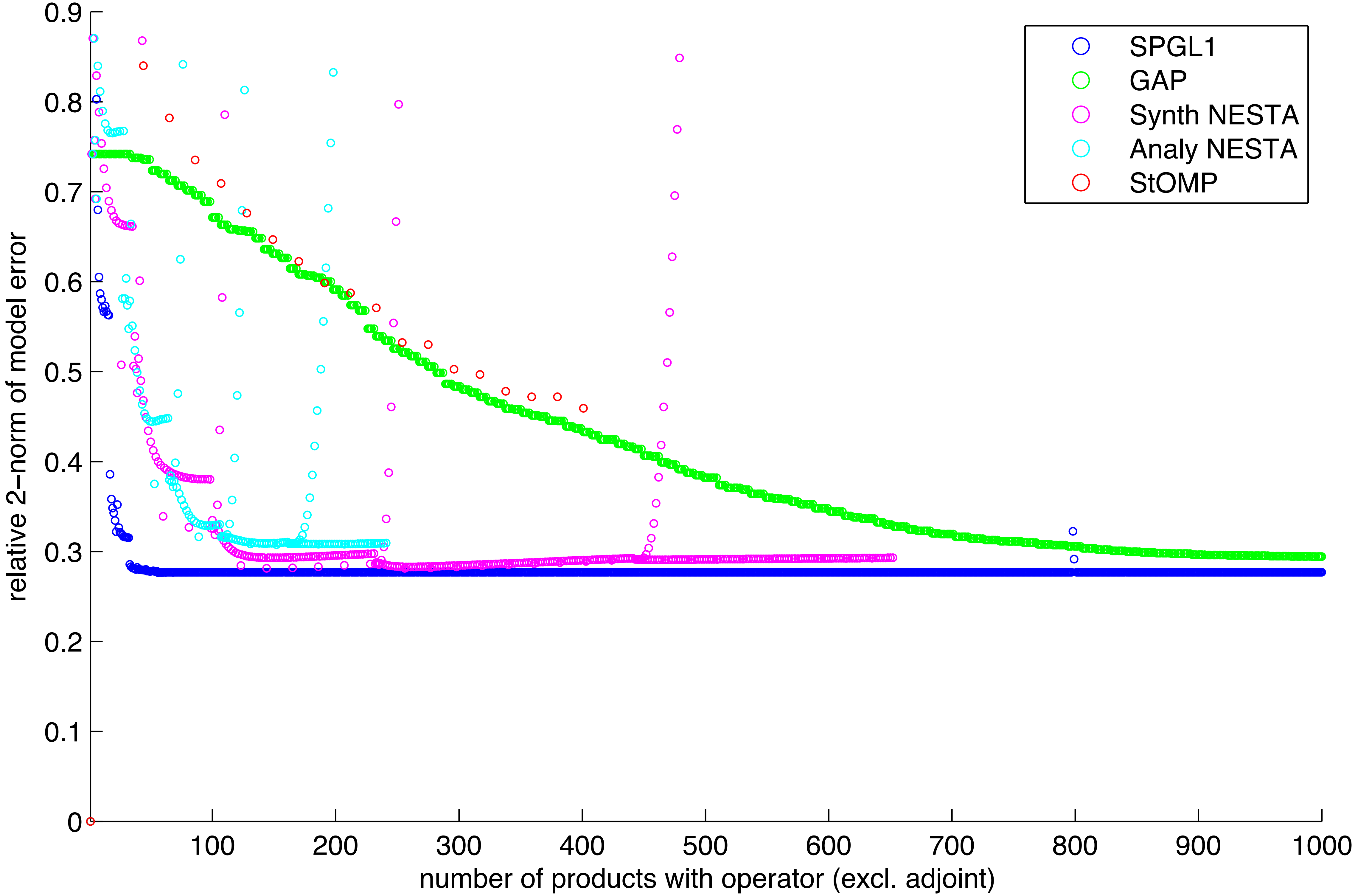
Model error vs computation cost
75% observed traces



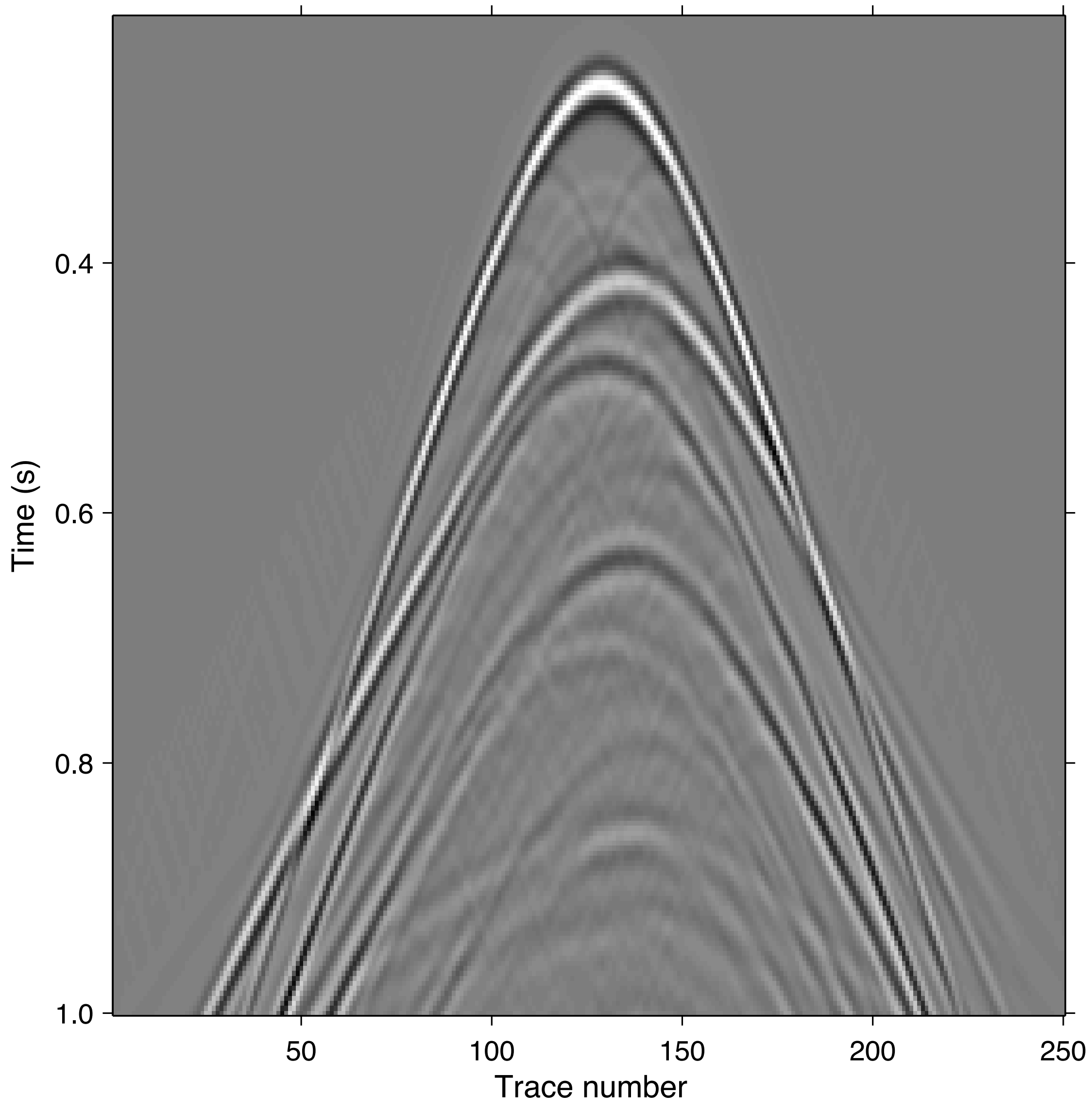
Model error vs computation cost
50% observed traces



Model error vs computation cost
25% observed traces

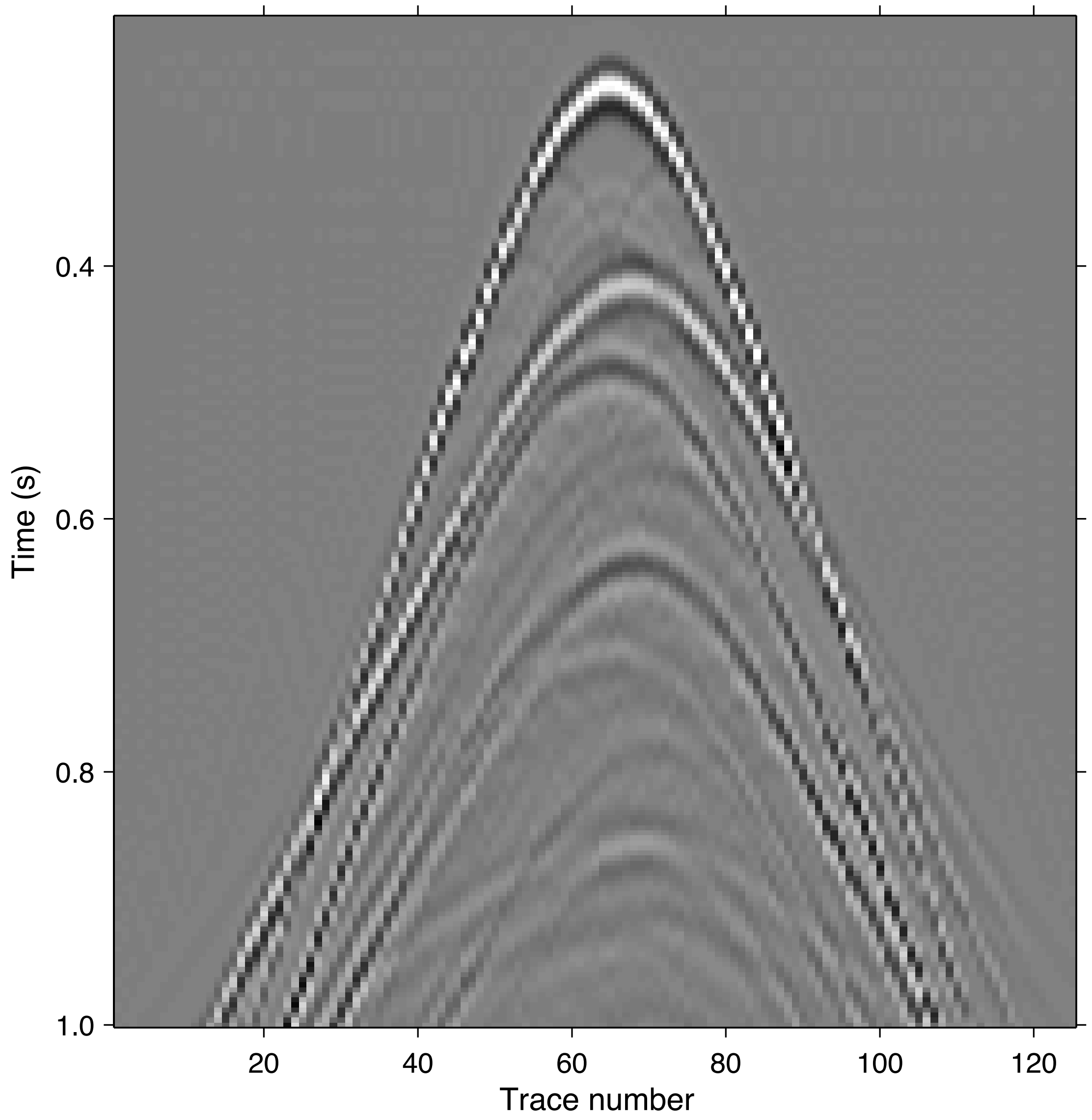


Regularization + Interpolation



Original data

Regularization + Interpolation



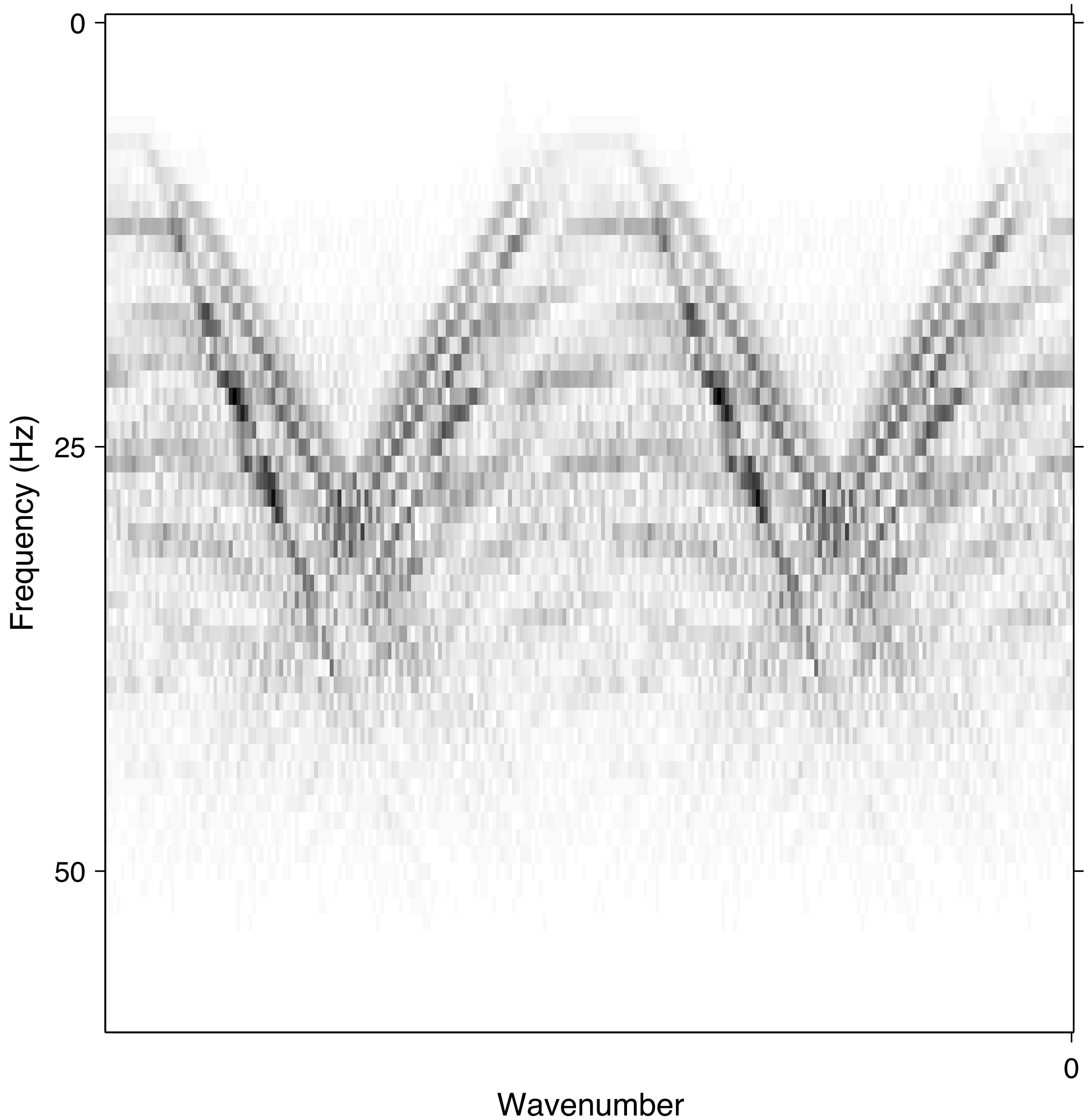
Decimated:

15m grid -> 30m grid

Perturbed:

uniformly random trace shift in the range [-8m, +8m] from gridpoint

Regularization + Interpolation



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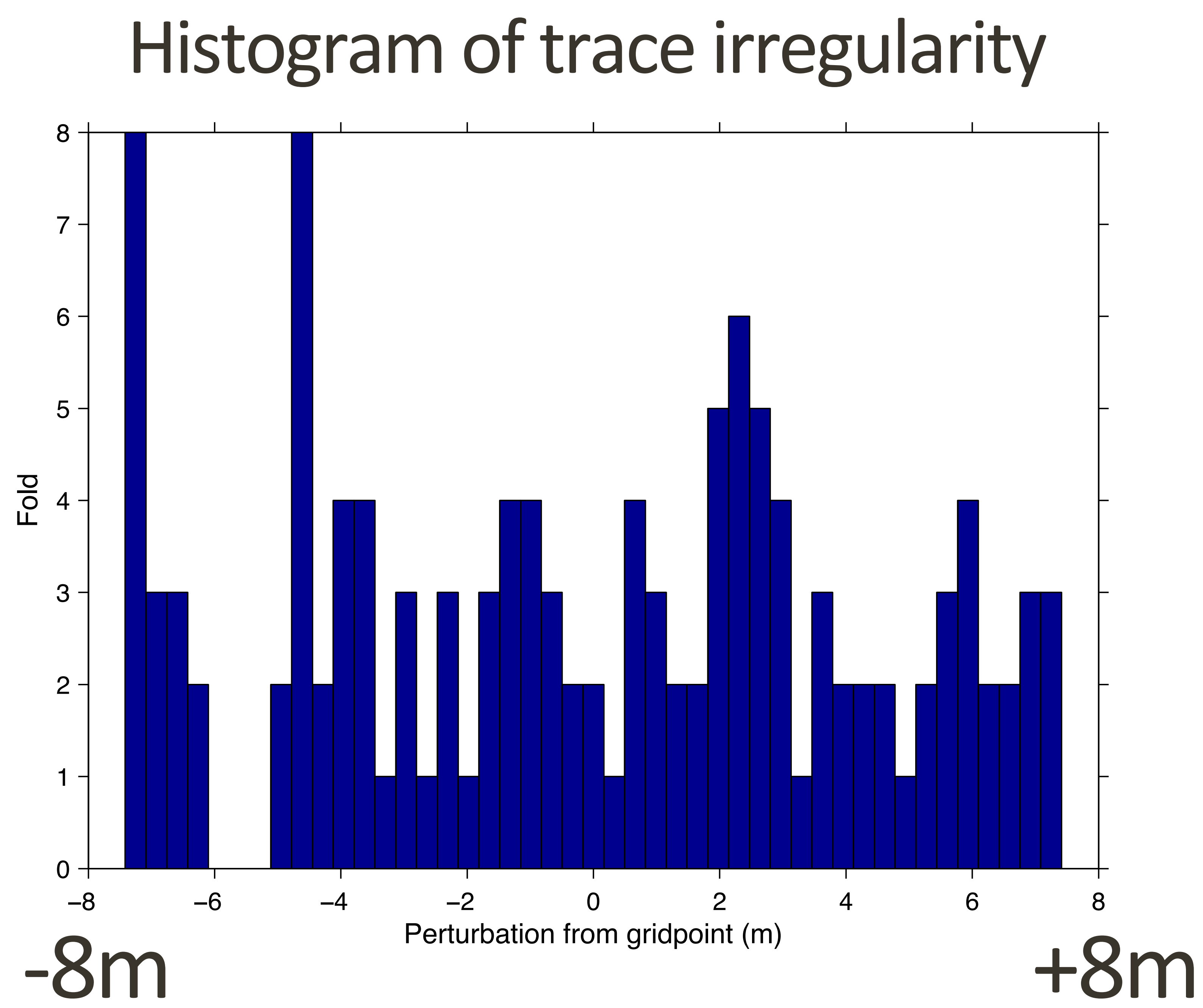
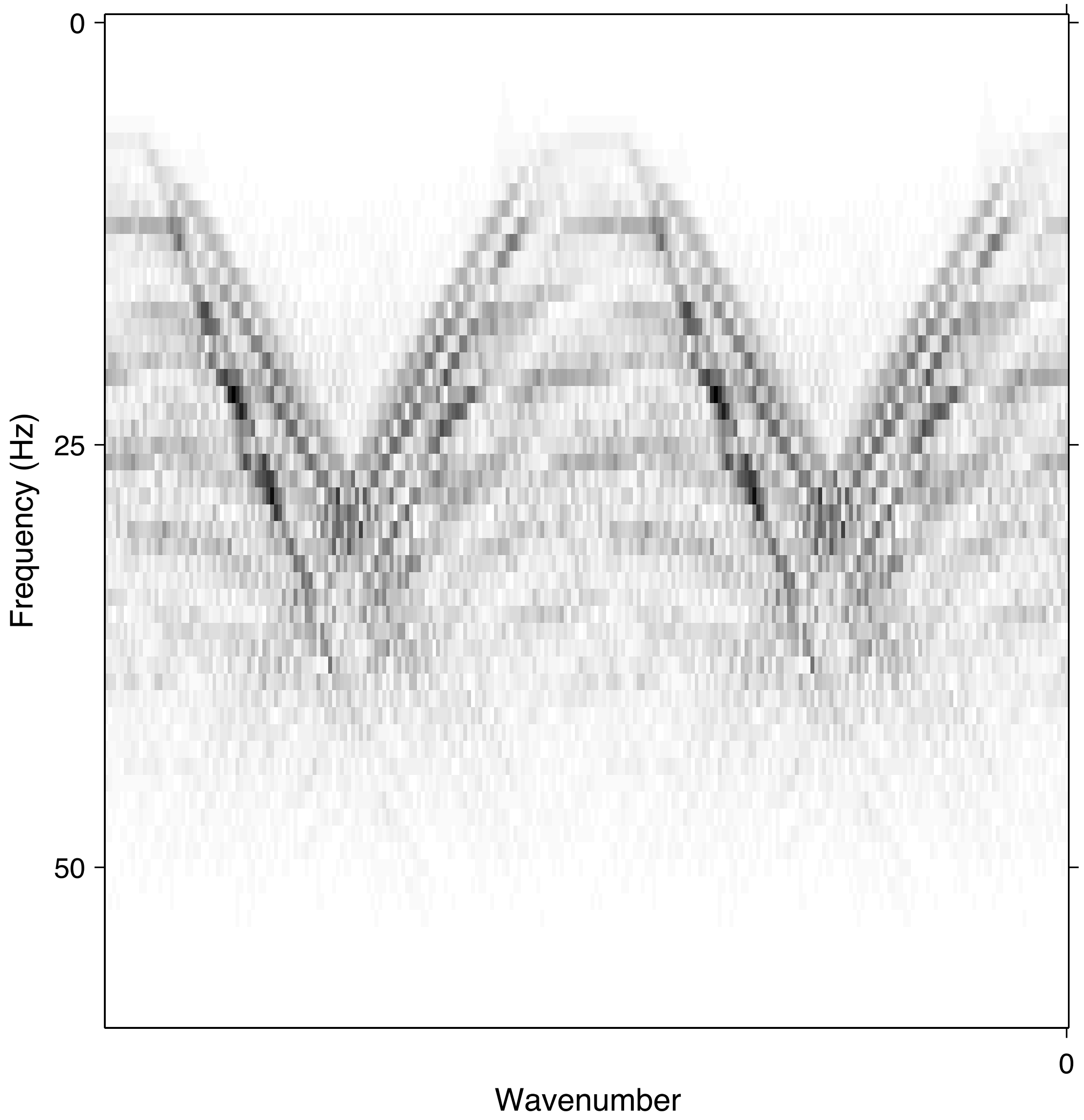


SLIM

Regularization + Interpolation



Decimated:

15m grid -> 30m grid

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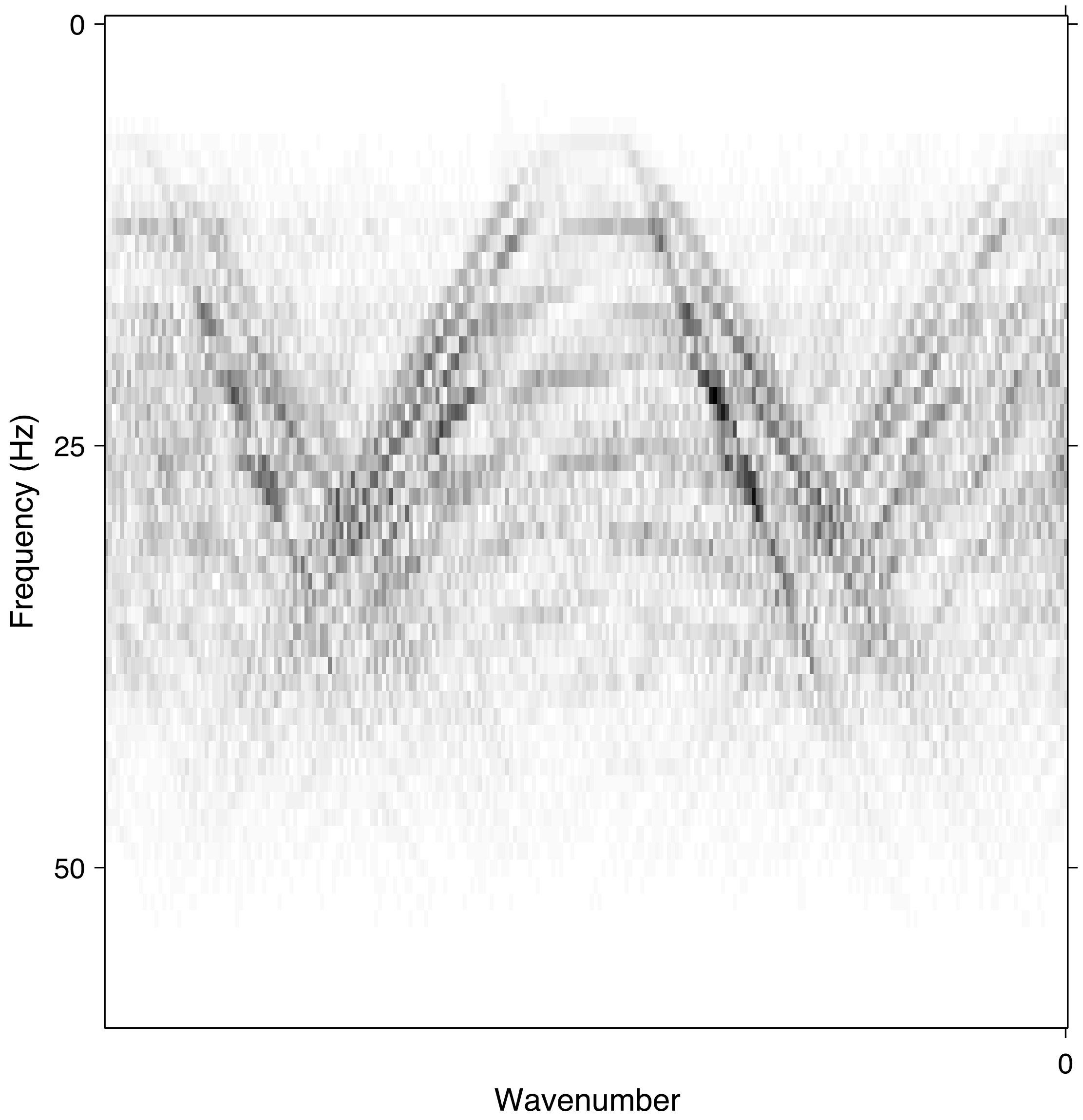
uniformly random trace shift in the range [-8m, +8m] from gridpoint

Regularization + Interpolation



Decimated:

15m grid -> 30m grid

Perturbed:

uniformly random trace shift in the range [-8m, +8m] from gridpoint

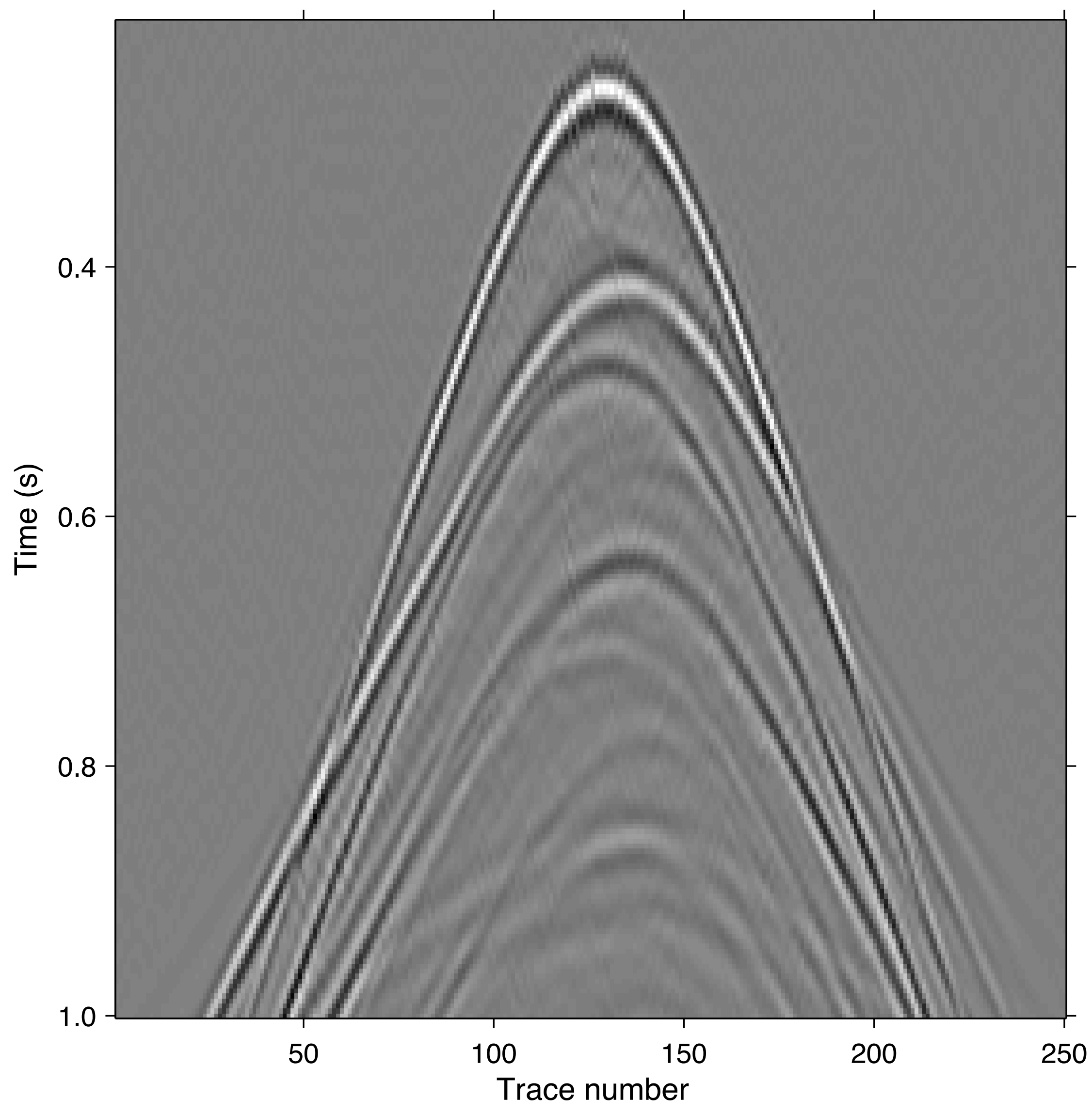
Regularization + Interpolation

- Using non-uniform FFT as measurement operator $\mathbf{A}$
(*non-uniform* physical grid -> *uniformly* spaced FK coefficient)

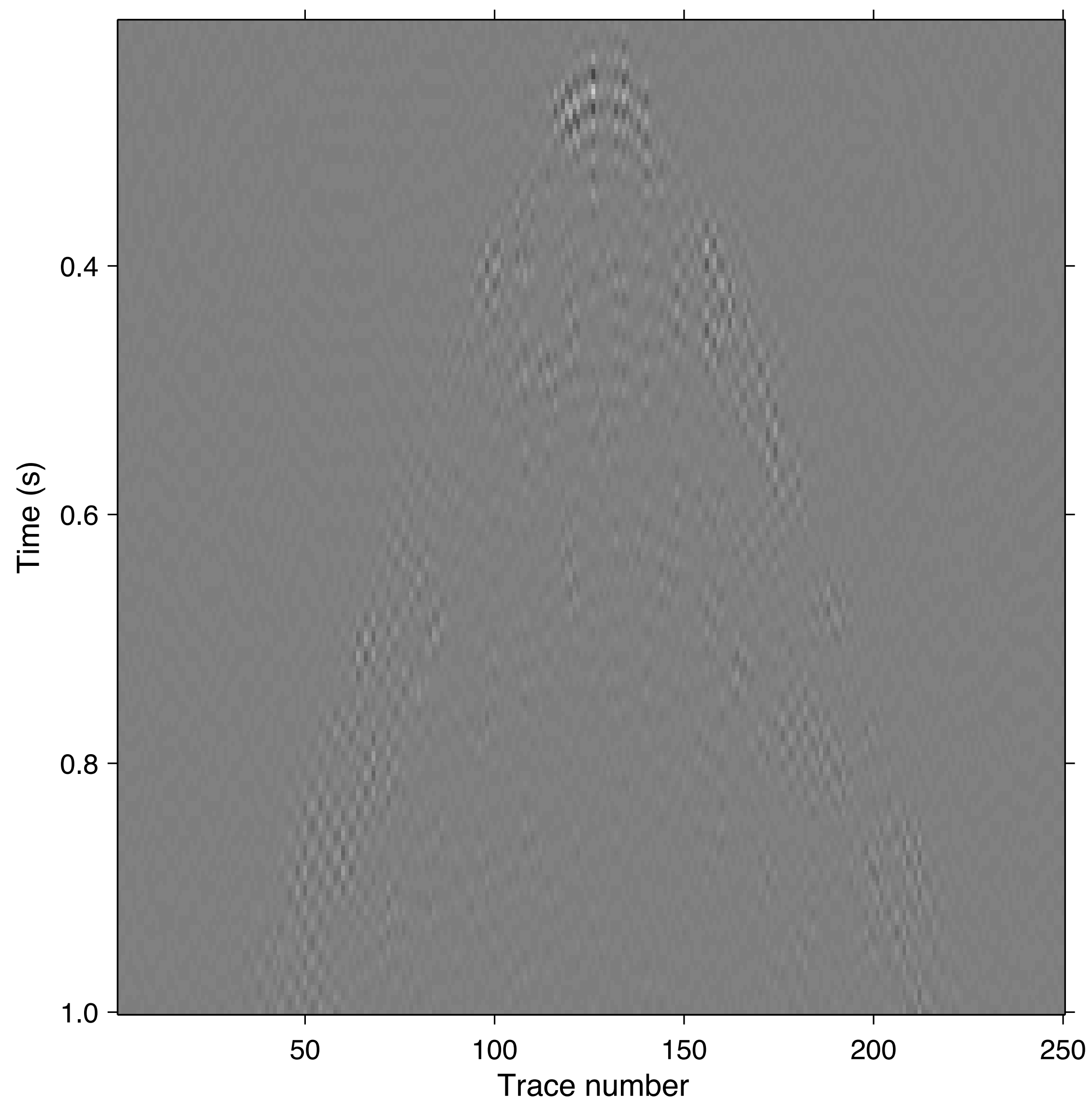
- Curvelet dictionary $\mathbf{D}$ constructed from FK domain instead of TX

Regularization + Interpolation

Synthesis (L1) solution using SPGL1

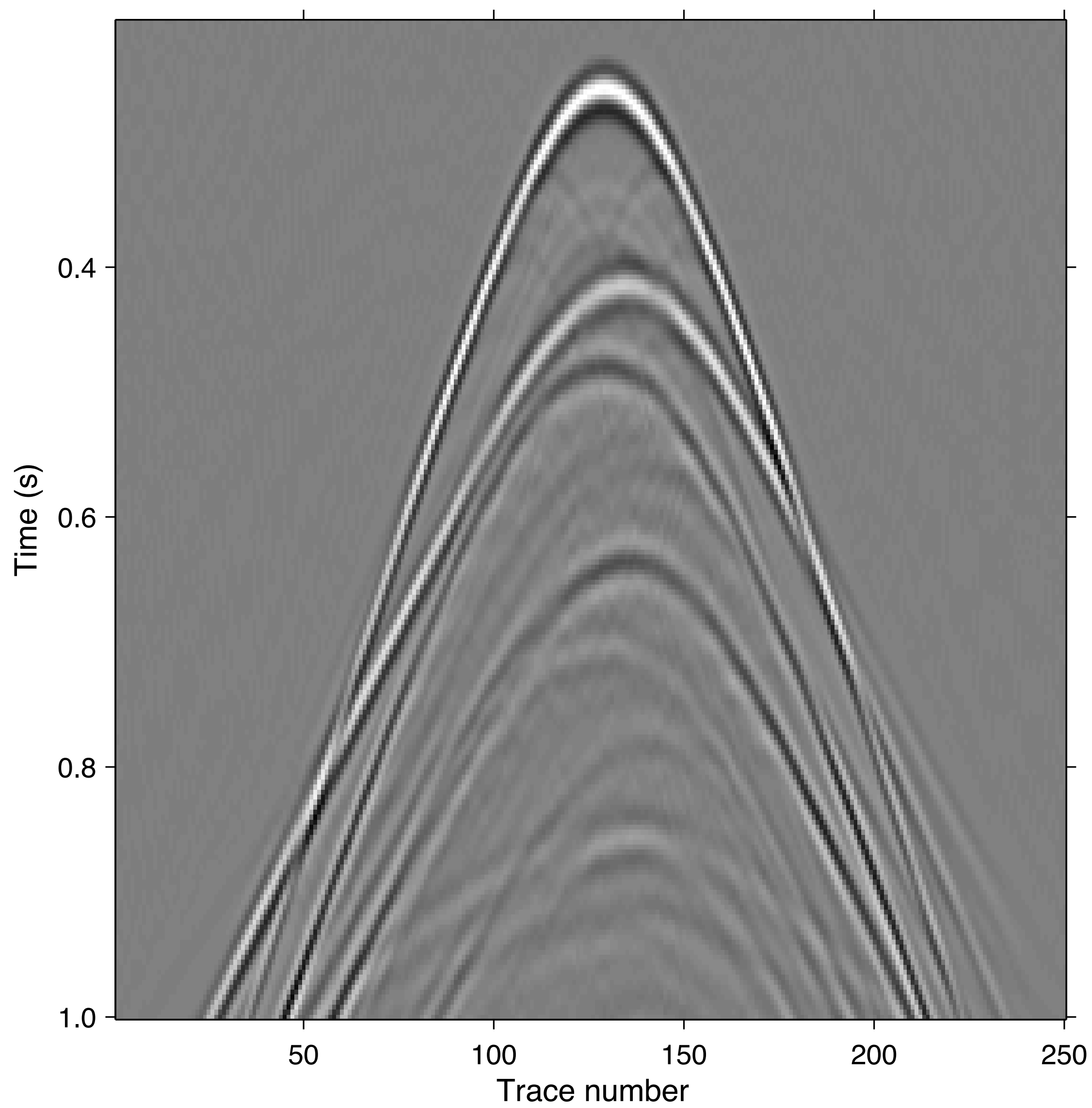


difference from truth

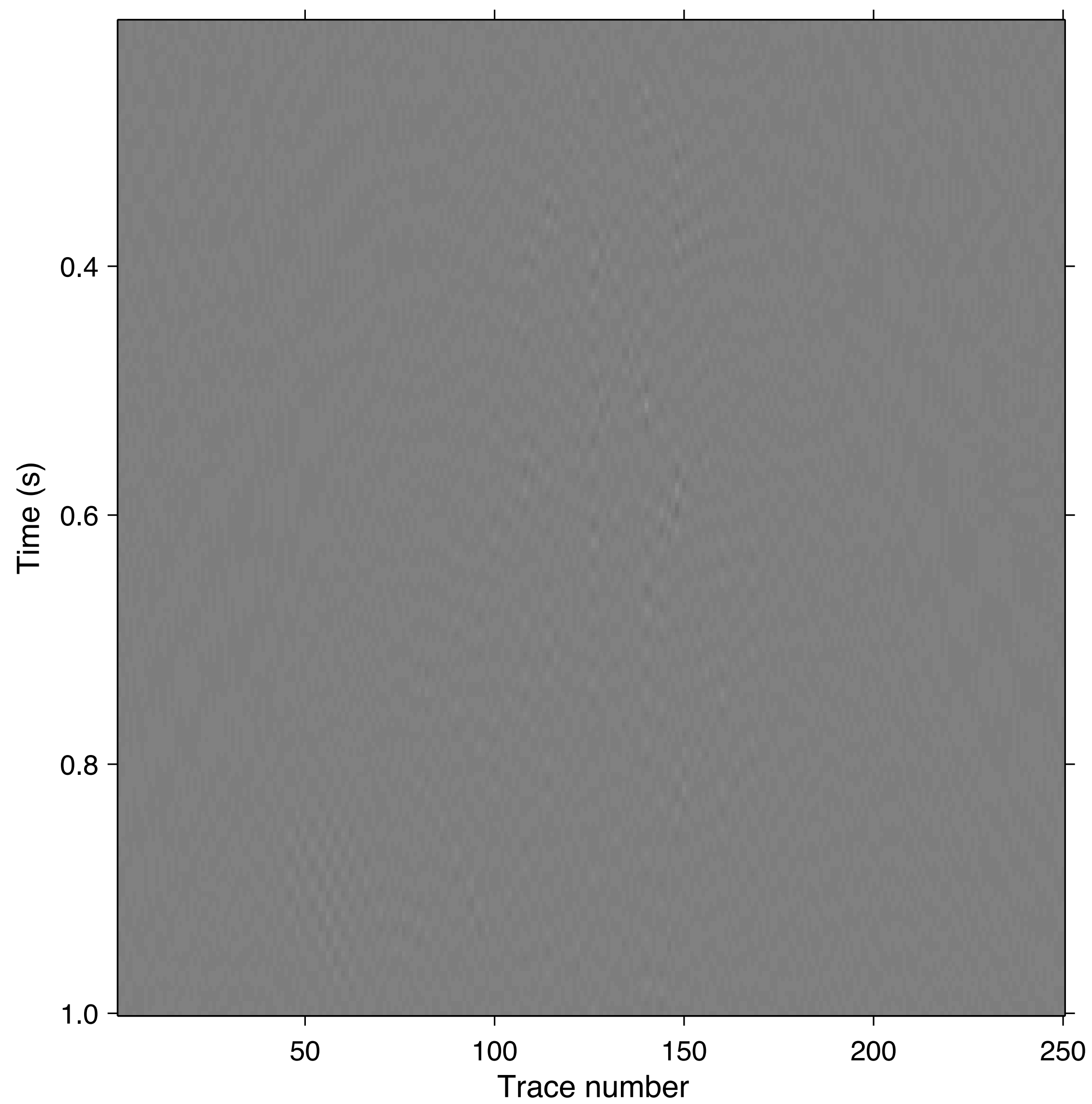


Regularization + Interpolation

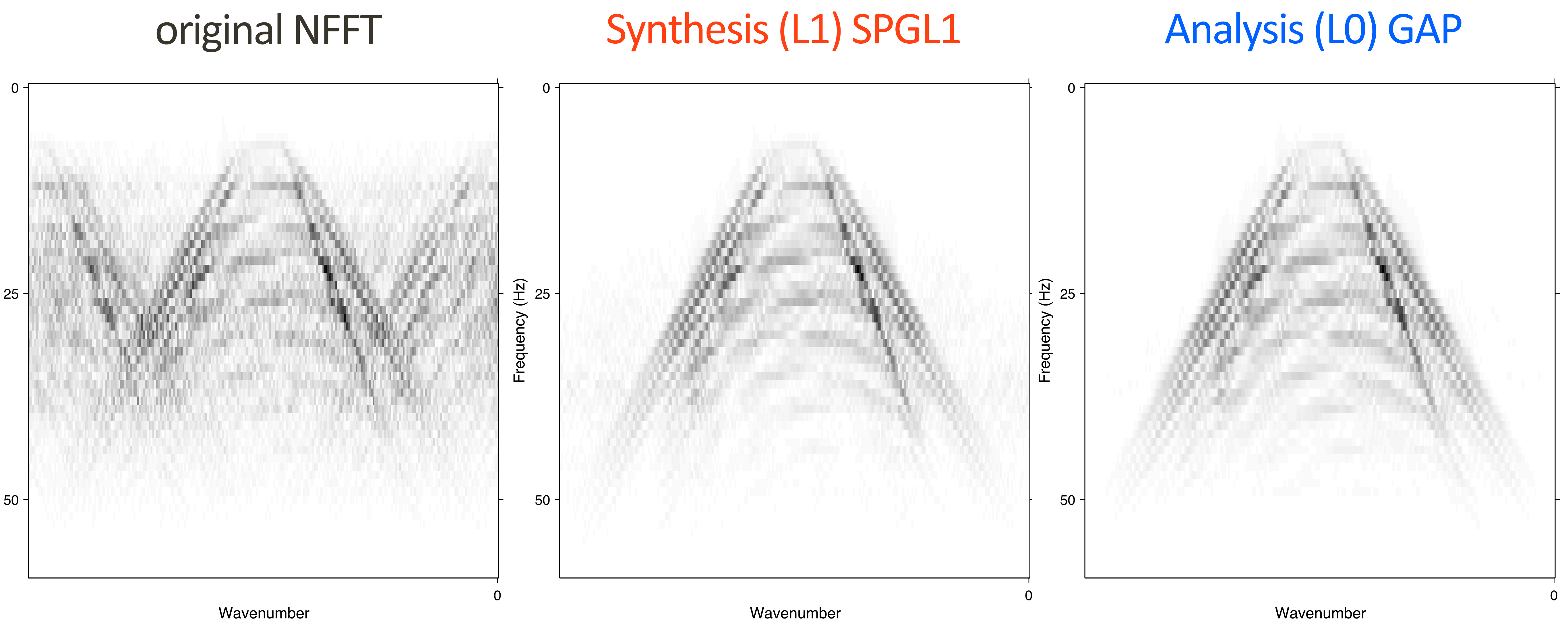
Analysis (L0) "solution" using GAP



difference from truth



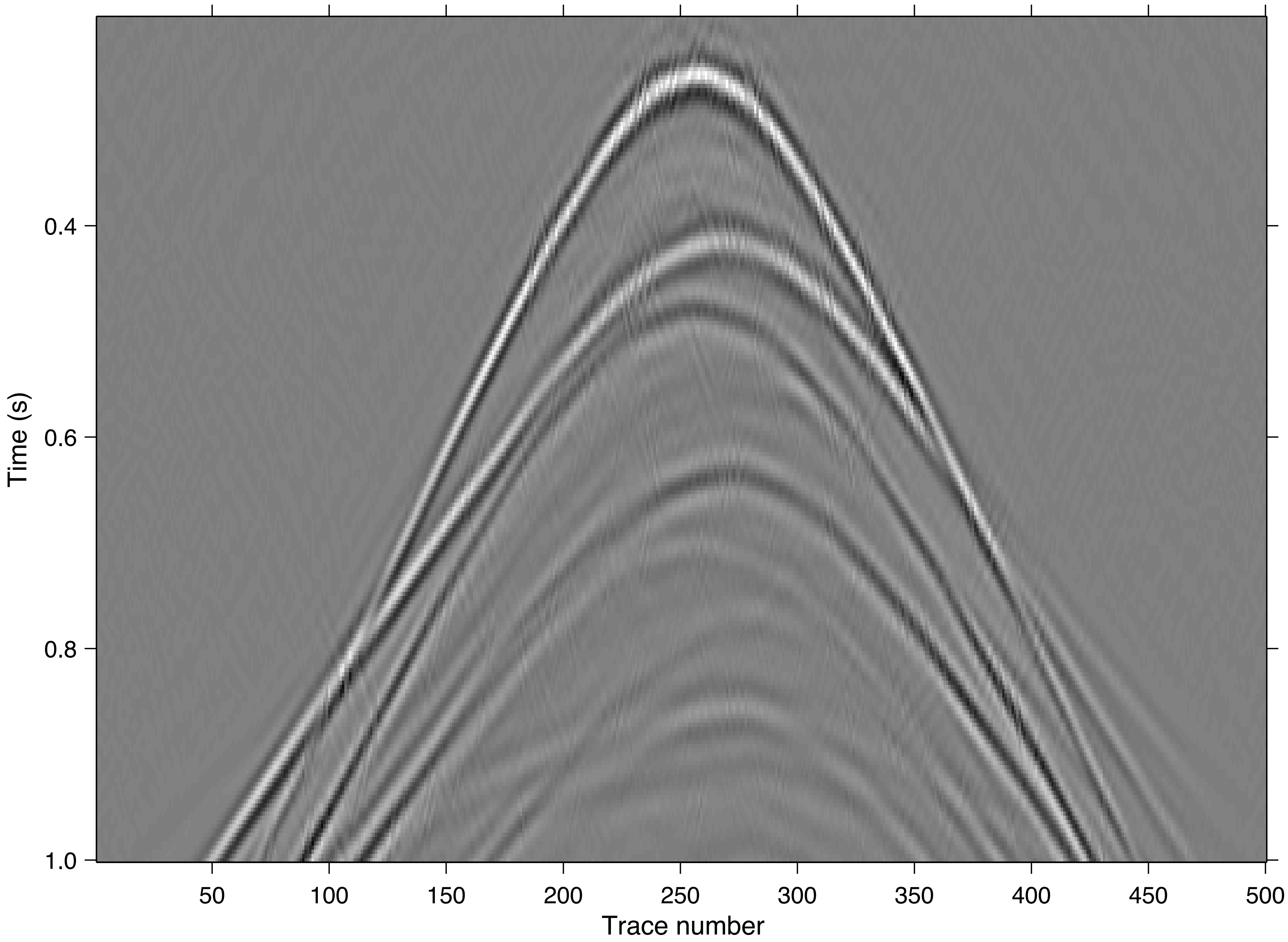
Regularization + Interpolation



Regularization + Interpolation

15m -> 3.75m (4 to 1)

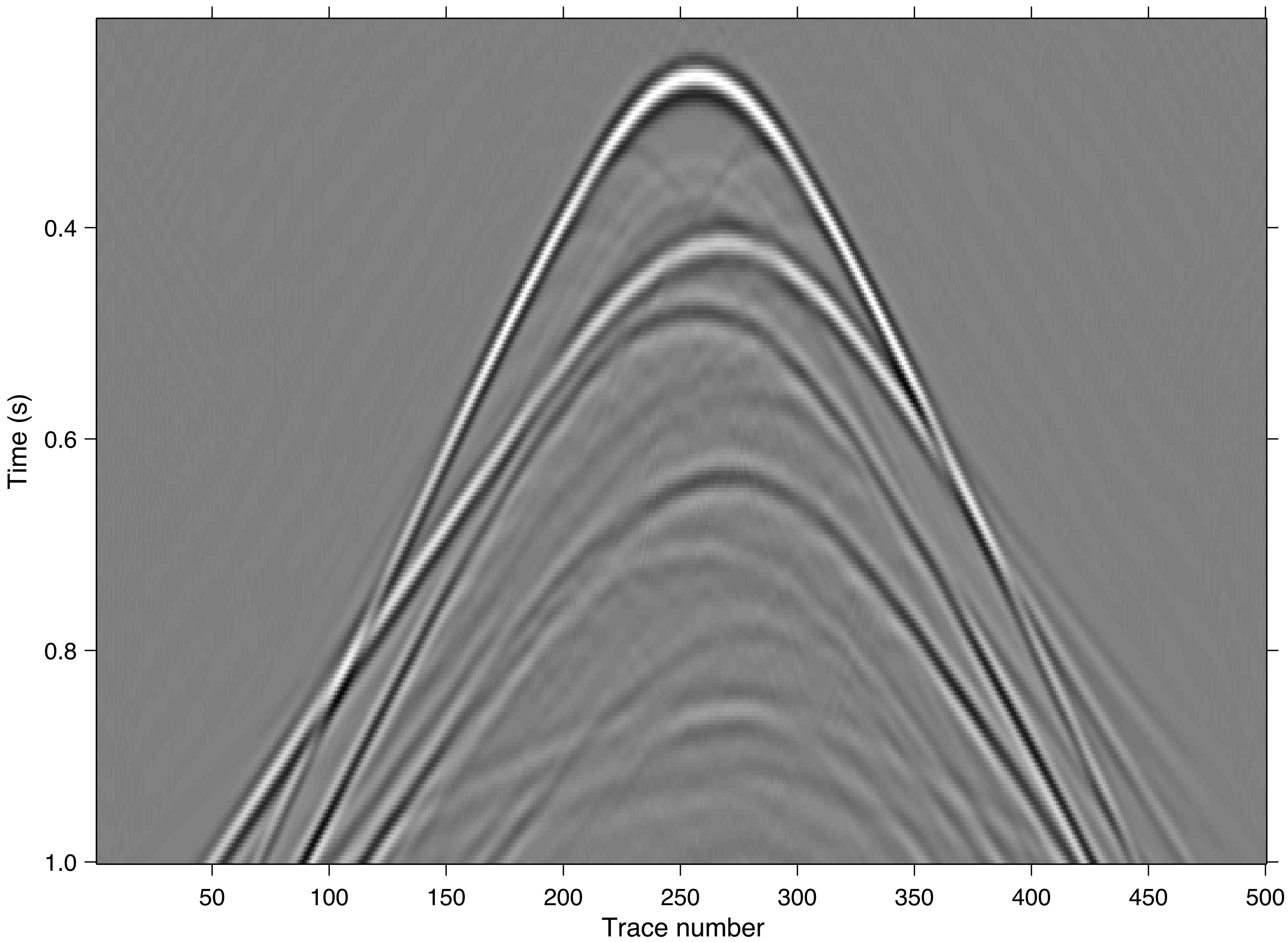
Synthesis (L1) SPGL1



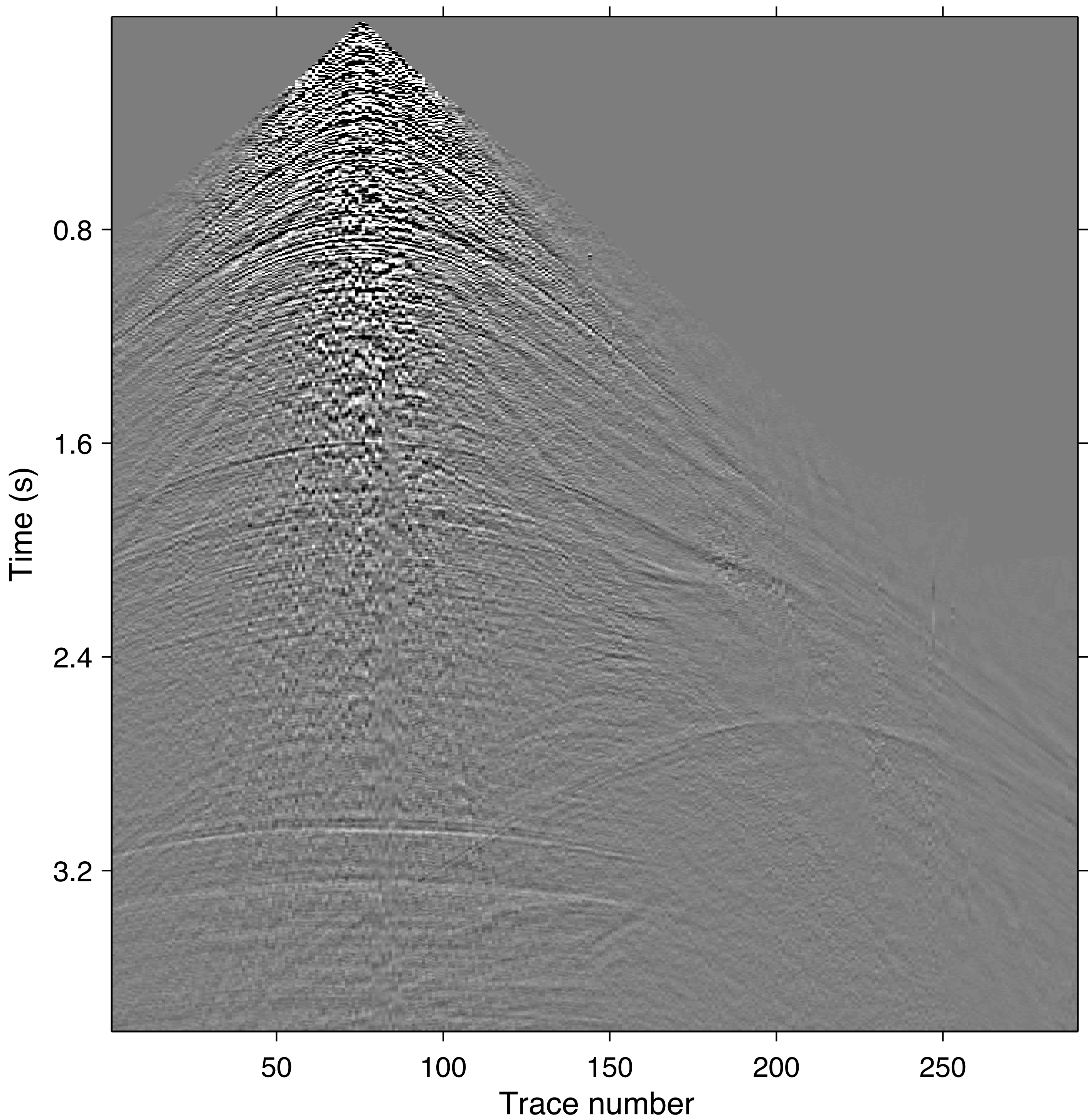
Regularization + Interpolation

15m -> 3.75m (4 to 1)

Analysis (L0) GAP



Machar dataset (courtesy BP)



Original Data

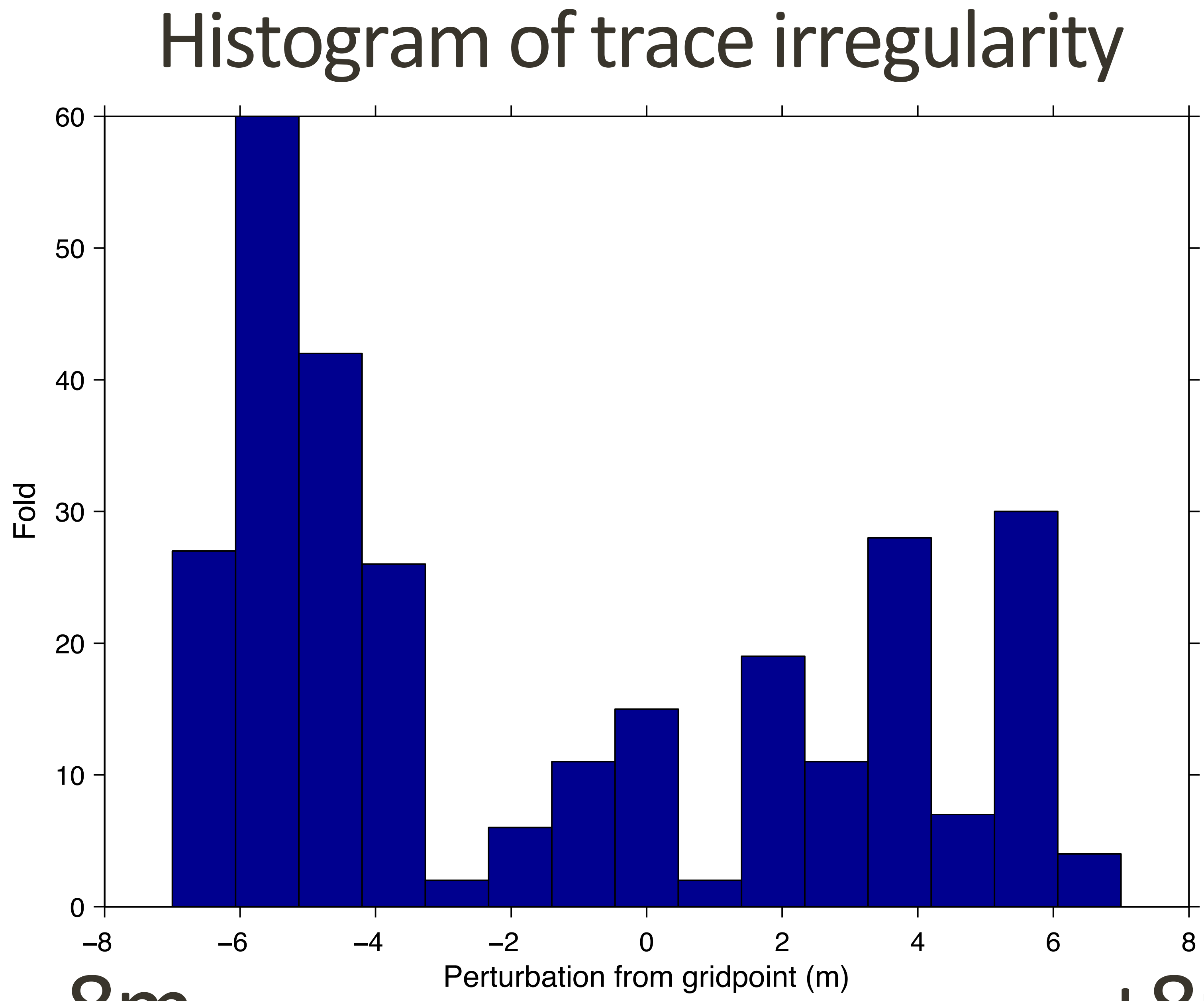
25m receiver grid

OBC

Summed P+Vz

Post-processing

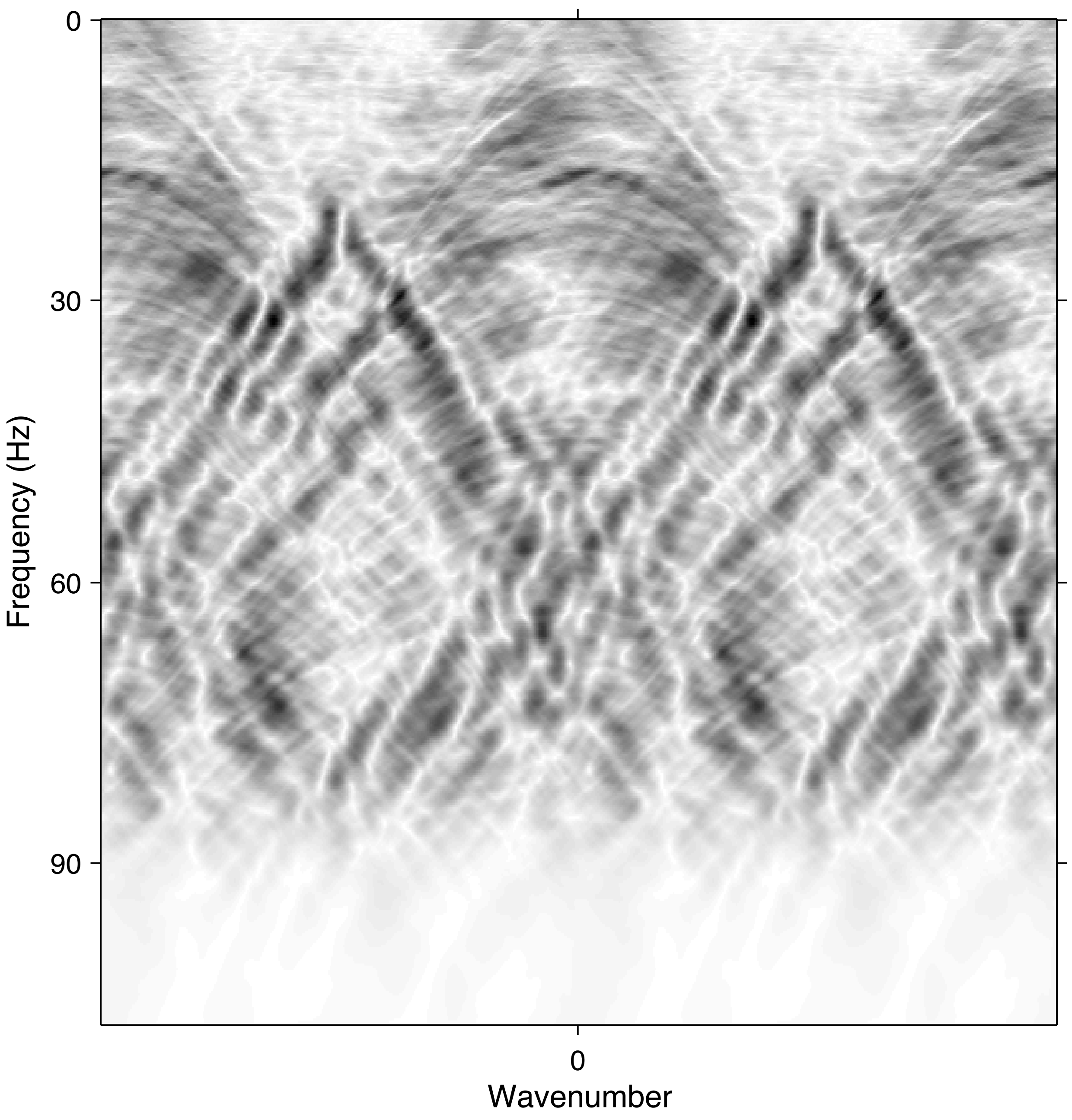
Machar dataset (courtesy BP)



-8m +8m

SLIM

Machar dataset (courtesy BP)



Original Data

25m receiver grid

OBC

Summed P+Vz

Post-processing

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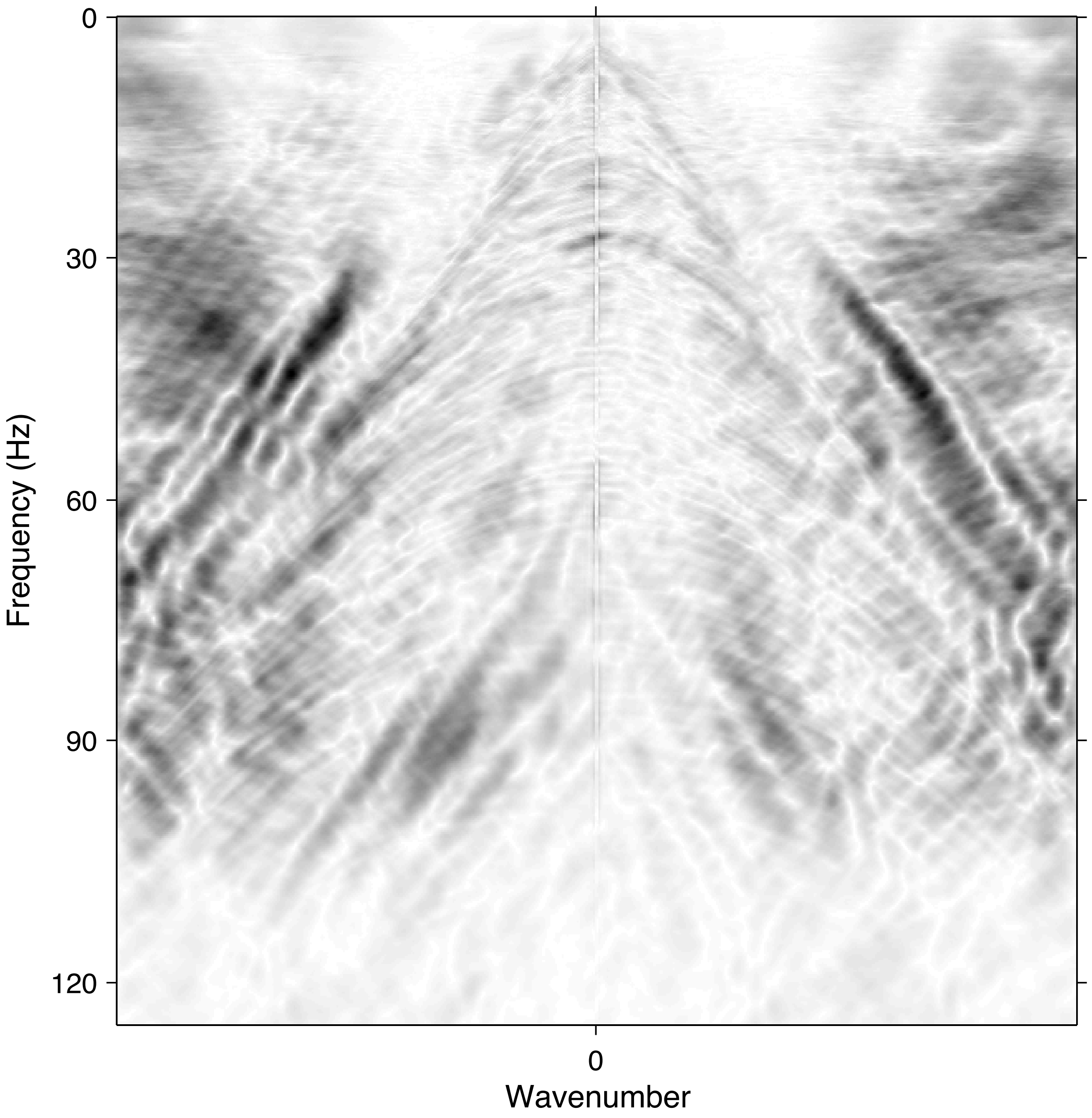


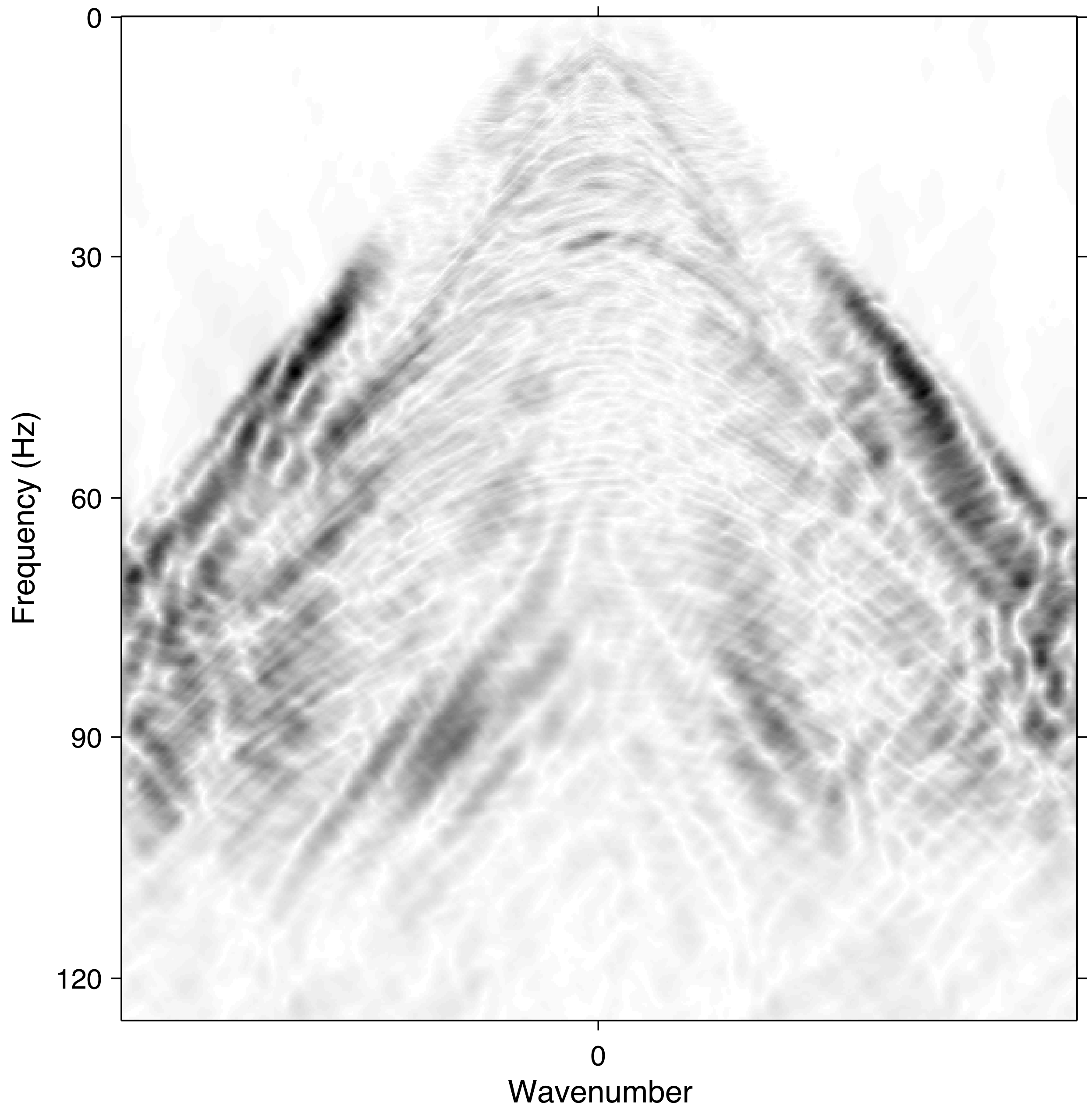
Regularized + Interpolated

12.5m nominal grid

nFFT + 2D Curvelet

Sparse regularization

Machar dataset (courtesy BP)



Regularized + Interpolated

12.5m nominal grid

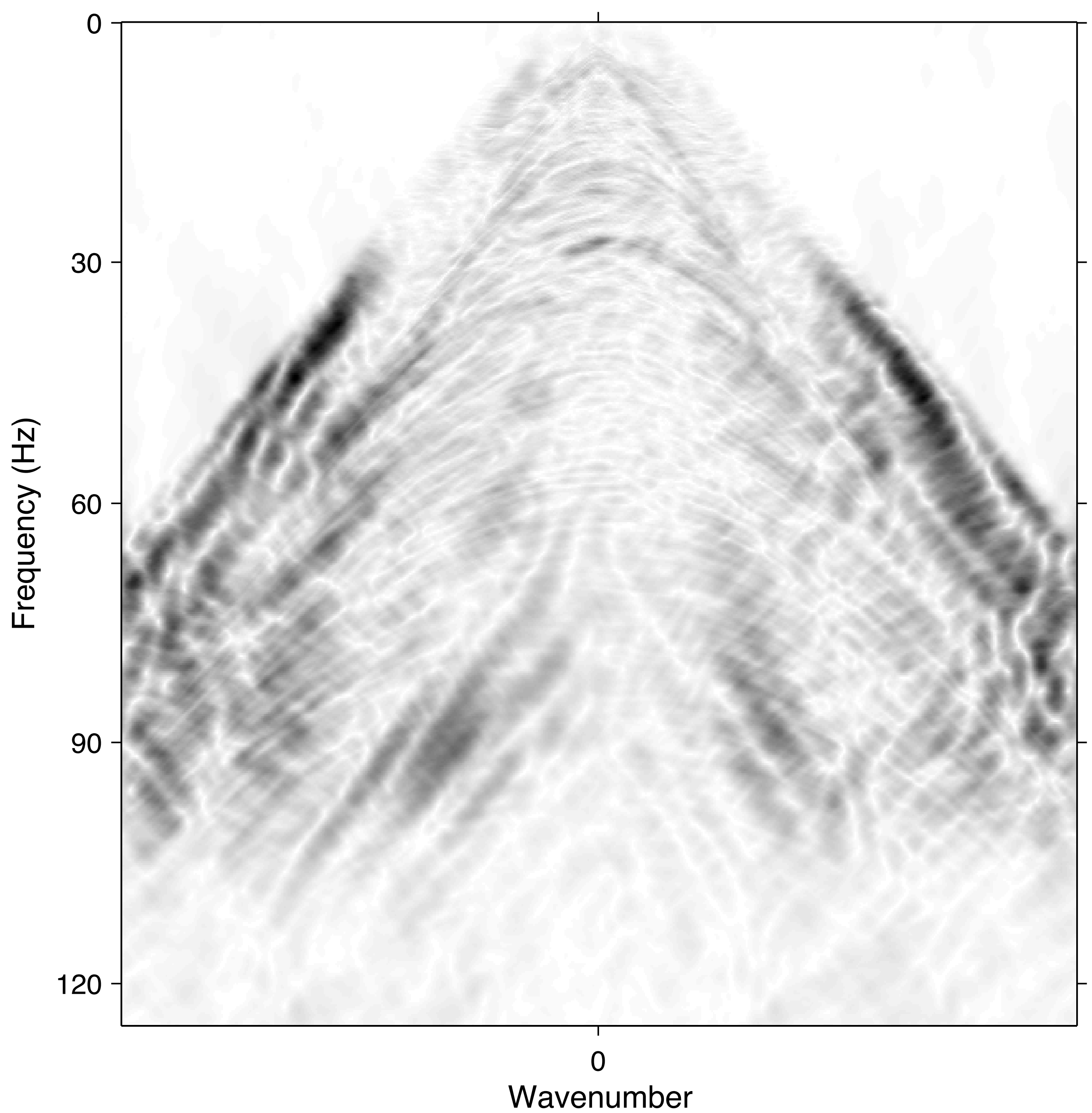
nFFT + 2D Curvelet

Sparse regularization

Velocity mute

Low-freq preserved

Machar dataset (courtesy BP)



Regularized + Interpolated

12.5m nominal grid

nFFT + 2D Curvelet

Sparse regularization

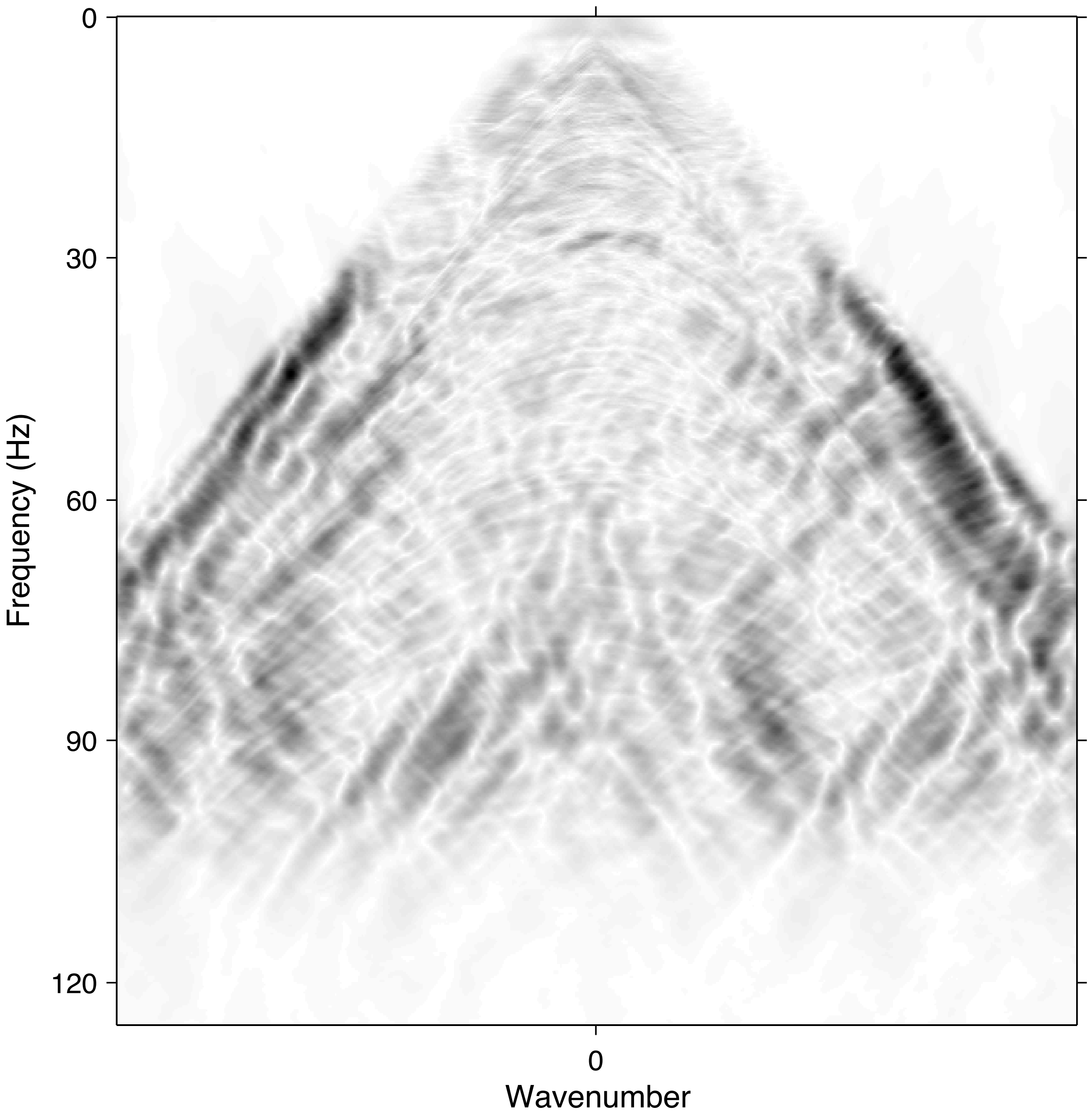
Velocity mute

Low-freq preserved

Synthesis (L1)

SPGL1

Machar dataset (courtesy BP)



Regularized + Interpolated

12.5m nominal grid

nFFT + 2D Curvelet

Sparse regularization

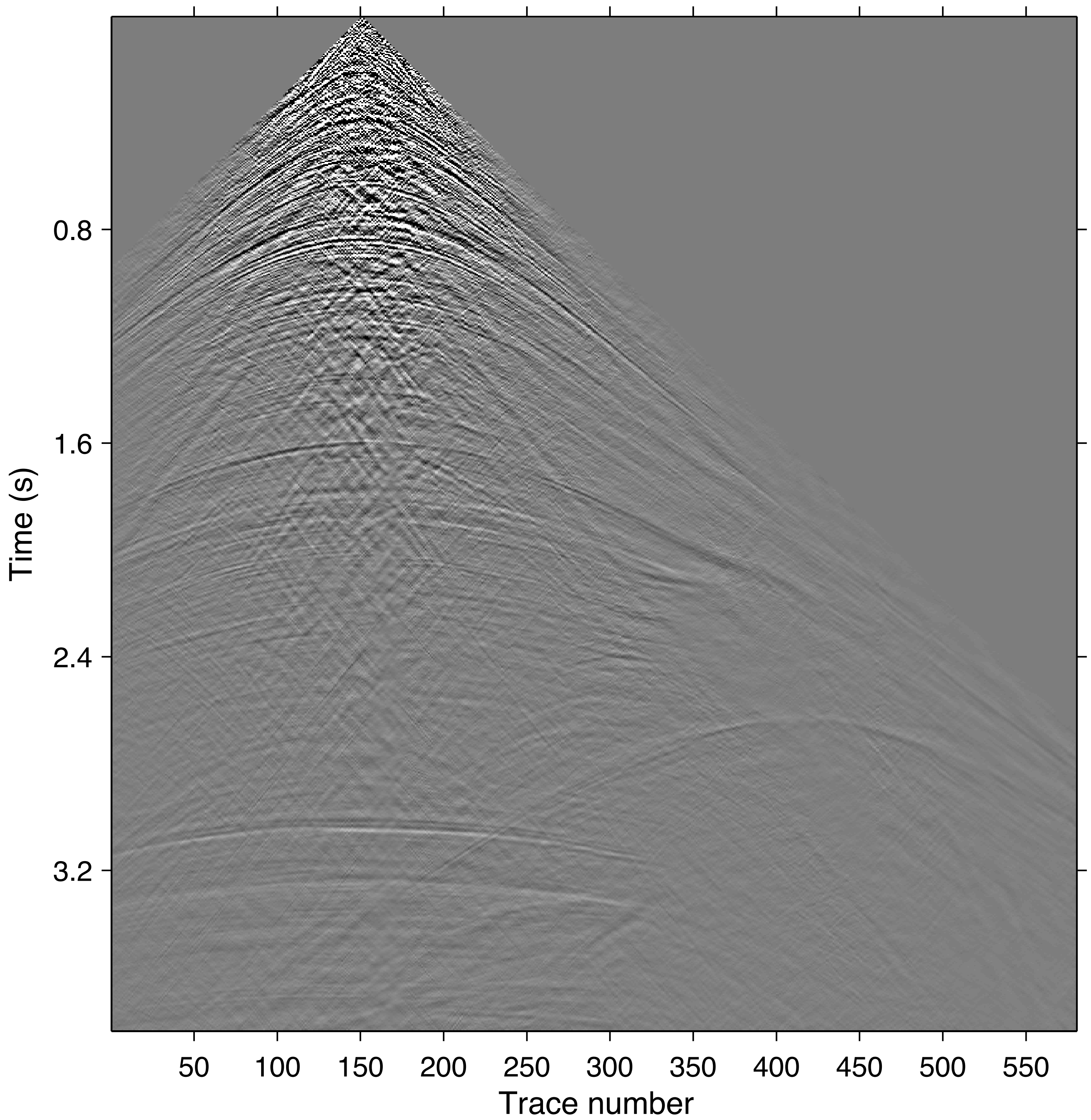
Velocity mute

Low-freq preserved

Analysis (L0)

GAP

Machar dataset (courtesy BP)



Regularized + Interpolated

12.5m nominal grid

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Sparse regularization

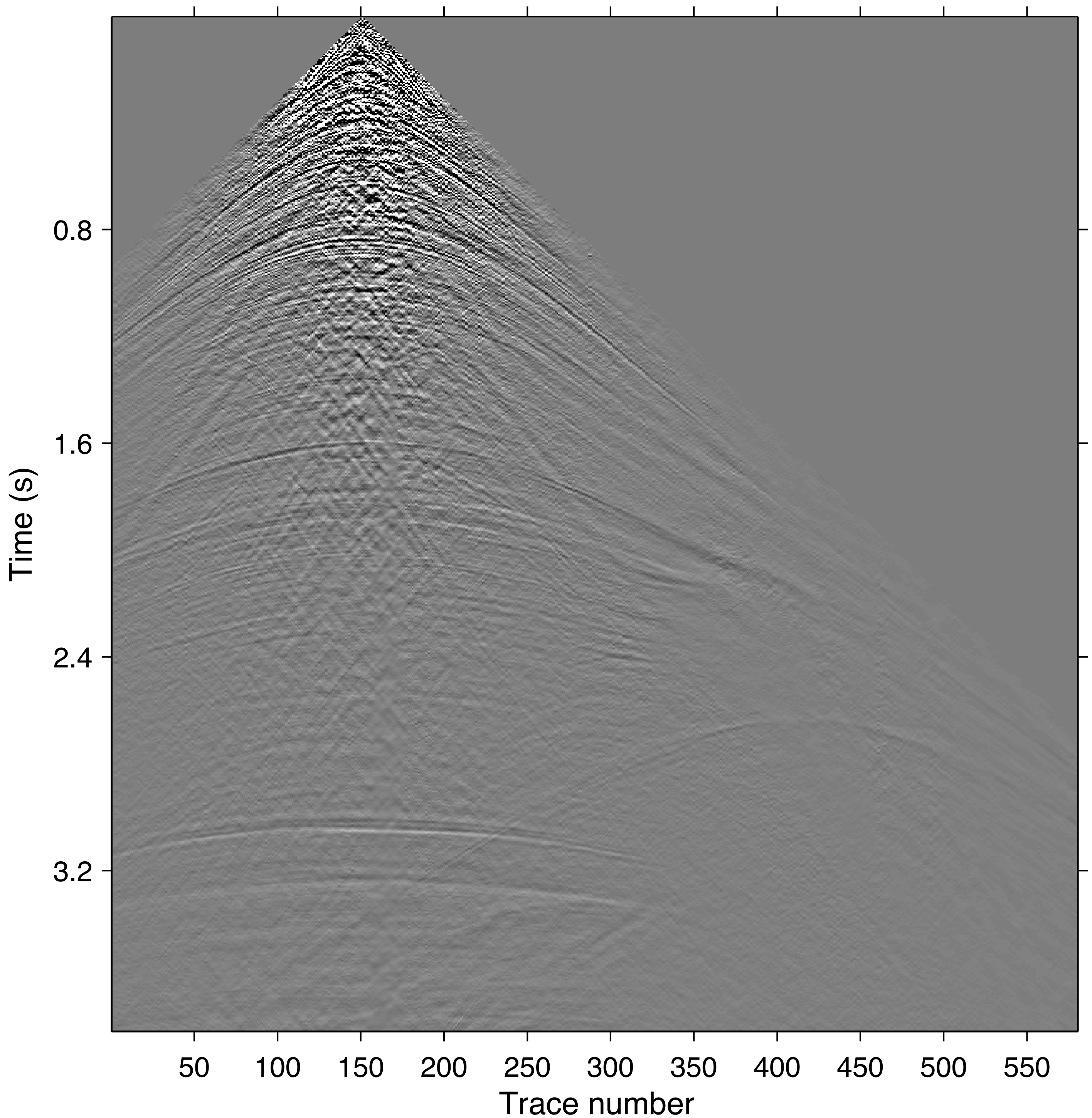
Velocity mute

Low-freq preserved

Synthesis (L1)

SPGL1

Machar dataset (courtesy BP)



Regularized + Interpolated

12.5m nominal grid

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Sparse regularization

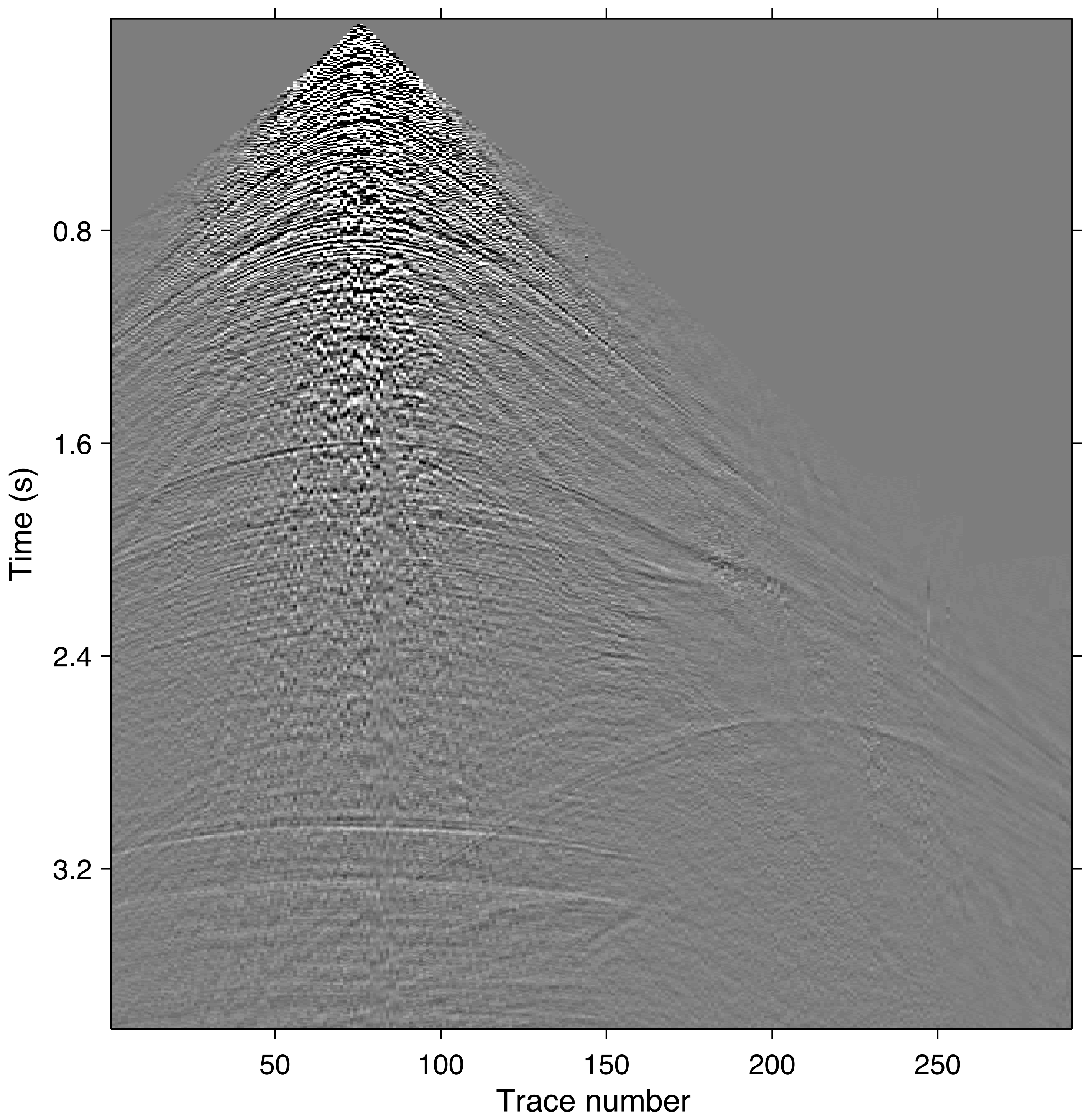
Velocity mute

Low-freq preserved

Analysis (L0)

GAP

Machar dataset (courtesy BP)



Original Data

25m receiver grid

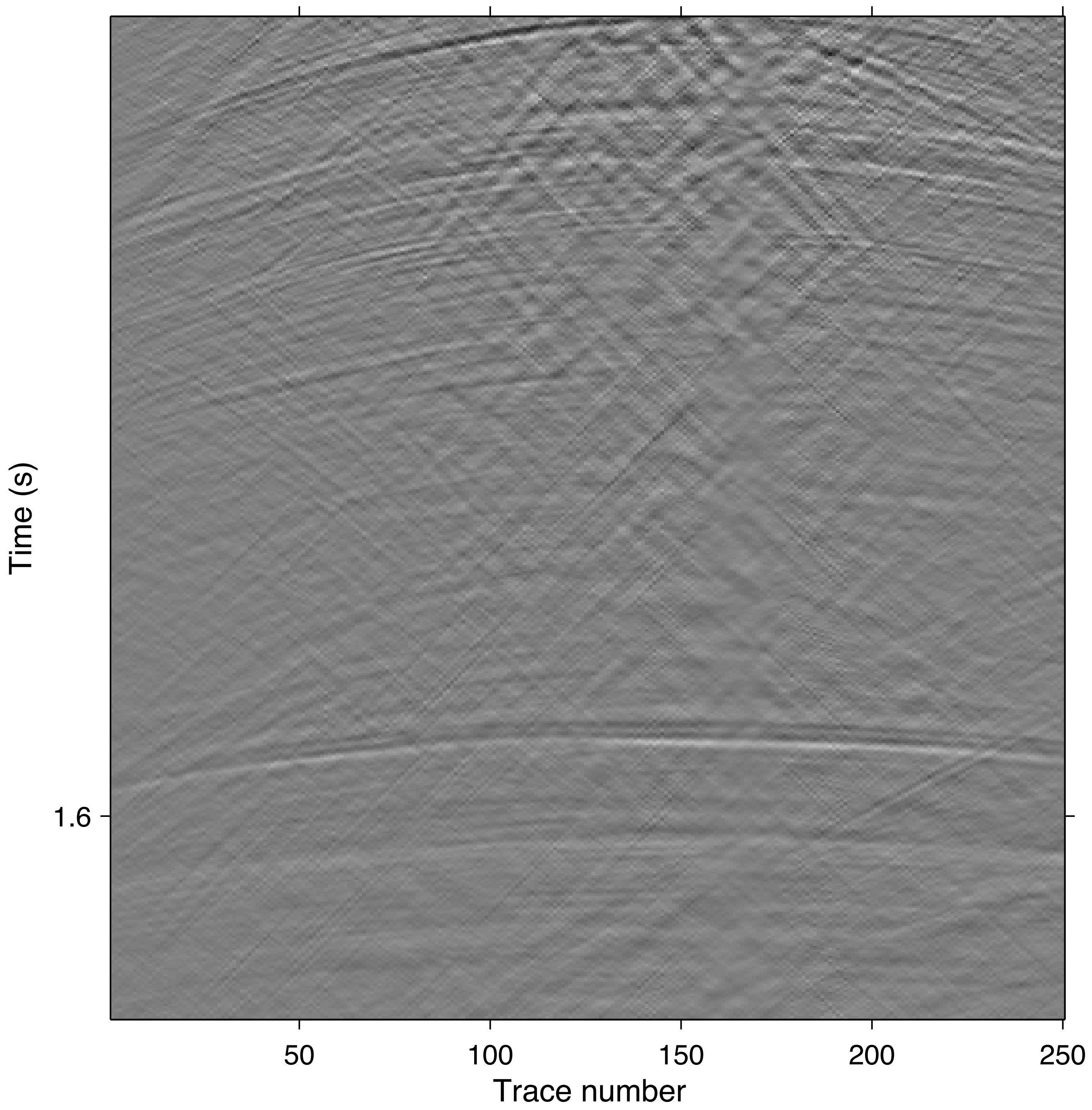
OBC

Summed P+Vz

Post-processing

SLIM

Machar dataset (courtesy BP)



Regularized + Interpolated

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nFFT + 2D Curvelet

Sparse regularization

Velocity mute

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Synthesis (L1)

SPGL1

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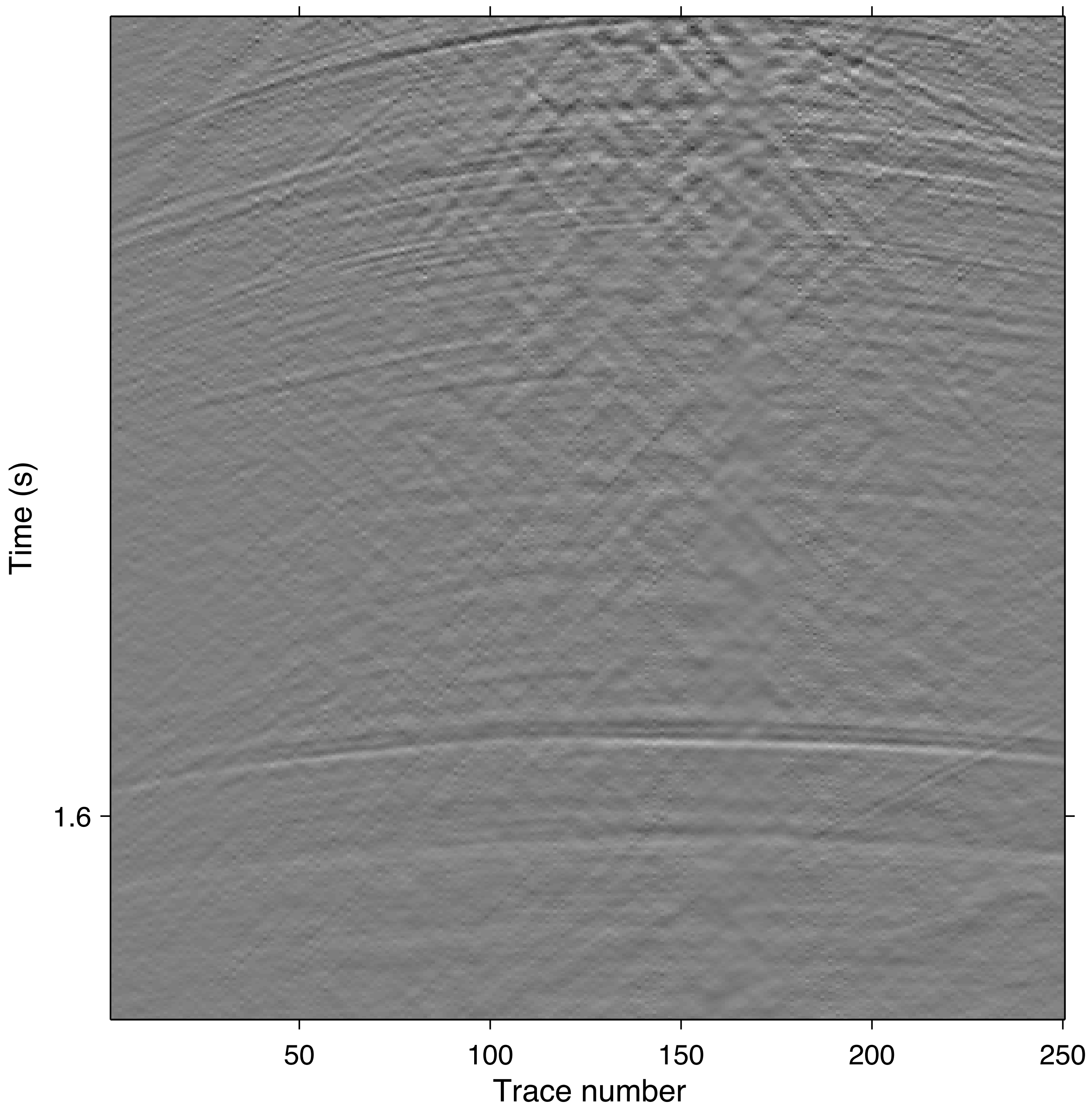

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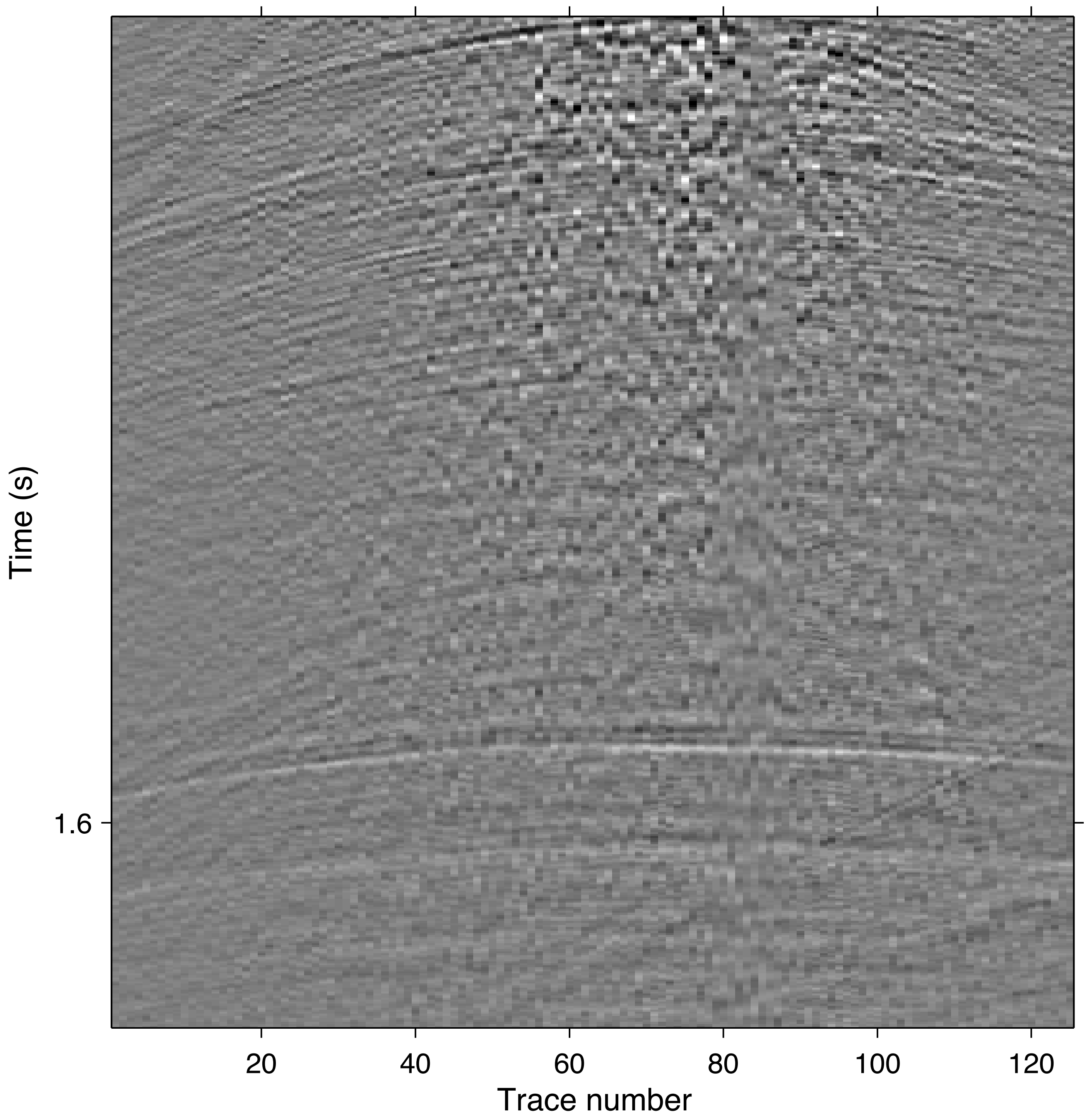


Original Data

25m receiver grid

OBC

Summed P+Vz

Post-processing

SLIM

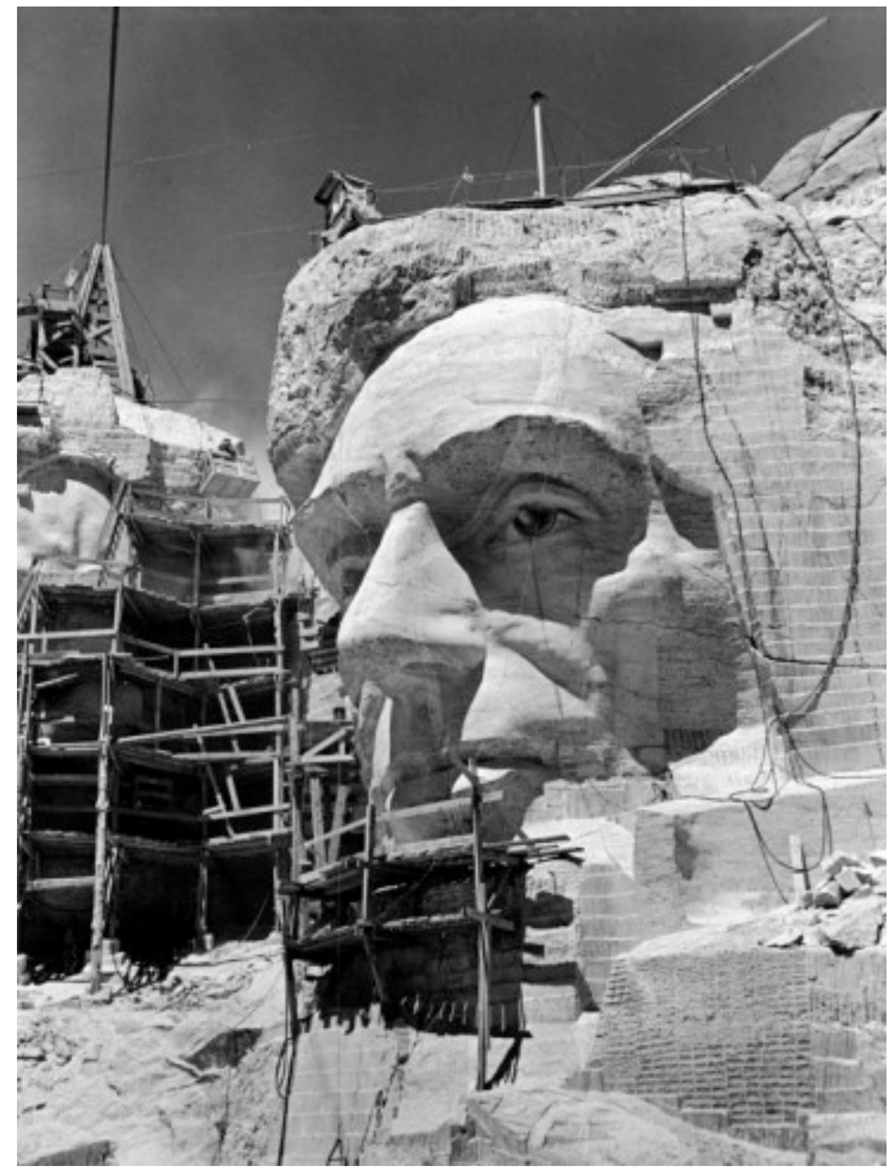
Summary

- choice of “sparsifying” algorithm is important
- Synthesis problem is not analysis problem
- the *zeroes* of a signal under transforms can be important in regularization
- cosparsity > sparsity for curvelet-domain seismic regularization/interpolation

SLIM

Constructing signals with…

Sparsity



Cosparsity

SLIM

Acknowledgements

- All cosparsity theoretical results, along with the GAP algorithm from S.Nam, M. Davies, M. Elad, and R. Gribonval

SINBAD

NSERC CRSNG

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