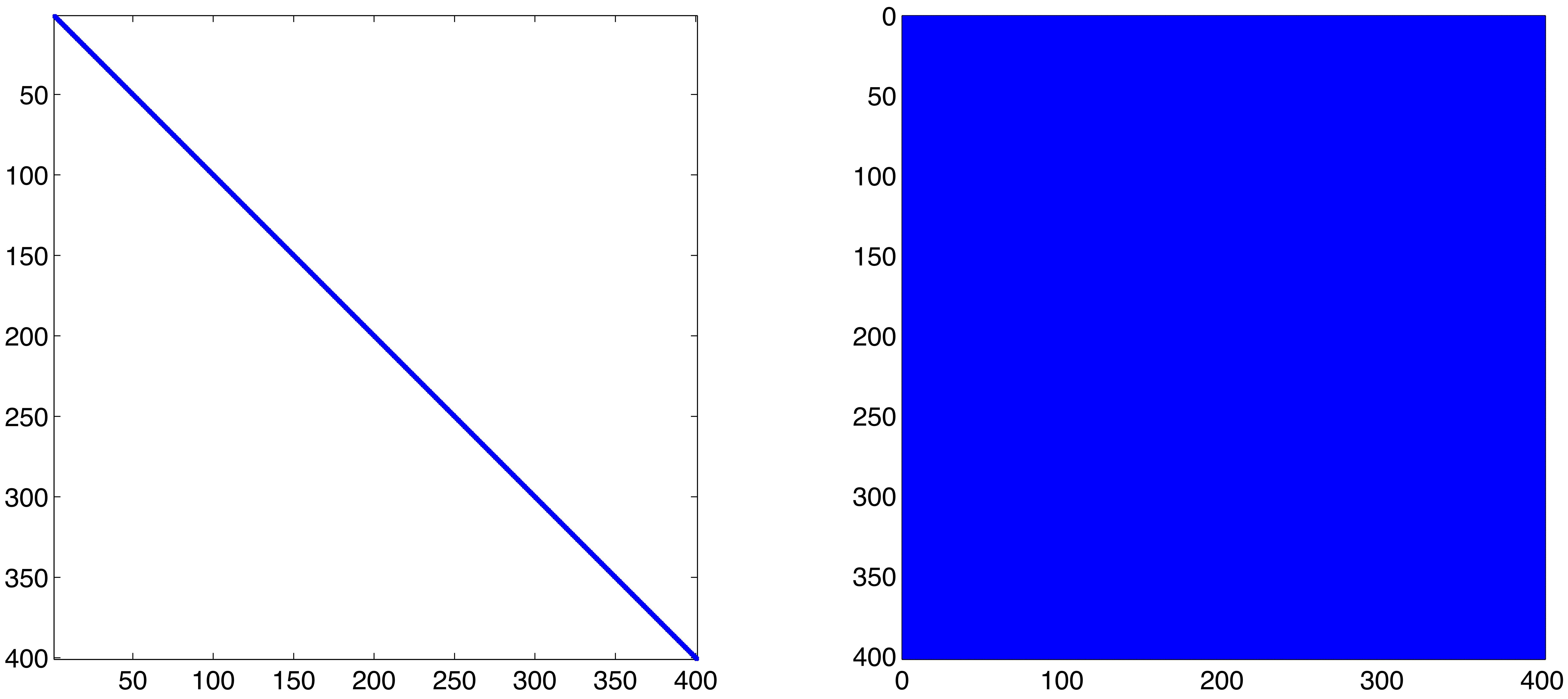
Imaging with Hierarchical Semi Separable Matrices

Polina Zheglova

Seismic Laboratory for Imaging and Modeling
University of British Columbia
December 1, 2013

Motivation



(a) 0.75% of non-zeros

(b) 100% of non-zeros

Figure 1: Structure of Helmholtz matrix and its inverse: 1-d case, simple finite differences

Motivation

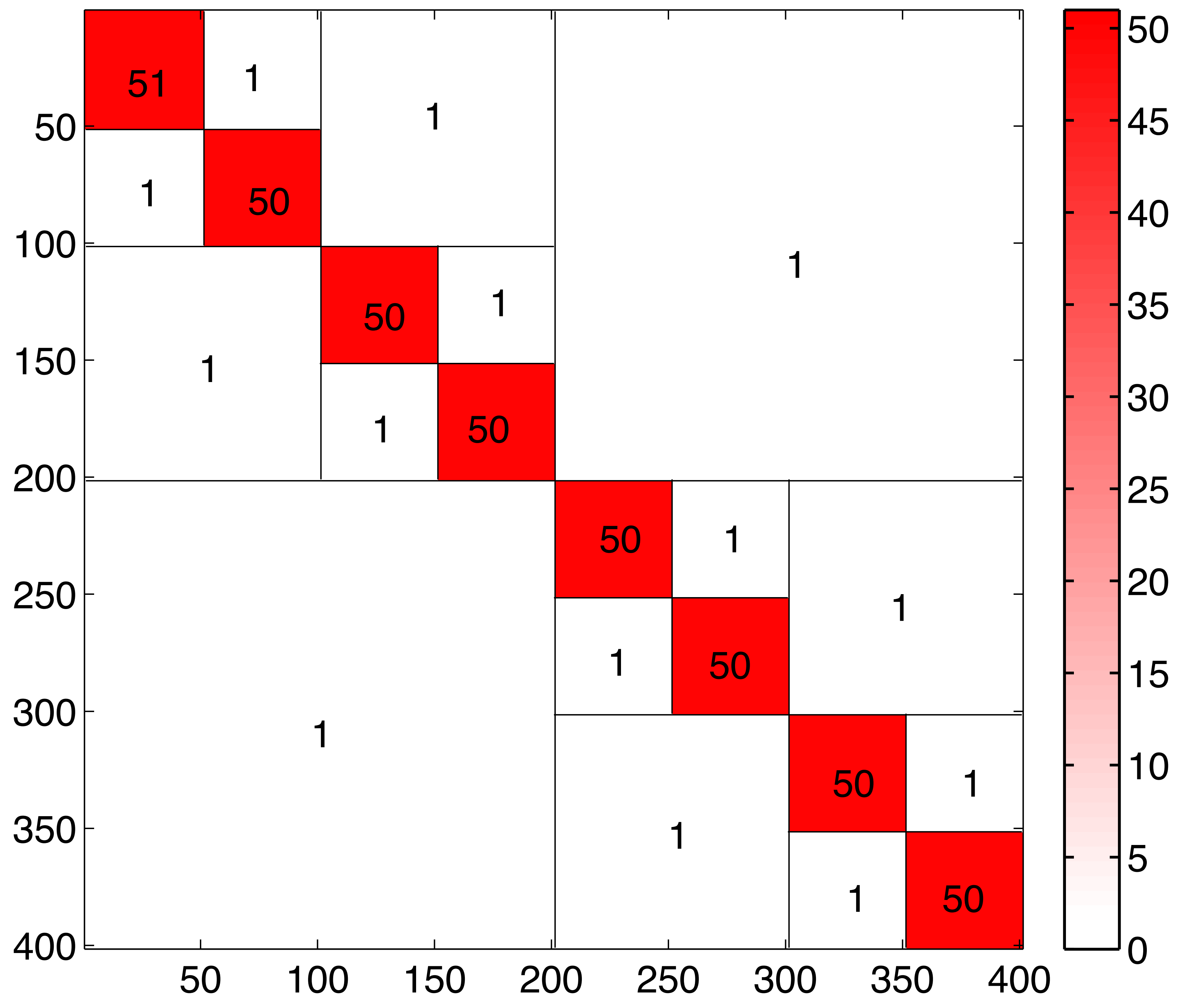


Figure 2: Rank representation of the inverse

- Diagonal blocks have full rank
- Off-diagonal blocks have rank 1

Motivation

Hierarchically semiseparable (HSS) representation of matrices allows us to

- Store dense matrices with less memory
- Do matrix operations - multiplication, addition, inversion, LU, etc. - fast (e.g. $O(n)$ vs $O(n^3)$ flops)
- Results are also HSS matrices
- HSS representation is approximate, but can be made arbitrarily accurate by increasing the rank of the block approximations

Motivation

Matrices that have HSS structure

- Have large blocks with low numerical rank
- Often arise in solutions of PDEs, e.g. integral operators:

$$u(x) = \int K(x, y)f(y)dy,$$

  where $K(x, y)$ decays fast away from $x = y$ or is smooth
- Discretized Helmholtz operator (and functions of thereof) have HSS structure. This has been proven for some functions (e.g. Beylkin et al., 1999 - sign function)
- Seismic data being the Green's function can also be represented with HSS (Kumar et al., 2013)

Motivation

Applications in geophysics:

- Compute functions of matrices efficienty
- Fast imaging algorithms:
  - spectral projection via matrix sign function $\Longrightarrow$ leads an efficient two way wave equation migration algorithm
  - square root of a matrix $\Longrightarrow$ one-way wave equation migration with up to 90° dip (current research)
- Missing data interpolation with matrix completion (Kumar et al., 2013)
- Possible application in FWI, .e.g. up-down field separation (future research)

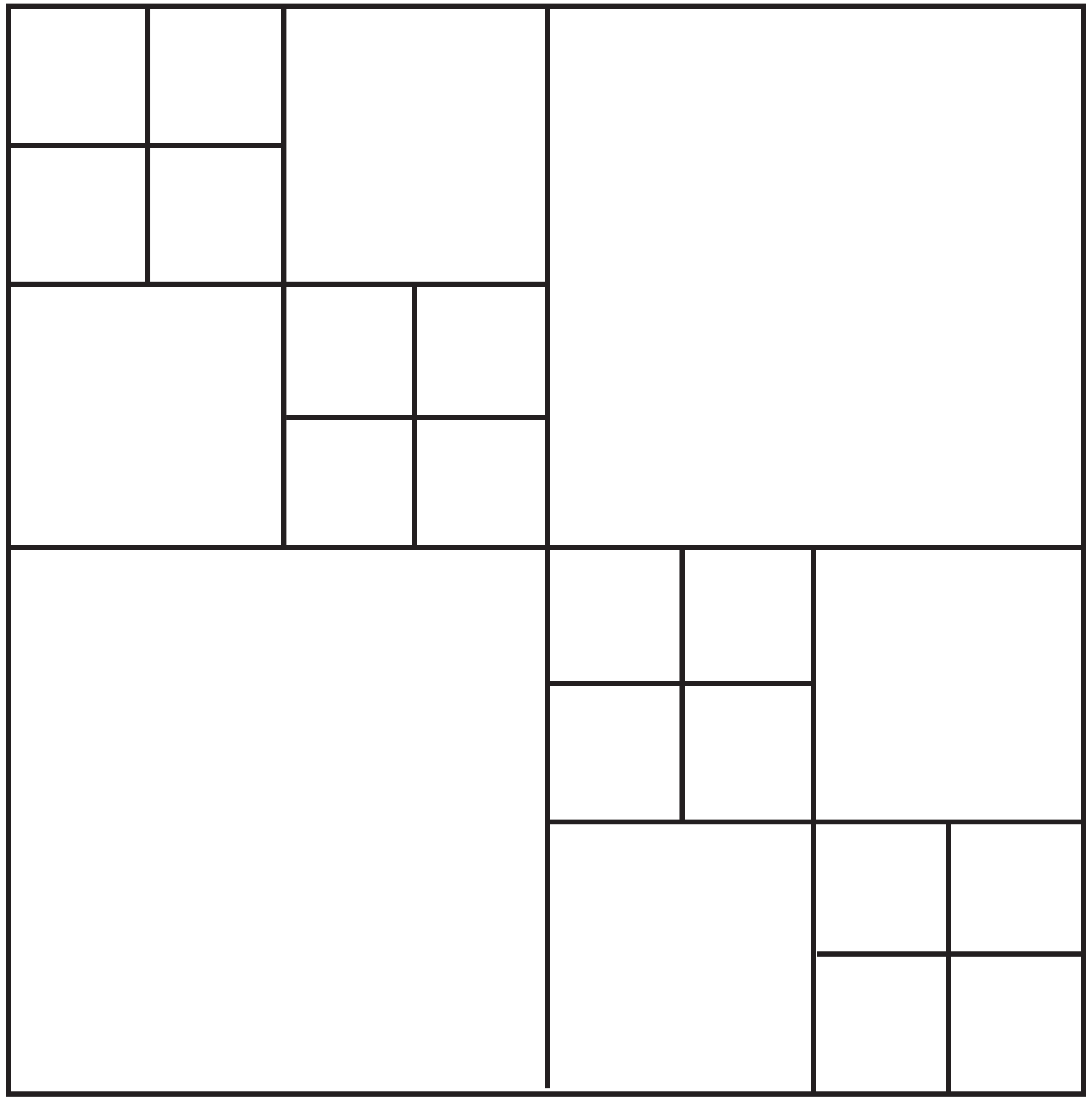
Other possible applications:

- Optimization: e.g. projection of a Hessian matrix on its invariant subspace

Outline

- HSS representation
- Applications
- Future work

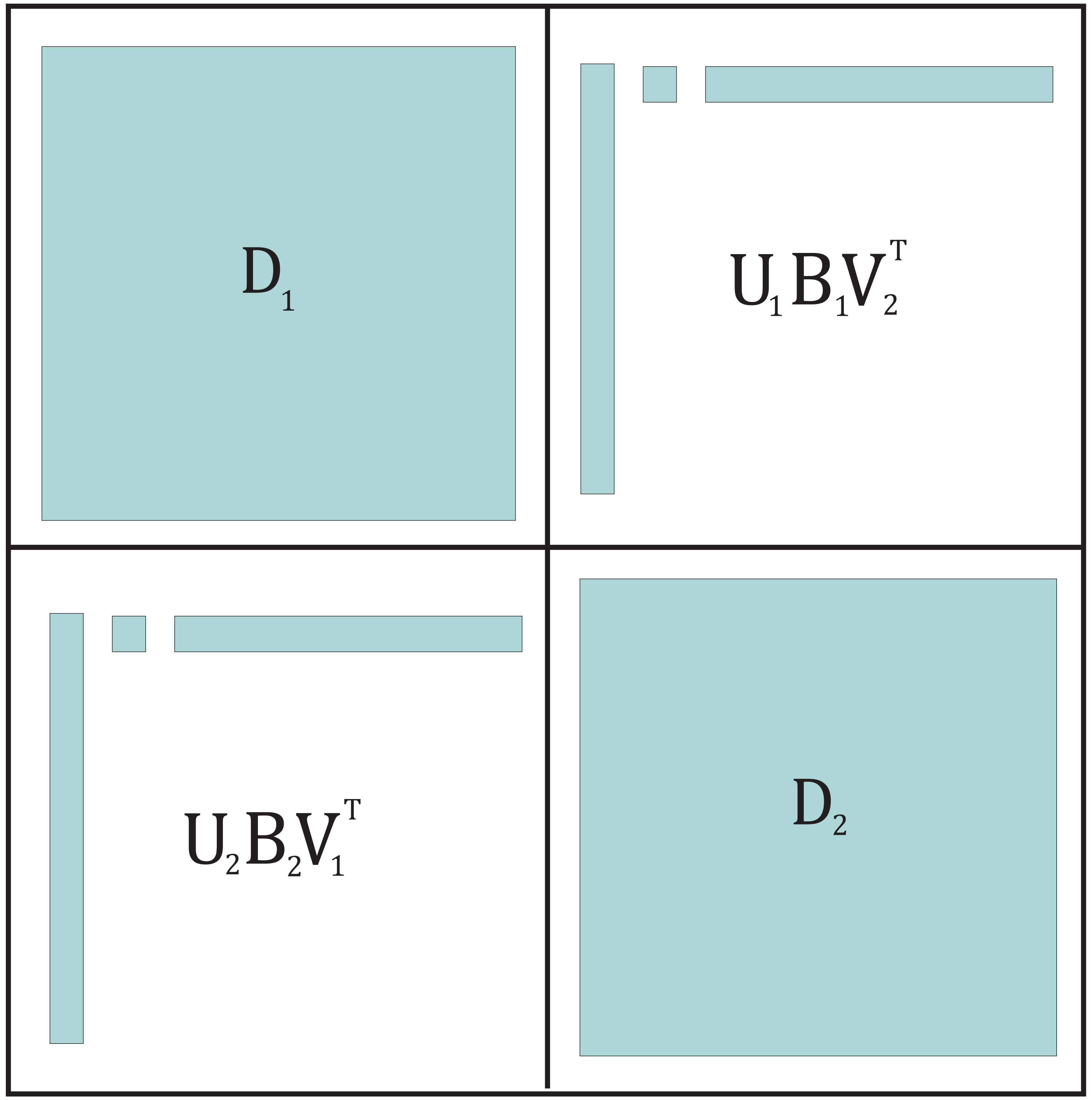
HSS representation

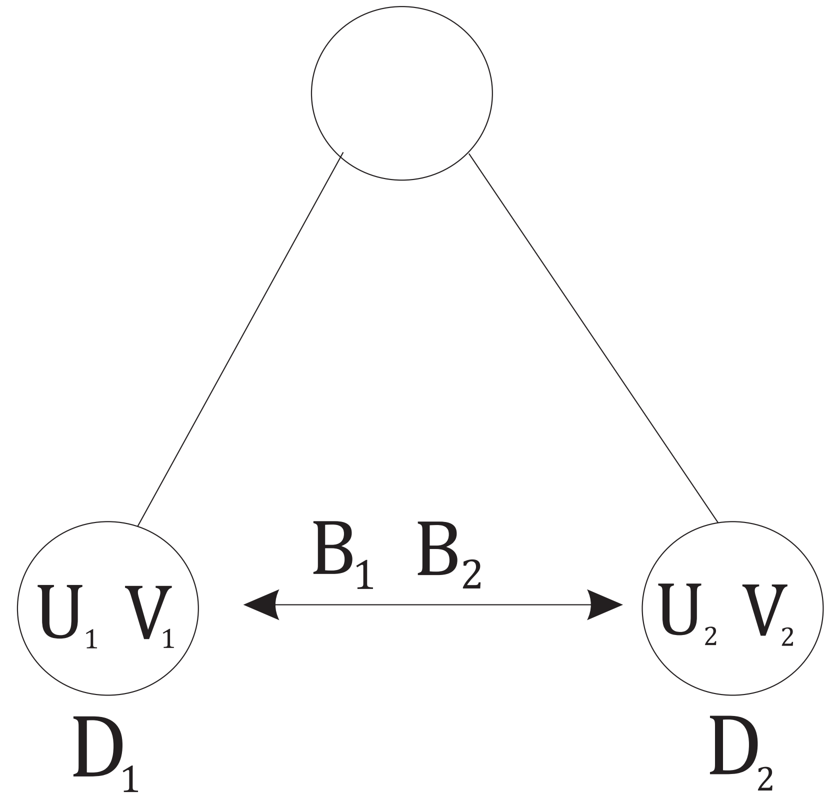


Xia, 2012, Lyons, 2005

- Off-diagonal blocks have low numerical rank
- Approximated by low rank matrices
- Each low rank approximation is a product of
  - a tall matrix
  - a small matrix and
  - a thin matrix
- Similar to reduced SVD
- The hierarchy is organized in a binary tree

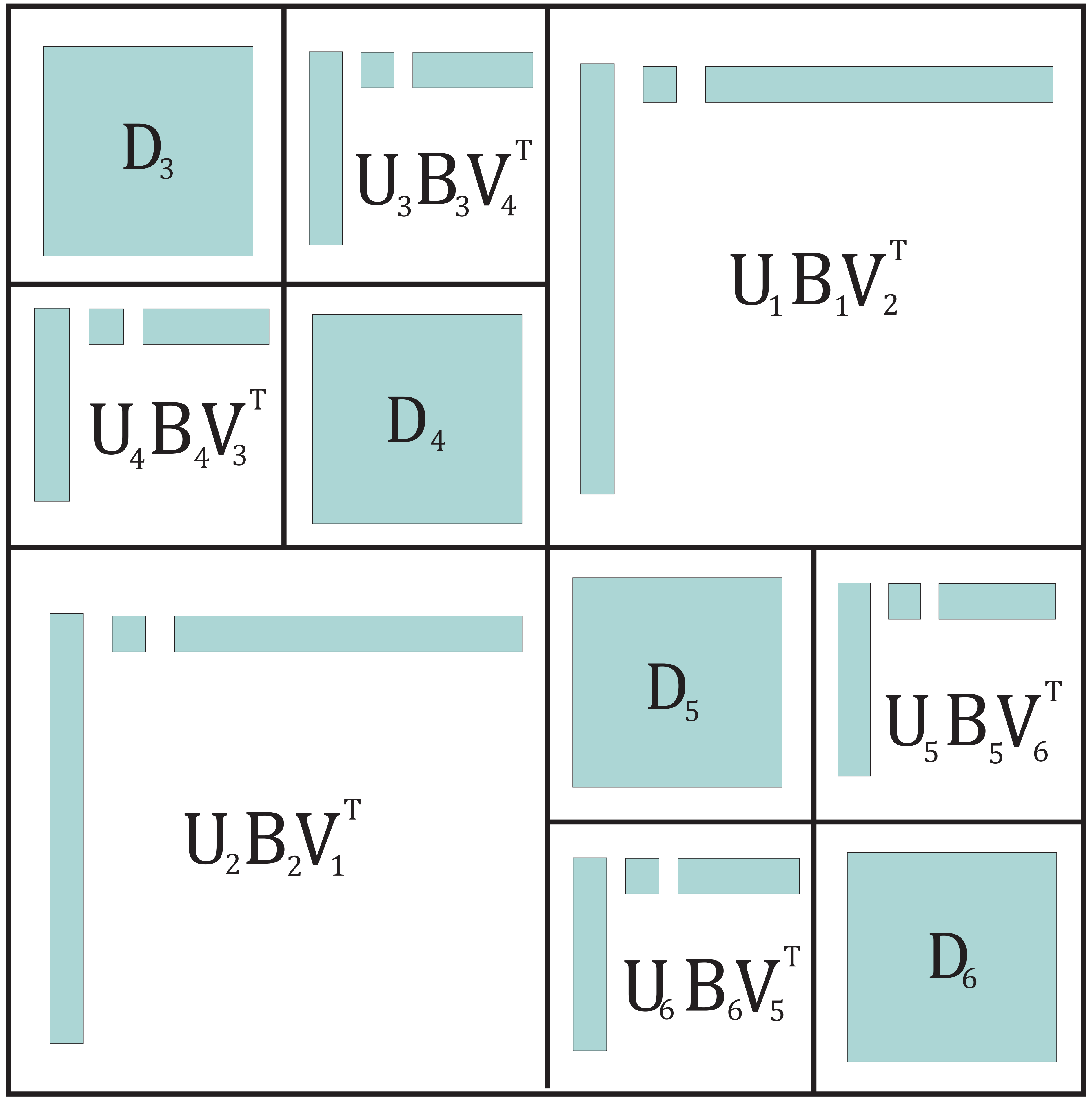
HSS representation

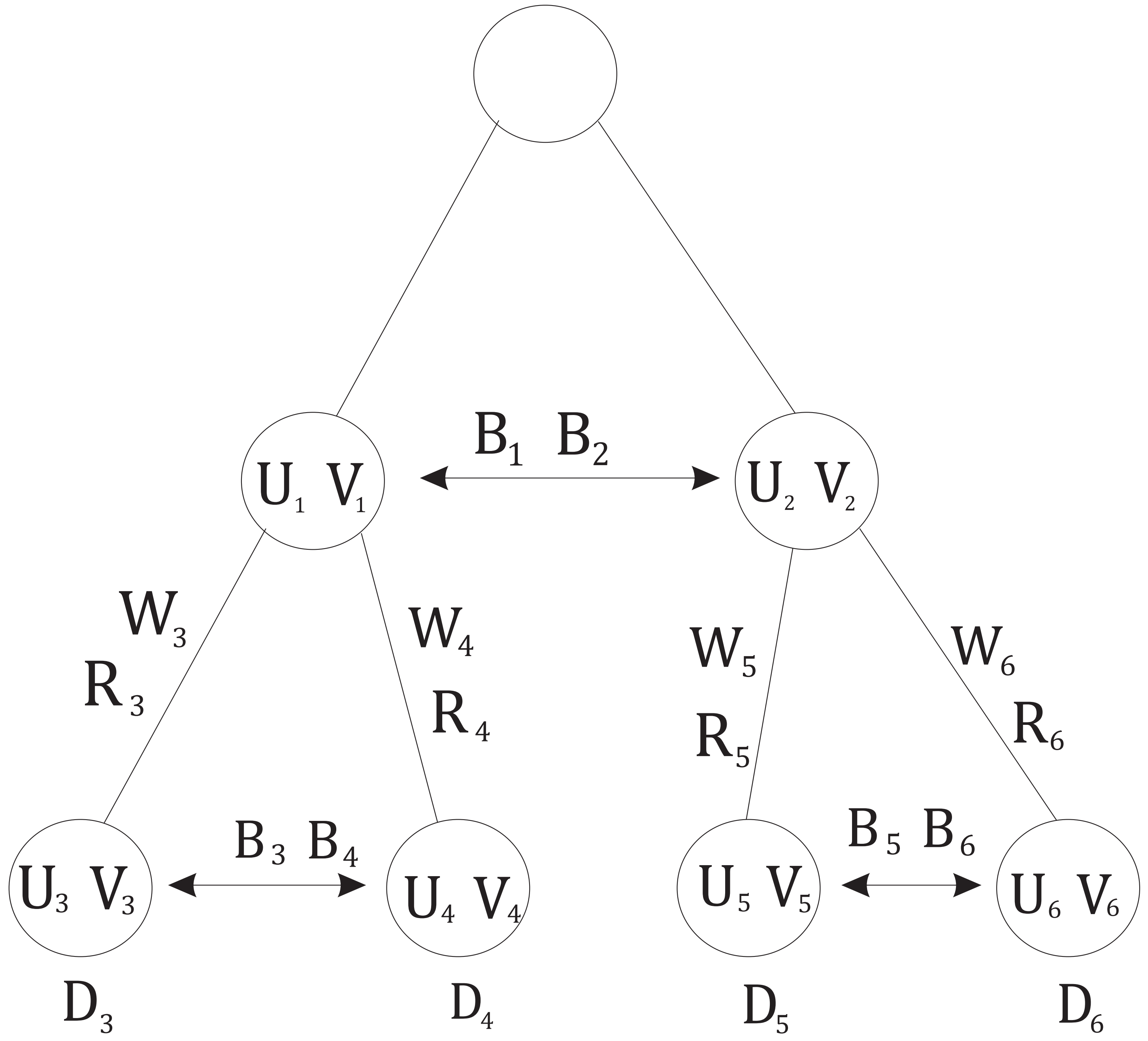




Xia, 2012, Lyons, 2005

HSS representation



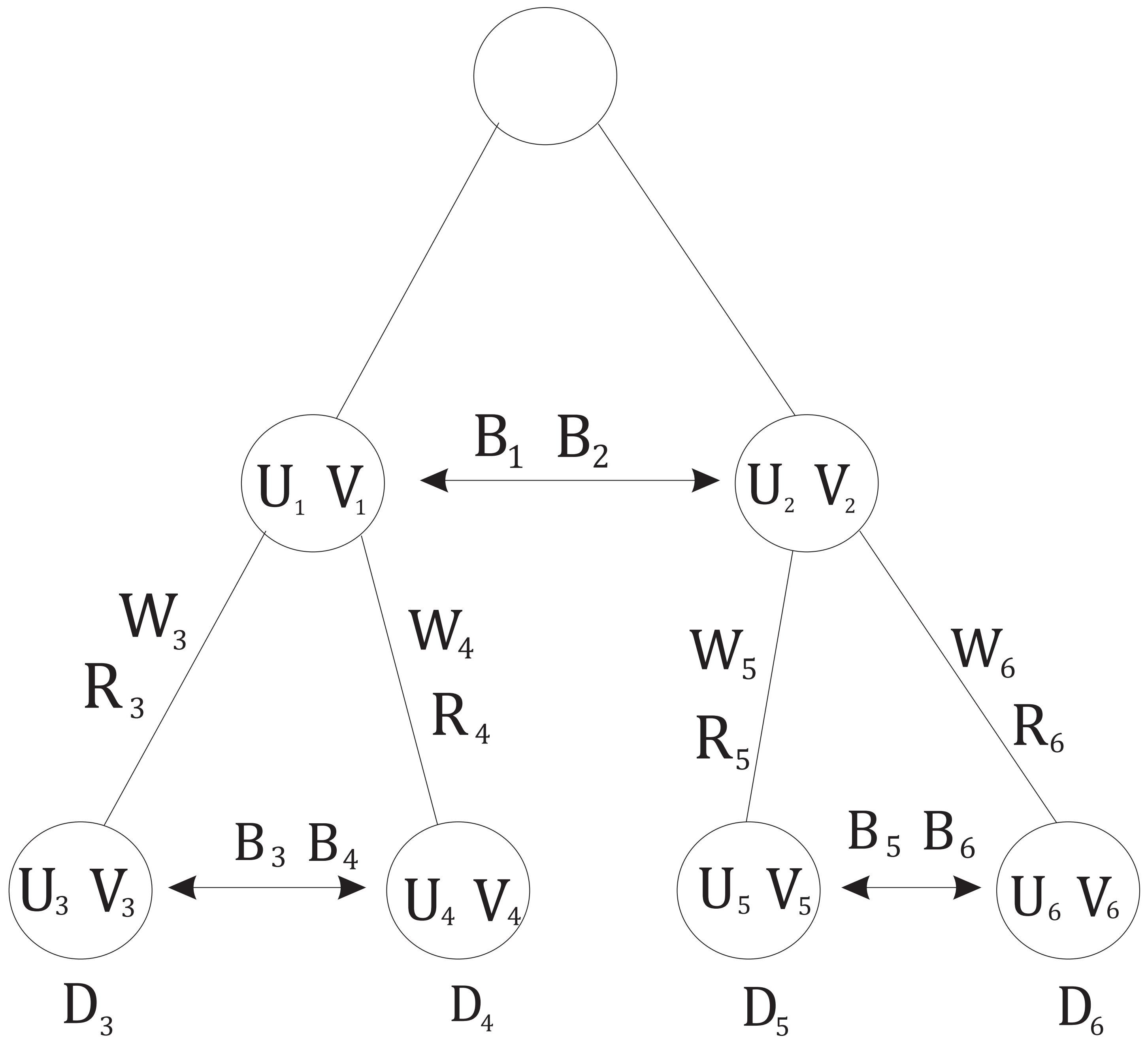


Xia, 2012, Lyons, 2005

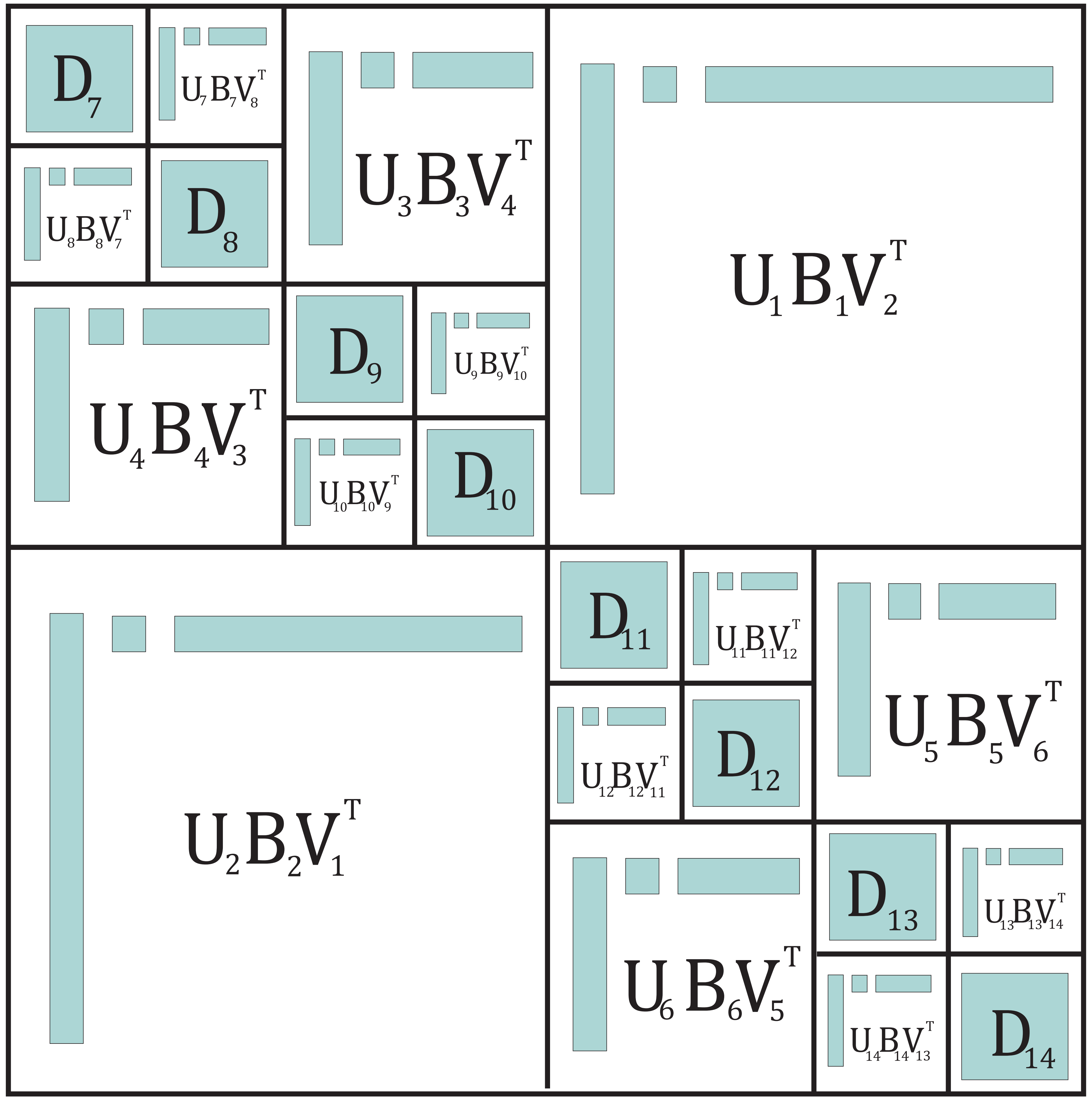
HSS representation

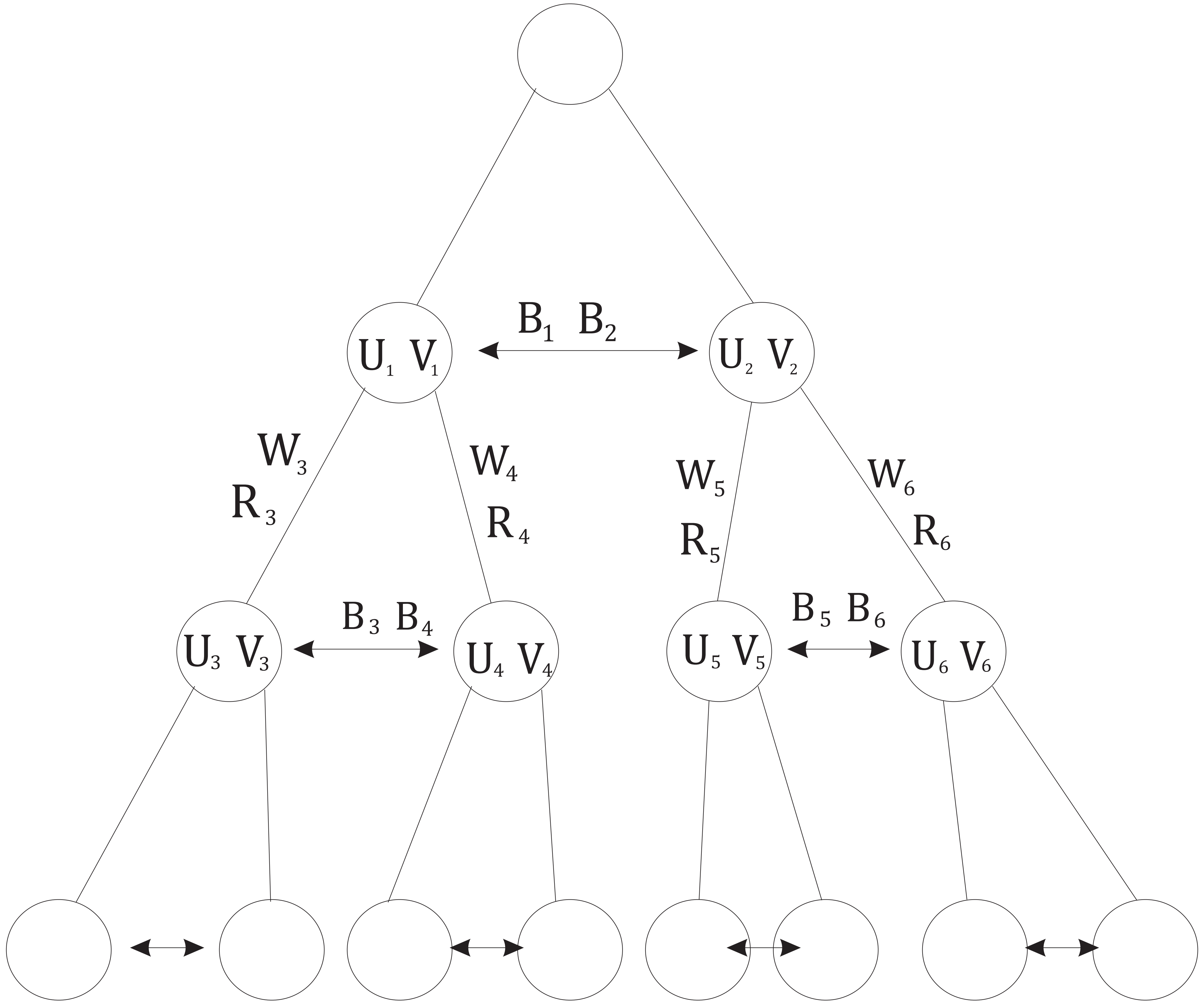
- Store only lowest U's and V's in the hierarchy
- Store B's, R's, W's for each level - small matrices, much smaller than U's and V's
- Higher U's and V's are determined from lower U's and V's via R's and W's
- Store the lowest D's in the hierarchy as dense matrices
- Optimized for matrix-vector multiplication

Xia, 2012, Lyons, 2005



HSS representation





Xia, 2012, Lyons, 2005

HSS representation

Complexity of algorithms for HSS matrix operations (Sheng et al., 2007):

Operation	Cost with HSS	Cost without HSS
Matrix-vector mutiplication	$O(nr^2)$	$O(n^2)$
Matrix-matrix multiplication	$O(nr^3)$	$O(n^3)$
Matrix addition	$O(nr^2)$	$O(n^2)$
LU decomposition	$O(nr^3)$	$O(n^3)$
Matrix inverse	$O(nr^3)$	$O(n^3)$
Transpose	$O(nr)$	$O(n^2)$
HSS construction	$O(nr)$	Not applicable

- r is maximum rank of off-diagonal blocks
- Efficient implementation is non-trivial

Applications: Two way wave equation migration

- Goal: march in depth stably:

$$\frac{\partial^2 p(\mathbf{x}, z)}{\partial z^2} = -\left(\frac{\omega^2}{c^2(\mathbf{x})} + \nabla^2_{\mathbf{x}}\right) p(\mathbf{x}, z) \qquad p(\mathbf{x}, z_0) = \text{given}$$

- The spectrum of the reduced Helmholtz operator $H = \frac{\omega^2}{c^2(\mathbf{x})} + \nabla^2_{\mathbf{x}}$ (Grimbergen et al., 1998)

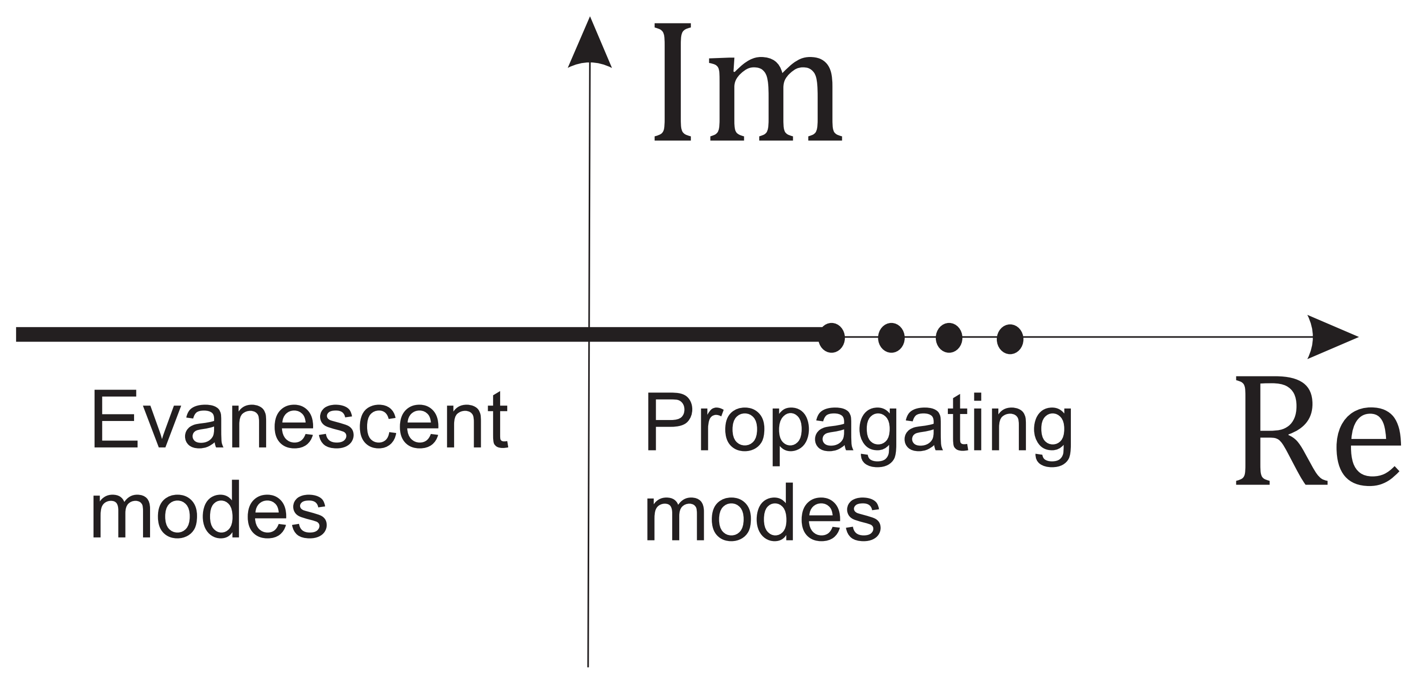


Figure 3: Spectrum of reduce Helmholtz operator

- Get rid of evanescent modes
- Project of $L = -H$ onto its invariant subspace corresponding to non-positive part of the spectrum of L (Sandberg & Beylkin, 2009)

Applications: Two way wave equation migration

- Spectral projector is defined as:

$$\mathcal{P} = \sum_{\lambda \in \Lambda, \lambda \leq 0} P_\lambda$$

  where P_λ is the projection onto subspace corresponding to λ
- Then propagating part of L is

$$\mathcal{P} L \mathcal{P}$$

Applications: Two way wave equation migration

- For self-adjoint operator L:

$$\mathcal{P} = \frac{1}{2}(I - \text{sign}(L))$$

where $\text{sign}(L)$ is found by following recursion

$$S_0 = \frac{L}{\|L\|_2}$$
$$S_{k+1} = \frac{3}{2}S_k - \frac{1}{2}S_k^3$$

(e.g. Kenney & Laub, 1995)

- Quadratic convergence, complexity with HSS: $\sim O(n)$
- Leads to a fast migration algorithm (more in Lina Miao's talk)

Applications: One way wave equation migration

- Use a polynomial recursion to compute $H^{1/2}$
- One way wave equation:

$$\frac{\partial p^{\pm}}{\partial z} = \mp i H^{1/2} p^{\pm}$$

  p^+, p^-: down and up going fields
- $H^{1/2}$ is usually approximated by a local operator $\Longrightarrow$ dip limitations
- Grimbergen et al. (1998): modal expansion of H to compute $H^{1/2}$:
  - requires eigen-value decomposition of H $\Longrightarrow$ expensive, only small problems

Applications: One way wave equation migration

- Instead $H^{1/2}$ can be computed by applying sign iteration to $\begin{bmatrix} 0 & H \\ I & 0 \end{bmatrix}$

  Shultz iteration (Higham, 2008):

$$Y_0 = \frac{H}{\|H\|_2}, Z_0 = I$$
$$Y_{k+1} = \frac{3}{2}Y_k - \frac{1}{2}Y_k Z_k Y_k$$
$$Z_{k+1} = \frac{3}{2}Z_k - \frac{1}{2}Z_k Y_k Z_k$$

- $Y_k \longrightarrow \left(\frac{H}{\|H\|_2}\right)^{1/2}, Z_k \longrightarrow \left(\frac{H}{\|H\|_2}\right)^{-1/2}$
- Quadratic convergence, complexity with HSS: $\sim O(n)$
- H can not have negative eigen-values $\Longrightarrow$ apply spectral projector first

Applications: One way wave equation migration

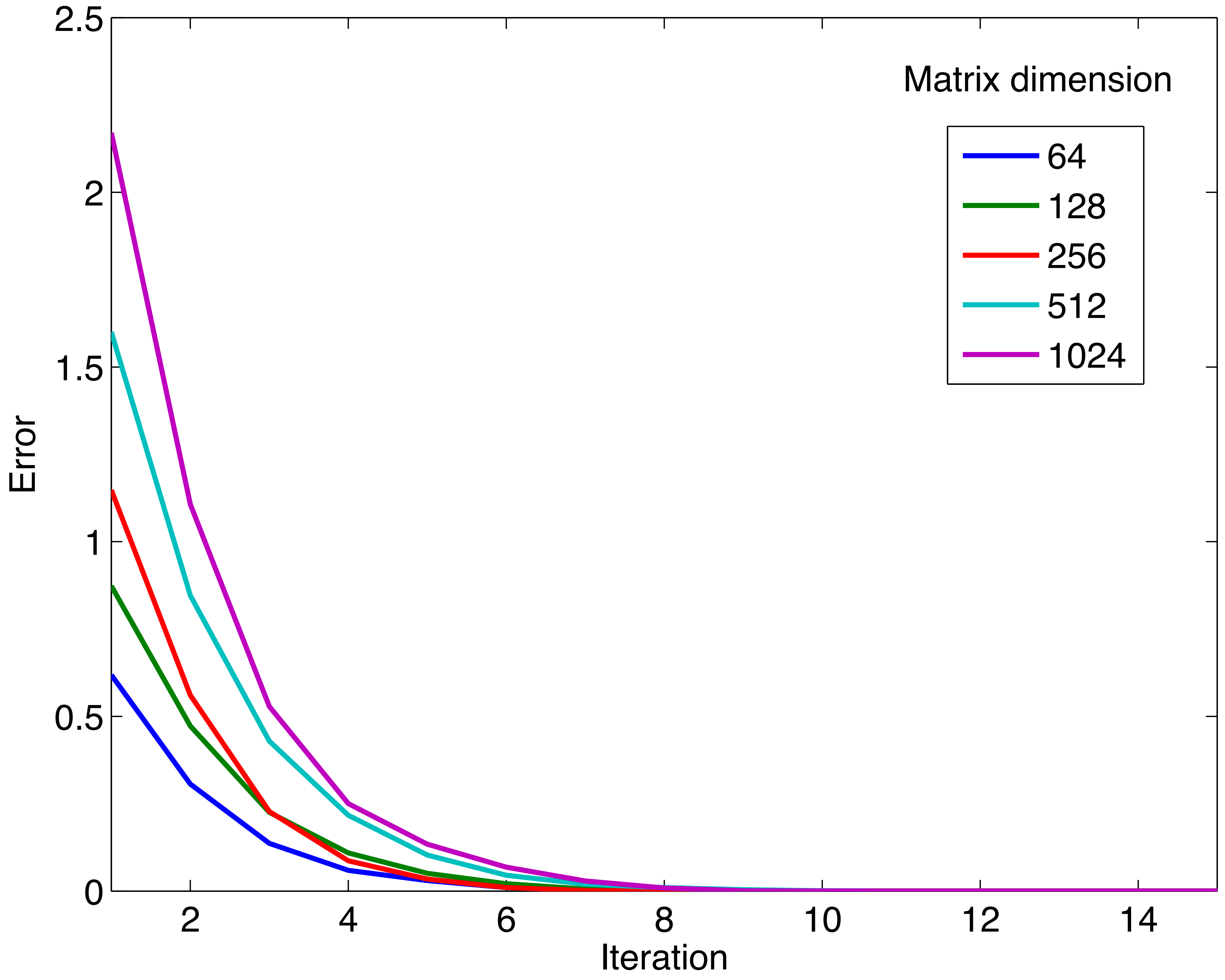


Figure 4: Convergence of Shultz iteration for matrices of different size. Error $= \left\| H^{1/2} - H_{eig}^{1/2} \right\|_F$, where $H_{eig}^{1/2}$ is computed by modal decomposition.

Future work

- Implement one-way wave equation migration
- Implement absorbing boundary conditions
- Possible applications in modeling for FWI
  - Coupling matrix can be expressed in terms of $H^{1/2}$ and $H^{-1/2}$
  - Should be expressible with HSS
  - Goal: model up and down going wave components separately
- Applications exploiting HSS structure of data and operators simultaneously

Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

SINBAD

NSERC CRSNG

This work was in part financially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant (22R81254) and the Collaborative Research and Development Grant DNOISE II (375142-08). This research was carried out as part of the SINBAD II project with support from the following organizations: BG Group, BGP, BP, CGG, Chevron, ConocoPhillips, ION, Petrobras, PGS, Statoil, Total SA, WesternGeco, and Woodside.

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