Compressed sensing, recovery of signals using random Turbo matrices

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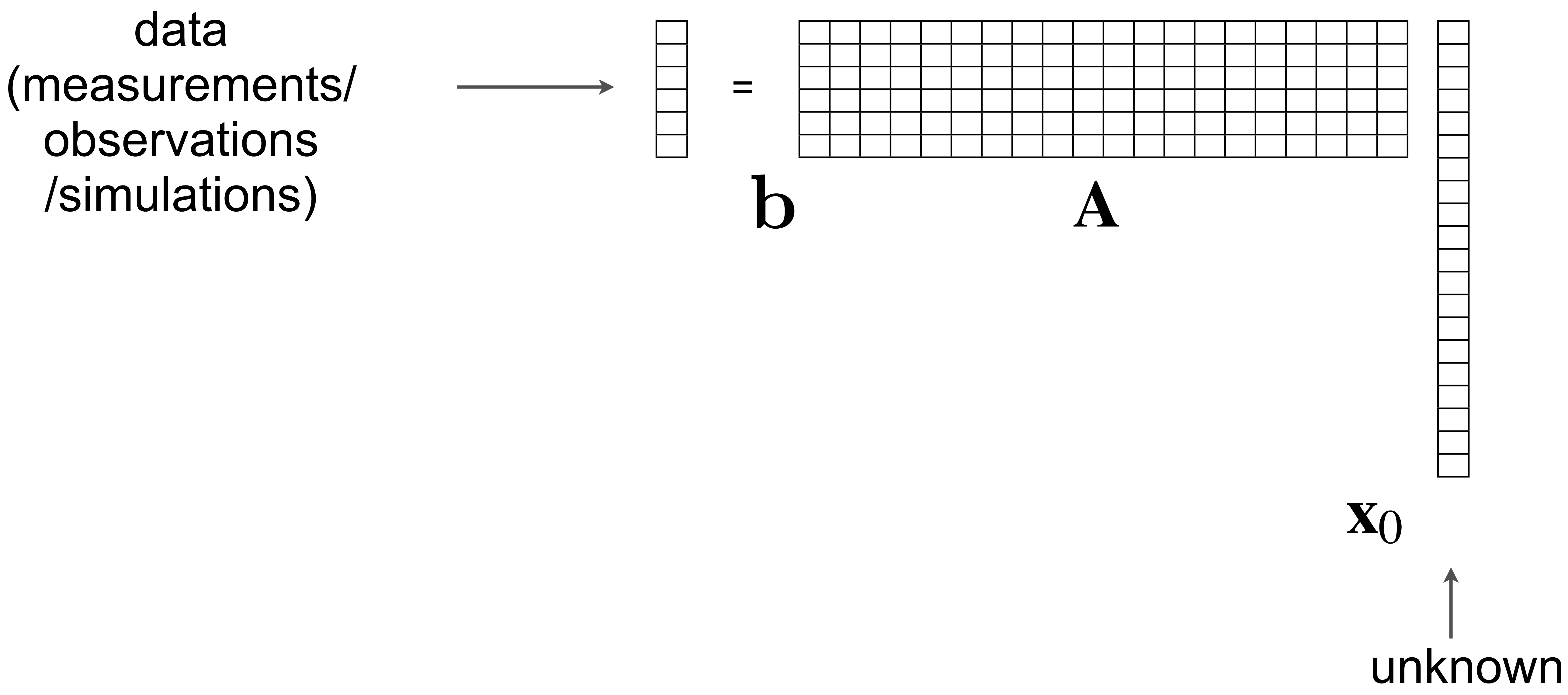
Introduction

Compressed sensing is a powerful technique to reconstruct sparse data.

- Can we do better than using Gaussian matrix?
- What do we mean by better results?

Compressed sensing

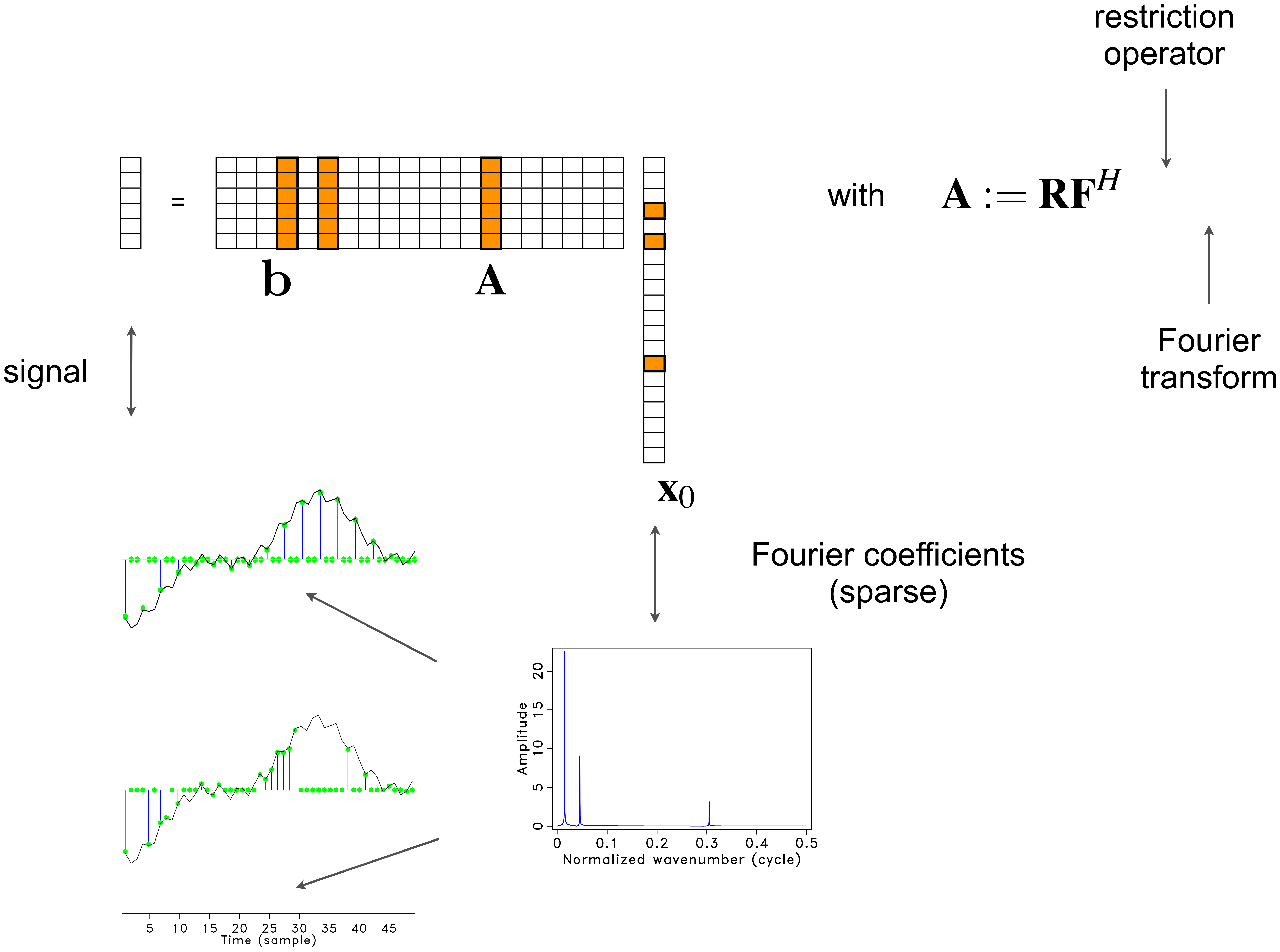
Consider the following (severely) *underdetermined* system of *linear* equations:



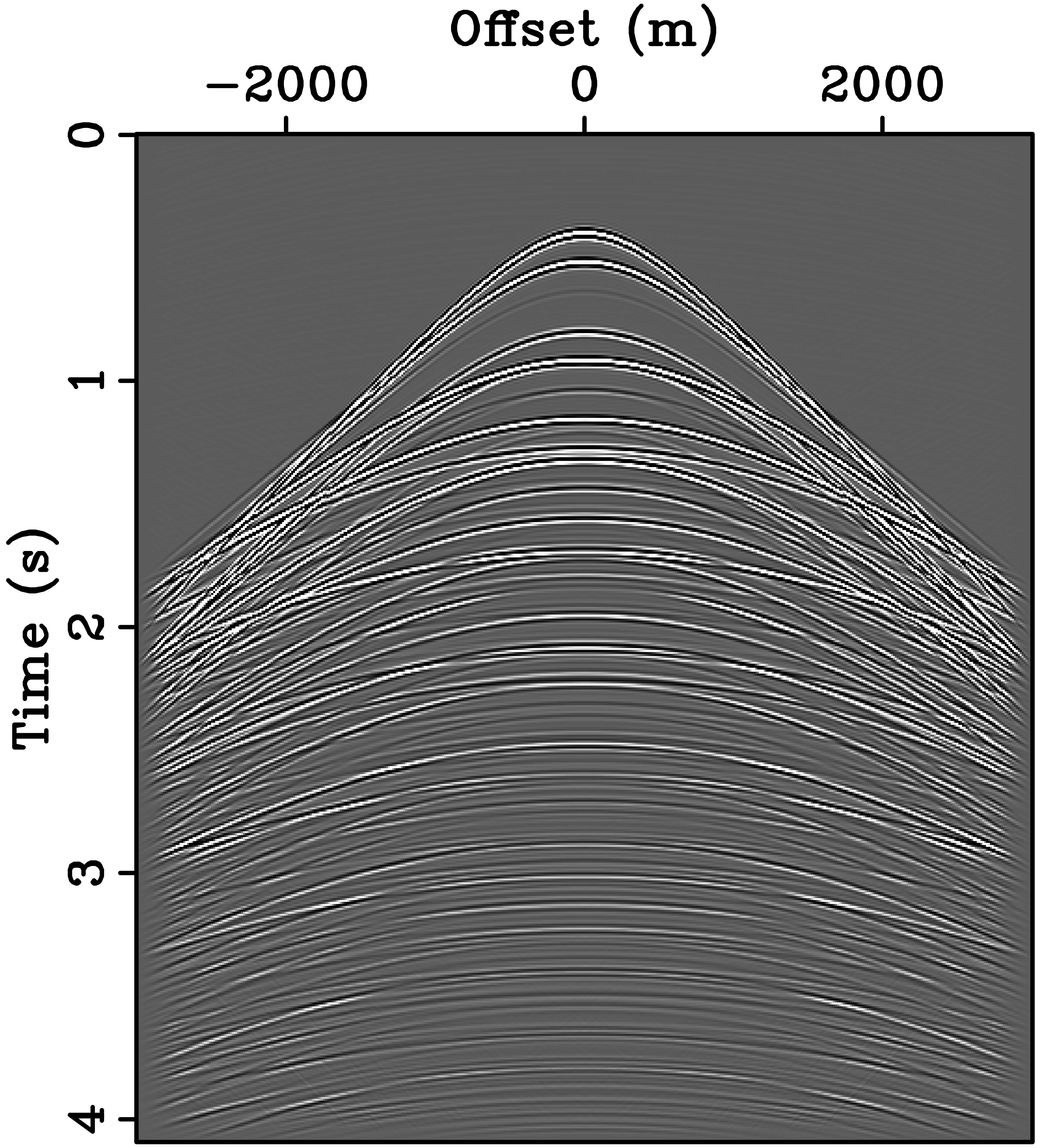
Is it possible to recover $\mathbf{x}_0$ accurately from $\mathbf{b}$?

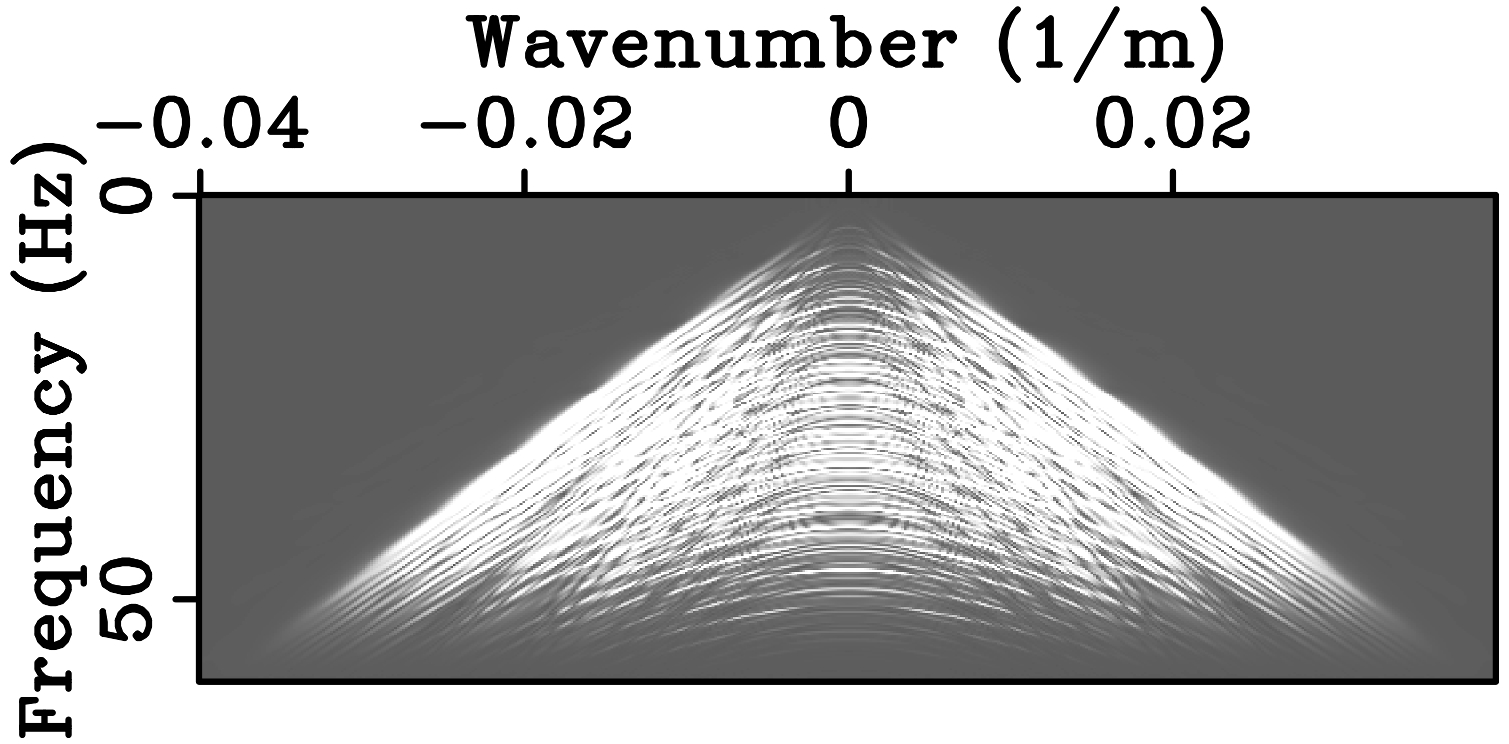
Compressed Sensing attempts to answer this questions rigorously.

Sparse recovery

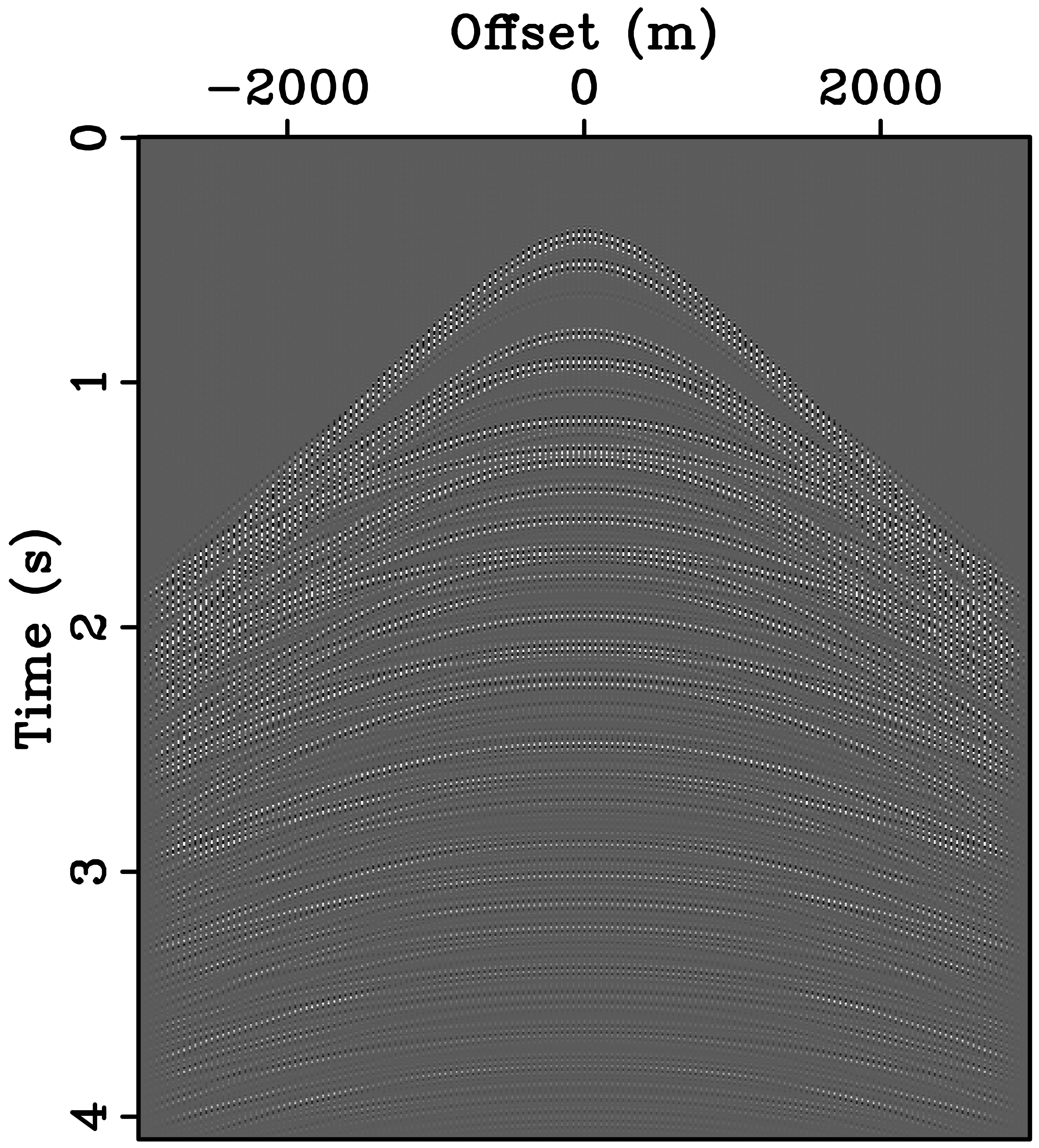


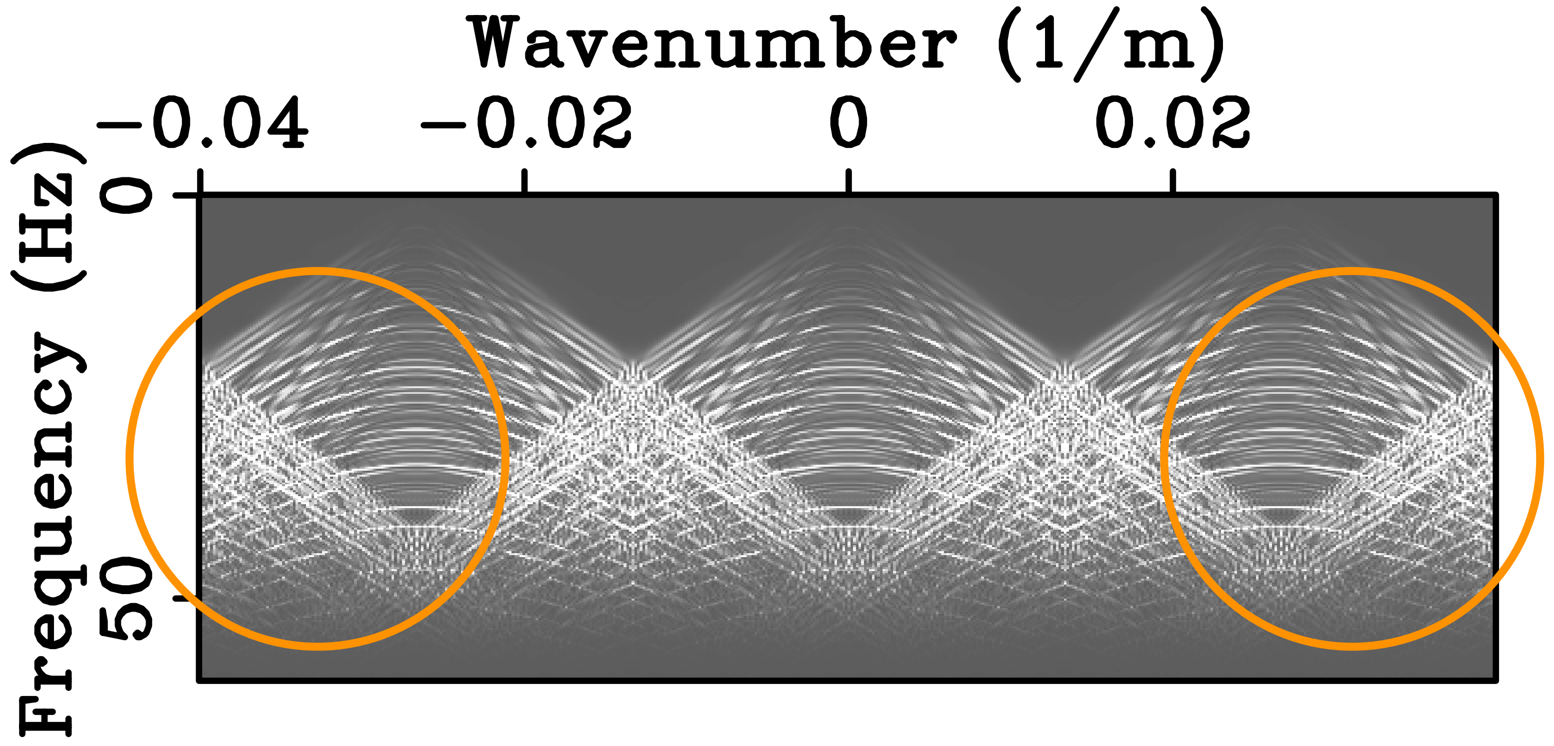
Model


Offset (m)
−2000
0
2000
Time (s)
0
1
2
3
4

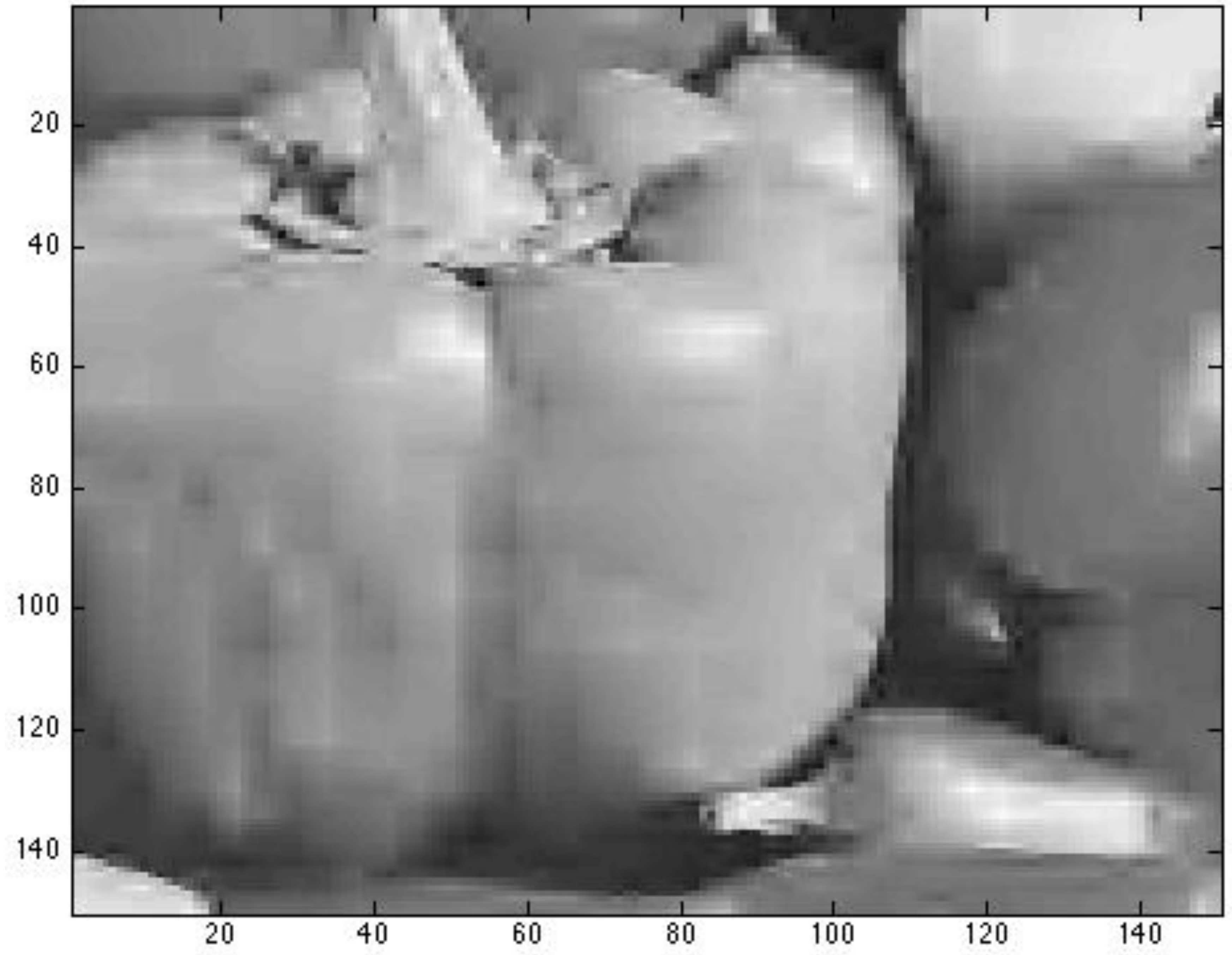

Wavenumber (1/m)
−0.04
−0.02
0
0.02
Frequency (Hz)
0
50

Regular 3-fold undersampling





Pepper (10% sparsity)



Throw away 90% of the wavelet coefficients

House

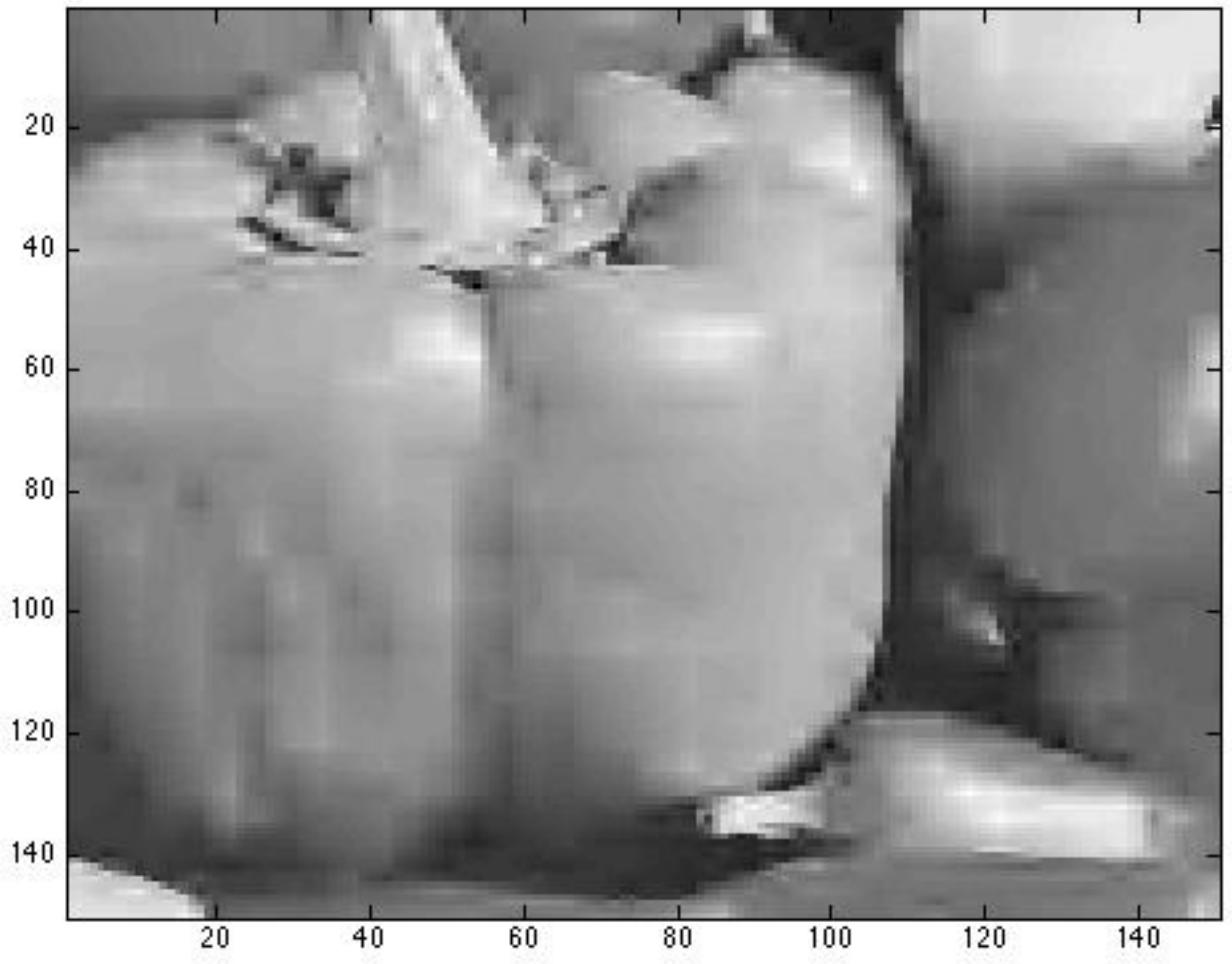
Original

Sparse

Why collect all the data?

25% of data

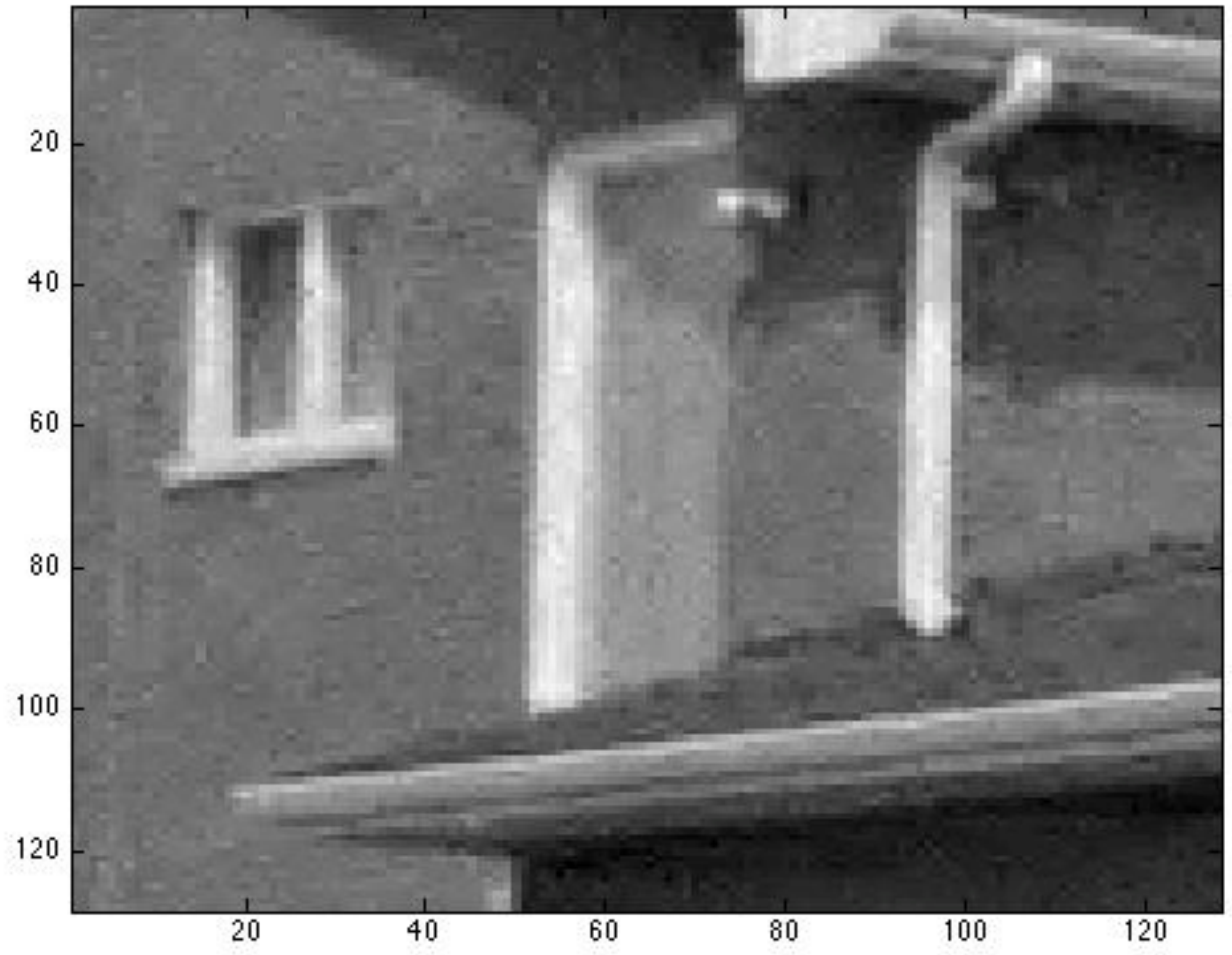
100% of data



SNR = 21

Why collect all the data?

20% of data



SNR = 25.8

100% of data

Sparse

Compressed sensing

How many measurements do we need to make?
Far less than what Shannon tells us.

$$y = Ax$$

x is a vector in $\mathbb{R}^N$
y is a vector in $\mathbb{R}^n$
y is observation
A is a measurement matrix
A has n rows and N columns, where $n \ll M$.

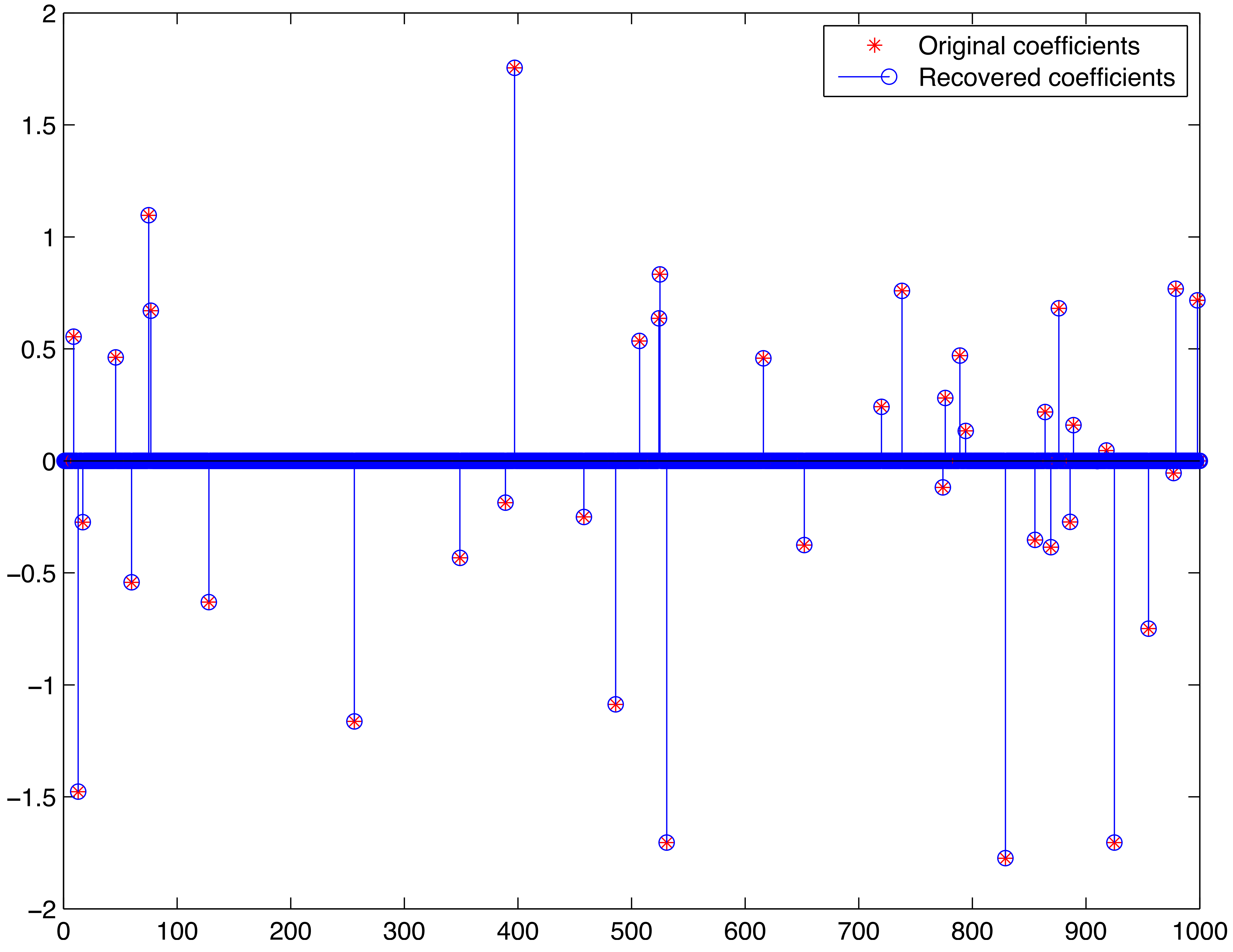
Recovery of sparse signal

N = 1000

n = 200

k-sparse

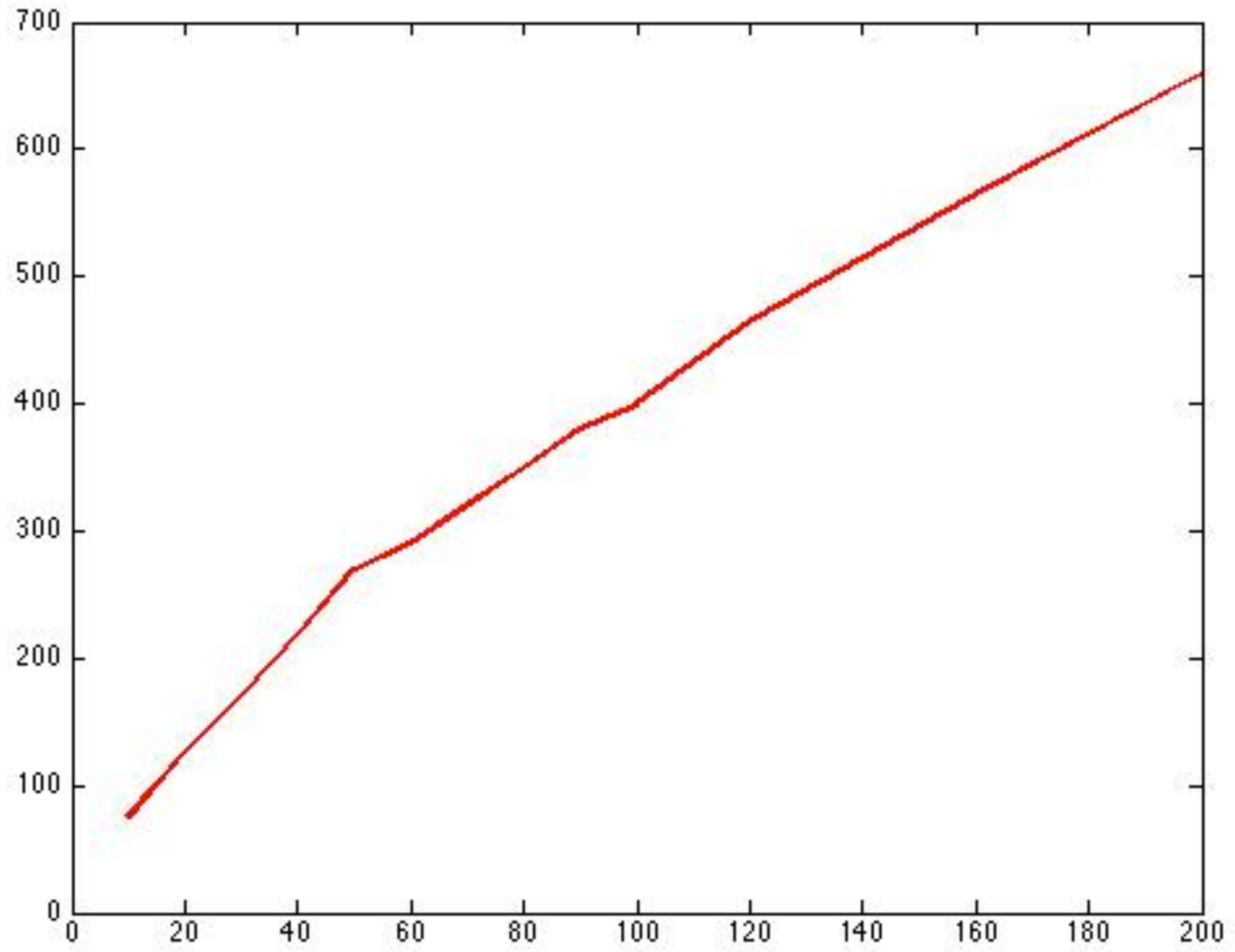
k = 40



sample number

How many measurements?

Number of measurements needed



Sparsity (k)

N = 2000
k-sparse

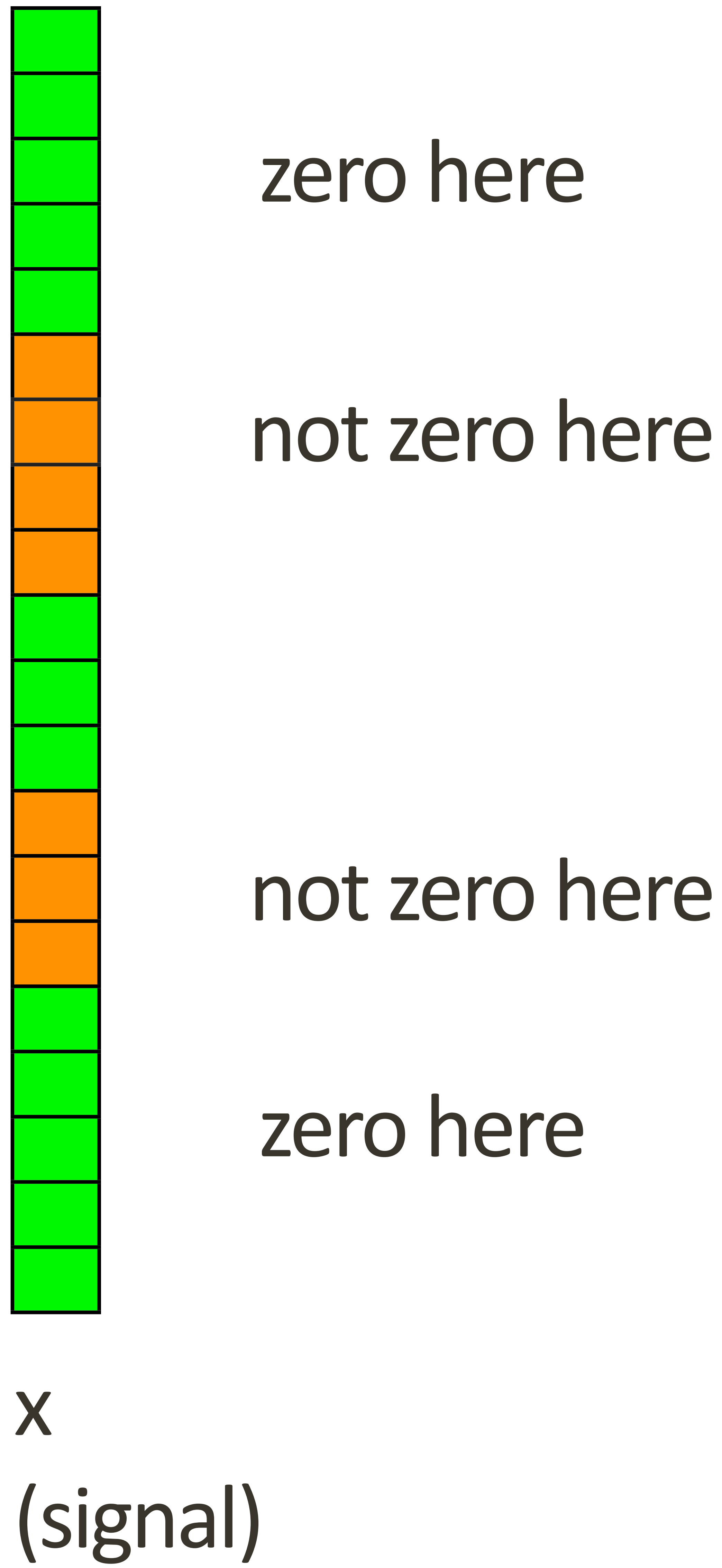
Sparsity

- Compressed sensing can be applied when a signal is sparse.

- What if you do not collect enough samples? (Below required minimum)

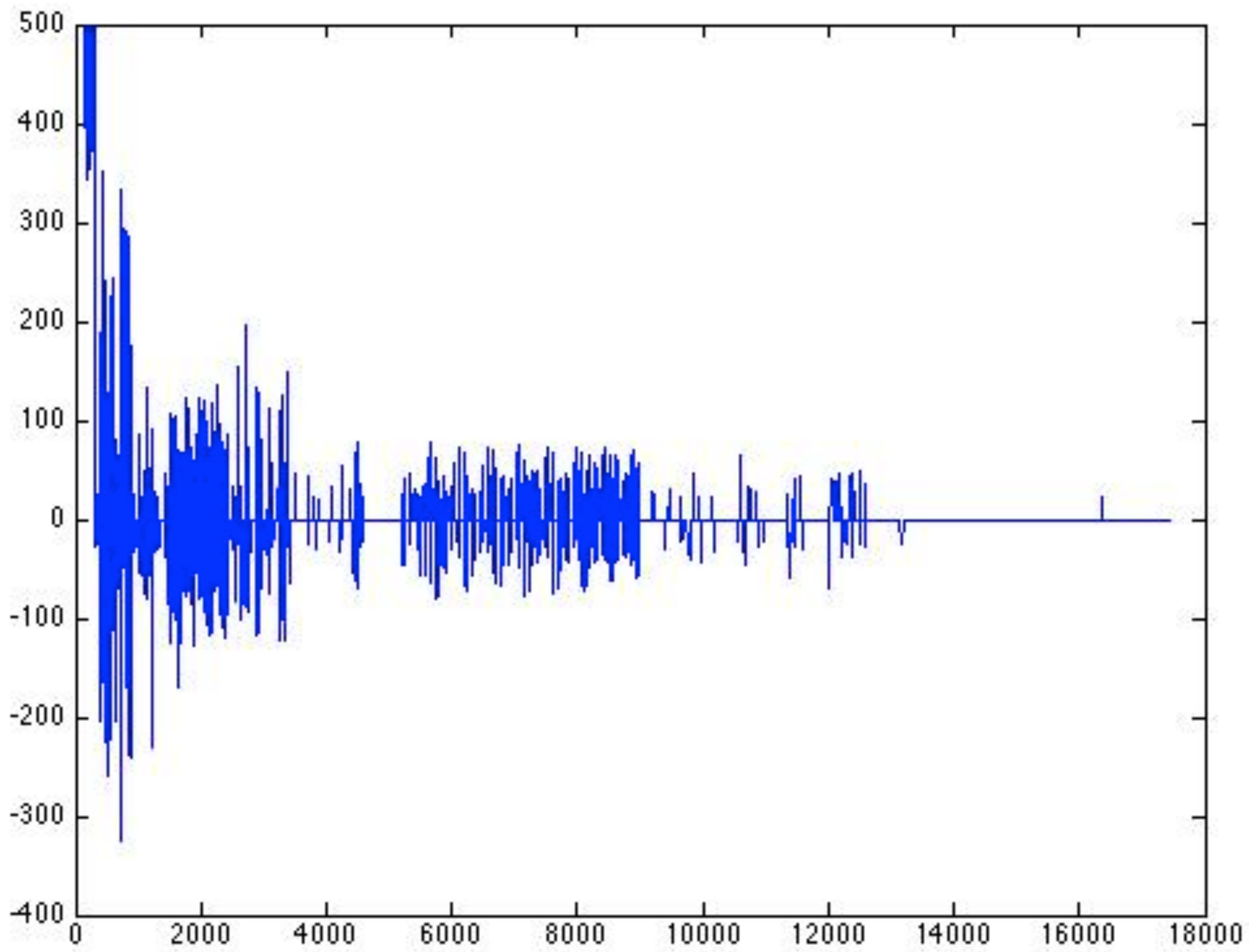
When you are seriously under-sampling, there will be reconstruction error.

But a signal may not be as random as we thought.



sparsity of house

wavelet coefficients



sample number

Measurement matrix

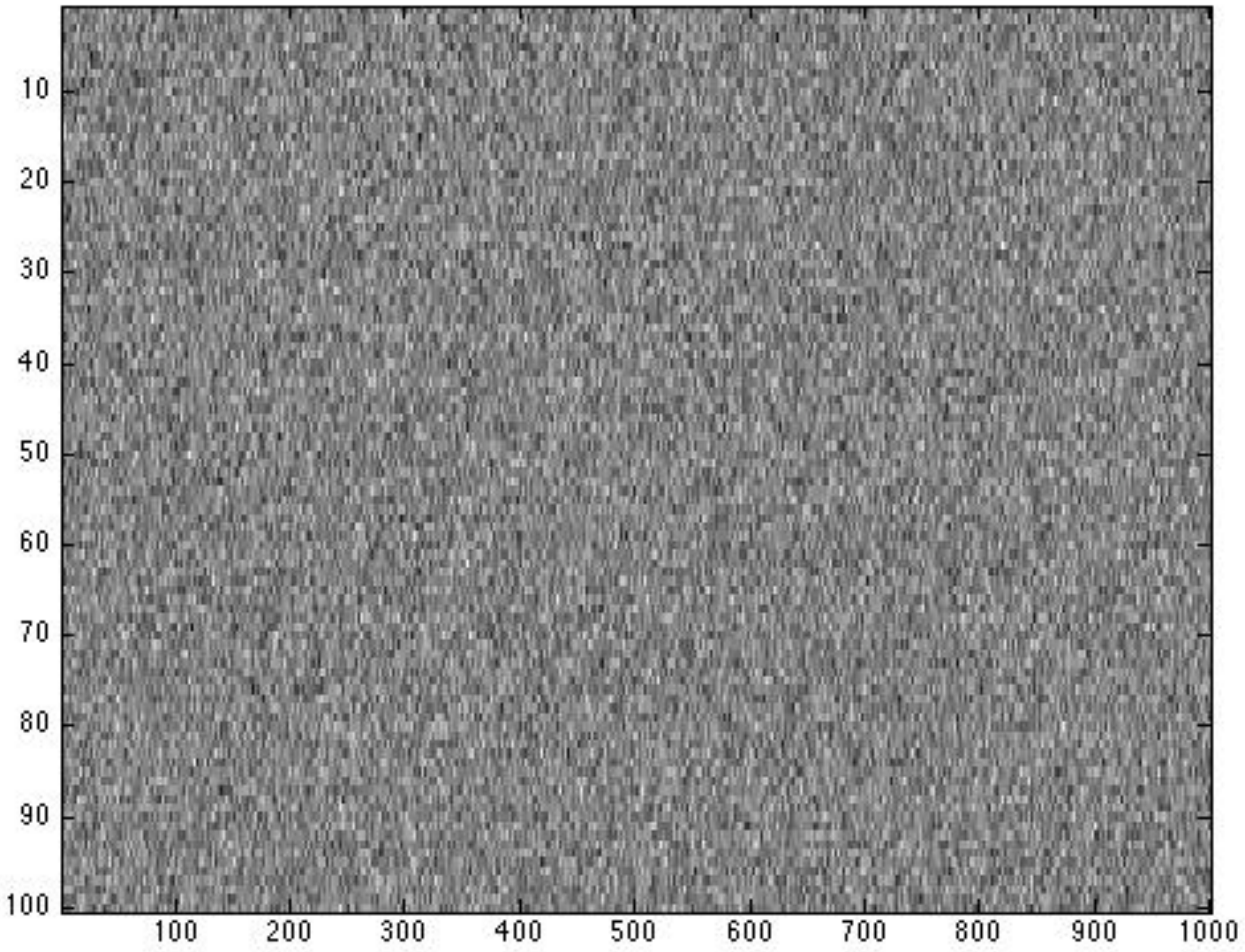
$$y = Ax \qquad A \text{ is a matrix}$$

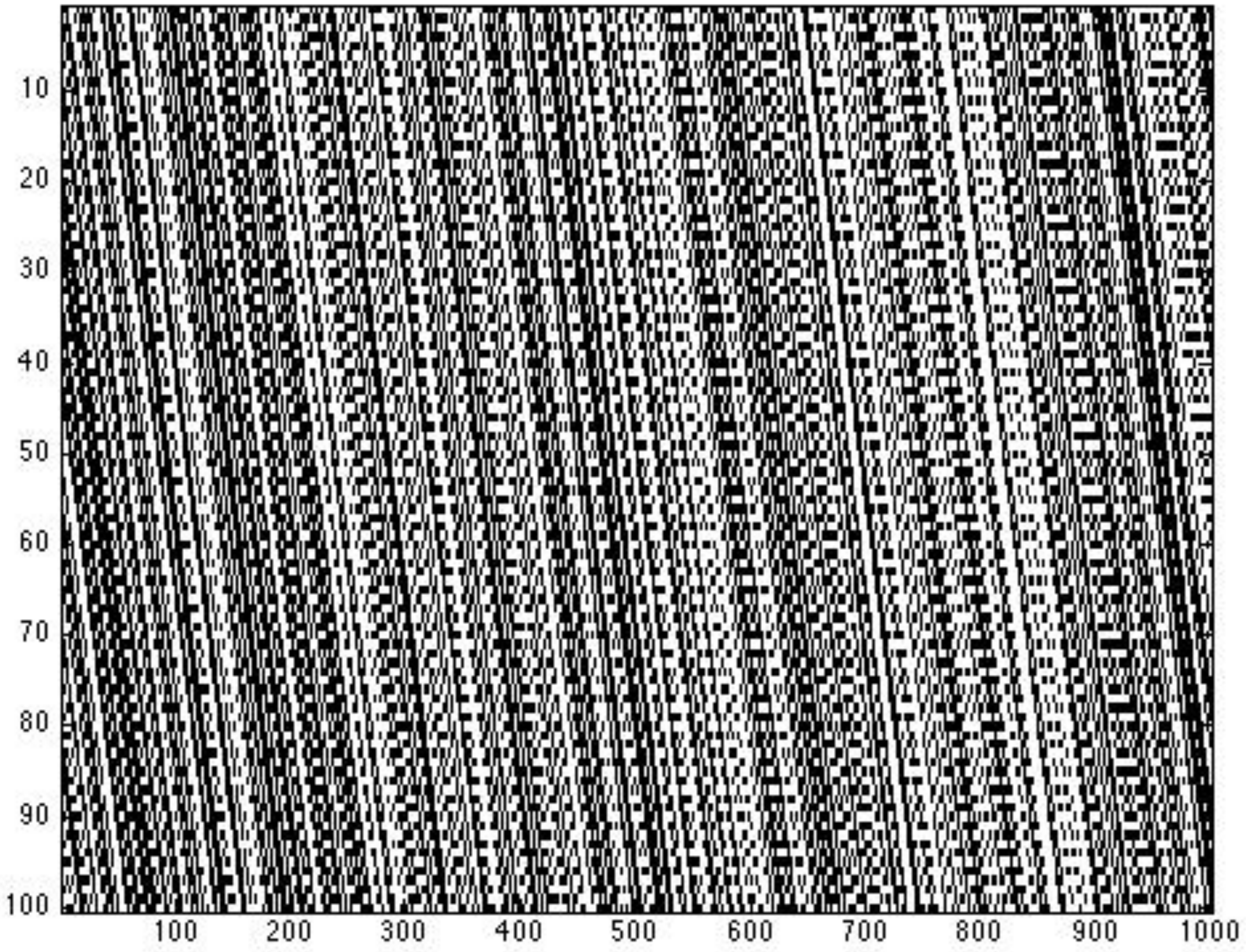
The matrix satisfies the RIP property:
nearly preserve length of sparse vectors.

Turbo matrix has some structure.

Gaussian matrix has no structure .

Gaussian vs Turbo

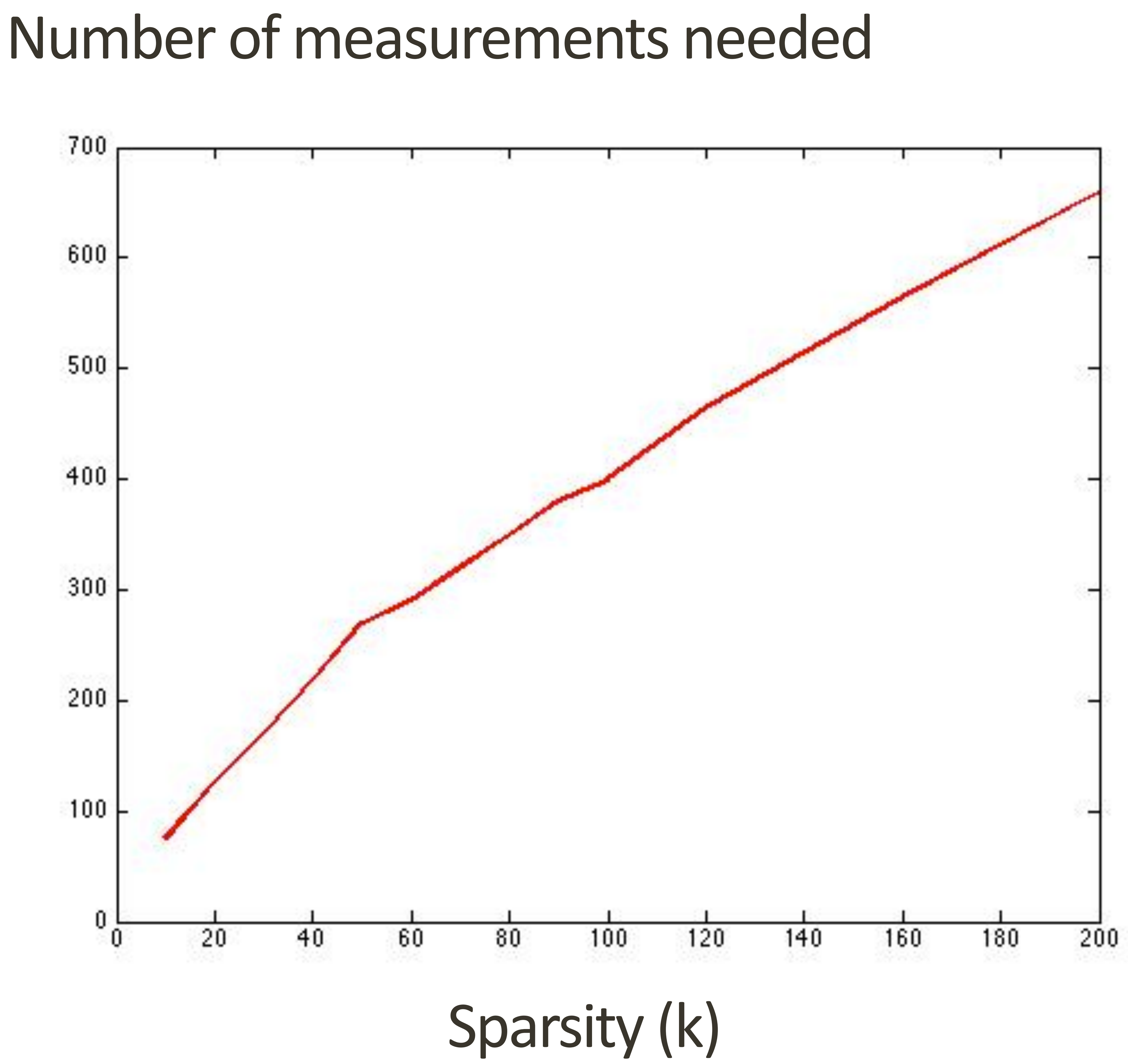




no structure

some structure

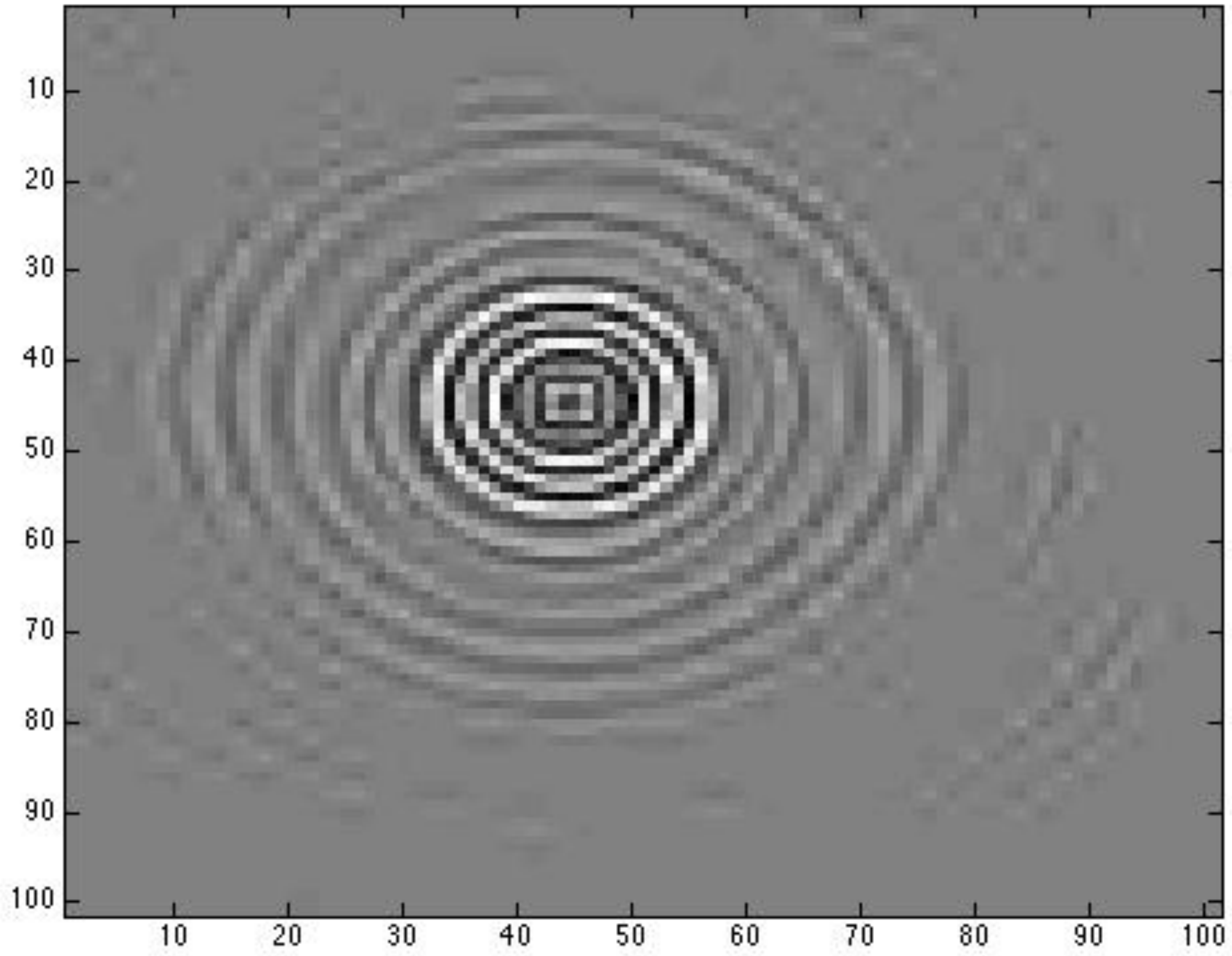
How many measurements?



N = 2000

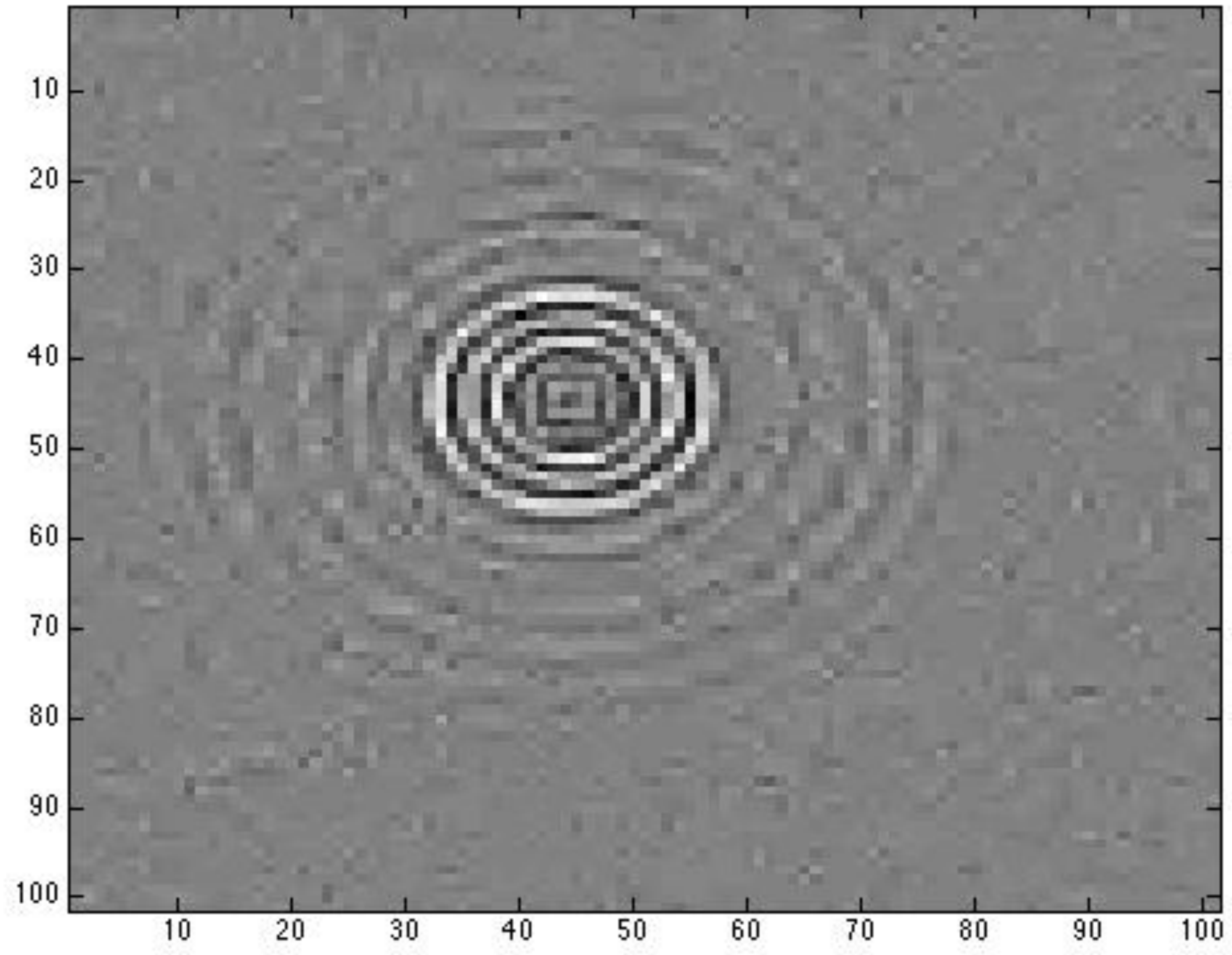
k-sparse

BG data (10% sparsity)



Throw away 90% of the wavelet coefficients

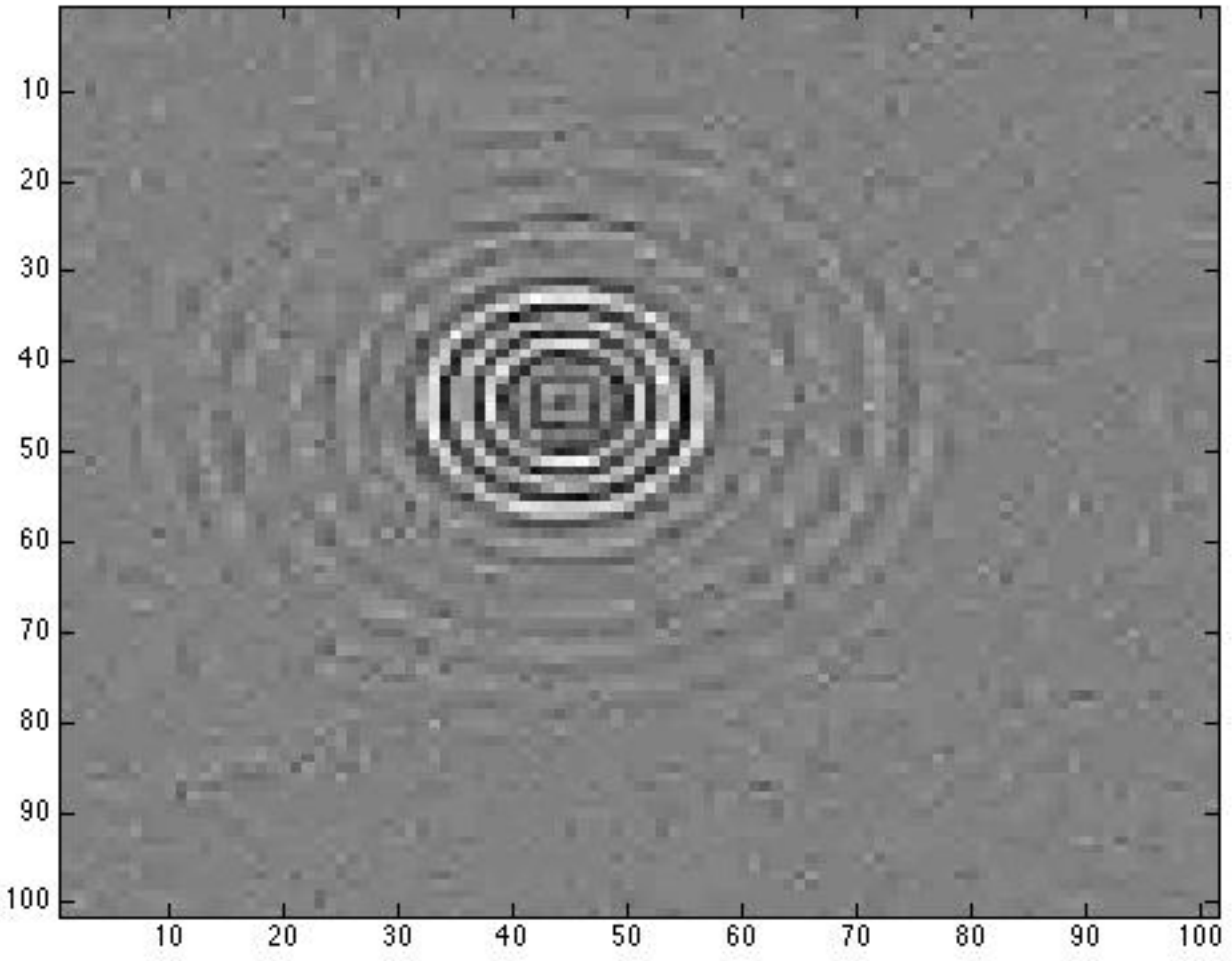
BG data (10% sparsity)



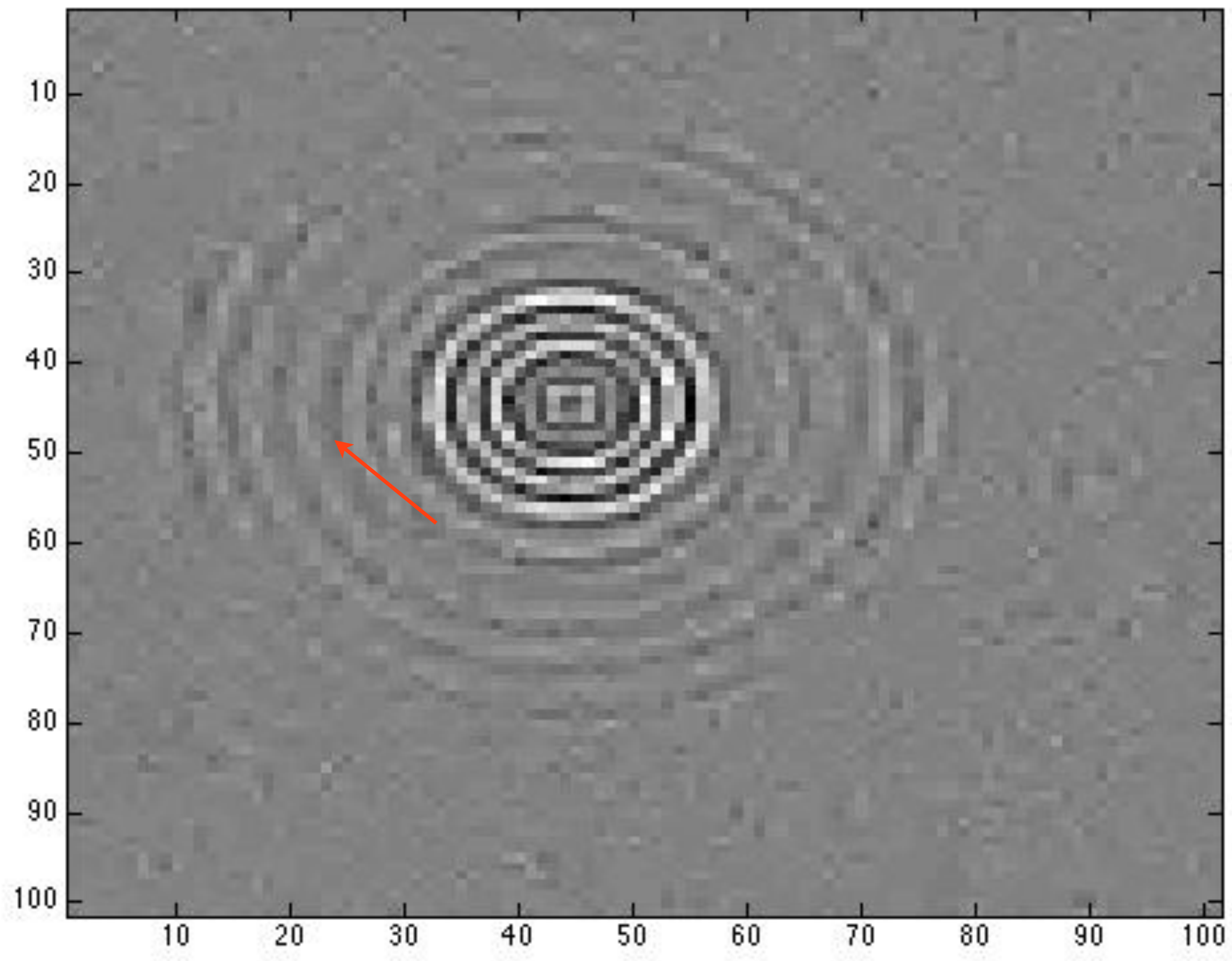
Throw away 90% of the wavelet coefficients

2.5 X sparsity (Under-sampling)

Gaussian vs Turbo



SNR = 6.1



SNR = 7.2

2.5 X Sparsity

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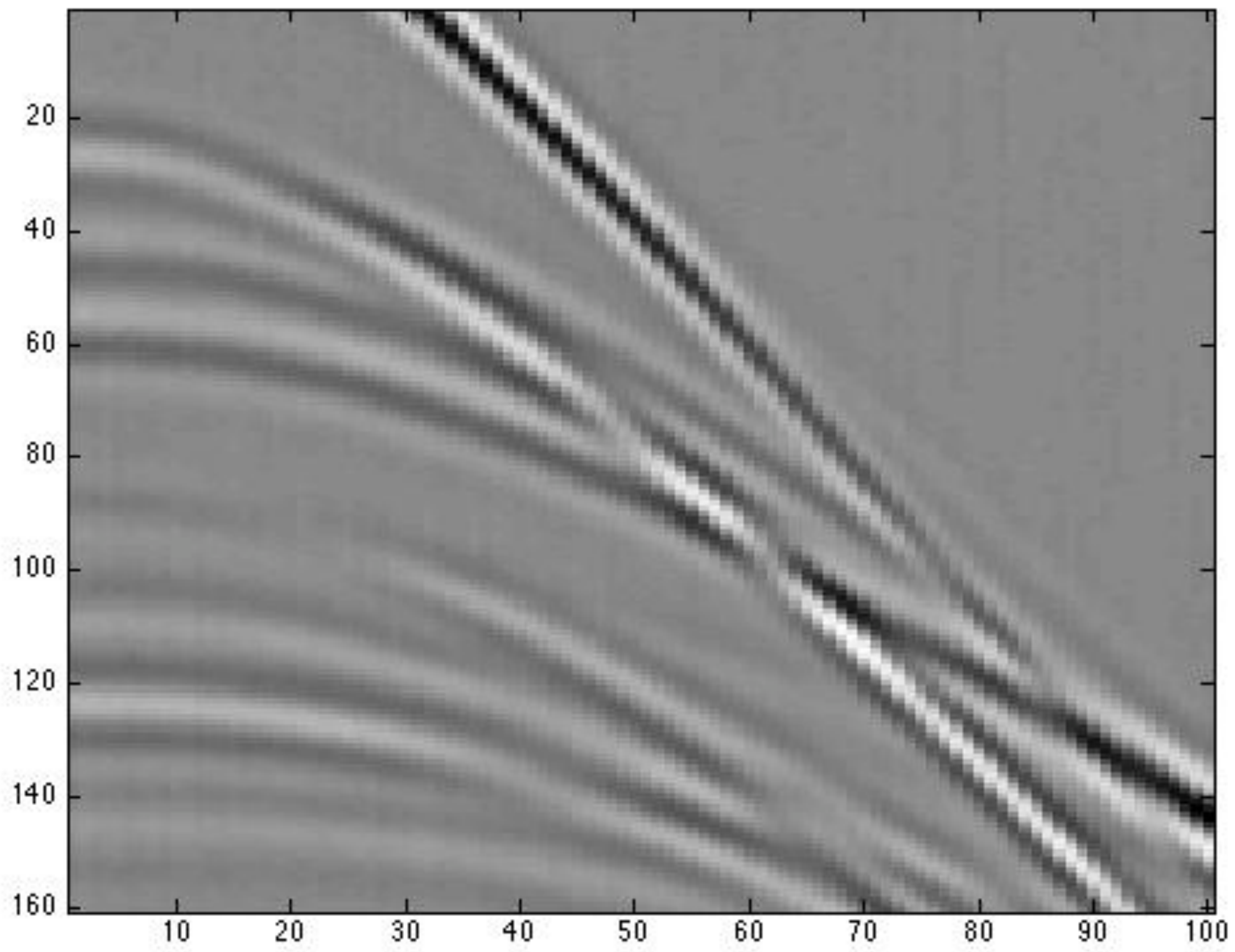
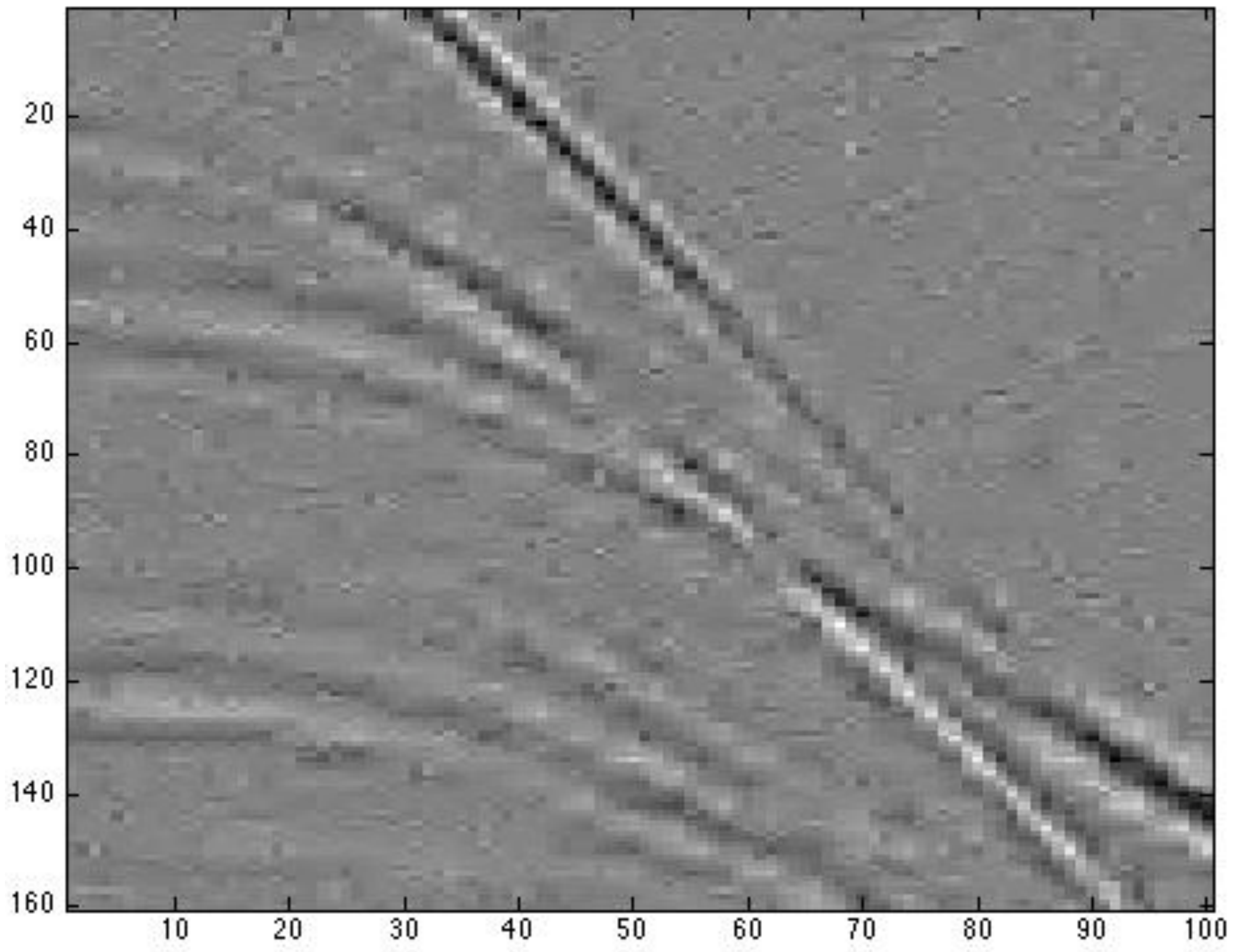
What do you gain?

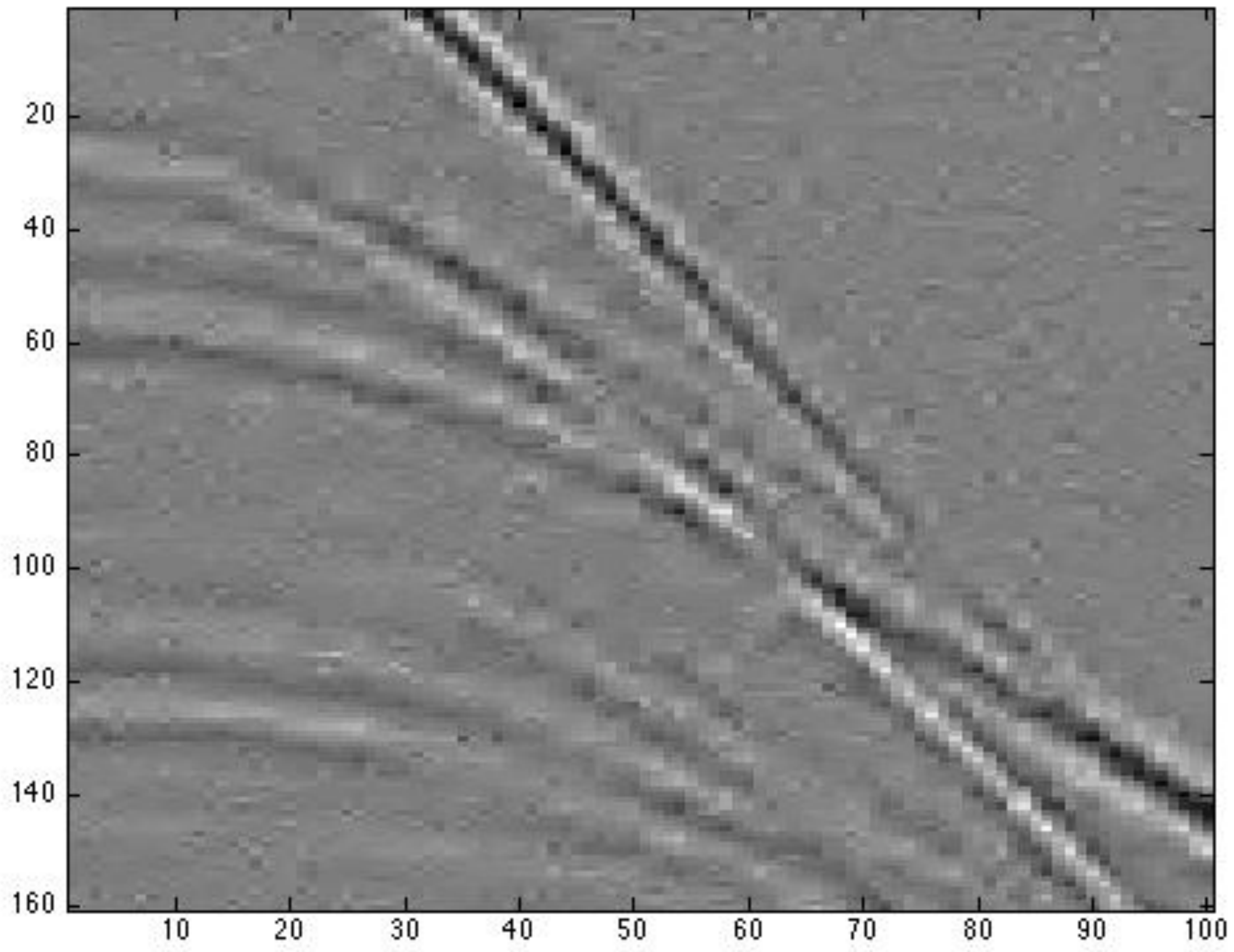
More clear edges on the image with Turbo matrix.

Seismic data with noise



Gaussian vs Turbo



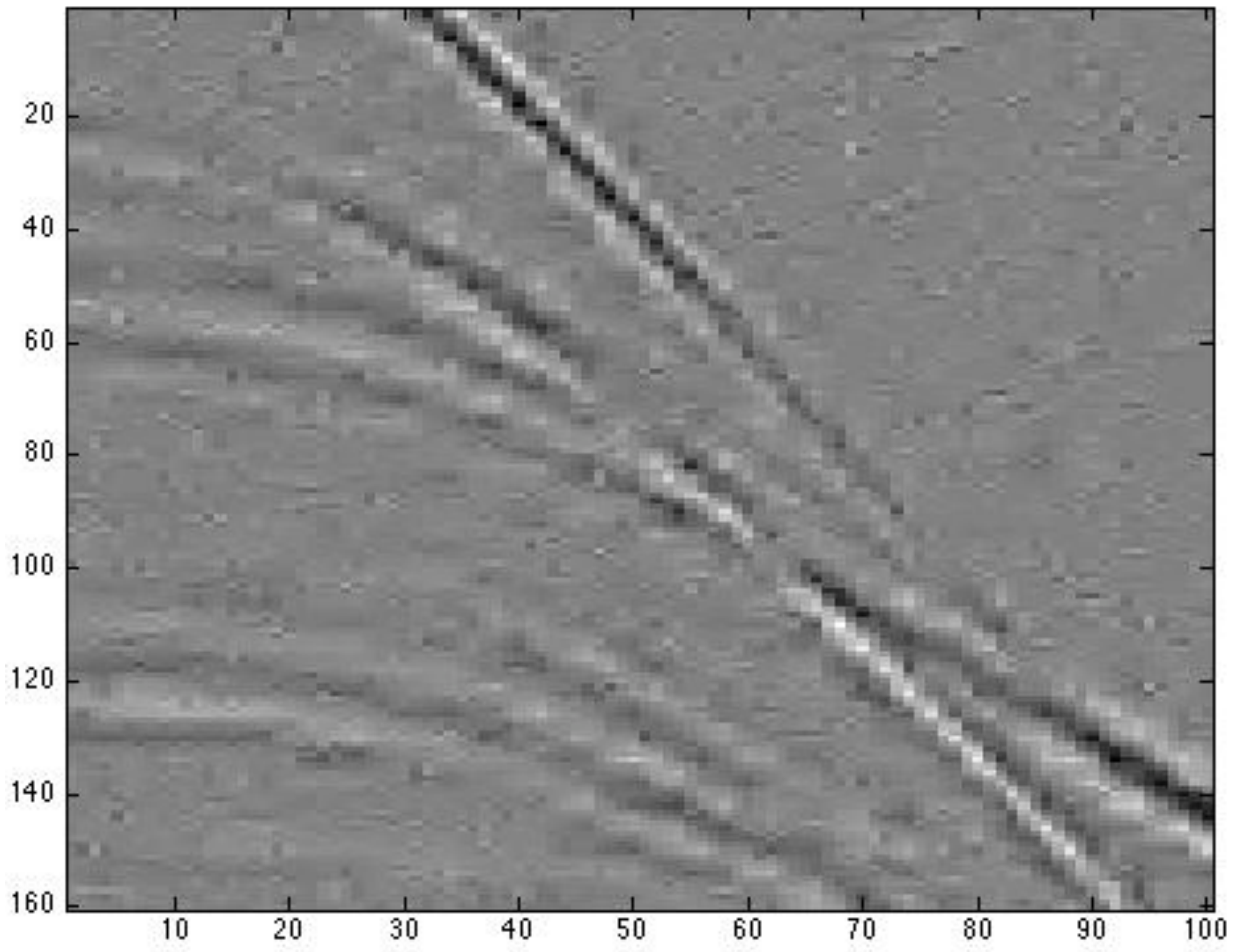


SNR = 5.5

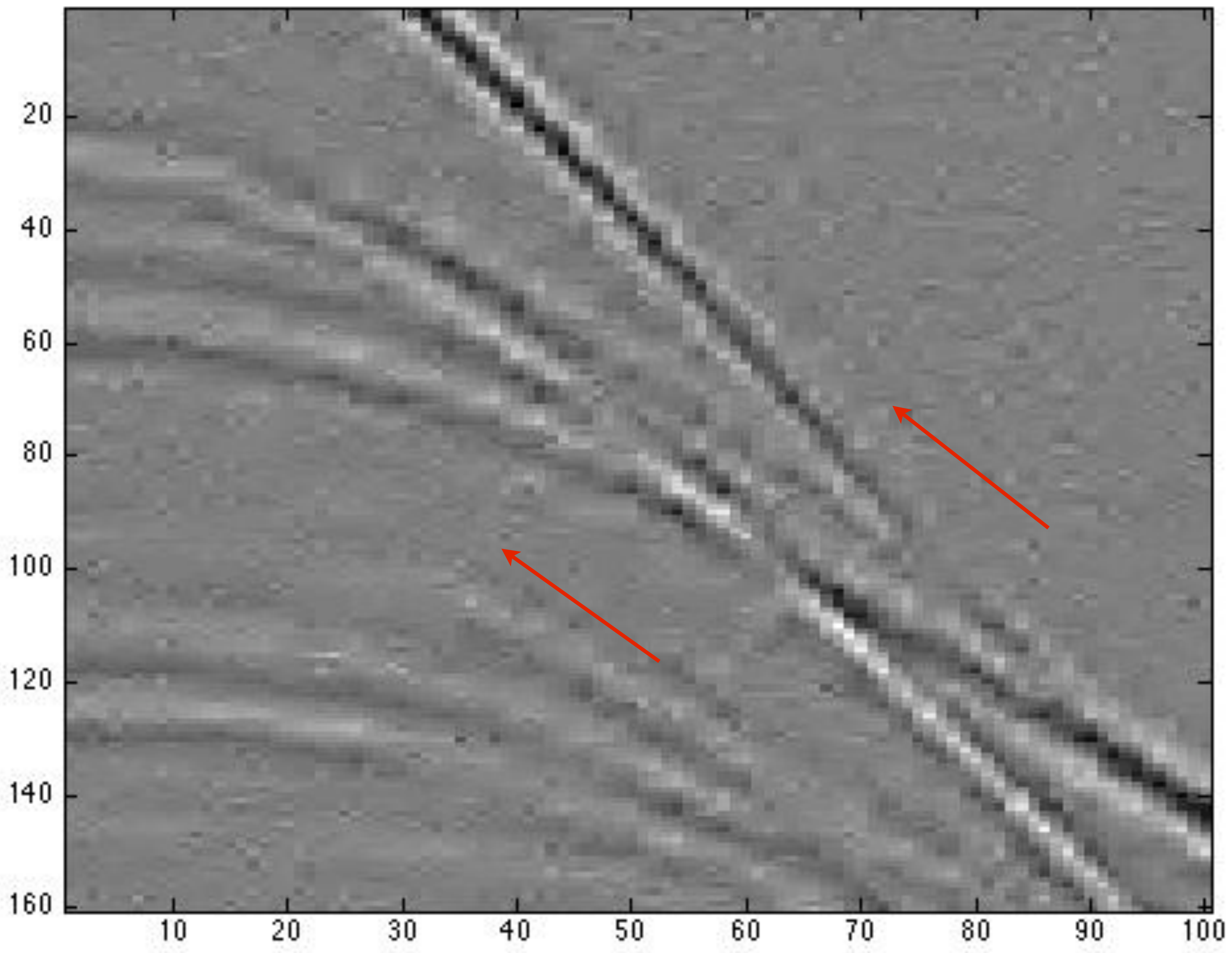
SNR = 6.9

Seismic data with noise

Gaussian vs Turbo



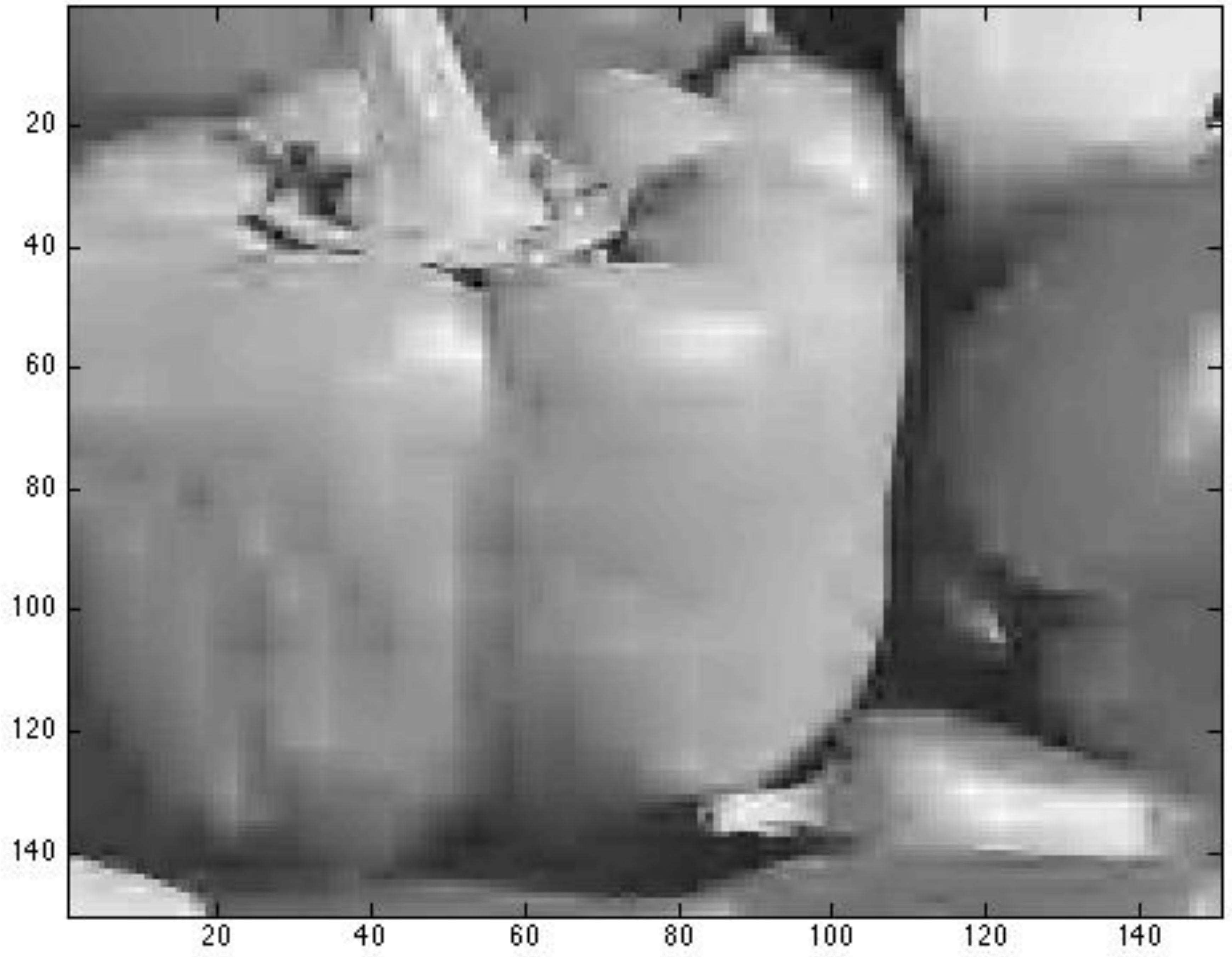
SNR = 5.5



SNR = 6.9

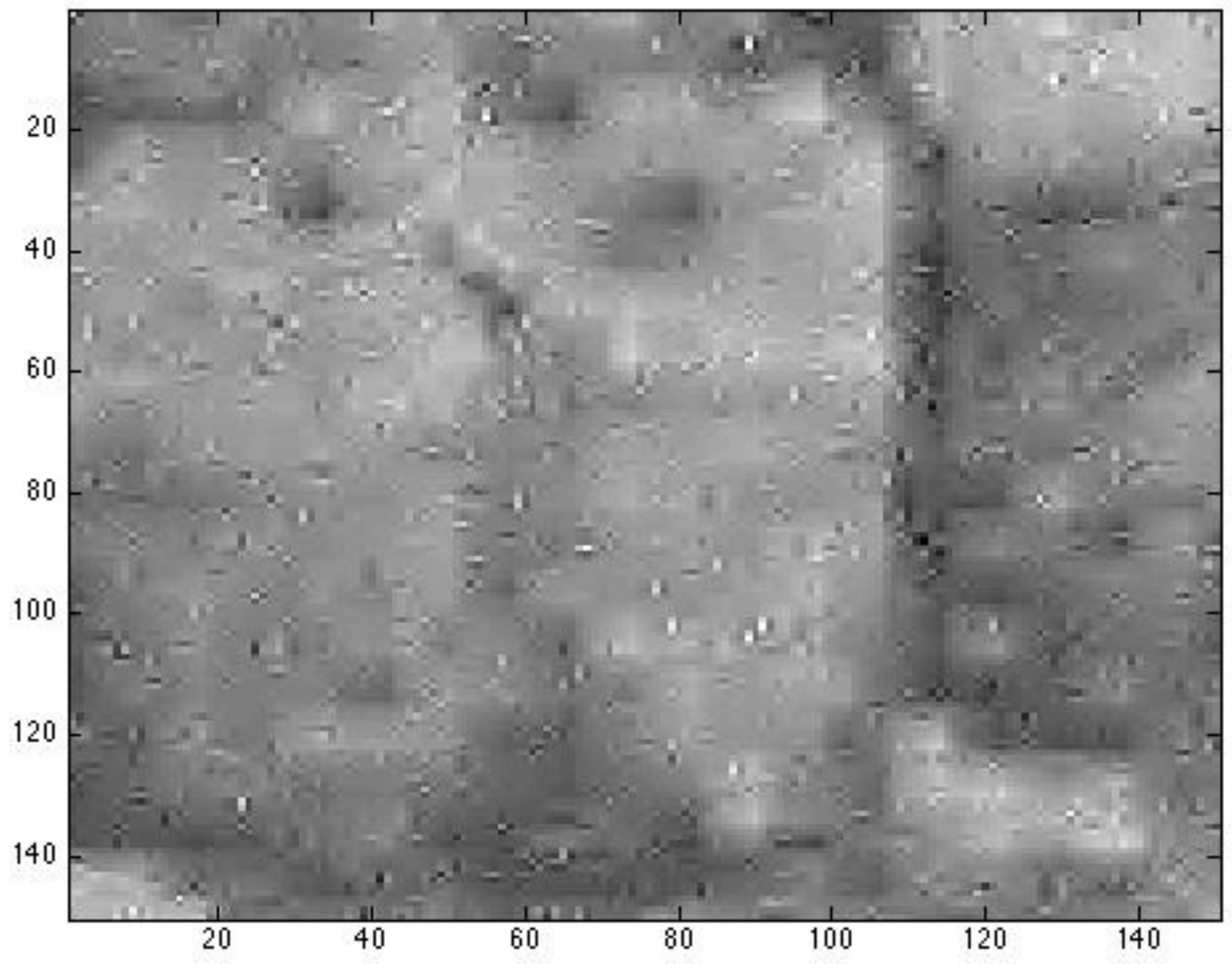
Seismic data with noise

Pepper (5% sparsity)

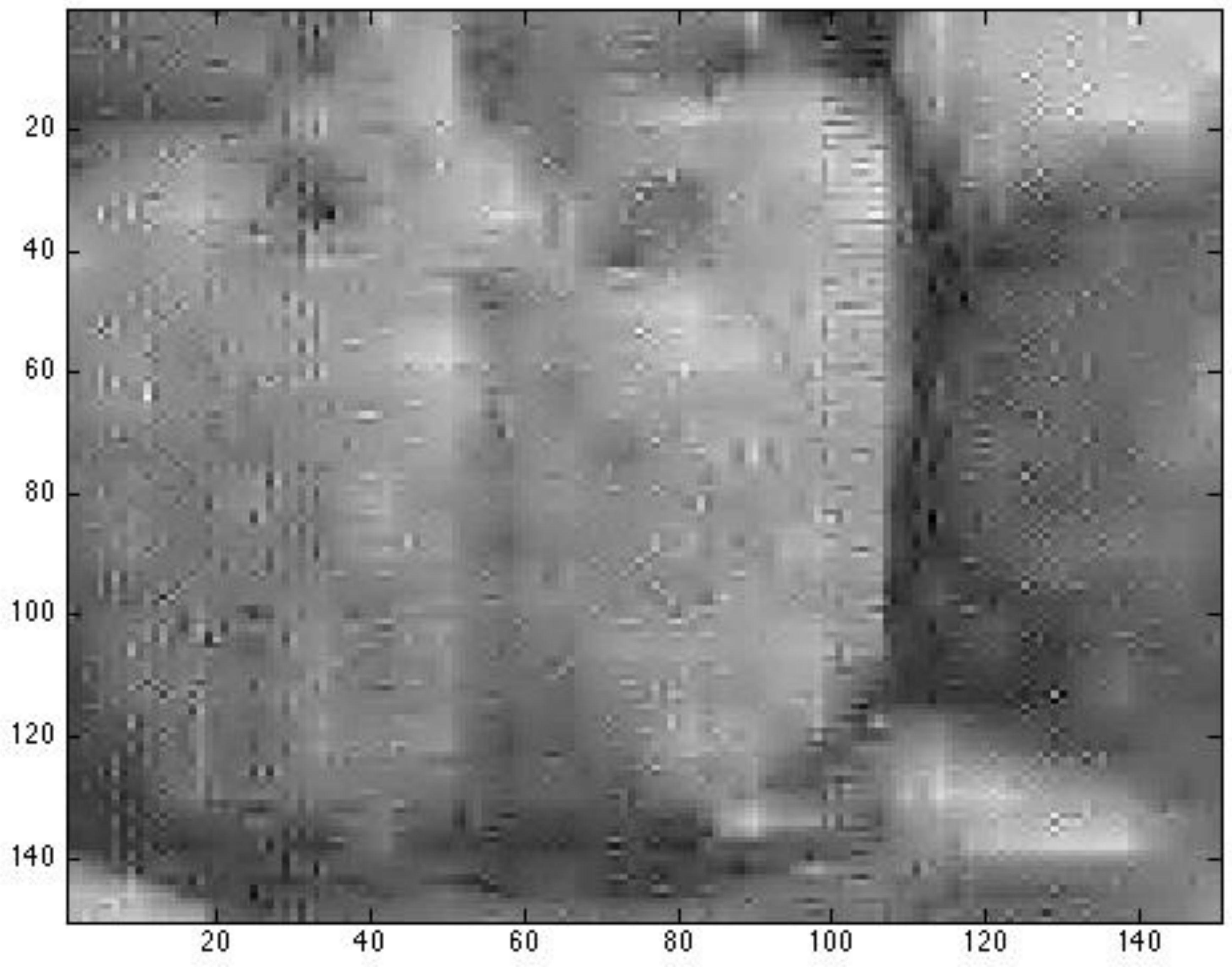


Throw away 95% of the wavelet coefficients

Gaussian vs Turbo

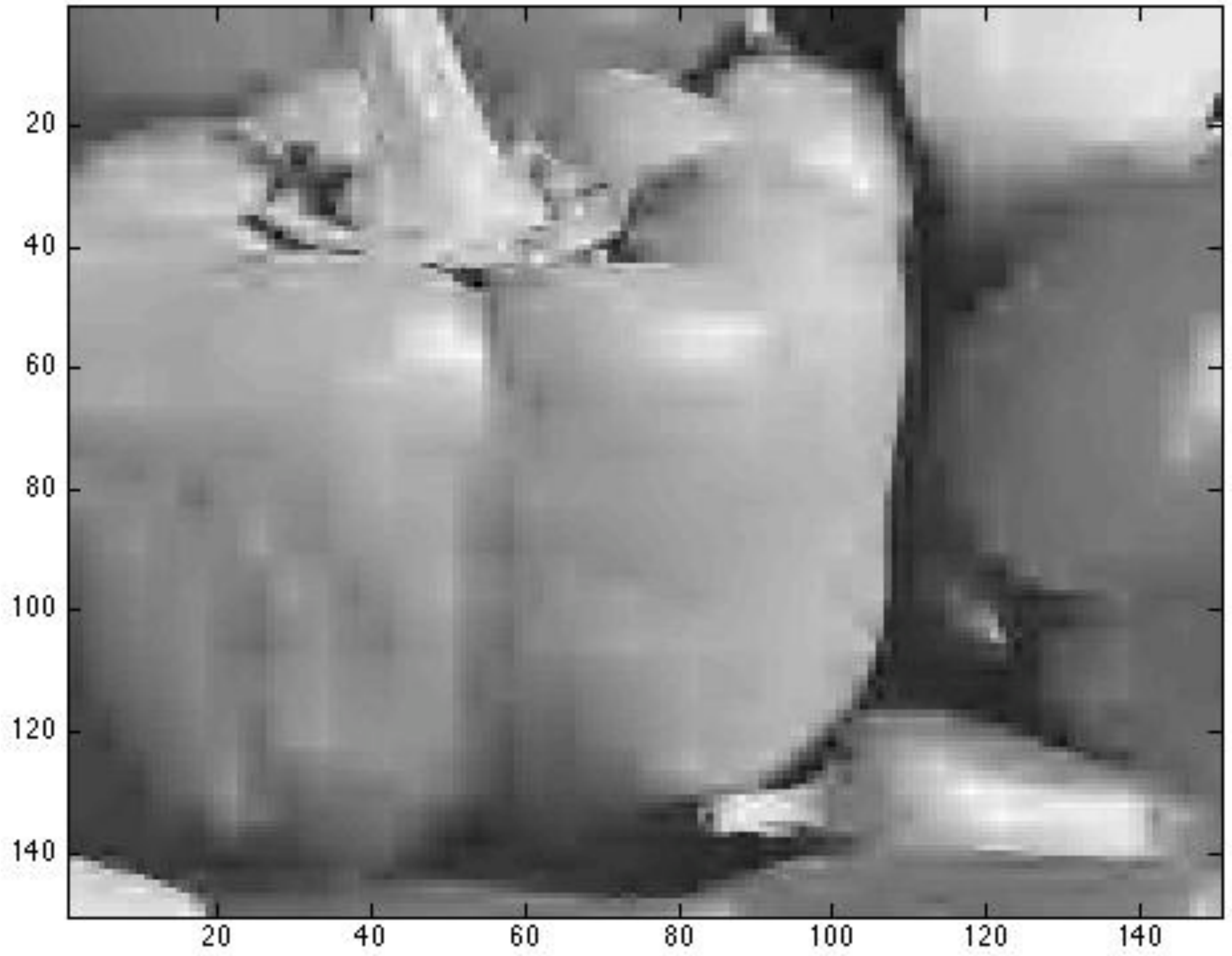


SNR = 10.3



SNR = 13.8

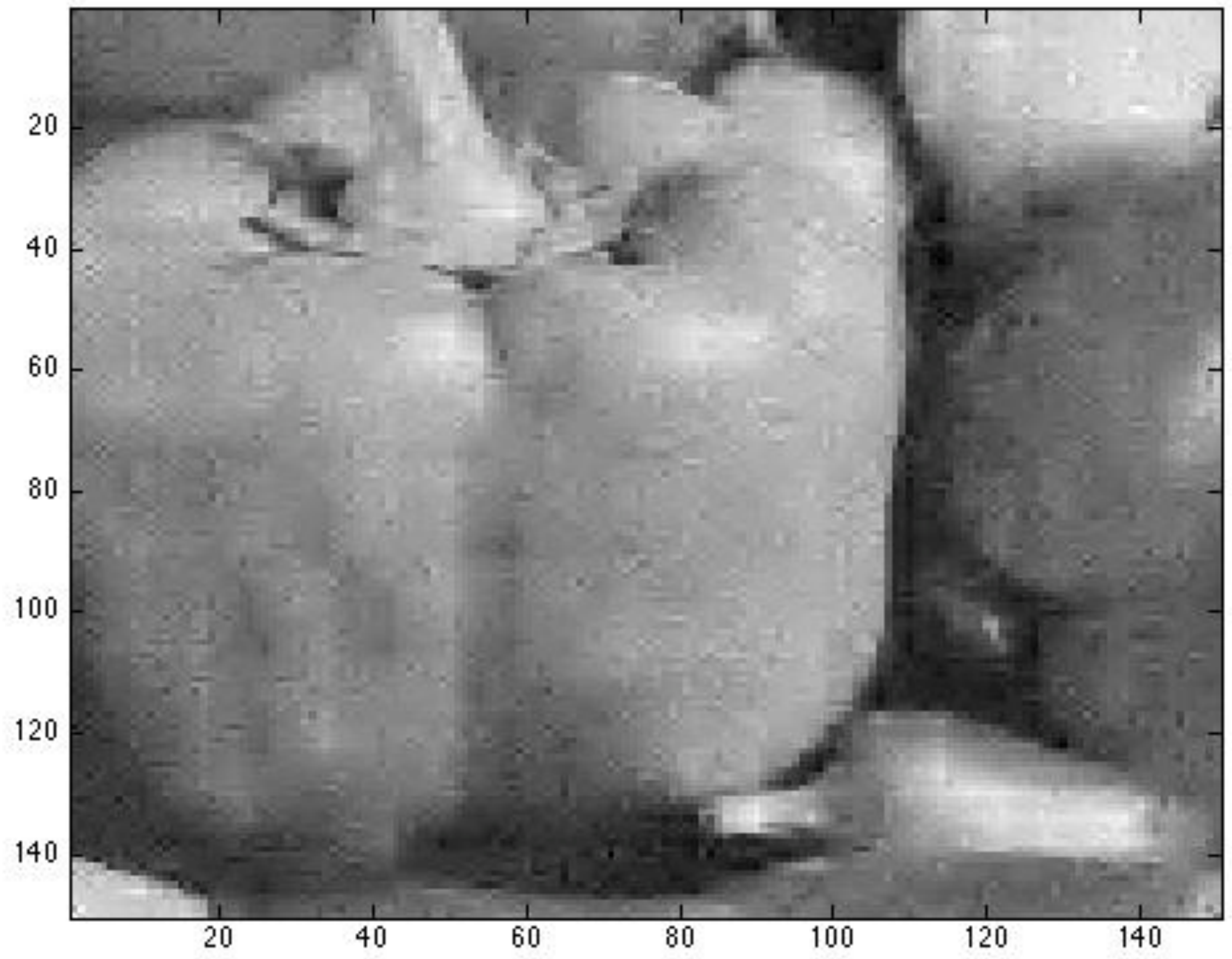
Pepper (10% sparsity)



Throw away 90% of the wavelet coefficients

Gaussian vs Turbo

SNR = 21



SNR = 21

Why do you need curvelet?

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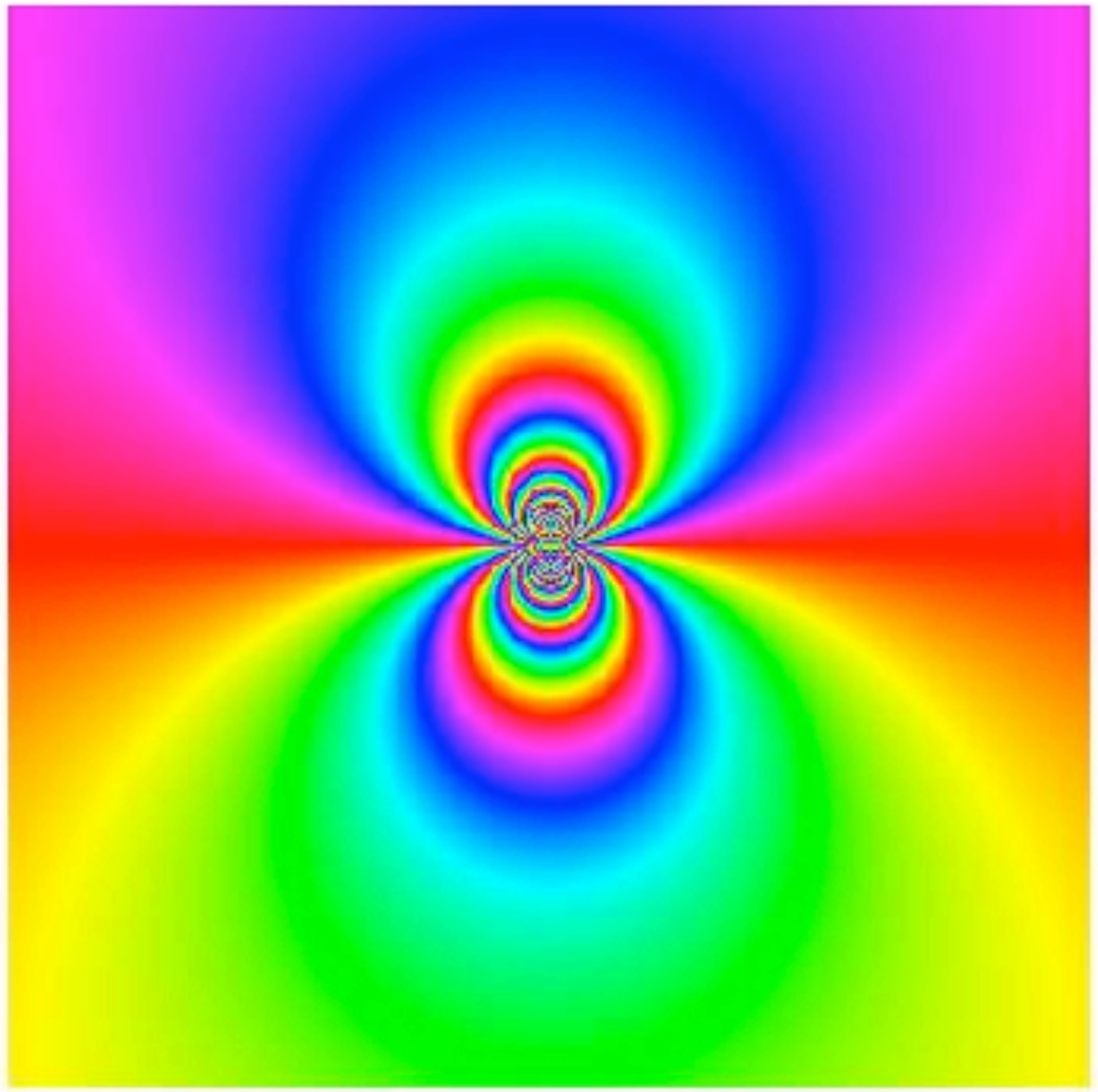

Not sparse if you use wavelet

Moral of the story

What is one main advantage of using a Turbo matrix?

Better reconstruction over Gaussian matrix
when you are ridiculously under-sampling.

Potential improvement

- Turbo matrix outperforms Gaussian matrix when you do not have enough samples
- Understand why
- For what structural data is this true?

Theory behind the scene

Let A be a matrix satisfying the Turbo condition.

Let n be the number of rows in matrix A
so that any s-sparse signal can be recovered.

Assume
$n \geq c_1 \cdot \delta^{-2} \cdot \beta \cdot s \cdot \log^2(4s/\epsilon), \quad \text{where } c_1 = 4\pi^2.$

Then with probability $1 - \epsilon$,

$$1 - \delta \leq \lambda_{\min}(A^*A) \leq \lambda_{\max}(A^*A) \leq 1 + \delta.$$

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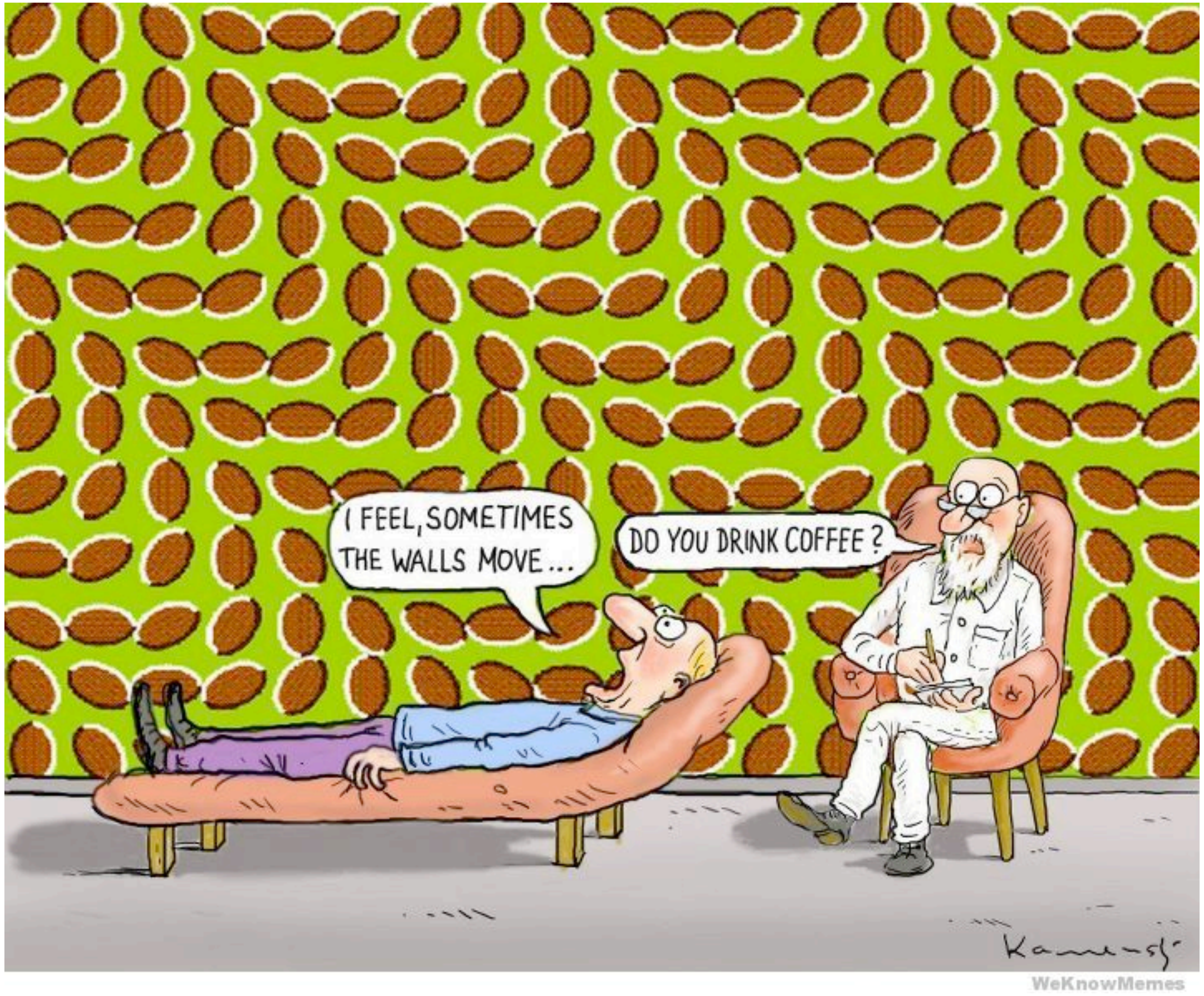
sometimes the walls move



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Conclusion

Turbo matrix leads to better reconstruction on real data while doing just as well when signal is completely random.

Acknowledgement

- Ozgur Yilmaz, Felix Herrmann
- Curt the MAN, Tim Lin
- All siblings from the SLIM family
- All sponsors

Thank You!

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Exploit sparsity structure

Data = sum of wave atoms

Each wave atom has the same shape.

$$X = c_1\psi_1 + c_2\psi_2 + \ldots + c_N\psi_N$$

Suppose coefficients are organized in a tree structure.

Sampling schemes

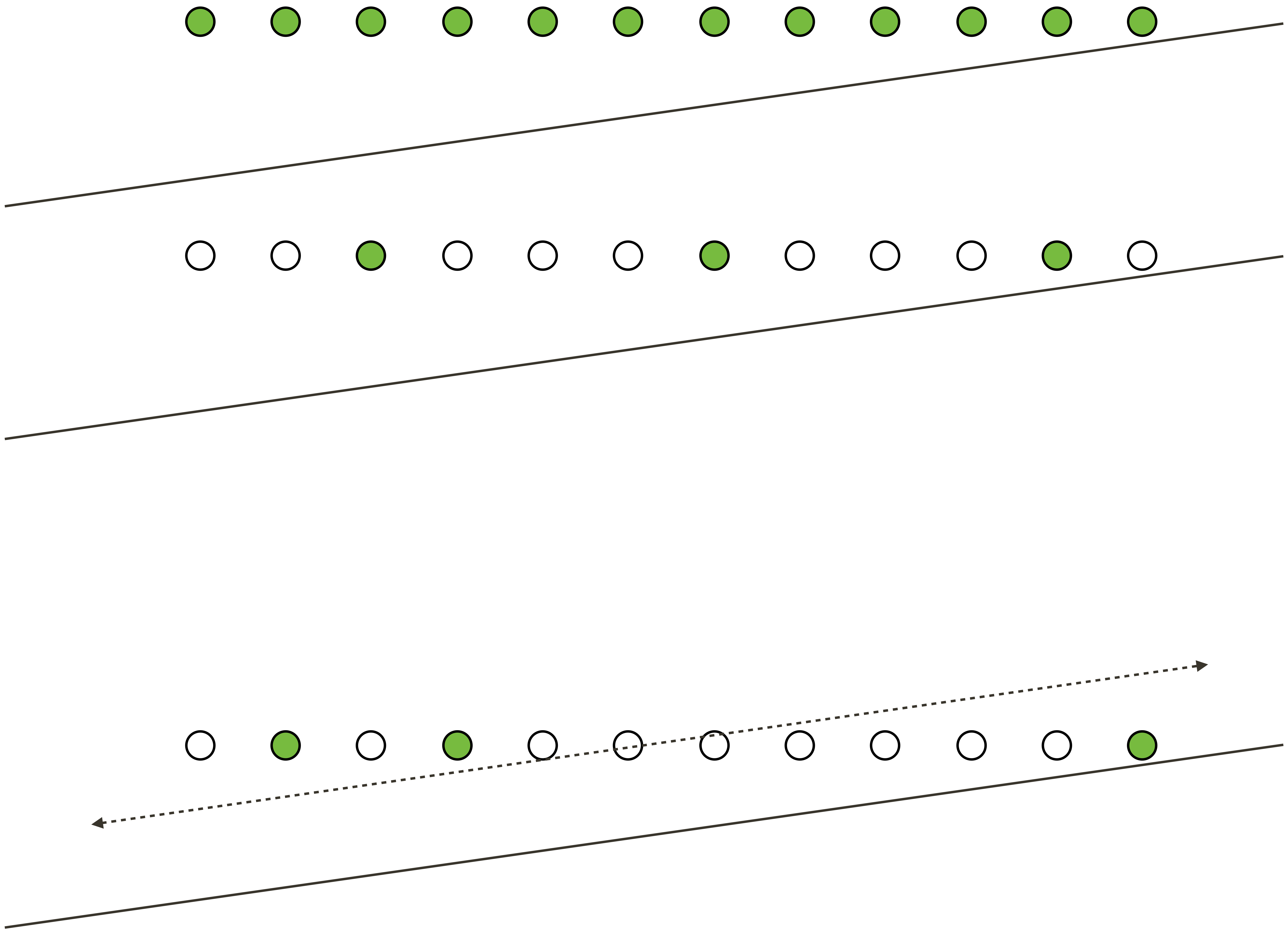
FULL SAMPLING

REGULAR UNDERSAMPLING

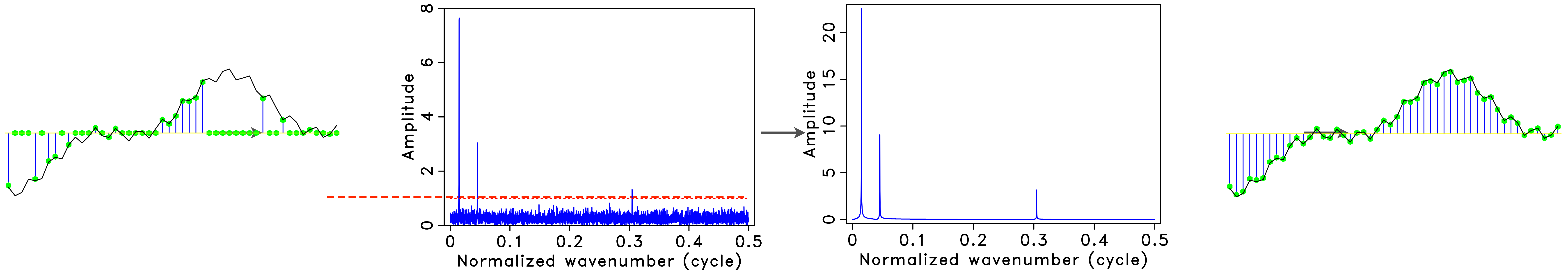
($\eta = 4$)

RANDOM UNDERSAMPLING

($\eta = 4$)



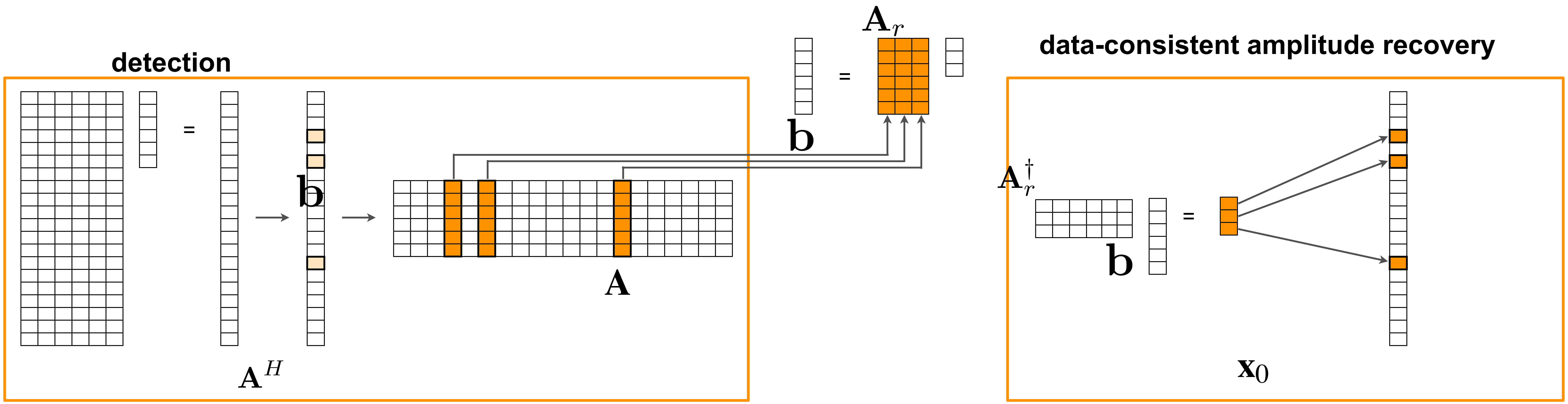
NAIVE sparsity-promoting recovery



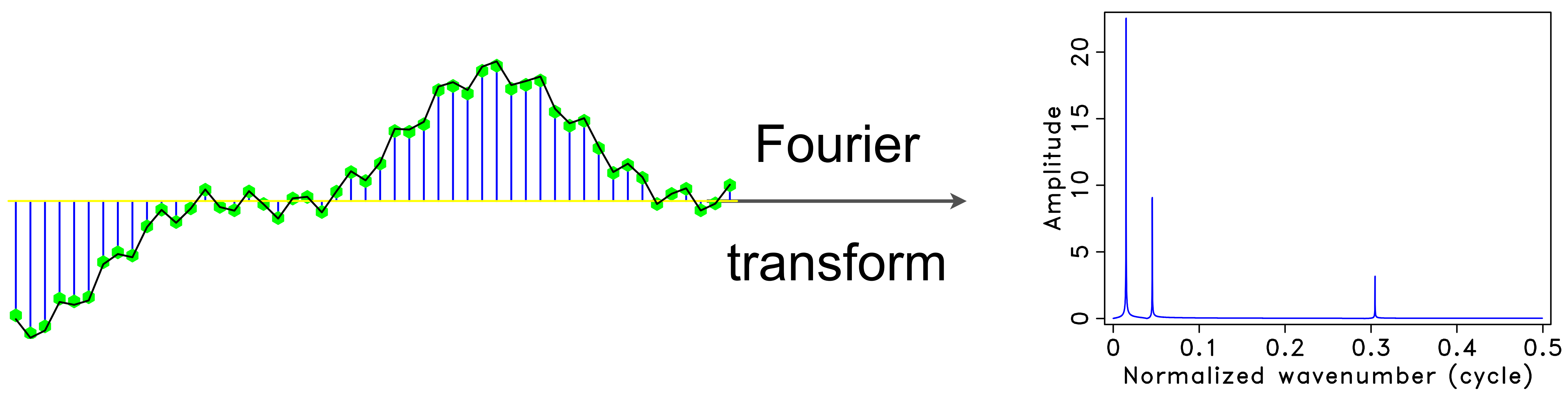
inverse
Fourier
transform

detection +
data-consistent
amplitude recovery

Fourier
transform

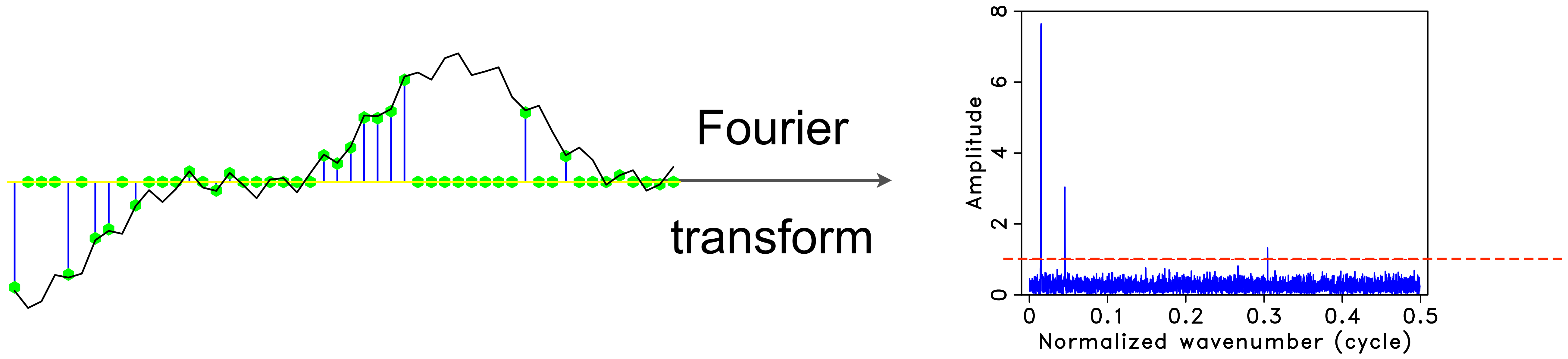


Coarse sampling schemes

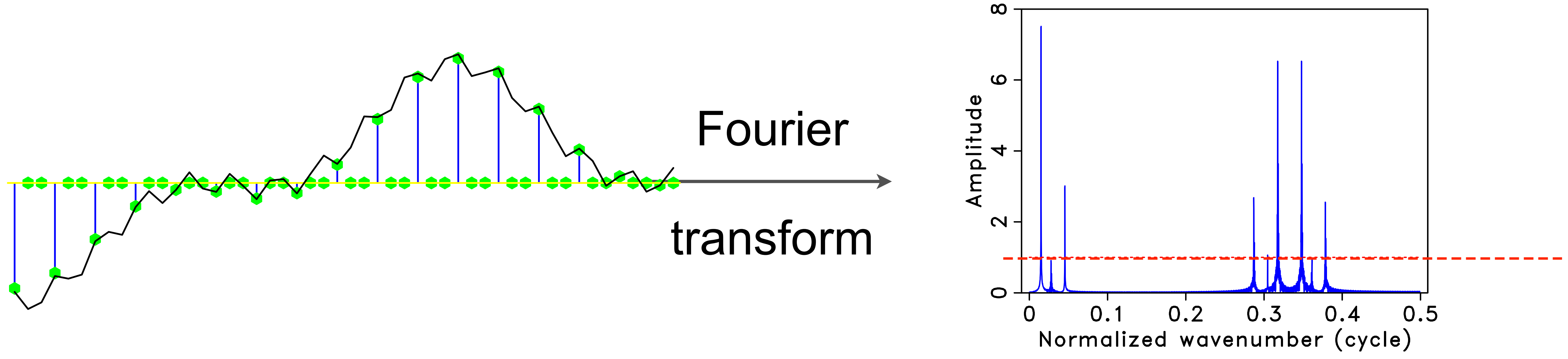


few significant coefficients

3-fold under-sampling



✓ significant coefficients detected



✗ ambiguity

[Hennenfent & Herrmann, ’08]