Solving the data-augmented WE

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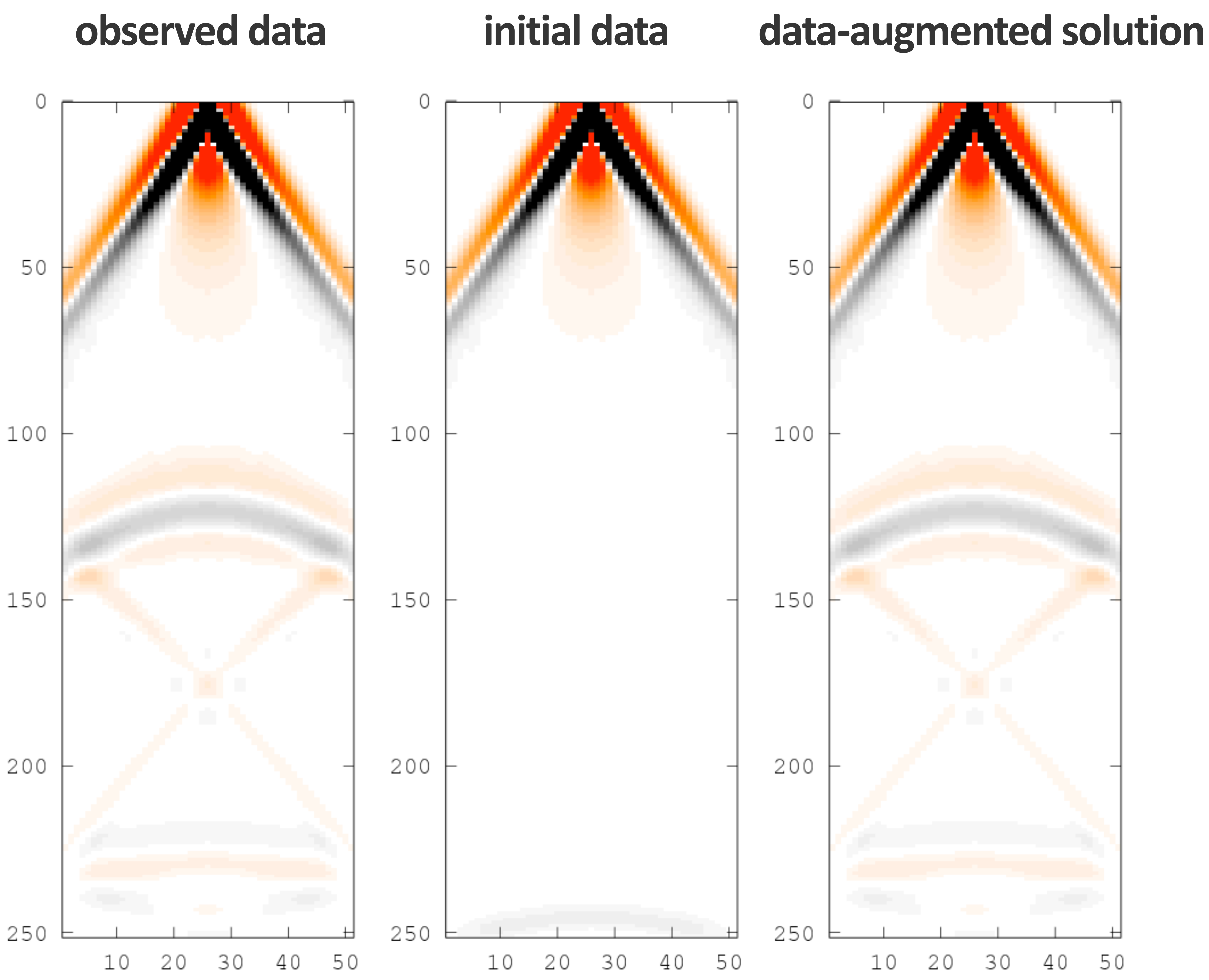
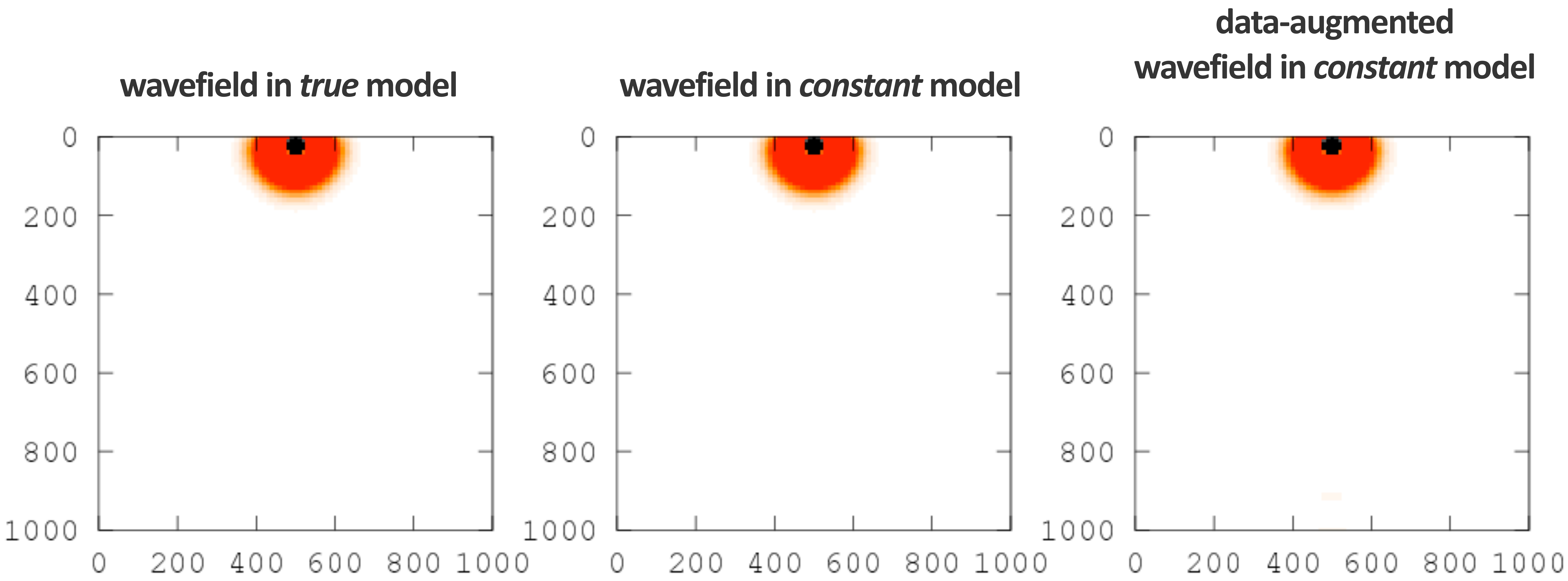
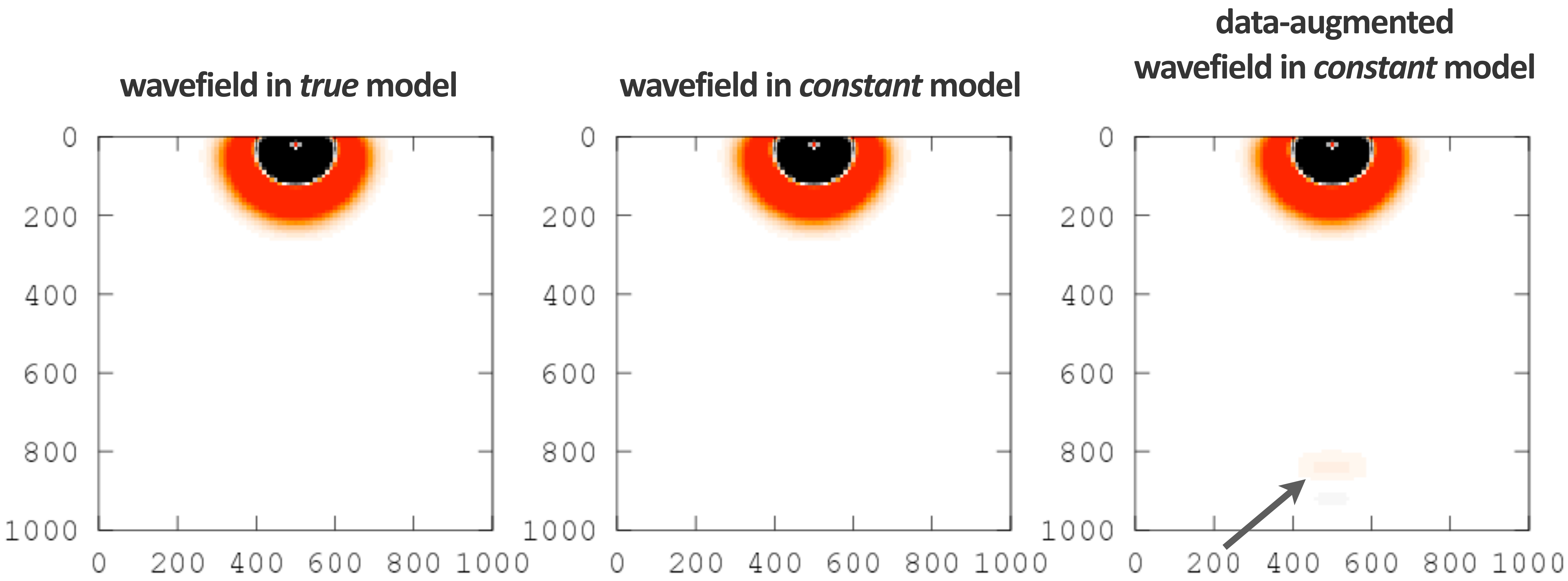
wave-equation X wavefield = source

versus

(wave-equation / sampling operator) X wavefield = (source / data)


observed data
initial data
data-augmented solution
0
50
100
150
200
250
10
20
30
40
50


wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
0
200
400
600
800
1000
0 200 400 600 800 1000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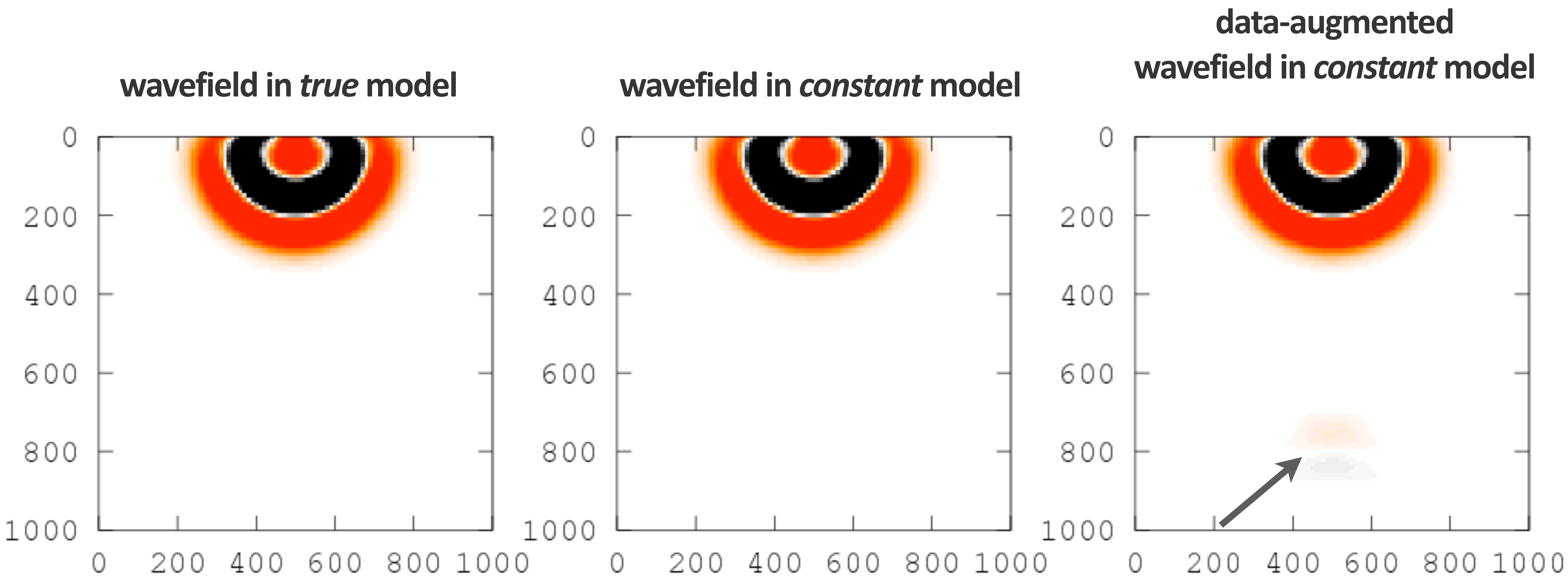

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wavefield in *constant* model
data-augmented
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600
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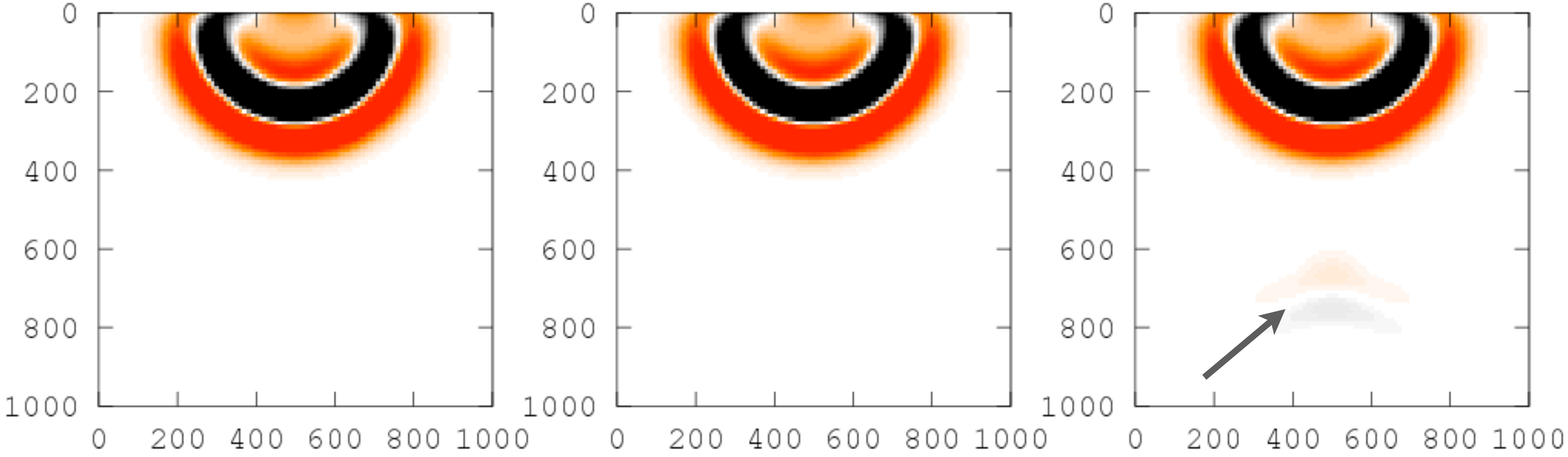
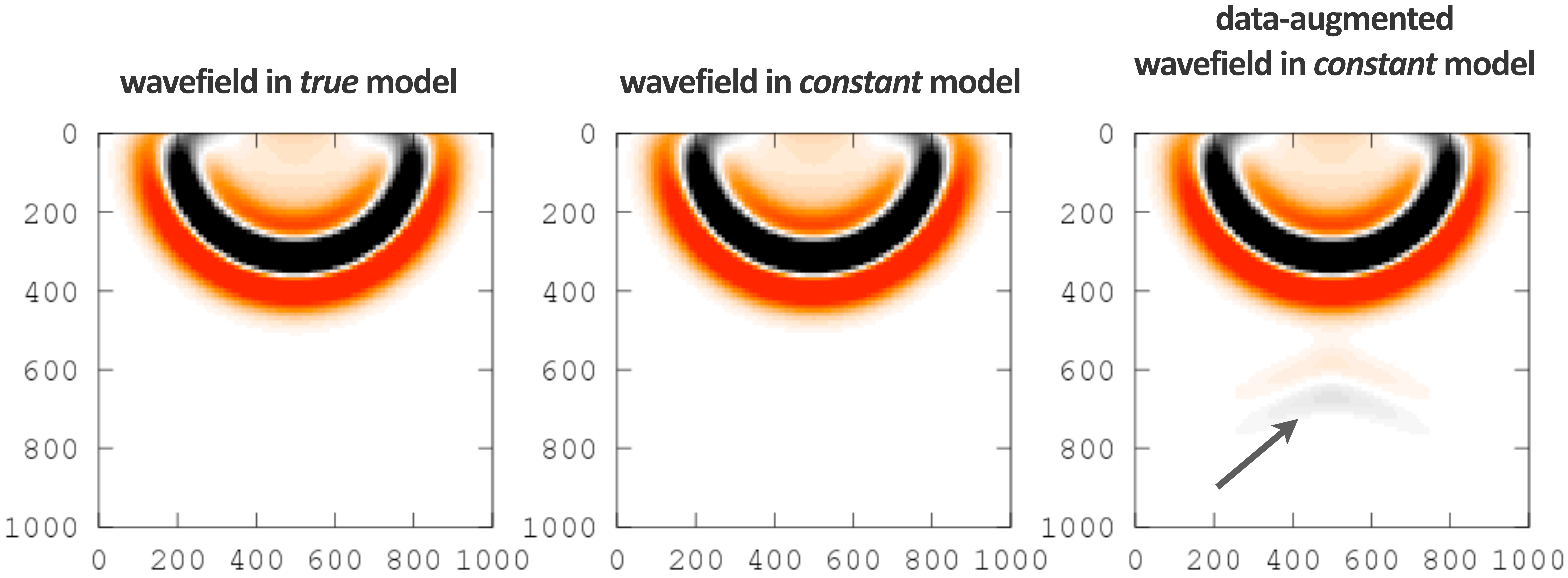
0 200 400 600 800 1000

wavefield in *true* model

wavefield in *constant* model

data-augmented wavefield in *constant* model





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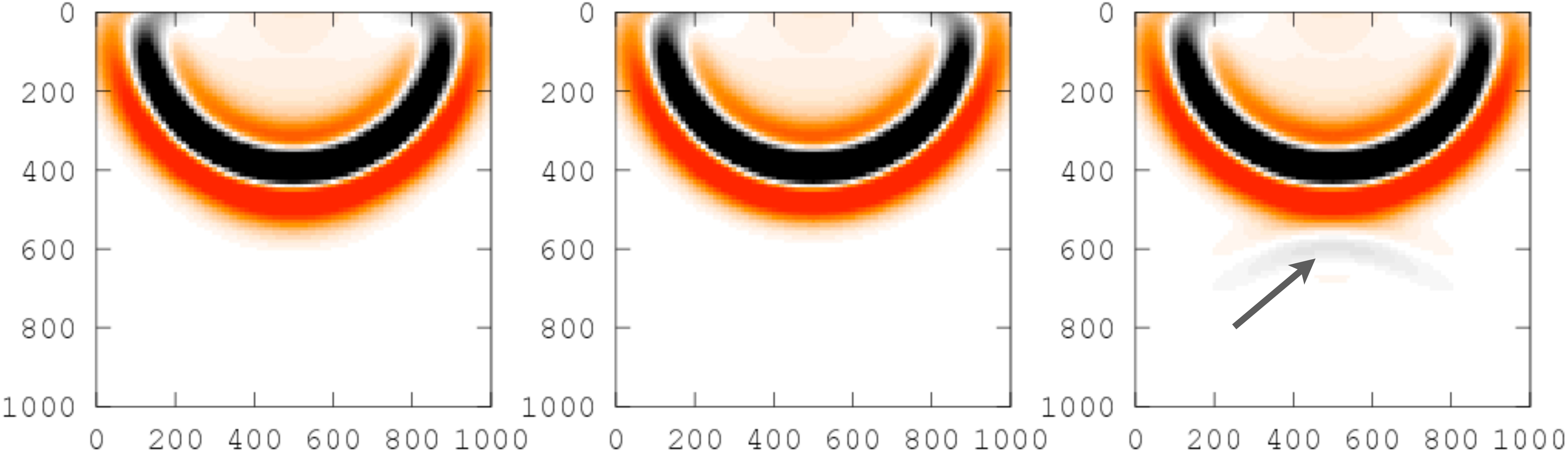
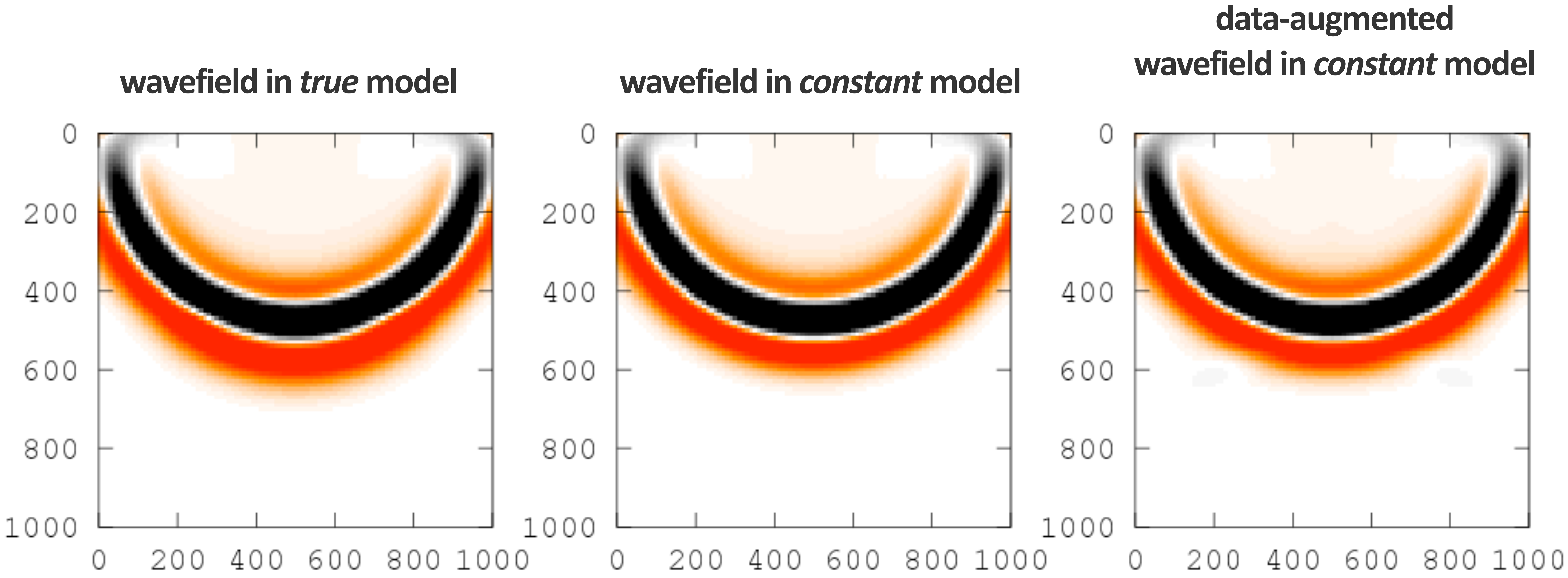
wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**

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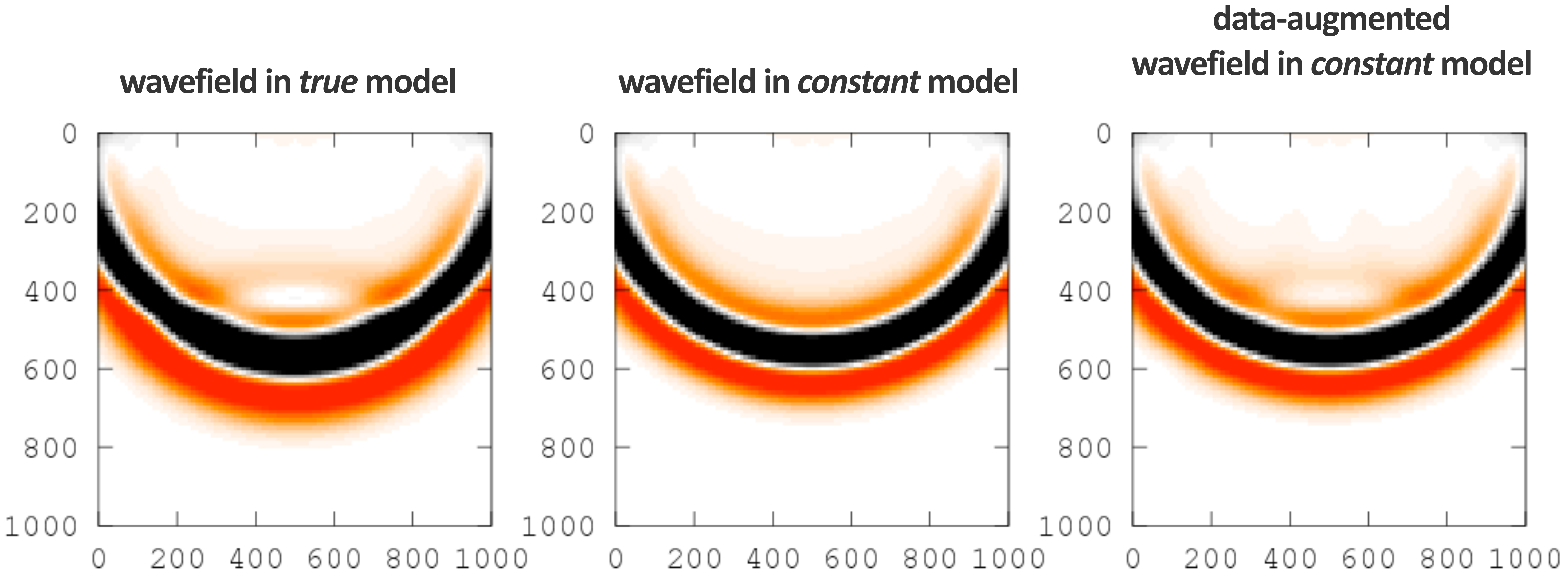
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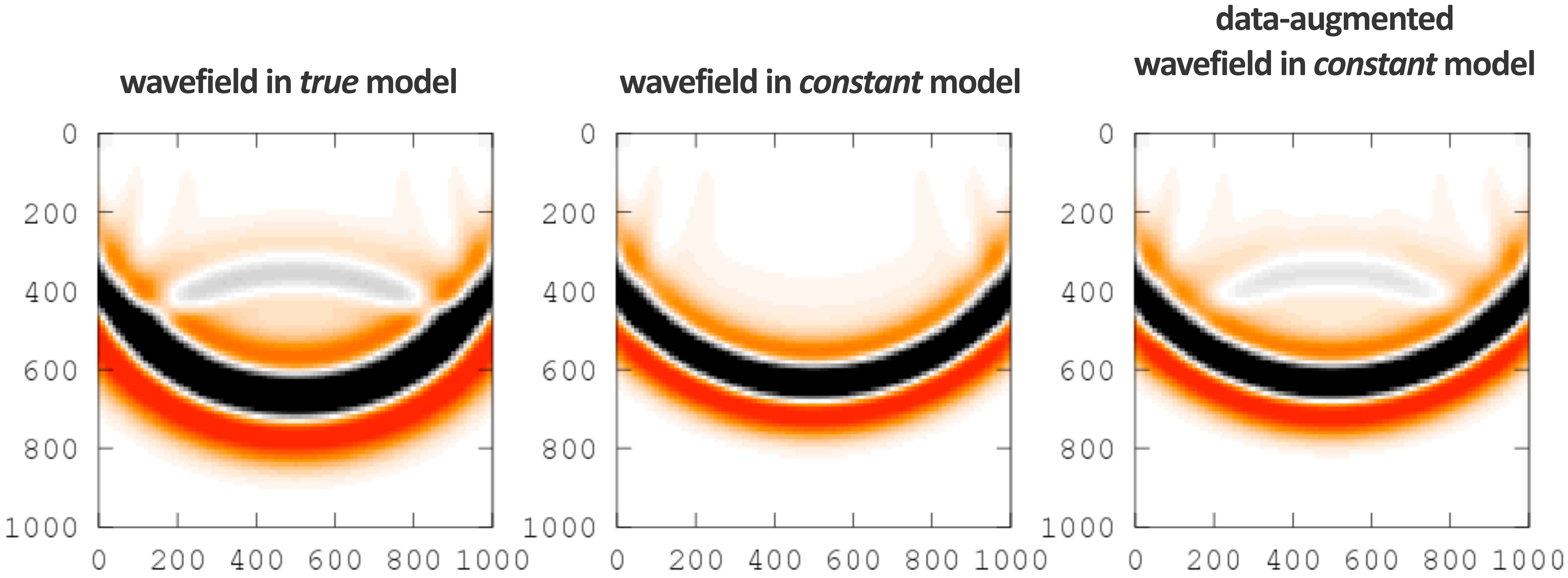
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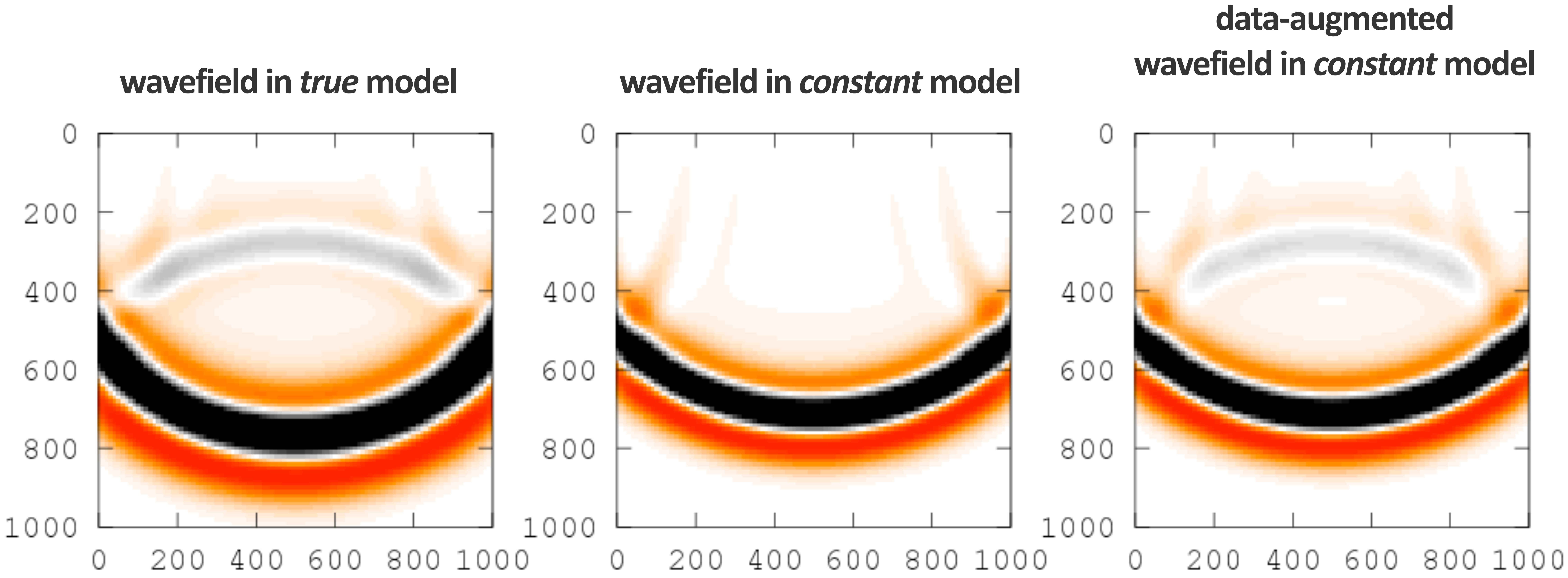
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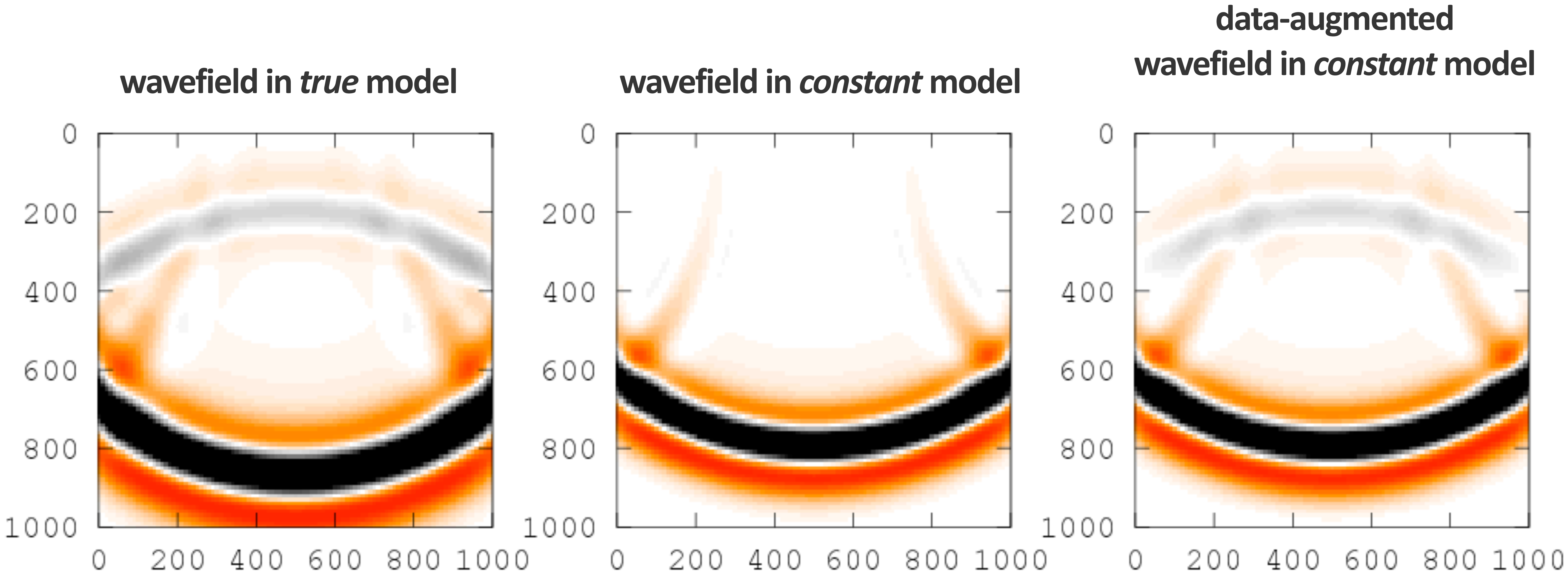

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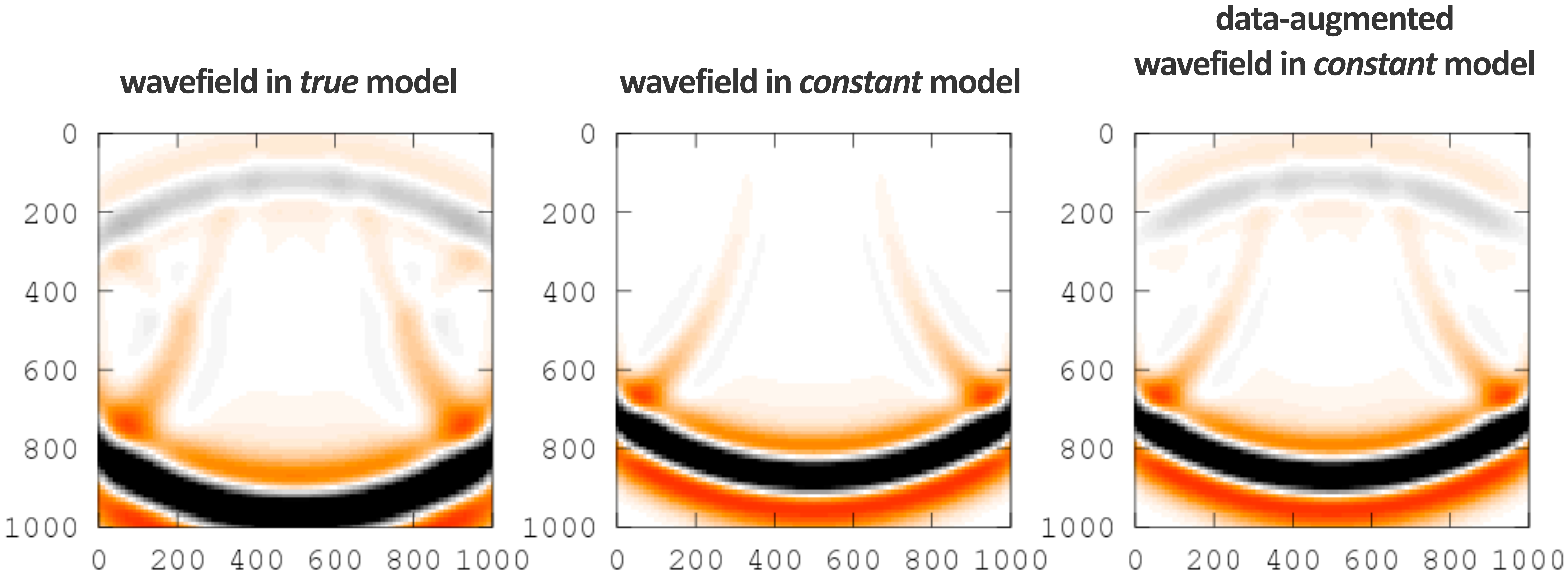

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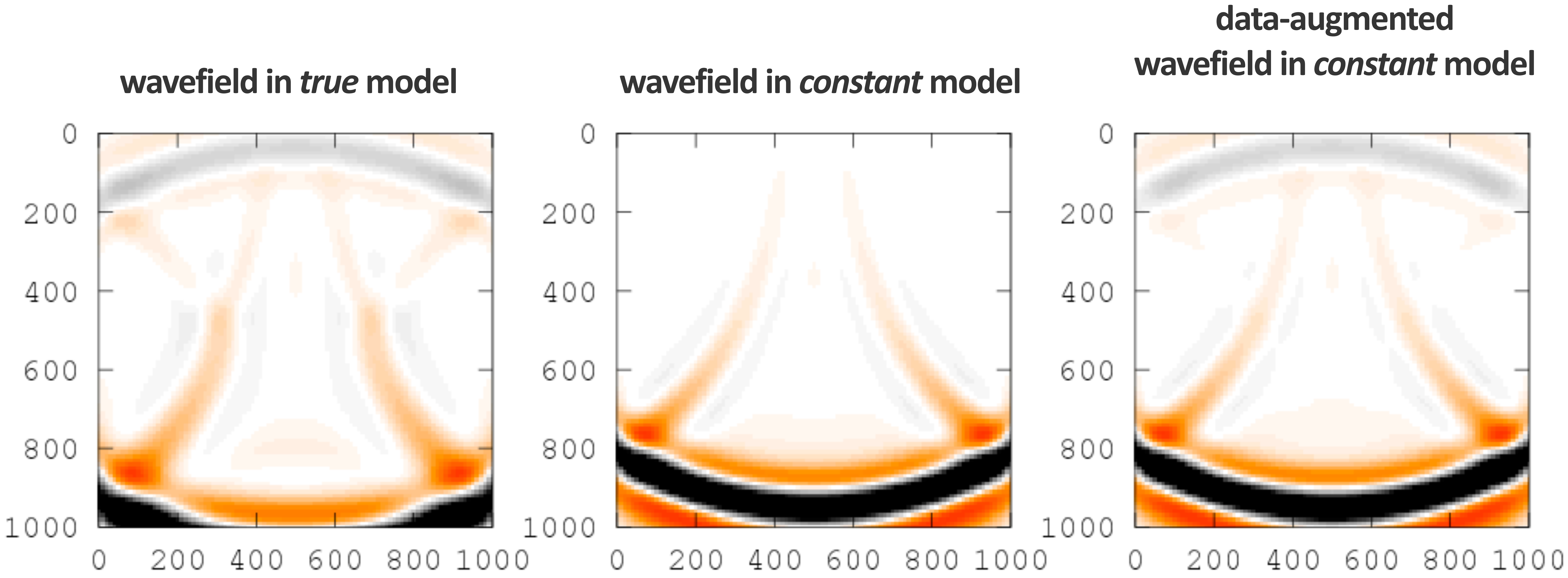

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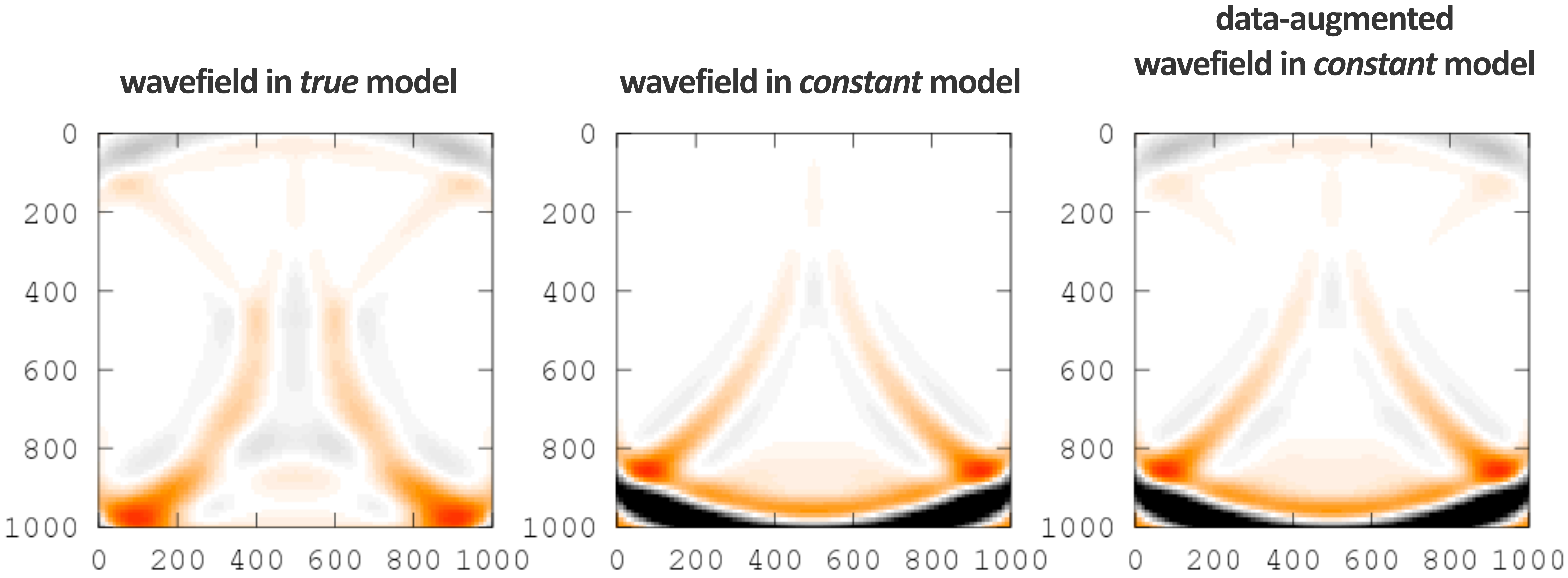
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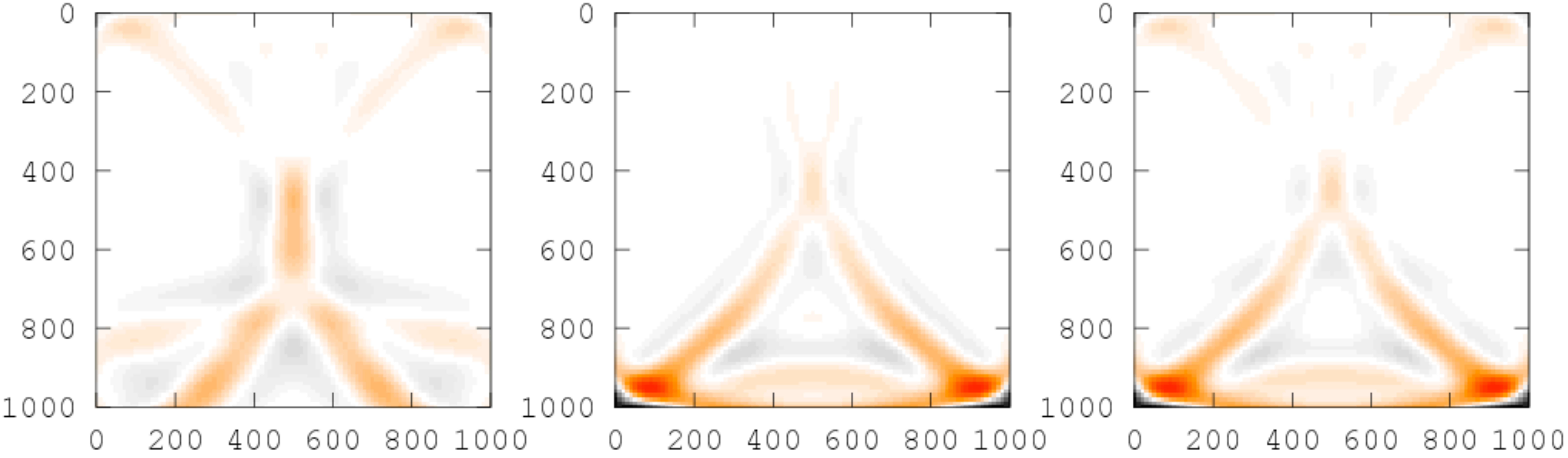

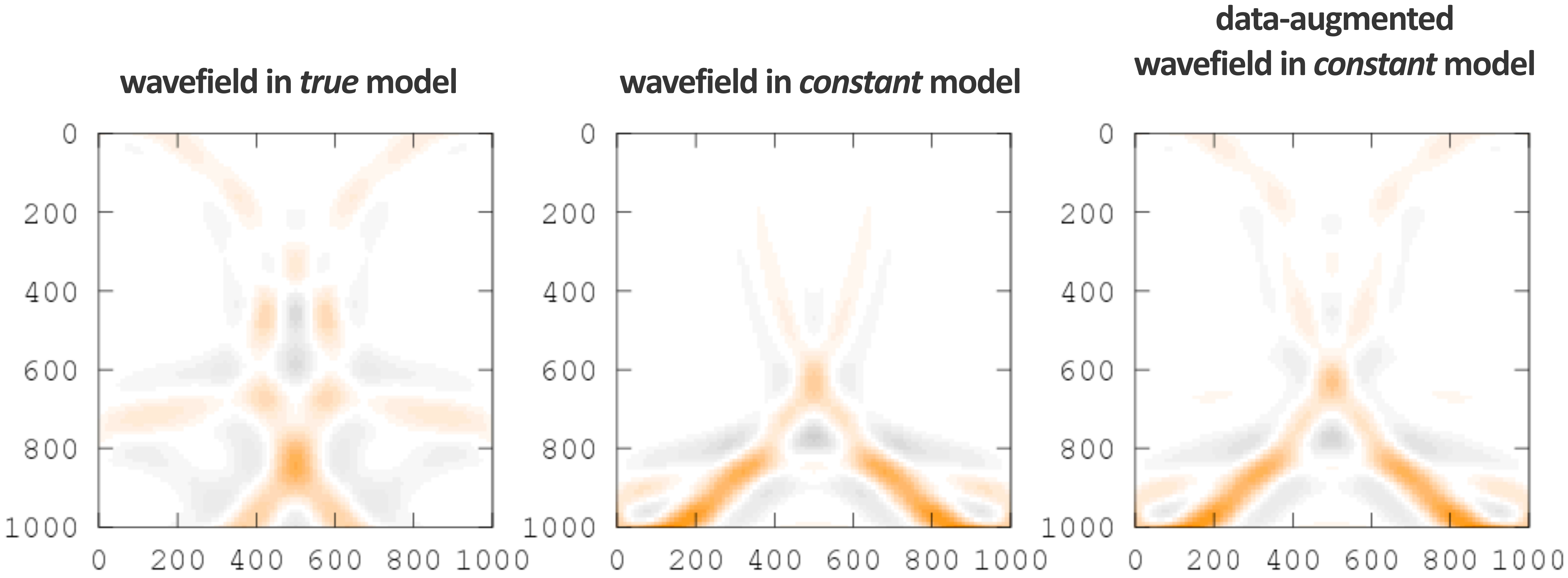
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Outline

Frequency-domain

- direct factorization
- iterative methods

Time-domain

- time-stepping for overdetermined ODEs

Frequency-domain

(mildly) Overdetermined system:

Helmholtz ($n \times n$) → A

Sampling ($nr \times n$) → P

$$\begin{pmatrix} A \\ P \end{pmatrix} \mathbf{u} \approx \begin{pmatrix} \mathbf{q} \\ \mathbf{d} \end{pmatrix}$$

$\mathbf{q}$ ← source

$\mathbf{d}$ ← data

Normal equations

$$(A^*A + P^*P)\,\mathbf{u} = \mathbf{A}^*\mathbf{q} + P^*\mathbf{d}$$

A^*A: full-rank

P^*P: rank nr

Direct factorization

QR factorization:

$$\begin{pmatrix} A \\ P \end{pmatrix} = QR$$

orthogonal (Q) lower-triangular (R)

Choleski:

$$(A^*A + P^*P) = LL^*$$

lower-triangular (L)

OK for 2D, prohibitive for 3D?!

Iterative methods

Solve overdetermined system

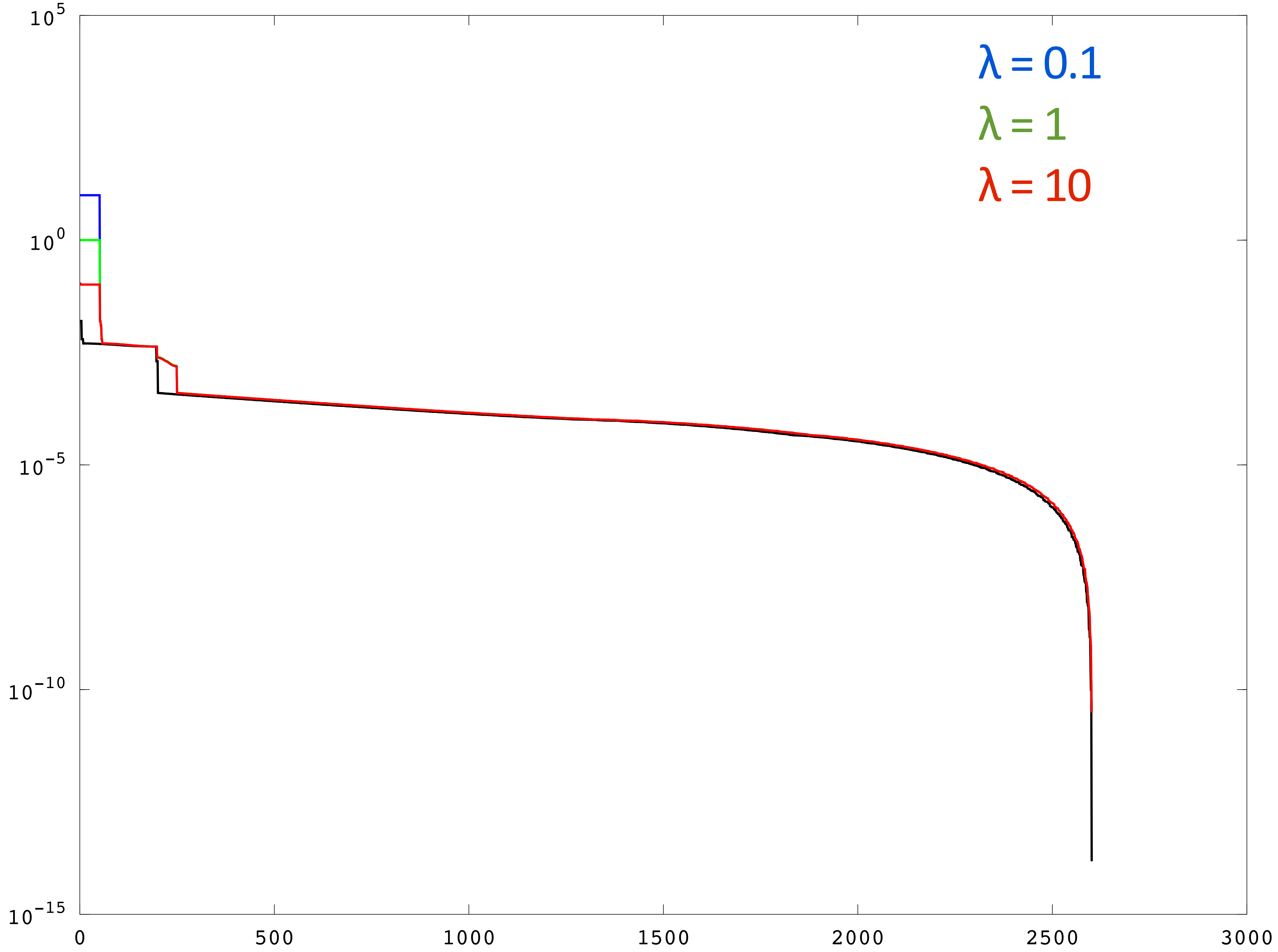
$$\begin{pmatrix} A \\ P \end{pmatrix} \mathbf{u} \approx \begin{pmatrix} \mathbf{q} \\ \mathbf{d} \end{pmatrix}$$ or

$$\begin{pmatrix} A \\ P \end{pmatrix} \delta\mathbf{u} \approx \begin{pmatrix} 0 \\ \mathbf{d} - P\mathbf{u}_0 \end{pmatrix}$$ where $$\begin{aligned} \mathbf{u}_0 &= A^{-1}\mathbf{q} \\ \mathbf{u} &= \mathbf{u}_0 + \delta\mathbf{u} \end{aligned}$$

- LSQR/LSMR
- CGLS?!

Conditioning

$$(\lambda A^*A + P^*P)$$



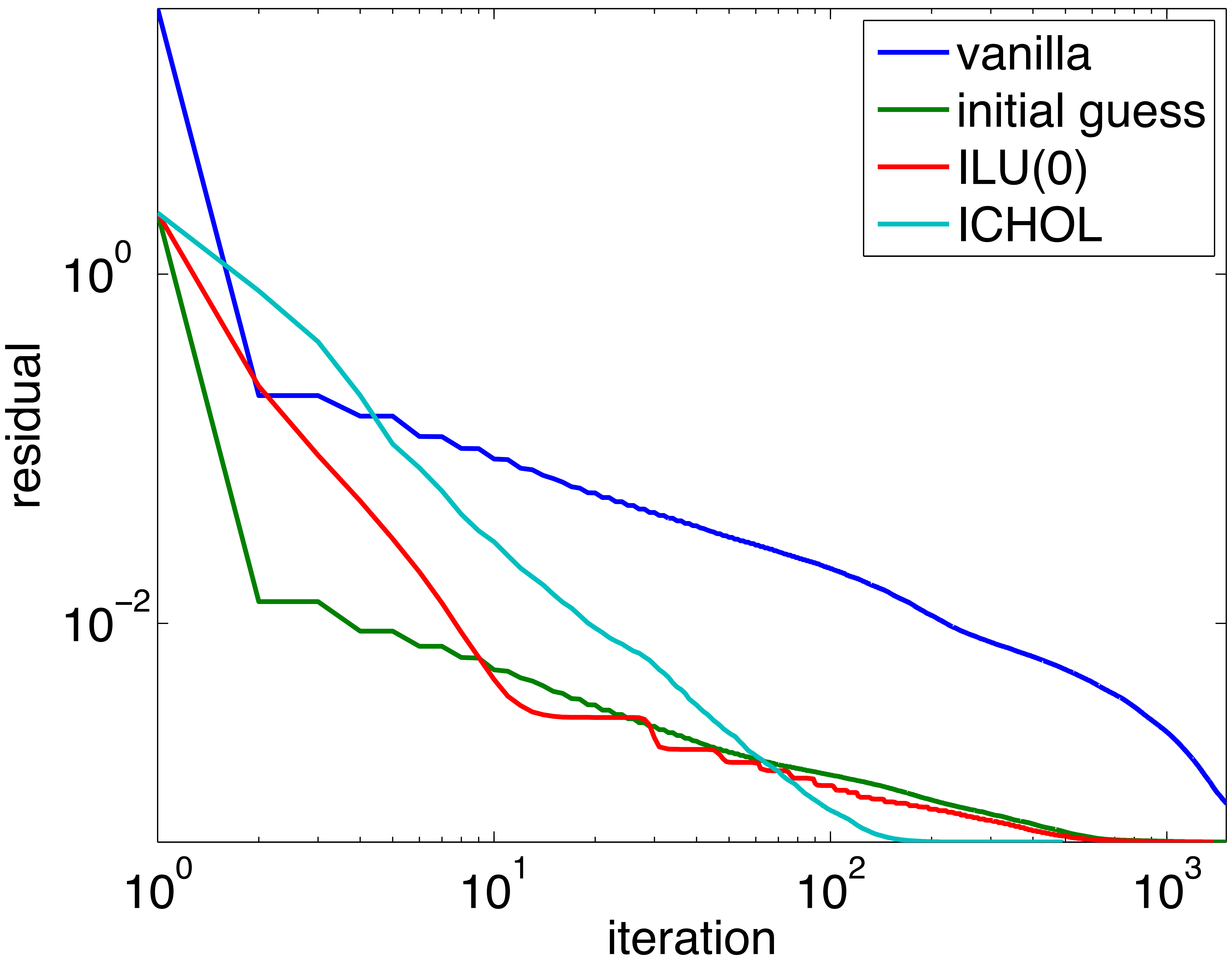
Preconditioners

Use right-preconditioner and solve

$$\begin{pmatrix} A \\ P \end{pmatrix} M^{-1}\mathbf{v} \approx \begin{pmatrix} \mathbf{q} \\ \mathbf{d} \end{pmatrix}$$

$$\mathbf{u} = M^{-1}\mathbf{v}$$

- incomplete LU of A
- incomplete Choleski of $A^*A + P^*P$
- ...


vanilla
initial guess
ILU(0)
ICHOL
residual
iteration
10^0
10^{-2}
10^0
10^1
10^2
10^3

Preconditioners

Helmholtz:

- Multigrid (Erlangga ’05)
- Sweeping (Enquist ’10, Pouson ’12)
- Factorization (Wang ’11)

Data-augmented (incomplete QR etc.)

- Multi-Level QR (Li & Saad ‘06)
- HSS-based methods

Time-domain

$$\begin{pmatrix} M \\ 0 \end{pmatrix} \mathbf{u}''(t) + \begin{pmatrix} S \\ P \end{pmatrix} \mathbf{u}(t) \approx \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{d}(t) \end{pmatrix}$$

Time-discretization

Standard Leap-frog leads to

$$\begin{pmatrix} M \\ 0 \end{pmatrix} \mathbf{u}^{n+1} + \begin{pmatrix} \Delta t^2 S - 2M \\ \Delta t^2 P \end{pmatrix} \mathbf{u}^{n} + \begin{pmatrix} M \\ 0 \end{pmatrix} \mathbf{u}^{n-1} \approx \Delta t^2 \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{d}(t) \end{pmatrix}$$

We need to compute the least-squares solution globally over all time-steps!

Time-discretization

$$
\begin{pmatrix}
M & & & & \\
\Delta t^2 S - 2M & M & & & \\
M & \Delta t^2 S - 2M & M & & \\
 & M & \Delta t^2 S - 2M & M & \\
 & & \ddots & \ddots & \ddots \\
P & & & & \\
 & P & & & \\
 & & P & & \\
 & & & \ddots &
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_N
\end{pmatrix}
\approx
\begin{pmatrix}
\mathbf{q}_0 \\ \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_N \\ \\ \mathbf{d}_0 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_N
\end{pmatrix}
$$

Time-discretization

Standard leap-frog leads to large (block) overdetermined system

lower-triangular →
diagonal →

$$\begin{pmatrix} L \\ D \end{pmatrix} \mathbf{u} \approx \begin{pmatrix} \mathbf{q} \\ \mathbf{d} \end{pmatrix}$$

Normal equations lead to penta-diagonal system

$$(L^T L + D^T D)\mathbf{u} = L^T \mathbf{q} + D^T \mathbf{d}$$

We can never form the complete matrices, have to solve by forward/ backward substitution!

Conclusions & Future work

Frequency-domain:

- A number of standard tools can be re-used
- preconditioning requires extra attention

Time-domain:

- need to solve LS problem globally over all time-steps
- forward substitution may require many sweeps

Acknowledgements

Thank you!

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