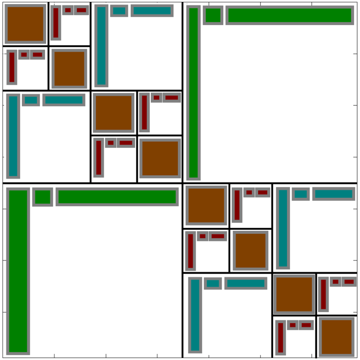
Krylov Solvers In Frequency Domain FWI

Rafael Lago

SINBAD Consortium Meeting, December 4, 2013

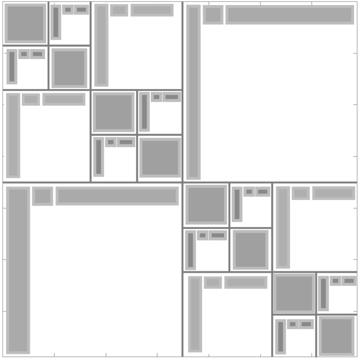
Multifrontal HSS

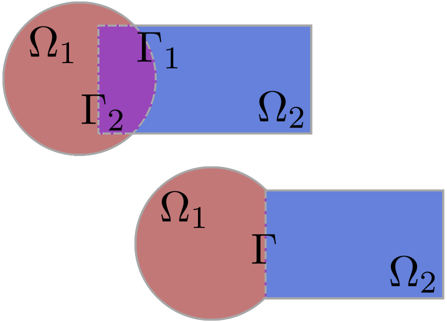


Wang et al. 2010, 2011, 2012

Multifrontal HSS

Schwarz Domain Decomposition





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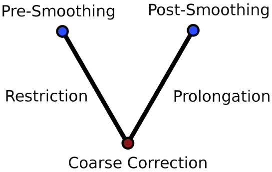
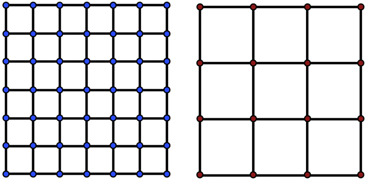
Haidar 2008

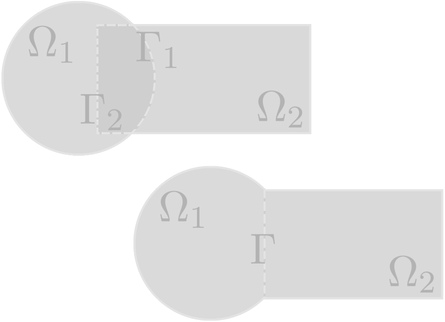
Multifrontal HSS

Schwarz Domain Decomposition

Complex Shifted Laplacian Multigrid







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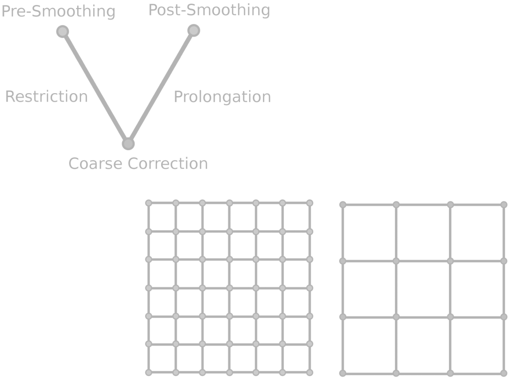
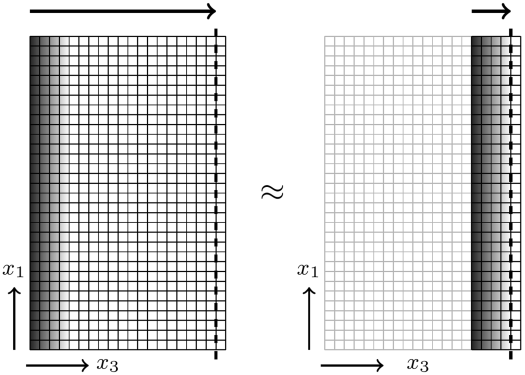
Parallel Sweeping Preconditioner





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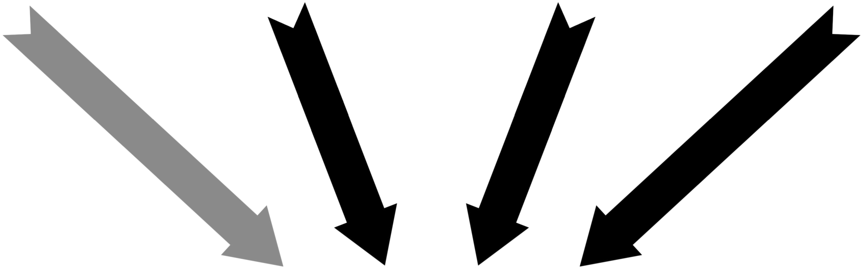

GMRES

Wang et al. 2010, 2011, 2012

Haidar 2008

Erlangga 2005, Calandra et al. 2013

Poulson et al. 2013

More Than a Linear System

$$A \qquad x \qquad = \qquad b$$

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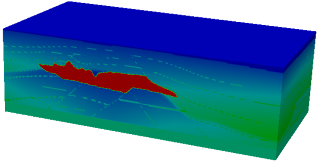

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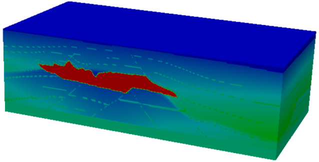

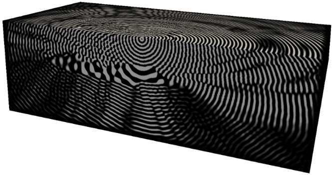

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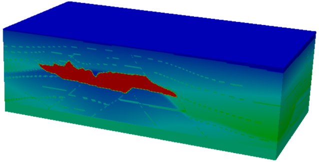
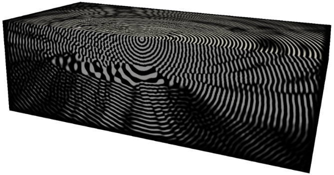
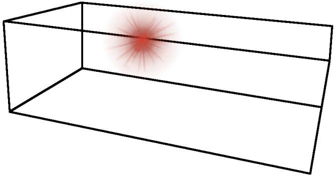
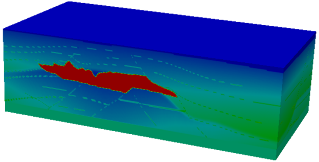
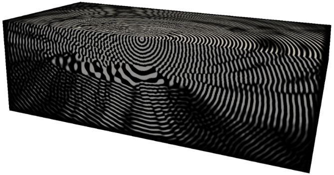
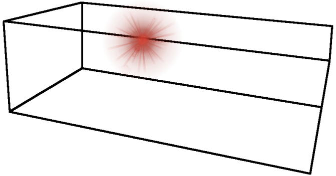






More Than a Linear System

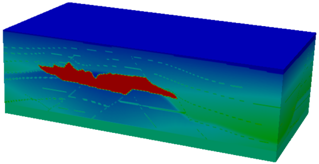
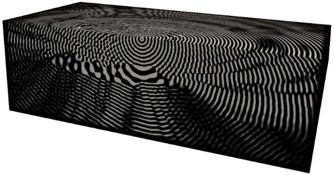
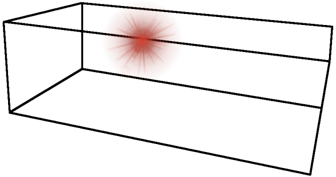
$$A \quad x^i \quad = \quad b^i$$

$\times n_s$ $\times n_s$

More Than a Linear System

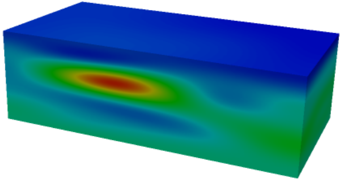
$$A^j \qquad x^{(i,j)} \qquad = \qquad b^i$$

$$\times n_f \qquad \times n_s \times n_f \qquad \times n_s$$

More Than a Linear System

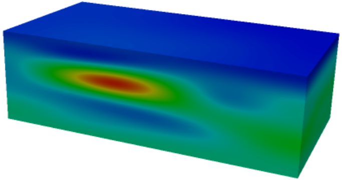
$$A^{\,j} \qquad x^{\,(i,j)} \qquad = \qquad b^i$$



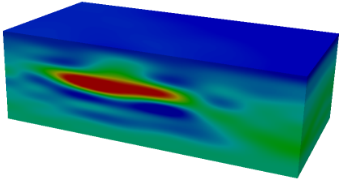
k

More Than a Linear System

$$A^{j} \quad x^{(i,j)} = b^{i}$$



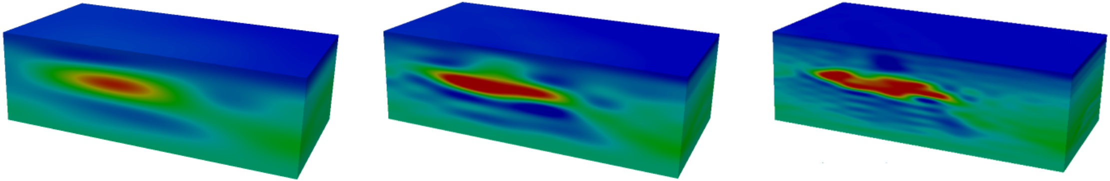
k



$k+1$

More Than a Linear System

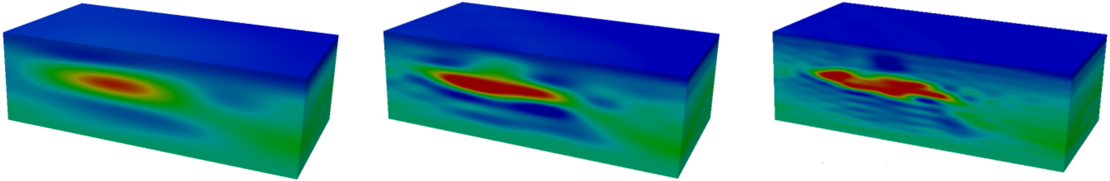
$$A^{j} \quad x^{(i,j)} \quad = \quad b^{i}$$



k $\quad$ $k+1$ $\quad$ $k+2$ $\quad \cdots$

More Than a Linear System

$$A_k{}^j \quad x_k{}^{(i,j)} = b^i$$



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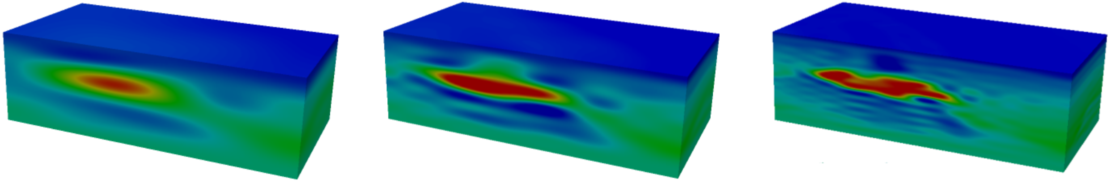


$k \qquad k+1 \qquad k+2 \qquad \dots$

$$(A_k^j)^H \qquad w_k^{(i,j)} \qquad = \qquad r(x_k^{(i,j)})$$

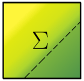
$$Ax^i = b^i \quad \Longrightarrow \quad AX = B$$

$$\mathcal{K}^{\square}(A, B) = \bigcup_{i}^{p} \mathcal{K}(A, Be_i)$$

$\exists i, j$ with $i \neq j$ s.t.

$$\mathcal{K}(A, Be_i) \bigcap \mathcal{K}(A, Be_j) \neq \emptyset$$

- Trivial generalization of GMRES

$R_j = U$  W^H

Vital (1990)

Block GMRES

$$Ax^i = b^i \quad \Longrightarrow \quad AX = B$$

$$\mathcal{K}^{\square}(A,B) = \bigcup_{i}^{p} \mathcal{K}(A, Be_i)$$

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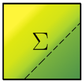
- Trivial generalization of GMRES
- Computes approximation X_j every iteration j

$R_j = U$  W^H

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Block GMRES

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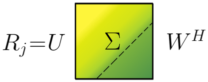
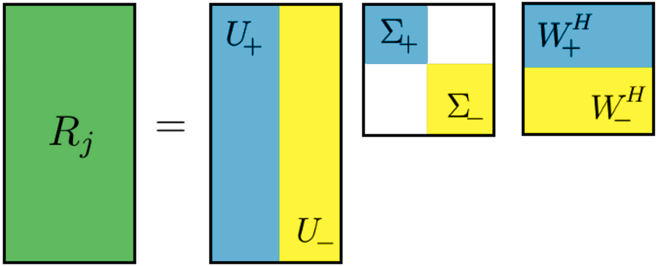
- Trivial generalization of GMRES
- Computes approximation X_j every iteration j
- X_j **minimizes** $\left\|B - AX_j\right\|_F$
- $R_j = B - AX_j$



Vital (1990)

Residual Deflation in a Nutshell

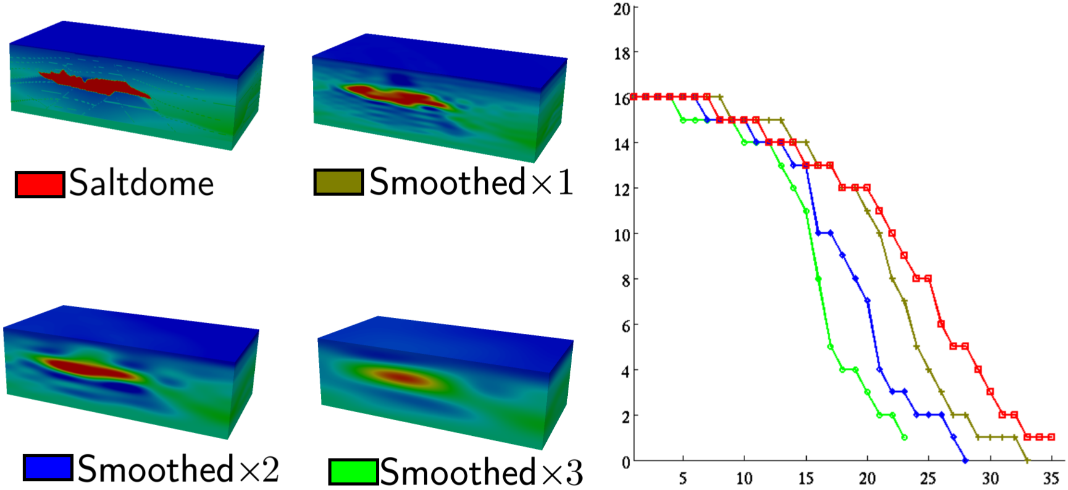


$$R_j = U_+ \Sigma_+ W_+^H + U_- \Sigma_- W_-^H$$

$$\left\| R_j W_+ \right\|_F \geq \varepsilon \qquad \left\| R_j W_- \right\|_F < \varepsilon$$

Langou (2003), Gutknecht (2008), Robbé and Sadkane (2006), Lago (2013), etc

Residual Deflation Numerical Examples



Residual Deflation Numerical Examples

Method	Sec	Ratio
GMRES	1187	1
BGMRES	1187	1
BGMRESD	509	0.42
BGMRES-RS	762	0.64
DMBR*	497	0.41
BGMREST	524	0.44
DMBR*	523	0.44

- **Helmholtz Equation:** $-\Delta u - k^2 u = s$
- SEG/EAGE Overthrust
- 3.64 Hz $(h = 50m)$
- 2.3×10^7 unknowns
- 128 shots
- Babel (IDRIS) BG/P
- 256 processors
- Same parameters; same preconditioner (multigrid); same memory used; same code

* Calandra et al (2013), Lago (2013)

Subspace Recycling

$$Ax = b$$

(Arnoldi relation) $$A\mathbf{Z} = \mathbf{V}H$$

(Invariant subspace) $$A\mathbf{V} = \mathbf{V}H$$

$$Av - \lambda v = 0$$

(Eigenvalues)

$$Av - \lambda v \perp AW$$

(Harmonic Ritz Values)

Morgan (1995), Morgan (2000), Morgan (2002), etc

Subspace Recycling

$$
\begin{array}{ccc}
Ax^1 = b^1 & \Longrightarrow & Av - \lambda v \perp AW \\
Ax^2 = b^2 & + & Av - \lambda v \perp AW \\
 & \cdots & \\
Ax^i = b^i & + & Av - \lambda v \perp AW
\end{array}
$$

Kilmer and de Sturler(2006), Parks et al. (2006), Wang et al. (2007), Lago (2013), etc

Subspace Recycling Numerical Examples

Method	MVP	Ratio
GMRES	1128	1
GMRES-DR	876	0.77
FGCRO-DR*	529	0.47

Matlab Illustration

- **Laplacian:** $\Delta u = s$
- 7.6×10^5 unknowns
- **12** right hand sides

* Carvalho et al (2011), Lago (2013)

Subspace Recycling

$$\begin{aligned} Ax^1 &= b^1 & \Longrightarrow && Av - \lambda v \perp AW \\ Ax^2 &= b^2 & + && Av - \lambda v \perp AW \\ && \cdots && \\ Ax^i &= b^i & + && Av - \lambda v \perp AW \end{aligned}$$

Kilmer and de Sturler(2006), Parks et al. (2006), Wang et al. (2007), Lago (2013), etc

Subspace Recycling

$$
\begin{aligned}
AX^1 &= B^1 & \implies && Av - \lambda v \perp AW \\
AX^2 &= B^2 & + && Av - \lambda v \perp AW \\
&& \cdots && \\
AX^i &= B^i & + && Av - \lambda v \perp AW
\end{aligned}
$$

Subspace Recycling

$$
\begin{array}{lcl}
A^j \quad x^j \quad = b & \Longrightarrow & A^j v - \lambda v \perp A^j W \\
A^{j+1} x^{j+1} = b & + & A^j v - \lambda v \perp A^j W \\
& \dots &
\end{array}
$$

(assuming $A^j \approx A^{j+1}$)

Parks et al. (2006)

Subspace Recycling

$$\begin{aligned} A_k \quad x_k \quad &= b & \implies \quad & A_k v - \lambda v \perp A_k W \\ A_{k+1} x_{k+1} &= b & + \quad & A_k v - \lambda v \perp A_k W \end{aligned}$$

$$\dots$$

Could this work? Open question!

Inexact Matrix-Vector Product

$$z = Av$$

Bouras and Frayssé (2005)

Inexact Matrix-Vector Product

$$z = Av$$

$$z \approx f_\varepsilon(v)$$

- Cost of $f_\varepsilon(.)$ depends on ε
- ε can be chosen for each multiplication

Bouras and Frayssé (2005)

Inexact Matrix-Vector Product

$$z = Av$$

$$z \approx f_\varepsilon(v)$$

$$z \approx (A + E_j)v$$

- Cost of $f_\varepsilon(.)$ depends on ε
- ε can be chosen for each multiplication
- $\|E_j\|_2$ can be chosen for each multiplication

Bouras and Frayssé (2005)

Inexact Matrix-Vector Product

$$\left\|E_j\right\|_2 \leq \frac{\sigma_m(H_m)}{m} \frac{1}{\left\|\boldsymbol{\beta e_1 - H_j y_j}\right\|_2} \varepsilon^g$$

Valeria Simoncini and Daniel Szyld (2008)

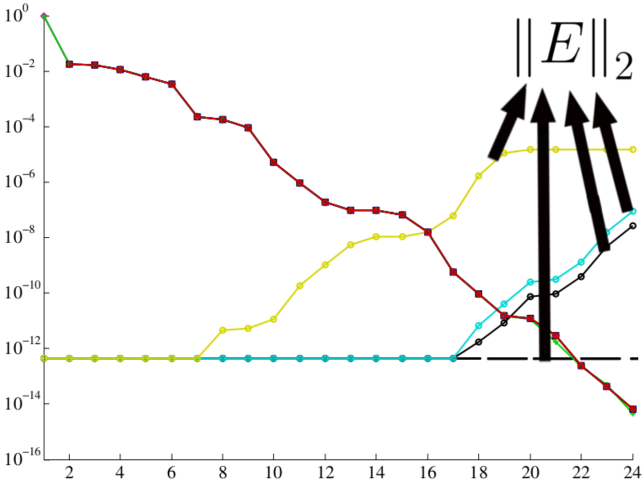
Inexact Matrix-Vector Product

$$\left\|E_j\right\|_2 \leq \frac{\sigma_m(H_m)}{m} \frac{1}{\left\|\boldsymbol{\beta e_1 - H_j y_j}\right\|_2} \varepsilon^g$$

$$\left\|E_j\right\|_2 \leq \frac{\boldsymbol{c}}{\left\|\boldsymbol{\tilde{r}_j}\right\|_2}$$

Valeria Simoncini and Daniel Szyld (2008)

Inexact GMRES - Numerical Illustration



Matlab Illustration

- **e05r0400** - fluid flow in a driven cavity
- 5 right hand sides
- ILU(10^{-3}) left preconditioner
- Restart Size: 30

Preconditioner

$$Ax = b$$

$$A\boldsymbol{M}^{-1}\boldsymbol{M}x = b$$

$$A\boldsymbol{M}^{-1}y = b$$

Preconditioner

$$Ax = b$$

$$AM^{-1}Mx = b$$

$$AM^{-1}y = b$$

$$(A + E)M^{-1}y = b$$

Inexact... Preconditioner?!

$$Ax = b$$

$$AM^{-1}Mx = b$$

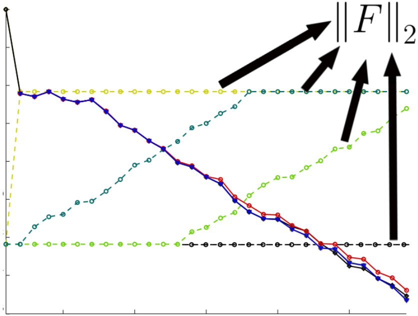
$$AM^{-1}y = b$$

$$(A + E)M^{-1}y = b$$

$$A(M^{-1} + F)y = b$$

Inexact GMRES

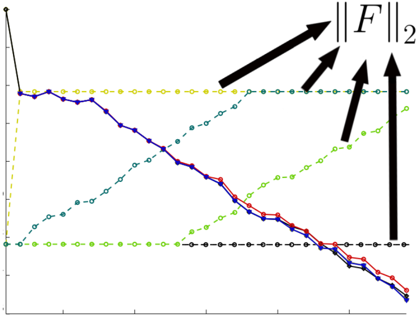
$$A(\boldsymbol{M}^{-1} + \boldsymbol{F})y = b$$



- Inexact GMRES for $(A + \boldsymbol{E})$ is known - Bouras and Frayssé (2005)

Inexact GMRES

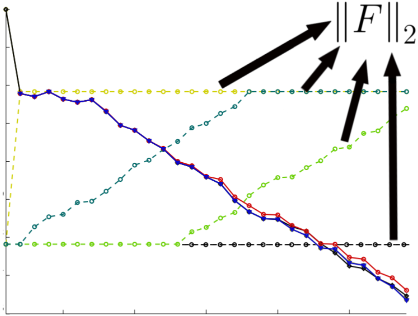
$$A(M^{-1} + \boldsymbol{F})y = b$$



- Inexact GMRES for $(A + \boldsymbol{E})$ is known - Bouras and Frayssé (2005)
- ... should work for $(M^{-1} + \boldsymbol{F})$ - Simoncini and Szyld (2008)

Inexact GMRES

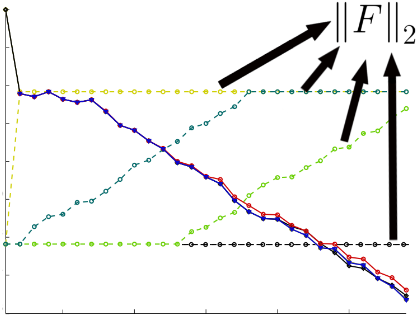
$$A(M^{-1} + F)y = b$$



- Inexact GMRES for $(A + E)$ is known - Bouras and Frayssé (2005)
- ... should work for $(M^{-1} + F)$ - Simoncini and Szyld (2008)
- But... what is an **inexact preconditioner**?

Inexact GMRES

$$A(M^{-1} + F)y = b$$



Speculation

Conclusions

- Our problem is more than $Ax = b$! We need to exploit that!
- Block Krylov methods provide significant speed up for **seismic application**
- Subspace recycling provide significant speed up; **needs to be investigated** for seismic application
- Inexact Preconditioning requires more flexible preconditioning techniques
- Is it worthwhile for seismic application?

Questions?