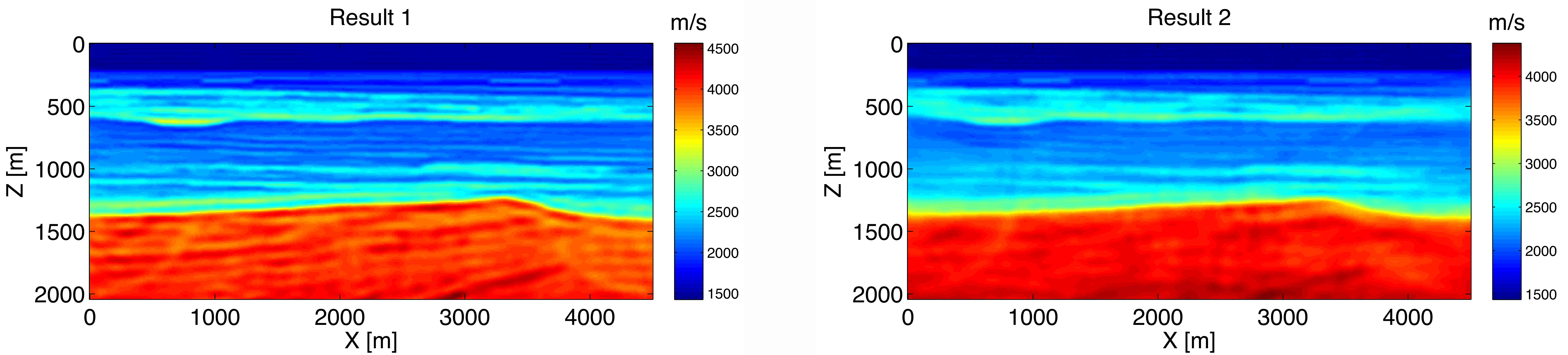
Uncertainty analysis for full waveform inversion

Zhilong Fang and Felix J. Herrmann
2013.12.02

SLIM
University of British Columbia

Motivation



Which one is better?

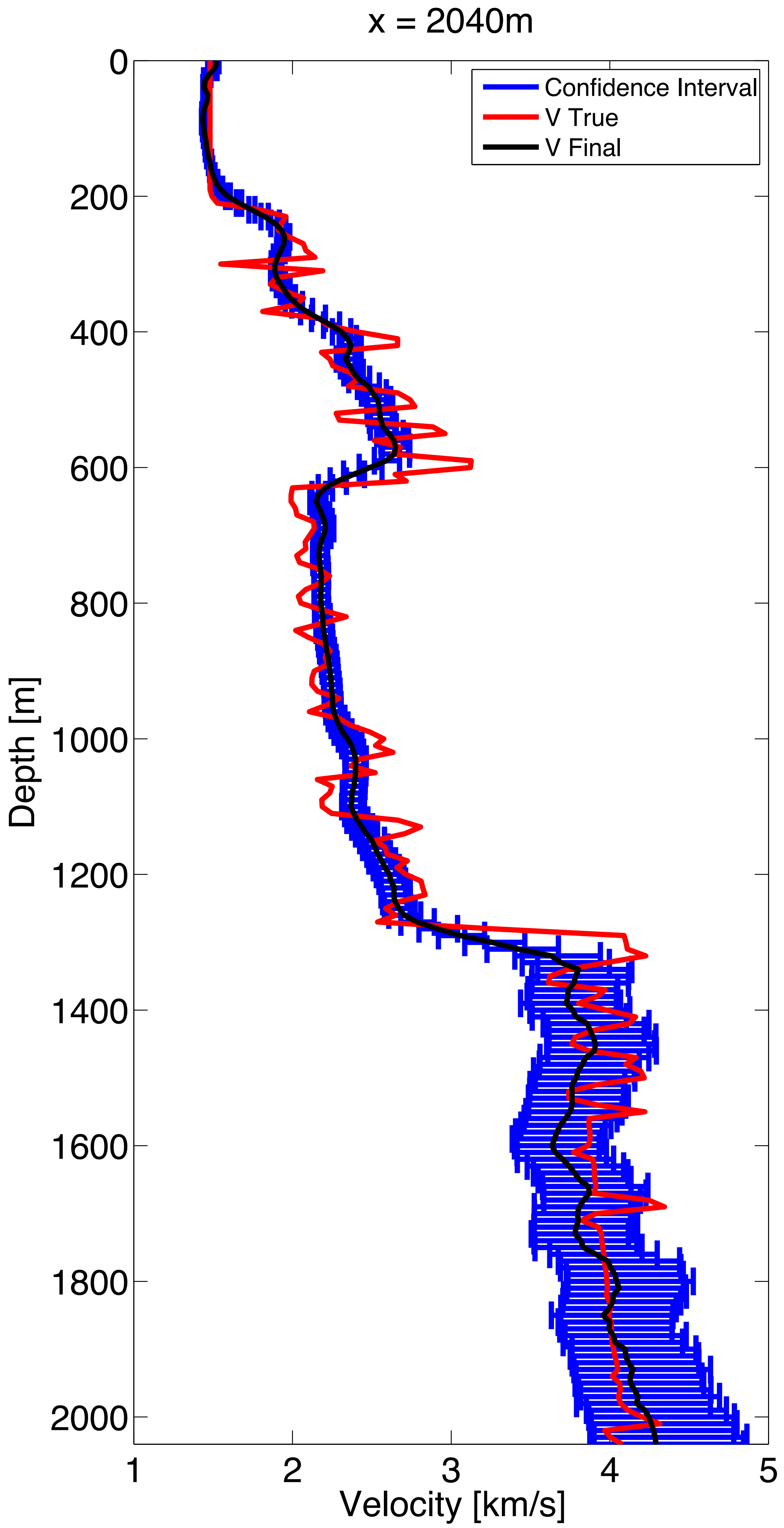
Misfit?

Eyeball norm?

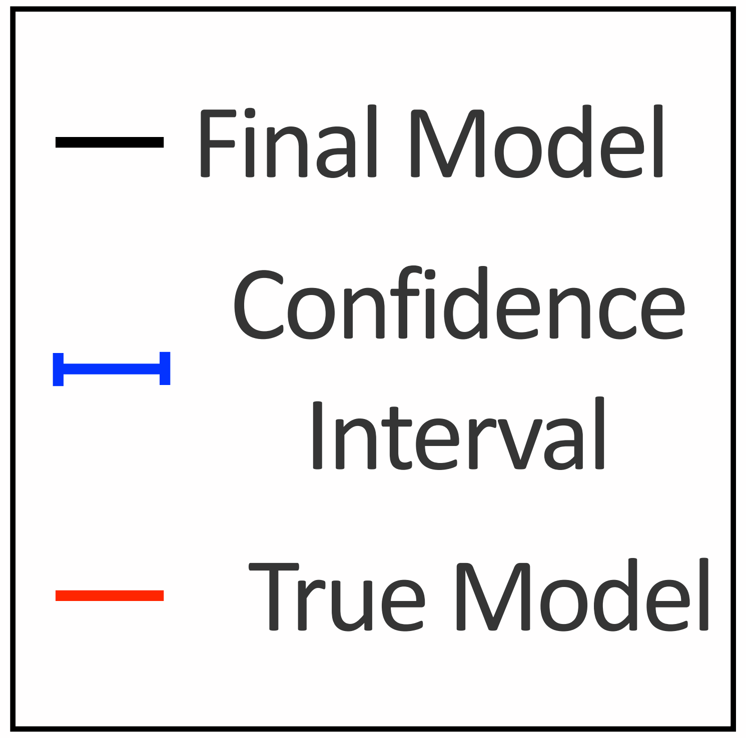
Uncertainty?

Standard Deviation?

Confidence interval



x = 2040m



Bayesian theory

Deterministic inverse problem:

$$\mathbf{m}^* = \arg\min(\frac{1}{2}\|f(\mathbf{m}) - \mathbf{d}_{obs}\|_W^2 + \frac{1}{2}\|\mathbf{m} - \overline{\mathbf{m}}\|_R^2)$$

Statistical inverse problem with Bayesian theory:

$$\pi_{post}(\mathbf{m}) := \pi(\mathbf{m}|\mathbf{d}_{obs}) \propto \pi_{prior}(\mathbf{m})\pi(\mathbf{d}_{obs}|\mathbf{m})$$

where $\mathbf{m}$ is the model parameter, and $\mathbf{d}_{obs}$ is the observed data

Bayesian theory

Assume:

$$\text{noise} \sim \mathcal{N}(0, \Gamma_{noise})$$
$$\text{prior model distribution} \sim \mathcal{N}(\mathbf{m}_{prior}, \Gamma_{prior}).$$

Negative log-posterior of the posterior pdf:

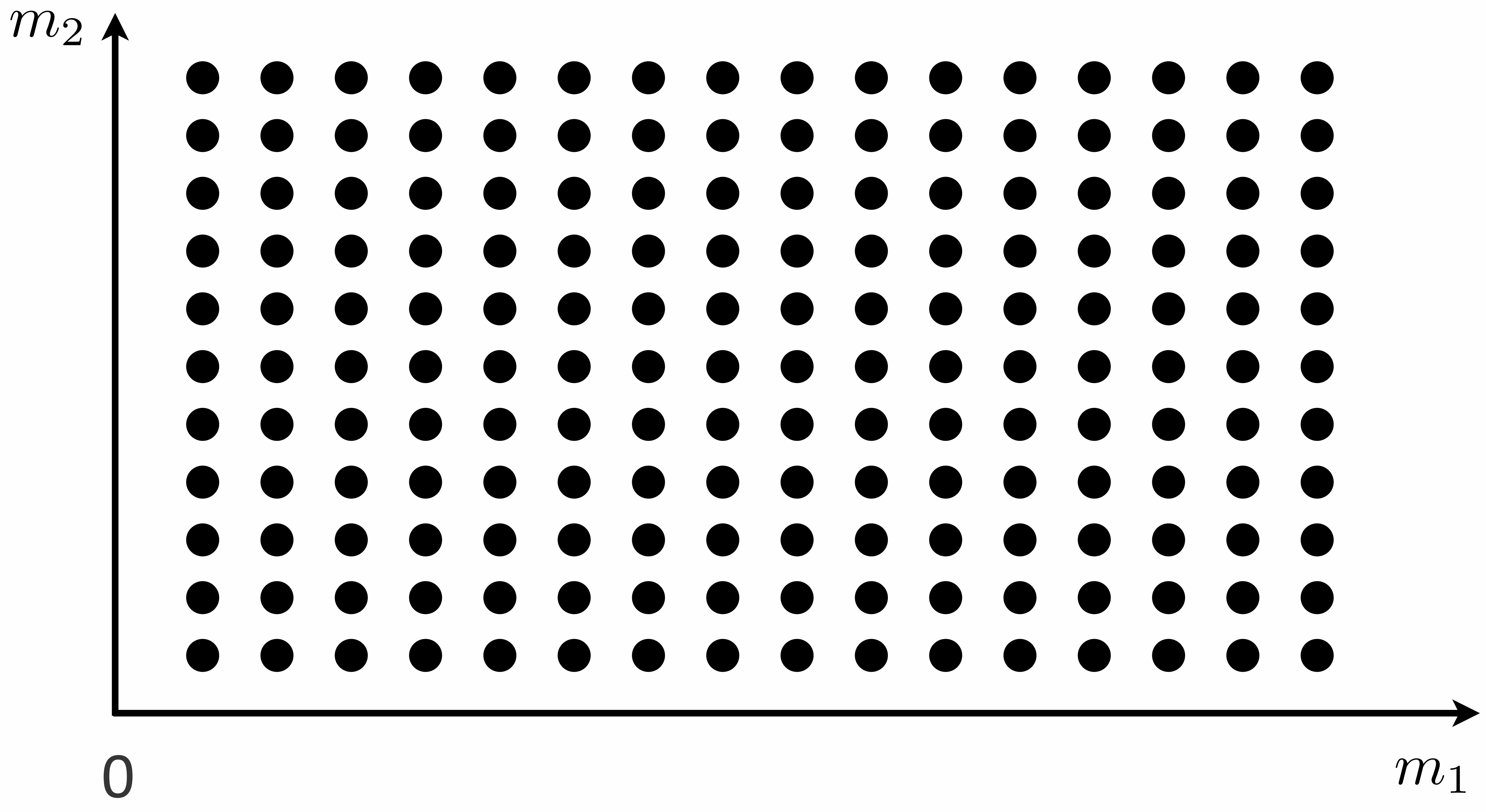
$$V(\mathbf{m}) := -\log \pi_{post}(\mathbf{m}) := \frac{1}{2}\|f(\mathbf{m}) - \mathbf{d}_{obs}\|^2_{\Gamma^{-1}_{noise}} + \frac{1}{2}\|\mathbf{m} - \mathbf{m}_{prior}\|^2_{\Gamma^{-1}_{prior}}$$
$$\mathbf{m}_{MAP} = \mathbf{m}^*$$

$$\|\mathbf{x}\|^2_{\Gamma^{-1}_{noise}} := \mathbf{x}^T \Gamma^{-1}_{noise} \mathbf{x}$$

Approximate the pdf

How to obtain the posterior probability density function?

- Uniformly sampling ?



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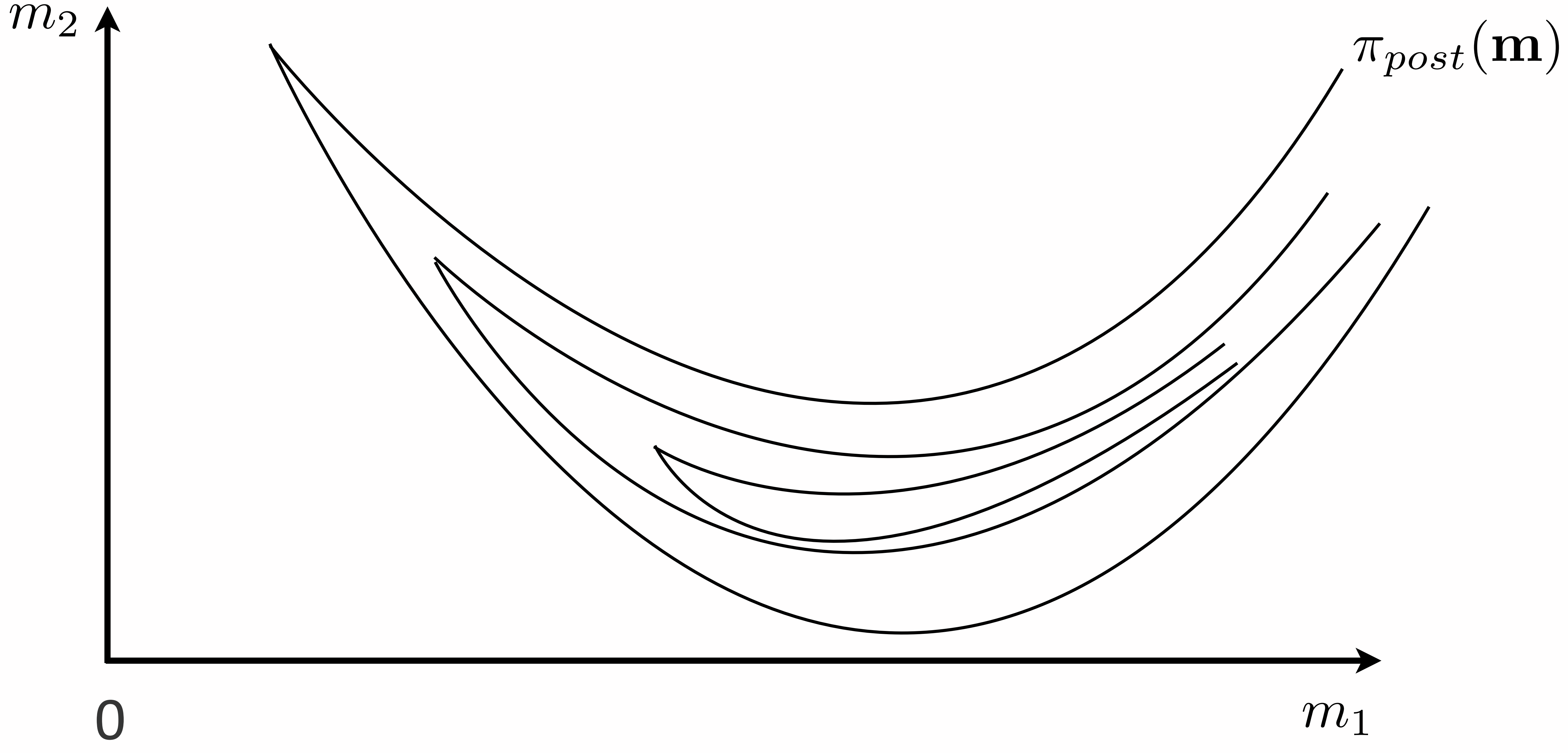
Approximate the pdf

How to obtain the posterior probability density function?

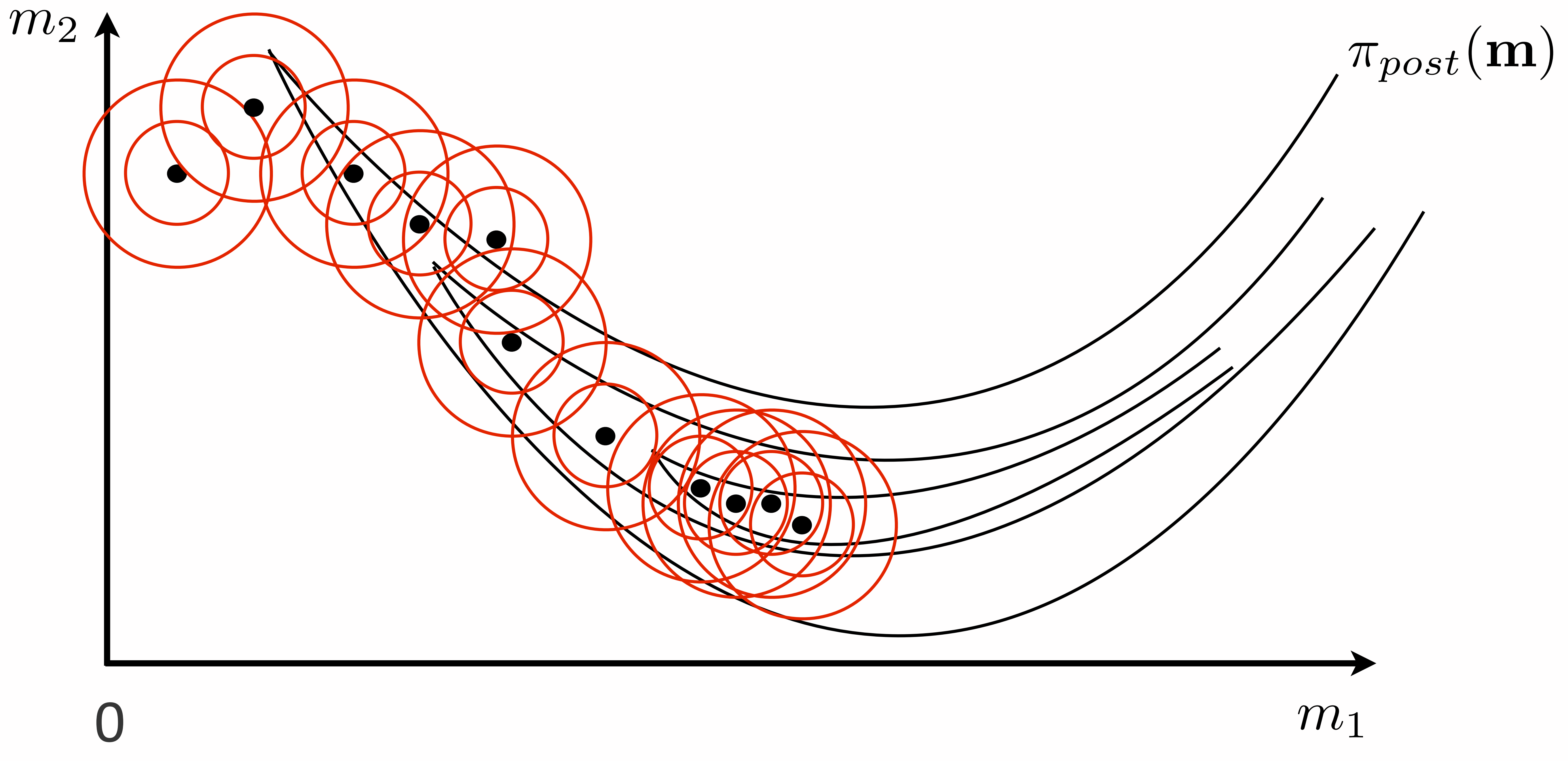
- Uniformly sampling ?
- N = Ns * Np ;



Approximate the pdf

How to obtain the posterior probability density function?

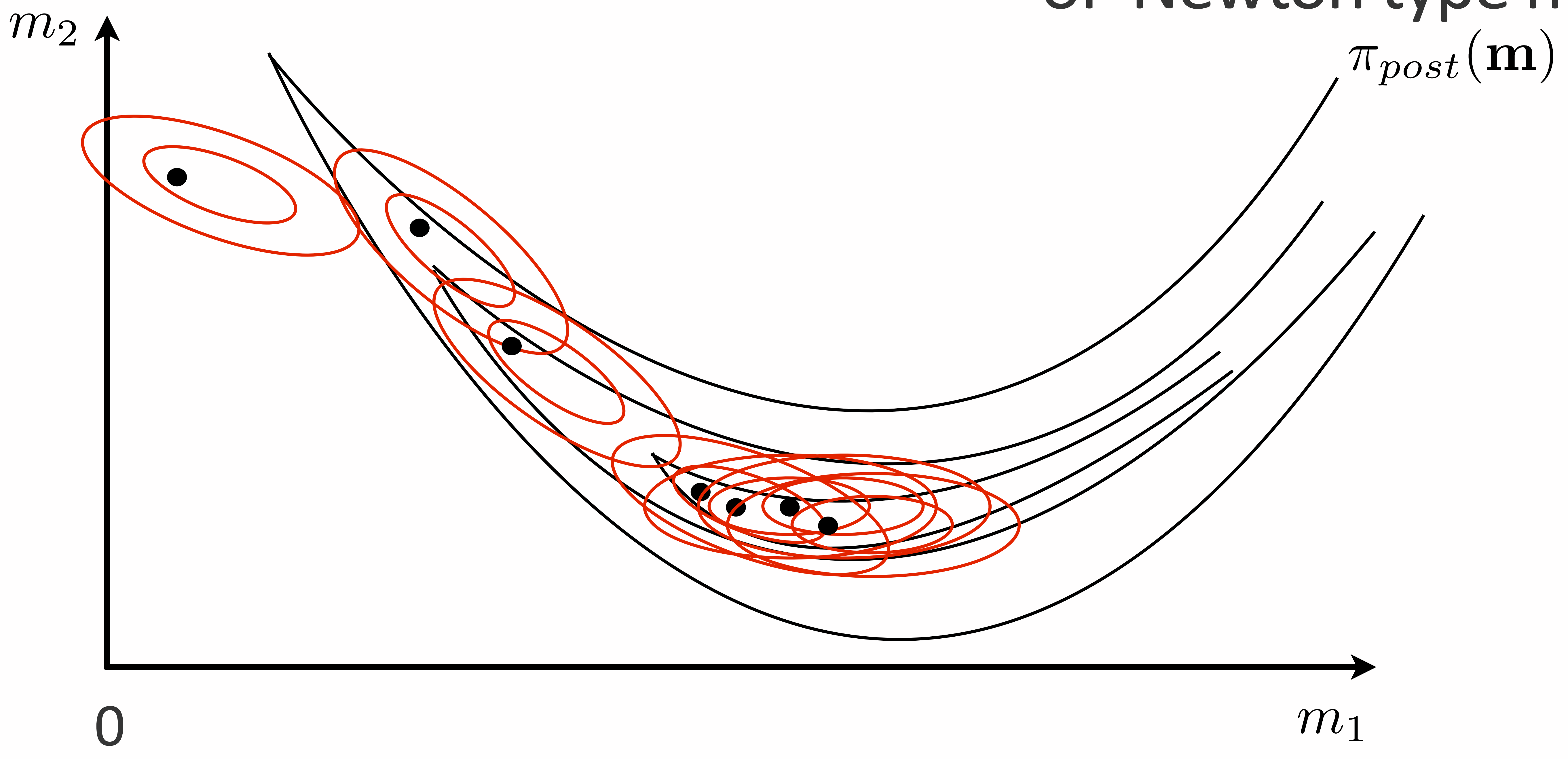
- Markov chain Monte Carlo method ?
- N is still large and needs burn-in period;



Approximate the pdf

How to obtain the posterior probability density function?

- Stochastic Newton MCMC ?
  (James Martin *et al*, 2012)
- Generating a sample equals to one iteration of Newton type method.



Approximate the posterior pdf

Possible Solution — use the *approximated* Gaussian distribution at the point of *optimal* solution as the *approximated* pdf.

Let $\mathbf{m}_{opt}$ be the optimal solution of the deterministic problem.

$$V(\mathbf{m}) \approx \tilde{V}(\mathbf{m}) := \frac{1}{2}(\mathbf{m} - \mathbf{m}_{opt})^T \mathbf{H}(\mathbf{m} - \mathbf{m}_{opt}) + \mathbf{g}^T(\mathbf{m} - \mathbf{m}_{opt}) + V(\mathbf{m}_{opt})$$

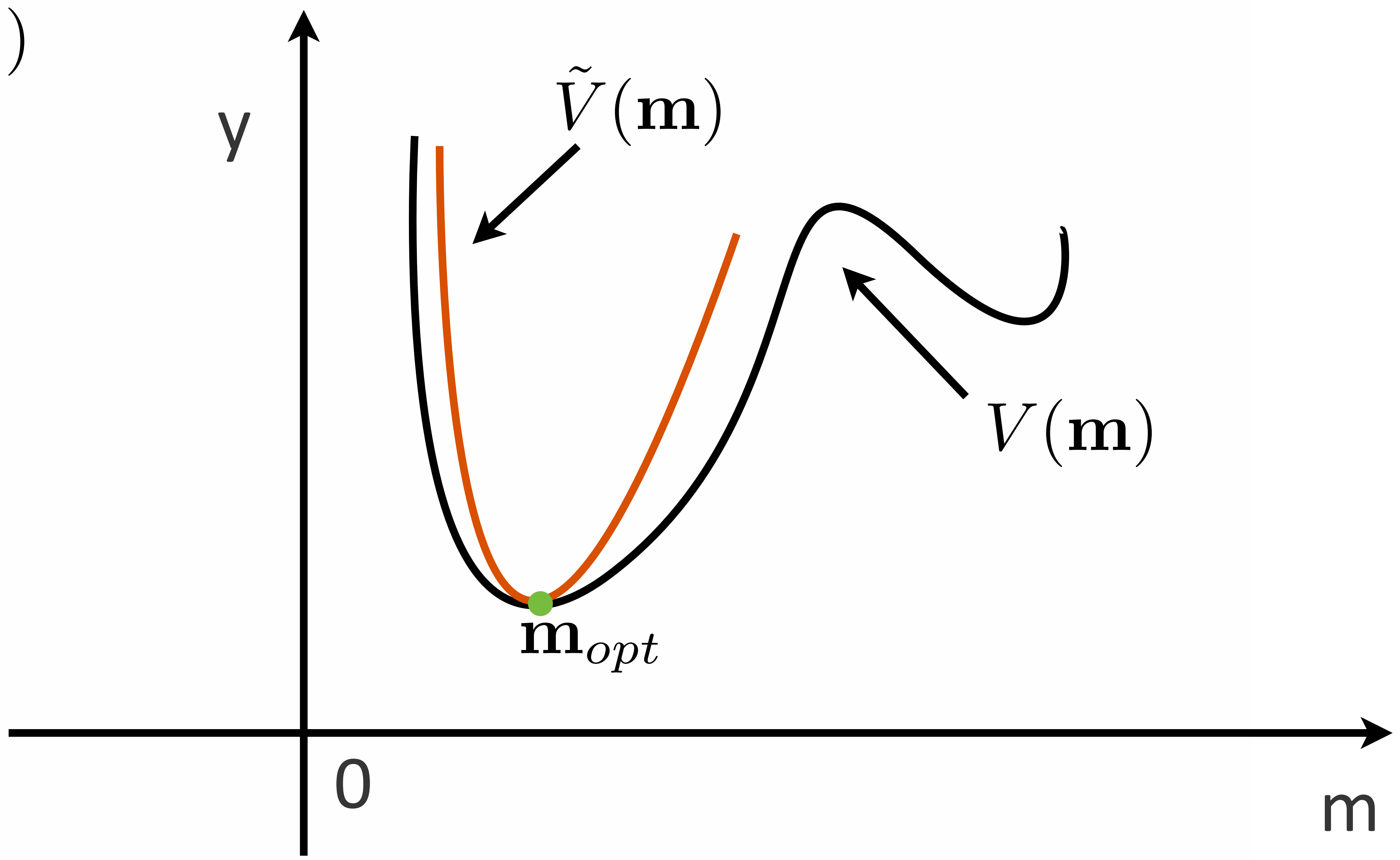
$$\tilde{V}(\mathbf{m}) = \frac{1}{2}(\mathbf{m} - \mathbf{m}_{opt} + \mathbf{H}^{-1}\mathbf{g})^T \mathbf{H}(\mathbf{m} - \mathbf{m}_{opt} + \mathbf{H}^{-1}\mathbf{g}) + \mathrm{const}$$

$$\pi_{post}(\mathbf{m}) \approx \tilde{\pi}(\mathbf{m}) \sim \exp\left(-\tilde{V}(\mathbf{m})\right) : \mathcal{N}(\mathbf{m}^*, \mathbf{H}^{-1})$$

where $\mathbf{m}^* = \mathbf{m}_{opt} - \mathbf{H}^{-1}\mathbf{g}$.

H is a linear operator!

Forming H *explicitly* is *impossible*!



Approximate the pdf

Possible solution: low-rank approximation

$$\mathbf{H} = \mathbf{H}_{\text{misfit}} + \boldsymbol{\Gamma}_{\text{prior}}^{-1} = \mathbf{L}^{-T}(\mathbf{L}^T\mathbf{H}_{\text{misfit}}\mathbf{L} + \mathbf{I})\mathbf{L}^{-1}$$

where $\mathbf{L} = \boldsymbol{\Gamma}_{\text{prior}}^{\frac{1}{2}}$.

r-rank approximation of $\mathbf{L}^T\mathbf{H}_{\text{misfit}}\mathbf{L}$:

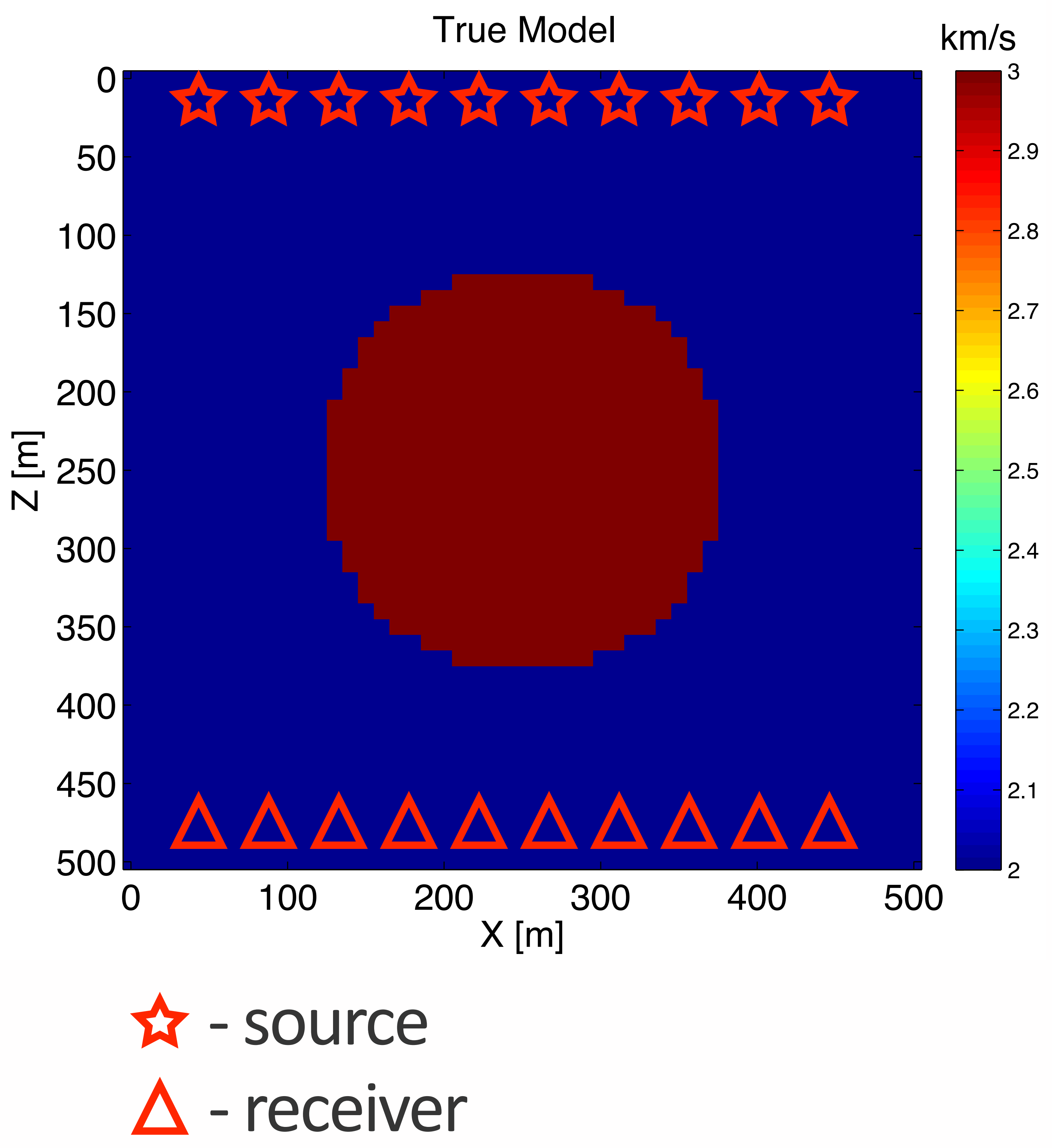
$$\mathbf{L}^T\mathbf{H}_{\text{misfit}}\mathbf{L} \approx \mathbf{V}_r\mathbf{D}_r\mathbf{V}_r^T$$

$$\underbrace{\left[\ \mathbf{L}^T\ \right]}_{n \times n}\underbrace{\left[\ \mathbf{H}_{\text{misfit}}\ \right]}_{n \times n}\underbrace{\left[\ \mathbf{L}\ \right]}_{n \times n} \approx \underbrace{\left[\mathbf{V}_r\right]}_{n \times r}\underbrace{\left[\mathbf{D}_r\right]}_{r \times r}\underbrace{\left[\ \mathbf{V}_r^T\ \right]}_{r \times n}$$

$$r \ll n$$

Numerical Experiments

Camambert model



Statistical parameter to be inverted:

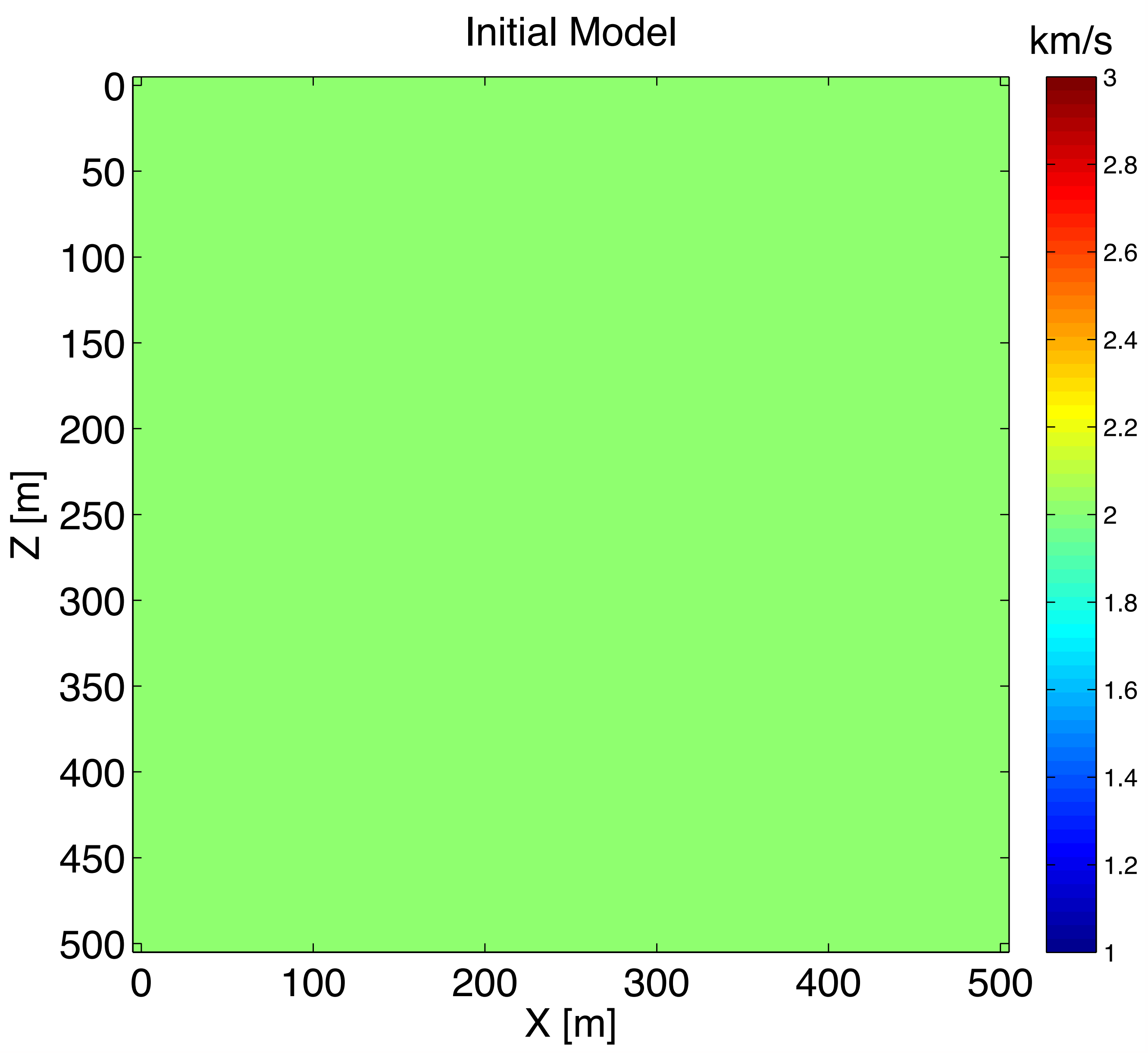
Standard deviation : σ

Confidence : $W = P((1-\lambda)\mathbf{m} \leq \mathbf{m} \leq (1+\lambda)\mathbf{m})$

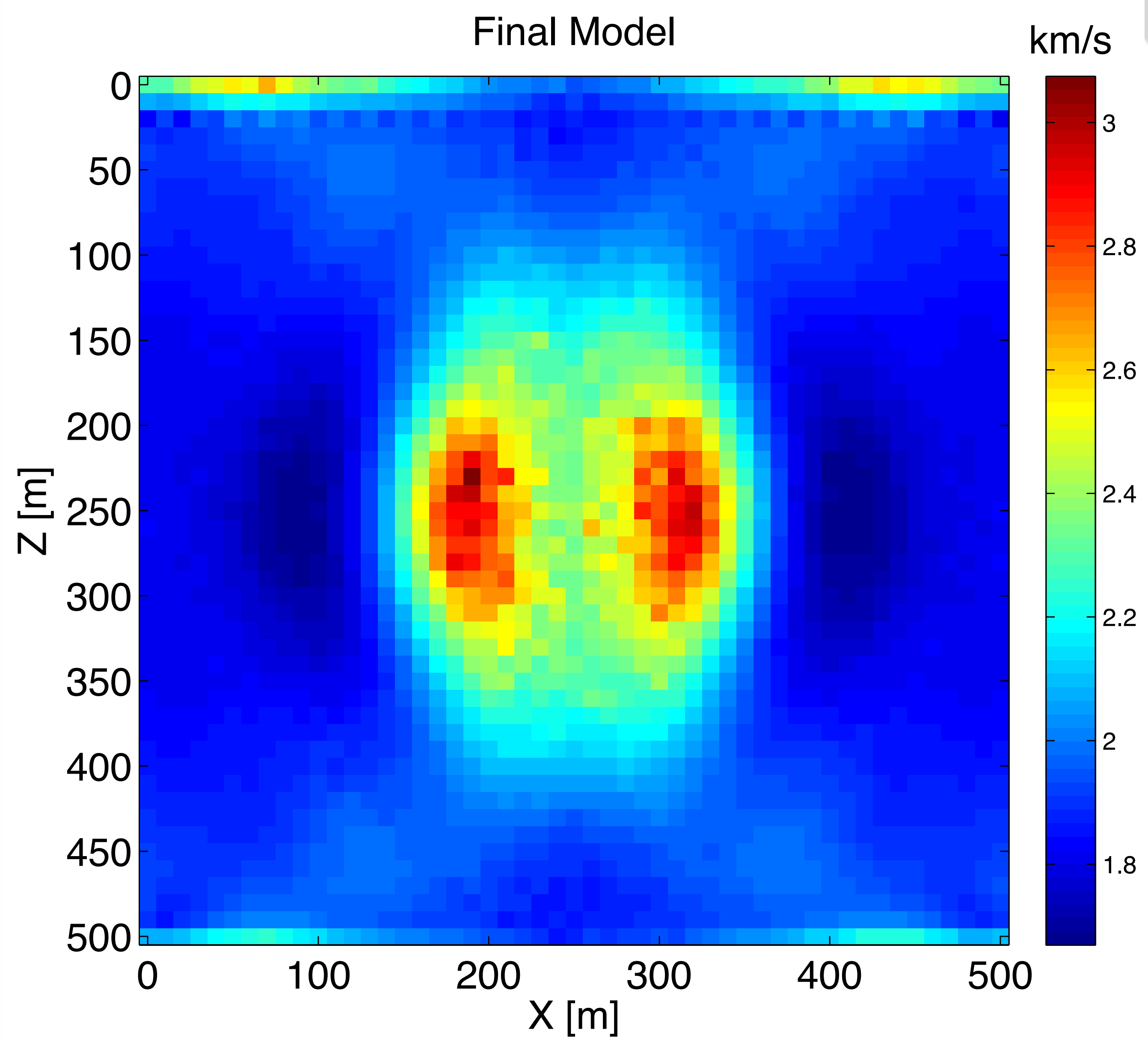
Confidence interval: $P(\mathbf{m} \in \mathrm{I}_{ci}) \geq \alpha$

Acquisition Geometry:
26 shots
51 receivers
10 frequencies

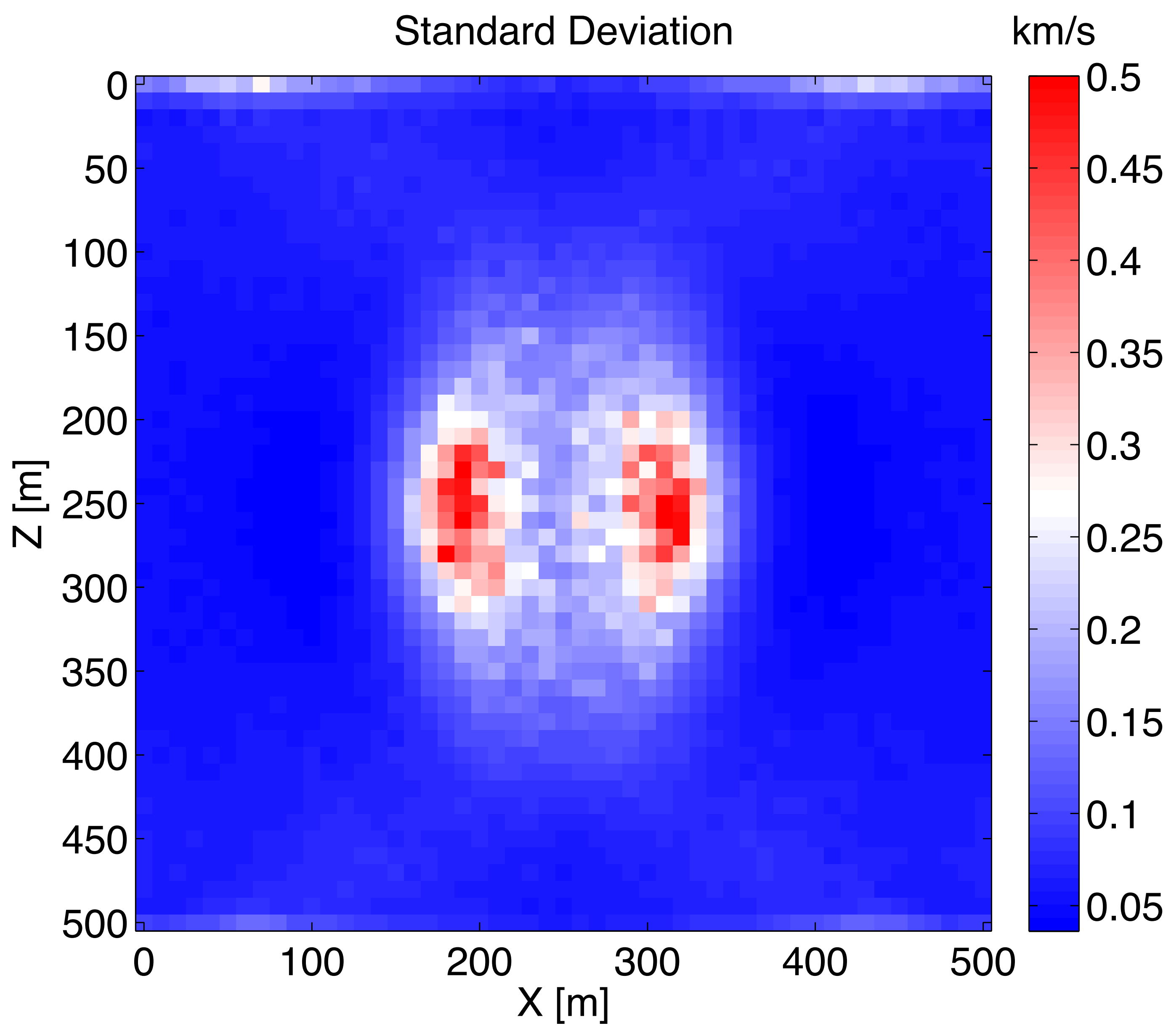
Initial model



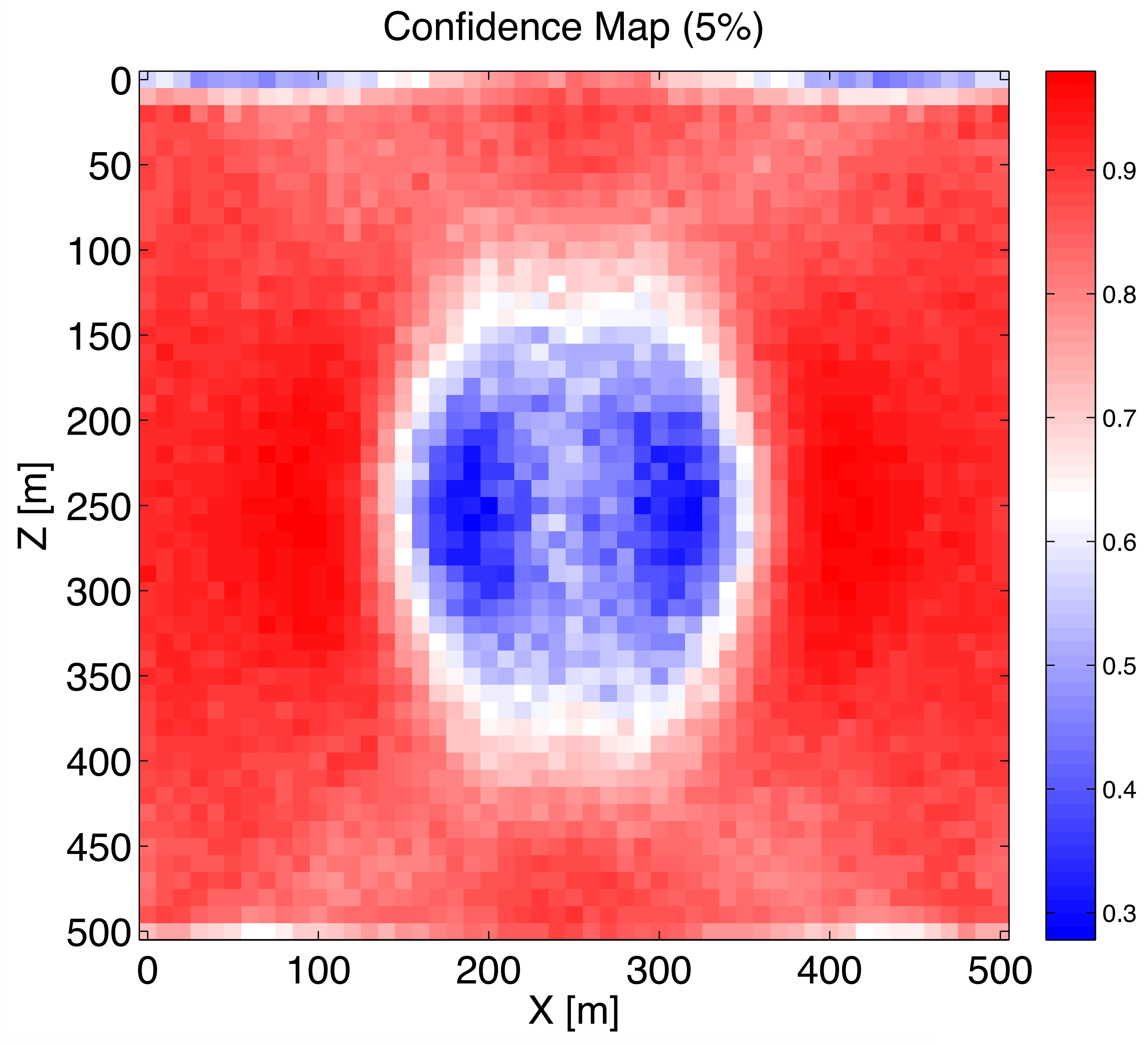
Final model



Standard deviation

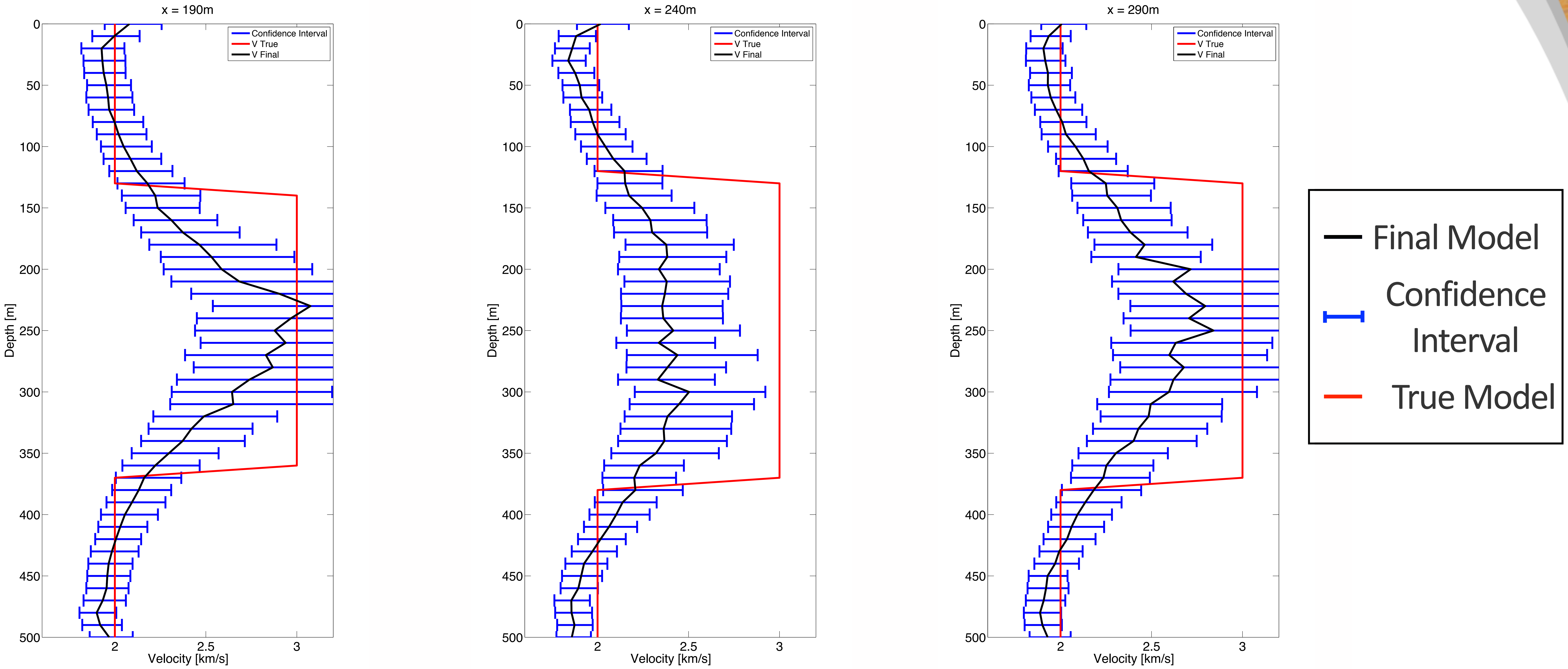


Confidence map



Confidence Interval

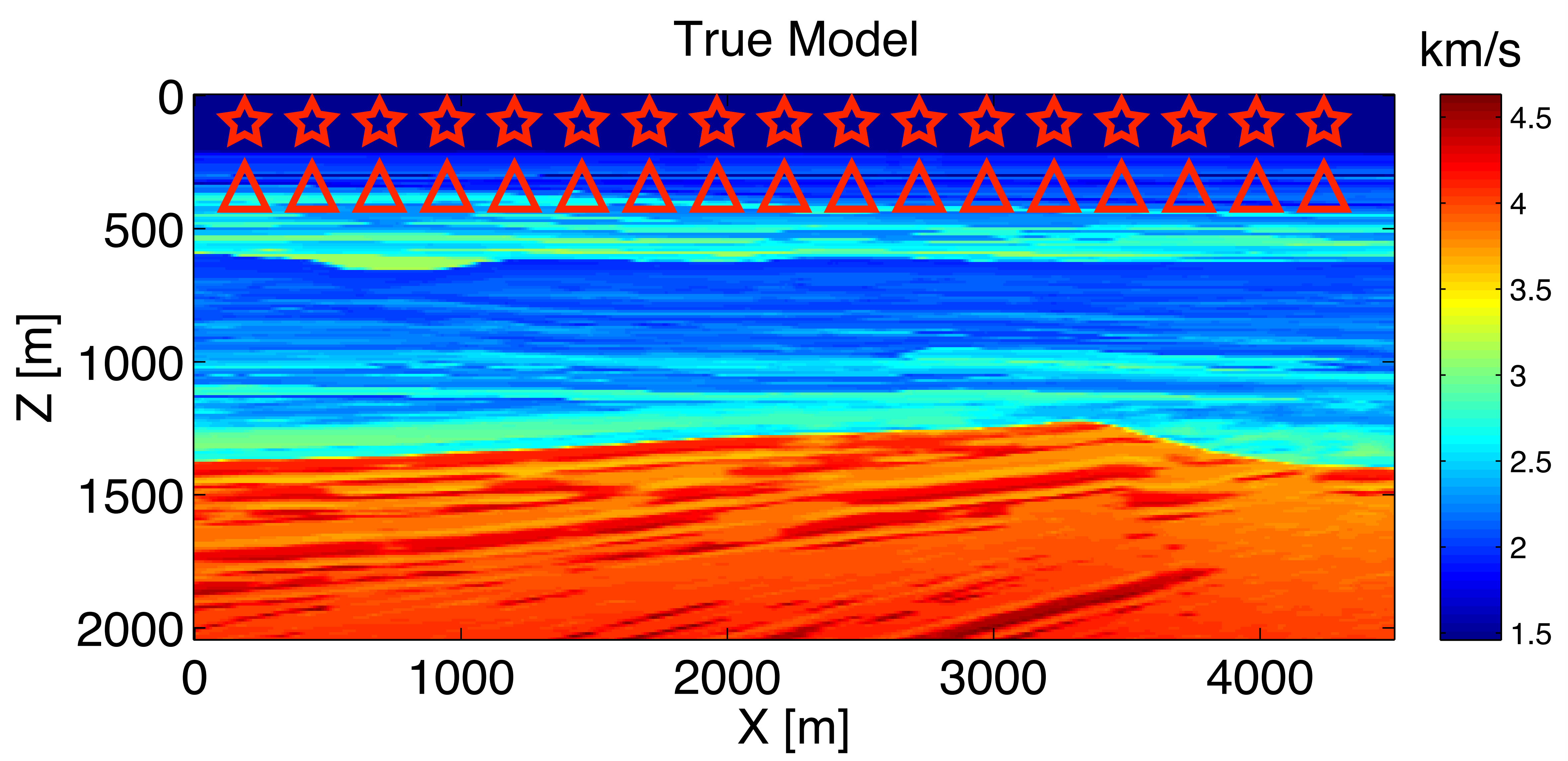
$\alpha = 0.9$



x = 190m   x = 240m   x = 290m

Numerical Experiments

BG model



☆ - source

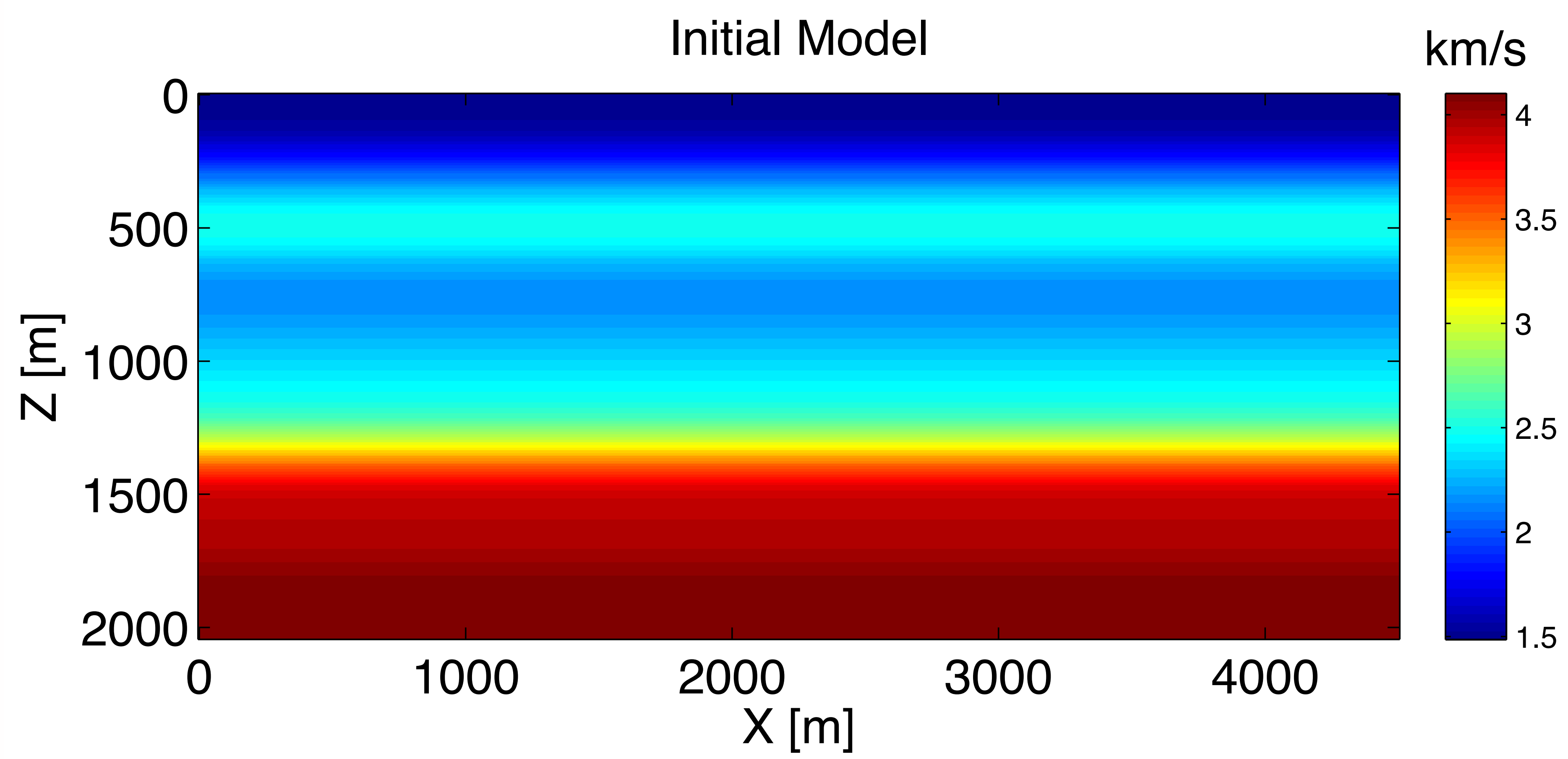
△ - receiver

Acquisition Geometry:

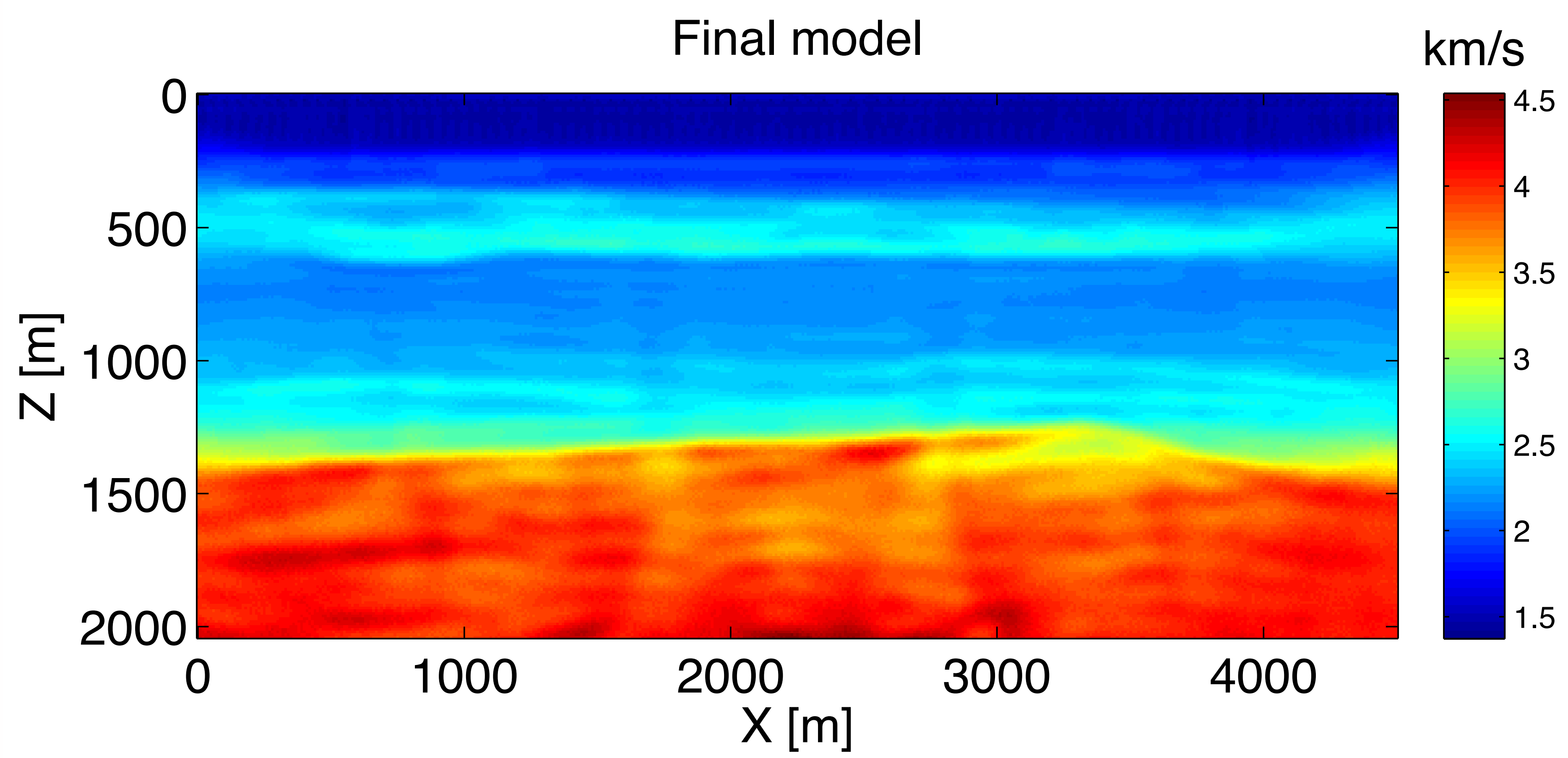
91 shots

451 receivers

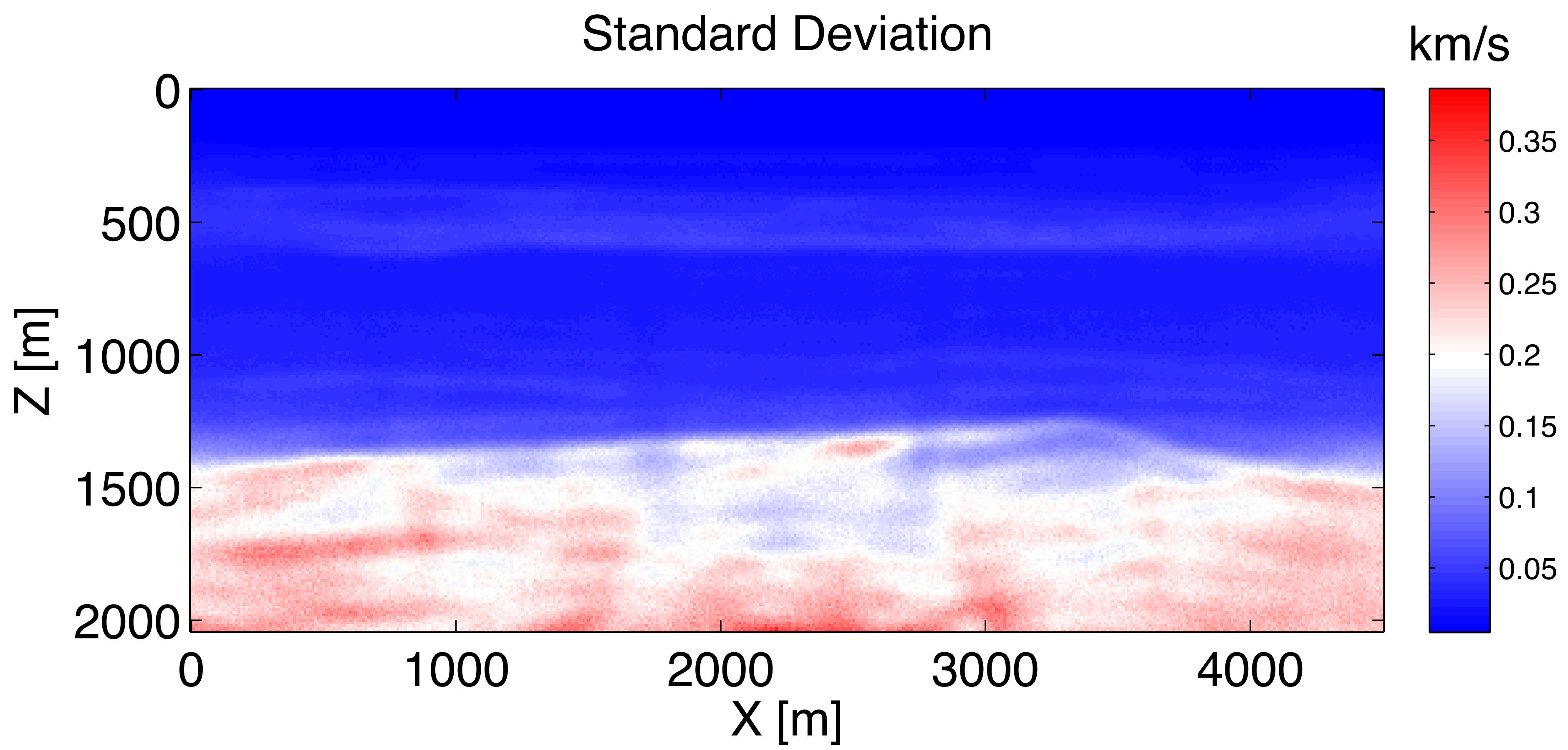
15 frequencies from 3Hz to 17Hz



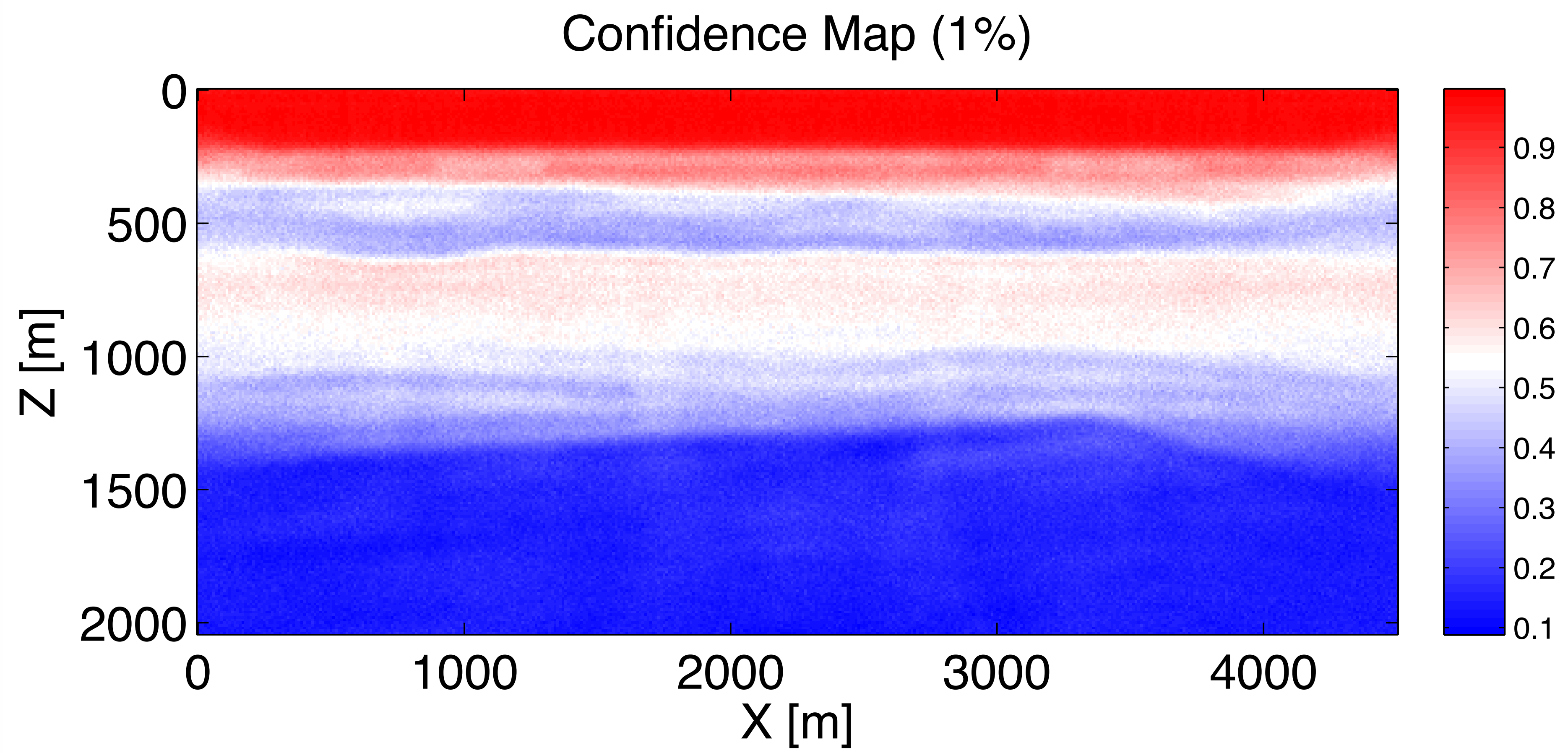
Initial model



Final model



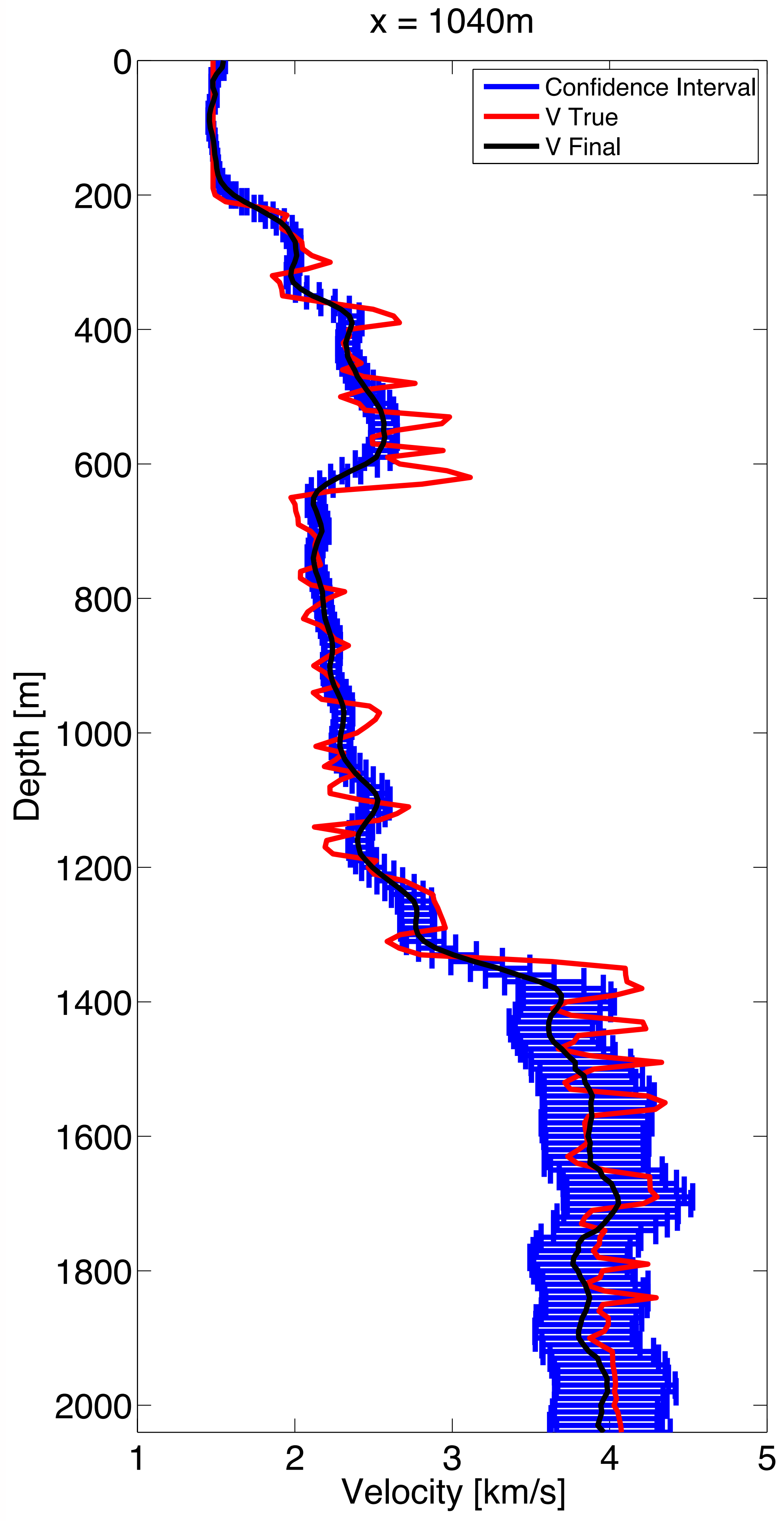
Standard deviation



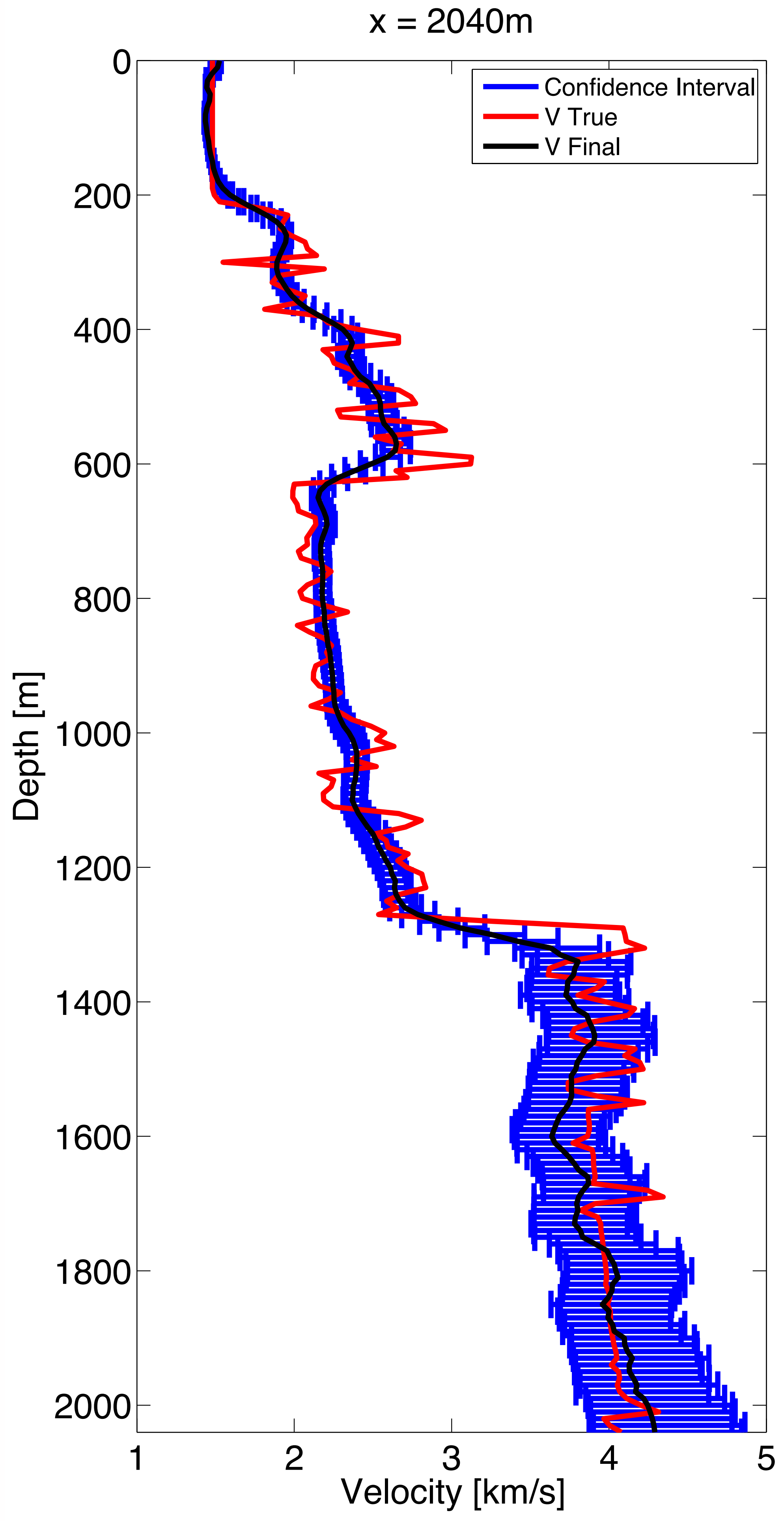
Confidence map

Confidence Interval

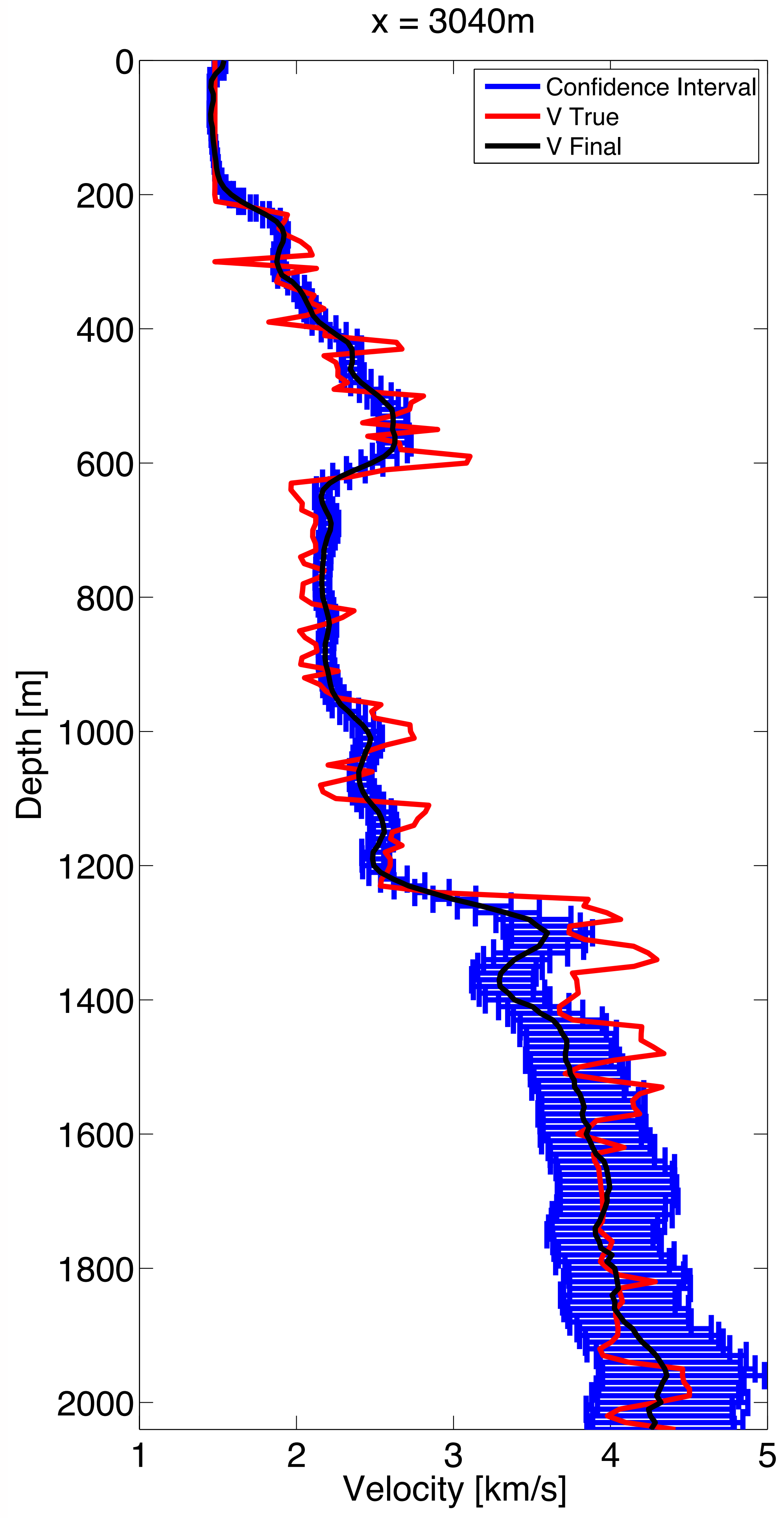
$\alpha = 0.9$



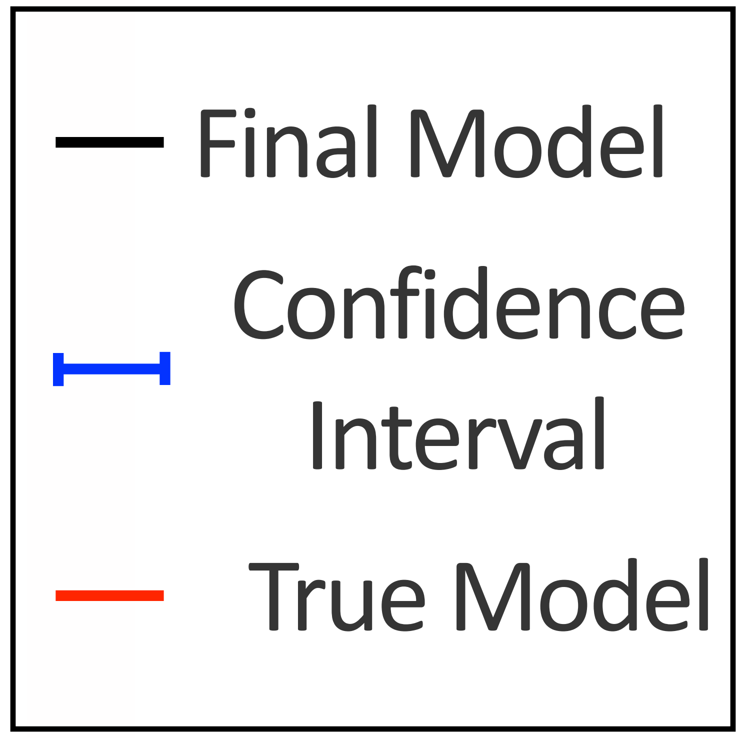
x = 1040m



x = 2040m



x = 3040m



Conclusions

- With *low*-rank approximation, the *evaluation* of *Hessian* becomes *tractable*.
- Problem is that the wave-equation Hessian is often NOT low-rank in exploration seismolog
- We only need to solve one Newton type method and evaluate Hessian once when using the *approximated* Gaussian distribution at the point of *optimal* solution.

Future Work

- Use (randomized) probing techniques to represent the Hessian
    - parametric high-frequency approximation of WE Hessian by Demanet
    - Hessian approximation by HSS matrices
- Use simultaneous shots to reduce the computational cost;
- Use LBFGS Hessian instead of Gauss-Newton Hessian to reduce the computational cost.
- Uncertainty analysis based on different assumption of the likelihood part and prior information;
- Uncertainty analysis based on the stochastic PDE constraint optimization problems

Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

SINBAD

NSERC CRSNG

This work was in part financially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant (22R81254) and the Collaborative Research and Development Grant DNOISE II (375142-08). This research was carried out as part of the SINBAD II project with support from the following organizations: BG Group, BGP, BP, CGG, Chevron, ConocoPhillips, ION, Petrobras, PGS, Statoil, Total SA, WesternGeco, and Woodside.