Structured tensor missing-trace interpolation in the Hierarchical Tucker format

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"Optimization on the Hierarchical Tucker manifold - applications to tensor completion". Submitted.

SLIM
University of British Columbia

Motivation

3D seismic experiments - 5D data

- expensive to acquire, store
- sample at *sub-Nyquist* rates

Data exhibits *low-rank* structure

- exploit structure for interpolation

Fully sampled data

- simultaneous sources in wave-equation based inversion
- mitigating multiples

[1] Kreimer, N, and Sacchi, M. D. "A tensor higher-order singular value decomposition for prestack seismic data noise reduction and interpolation." (2012)
[2] Gao, Jianjun, Vicente Oropeza, and Mauricio D. Sacchi. "Evaluation of a fast algorithm for the eigen-decomposition of large block Toeplitz matrices with application to 5D seismic data interpolation." *(2011)*

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Context

Low-rank matrix/tensor completion via *nuclear norm* projection [1]

- Require SVDs on huge data matrices
- Not scalable to large problem sizes

Data completion via Toeplitz embedding [2]

- Problem size - (# data points)^2

Goals

Generalization of Compressive Sensing to multiple dimensions

- what can we learn from 1D/2D recovery?

Randomized source/receiver acquisition

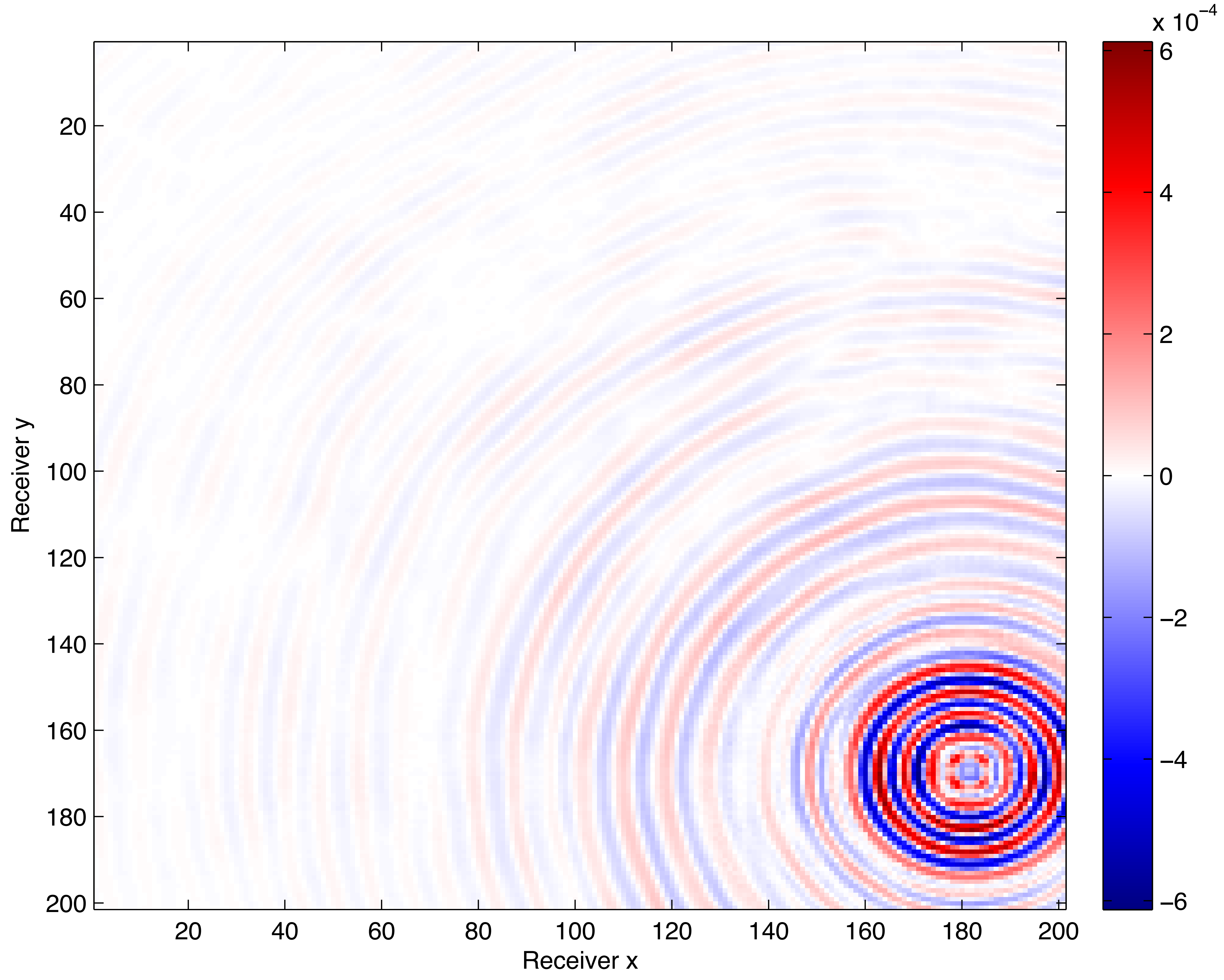
- reduce acquisition financial/time costs

Efficient solver

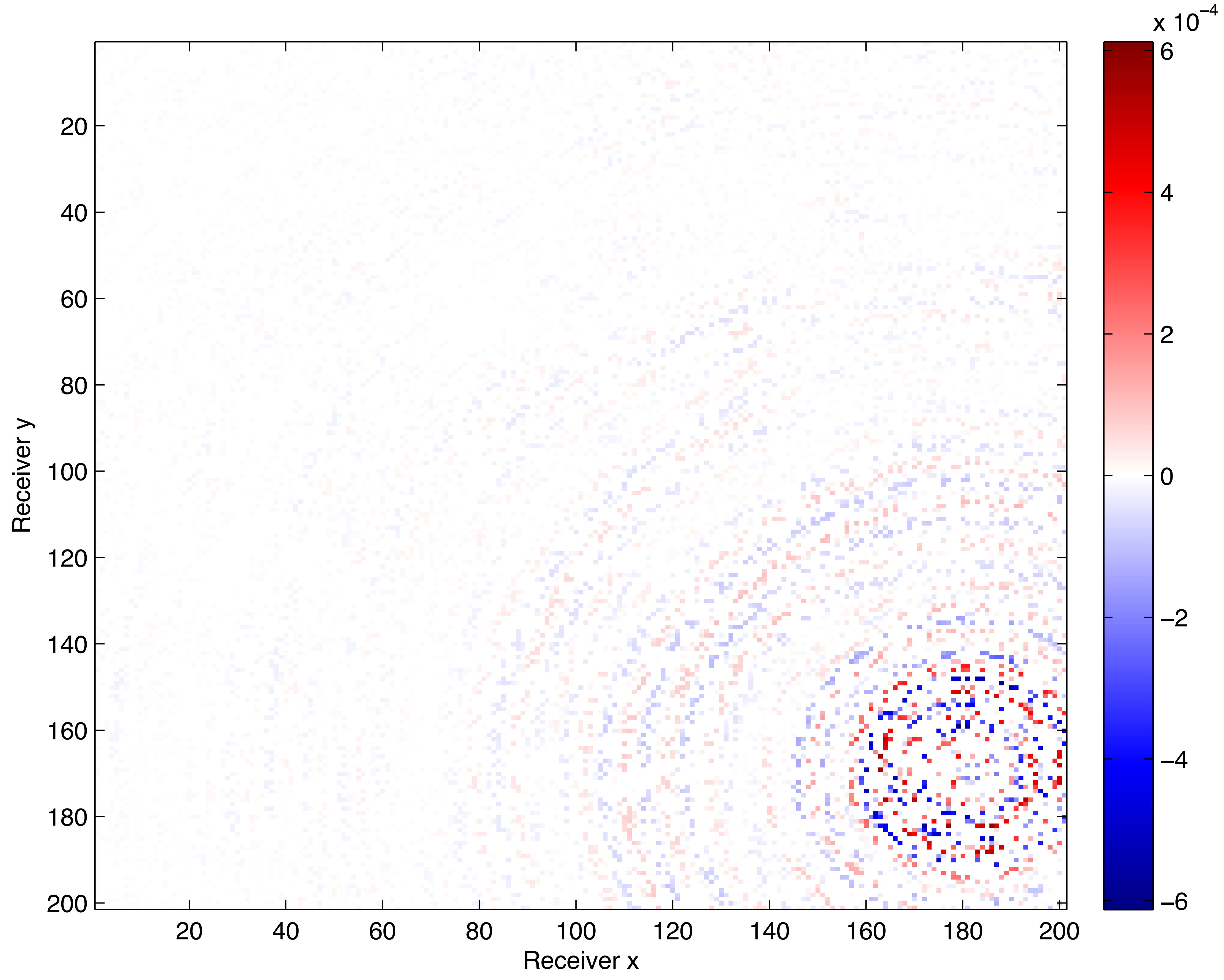
- SVD-free, parallelizable
- # parameters << # data points

7.34 Hz - 75% missing receivers

Common source gather



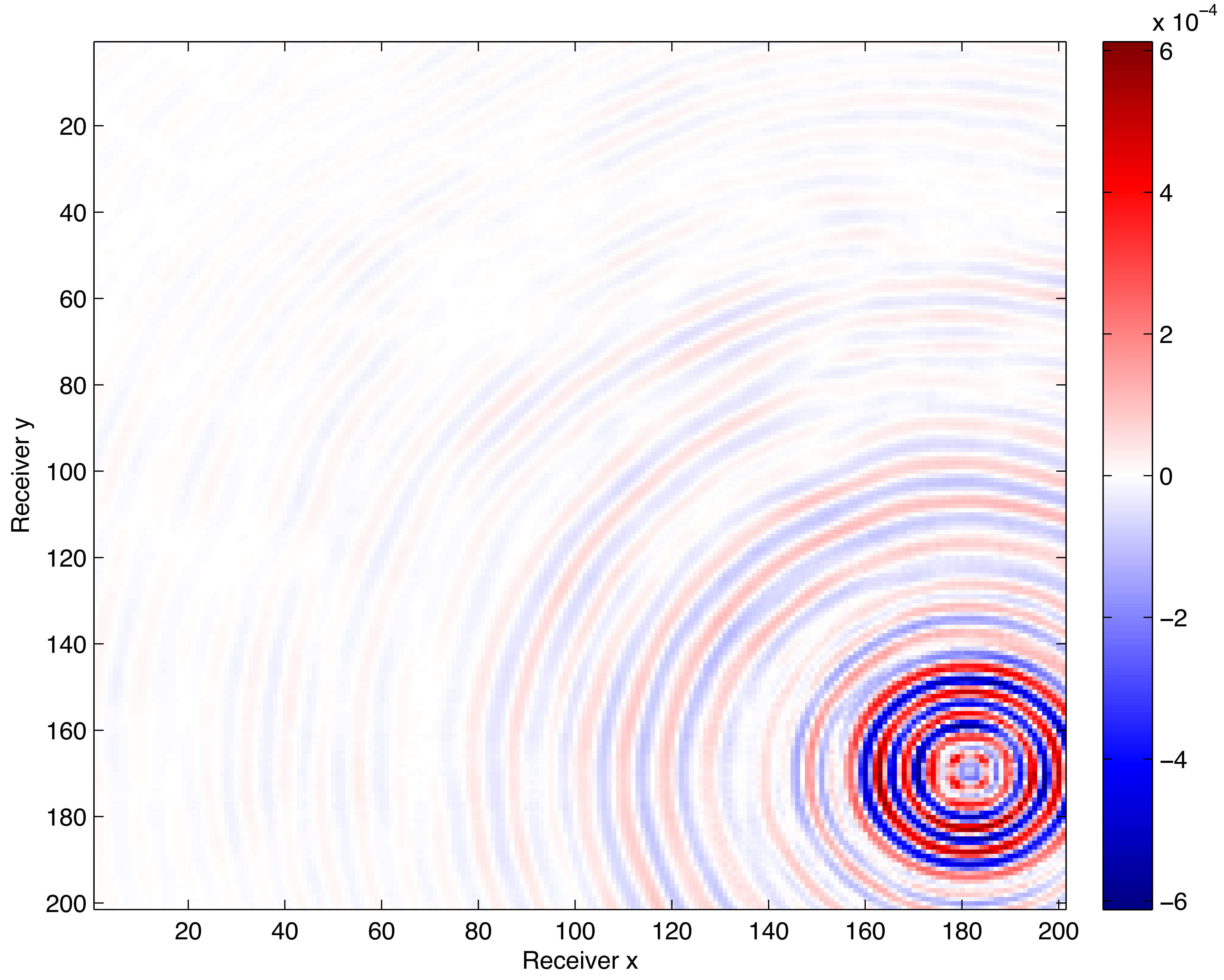
True data



Subsampled data

7.34 Hz - 75% missing receivers

Common source gather



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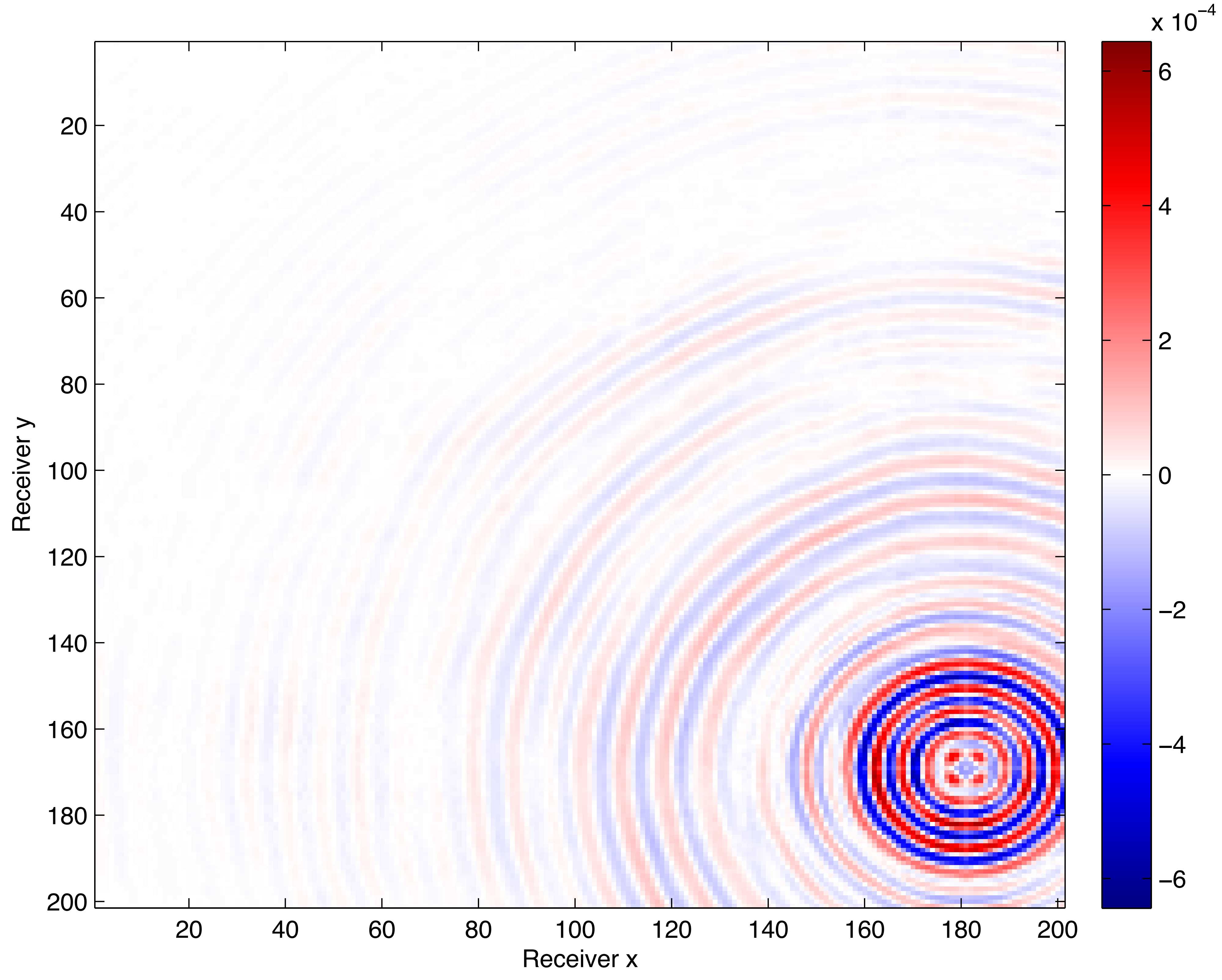

Recovered data - SNR 17.4 dB

Compressive sensing
with sparsity promotion

Successful reconstruction scheme

Signal structure

- sparsity

Sampling

- subsampling decreases sparsity

Optimization

- look for sparsest solution

Multidimensional interpolation

with Hierarchical Tucker

Successful reconstruction scheme

Signal structure

- ***Hierarchical Tucker***

Sampling

- subsampling increases hierarchical rank

Optimization

- fit data in the Hierarchical Tucker format

Matricization

The matricization of a tensor X with dimensions $1, \ldots, d$ along the dimensions $t = (t_1, \ldots, t_r)$ is the matrix formed by placing the dimensions t along the rows and dimensions t^c along the columns

Denoted $X^{(t)}$

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Example in Matlab

```
n1 = 20; n2 = 20; n3 = 20; n4 = 20;
% Tensor
x = randn(n1,n2,n3,n4);

% Matricization along dimensions 1 and 2
x12 = reshape(x,n1 * n2, n3 * n4);

% Matricization along dimensions 3 and 4
y34 = permute(x,[3 4 1 2]);
x34 = reshape(x, n3 * n4, n1 * n2);

% Matricization along dimensions 1 and 3
y13 = permute(x,[1 3 2 4]);
x13 = reshape(x,n1 * n3, n2 * n4);
```

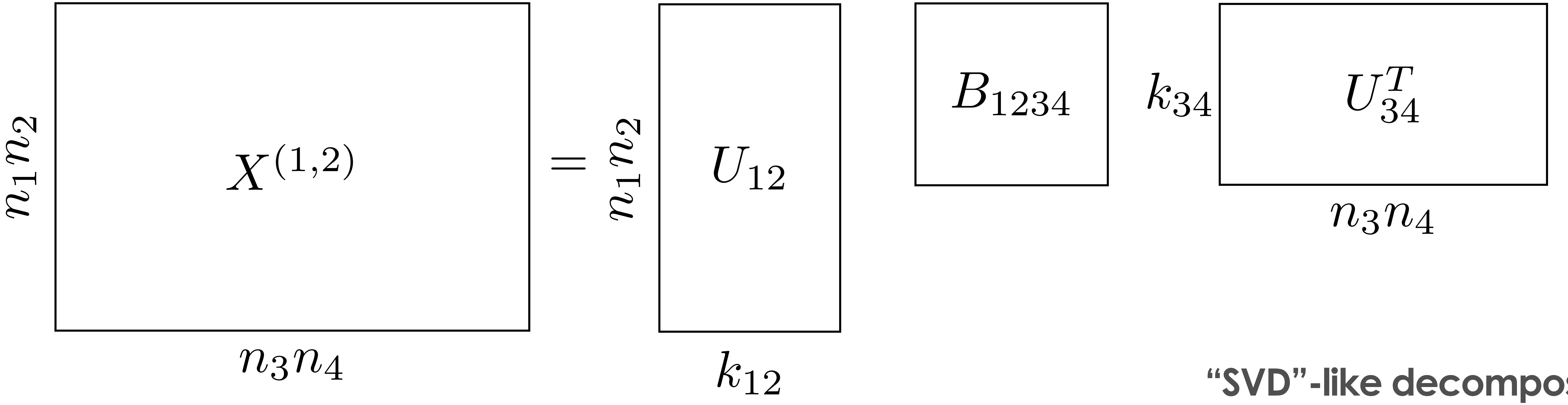
$X^{(1,2)}$: x12

$X^{(3,4)}$: y34, x34

$X^{(1,3)}$: y13, x13

Hierarchical Tucker format

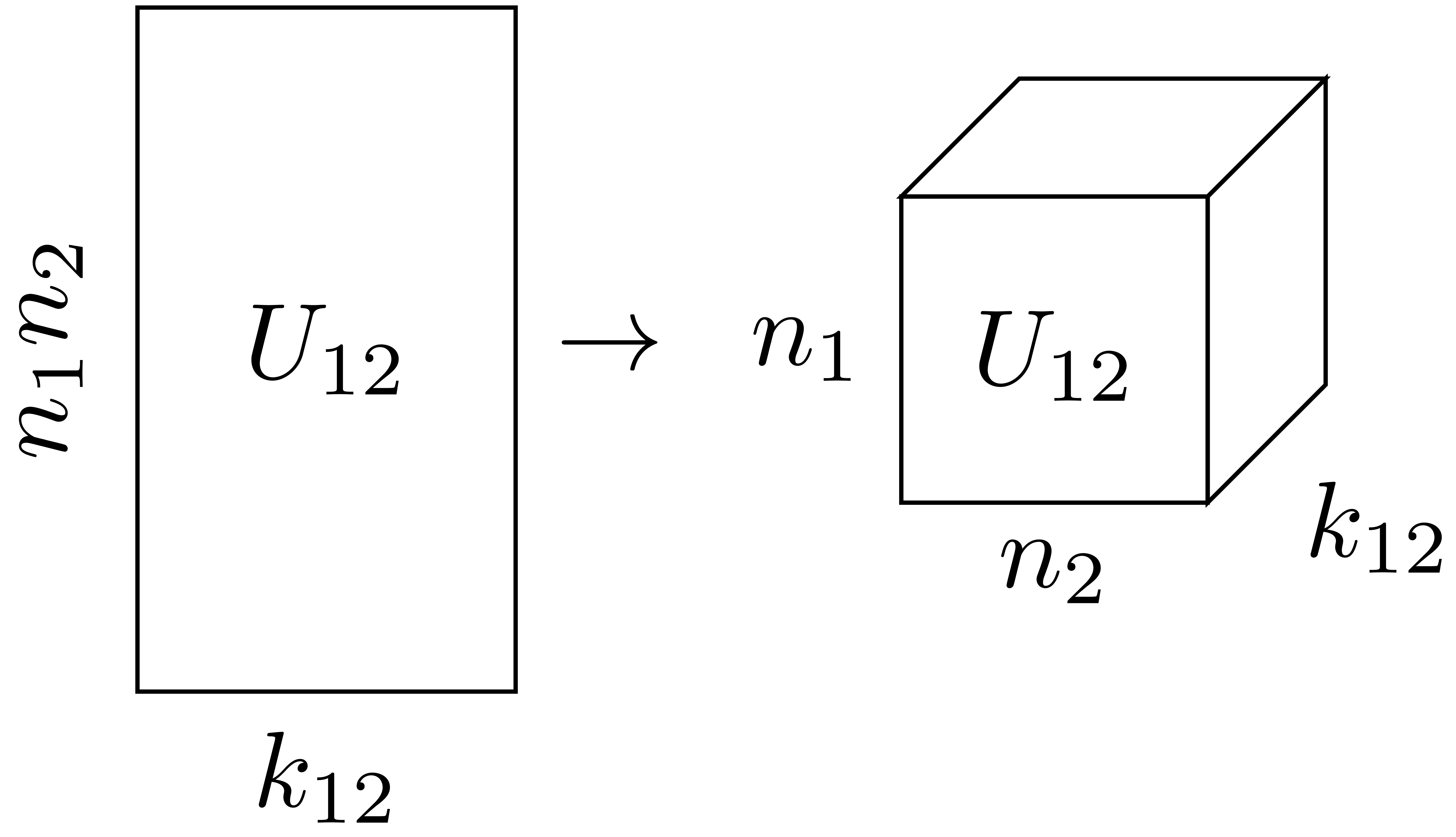
$X - n_1 \times n_2 \times n_3 \times n_4$ tensor



"SVD"-like decomposition

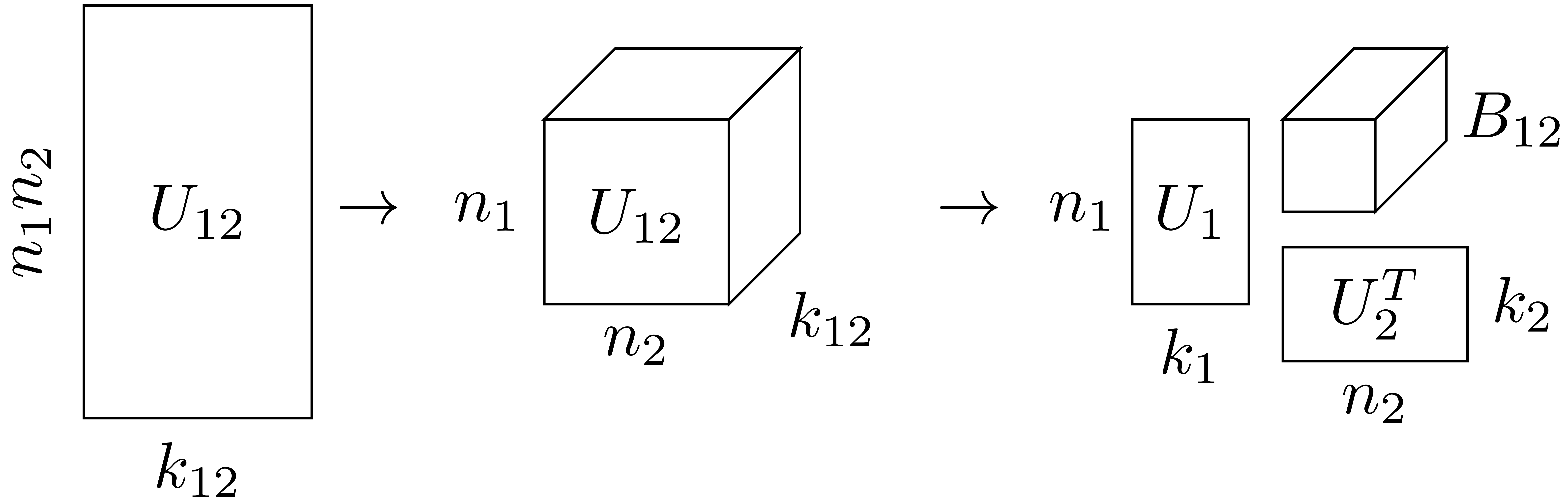
Hierarchical Tucker format

$X - n_1 \times n_2 \times n_3 \times n_4$ tensor



Hierarchical Tucker format

$X - n_1 \times n_2 \times n_3 \times n_4$ tensor



Hierarchical Tucker format

Intermediate matrices don't need to be stored

U_t, B_t - small parameter matrices

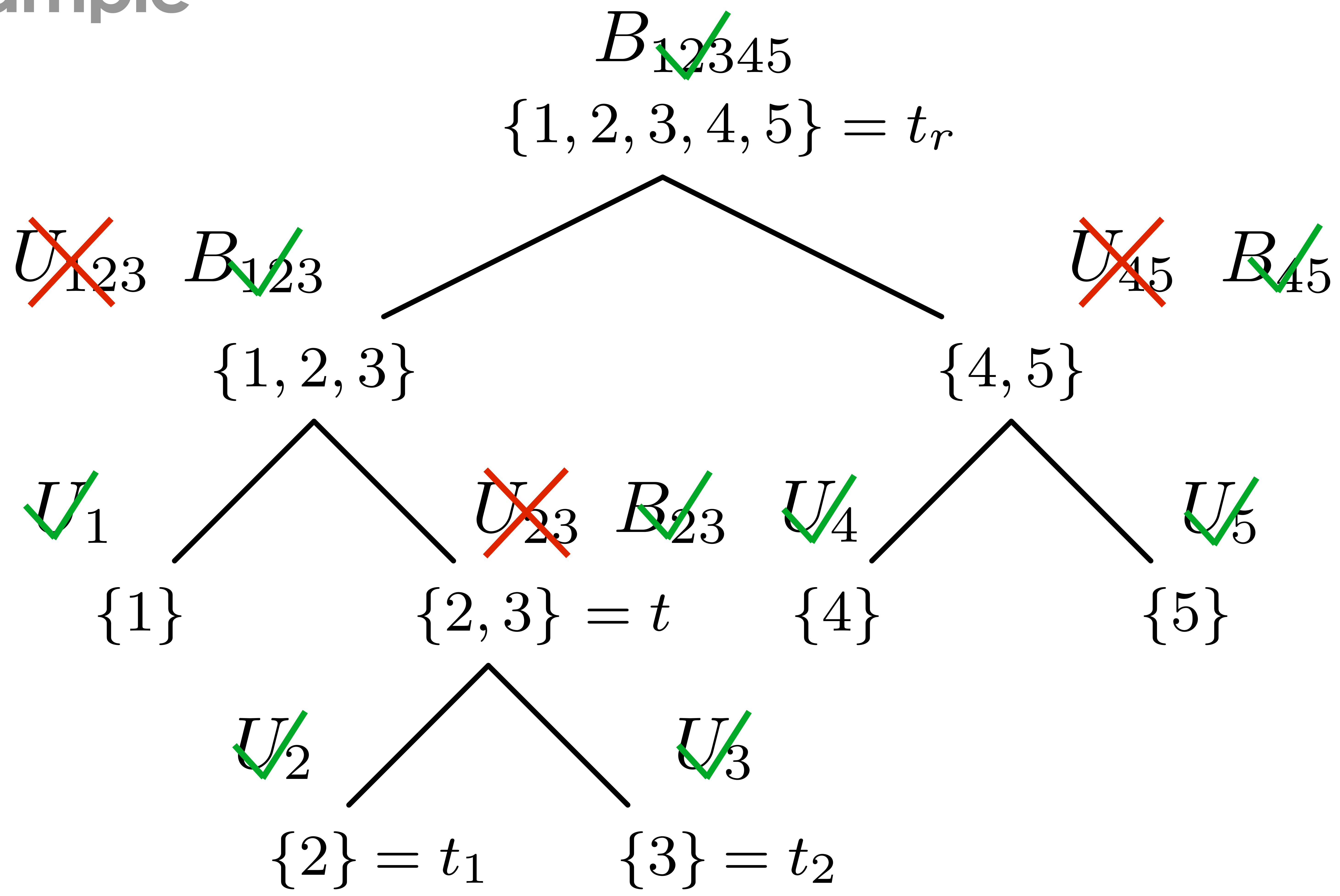
- specify the tensor completely

Separating groups of dimensions from each other

- dimension tree

A. Uschmajew, B. Vandereycken. The geometry of algorithms using hierarchical tensors. Linear algebra and its applications, 2013.

Example



Hierarchical Tucker format

Storage $\leq dNK + (d-2)K^3 + K^2$

Compare to N^d storage for the full tensor

Effectively breaking the curse of dimensionality when $K \ll N \quad d \geq 4$

Low frequency data compresses in HT

Hierarchical Tucker example

For a $100 \times 100 \times 100 \times 100$ cube with max rank 20

$N = 100, d = 4, K = 20$

Full storage: $N^d = 10^8$ values

HTucker storage: 24400 values

Compression of a factor of *99.97%*

Seismic Hierarchical Tucker

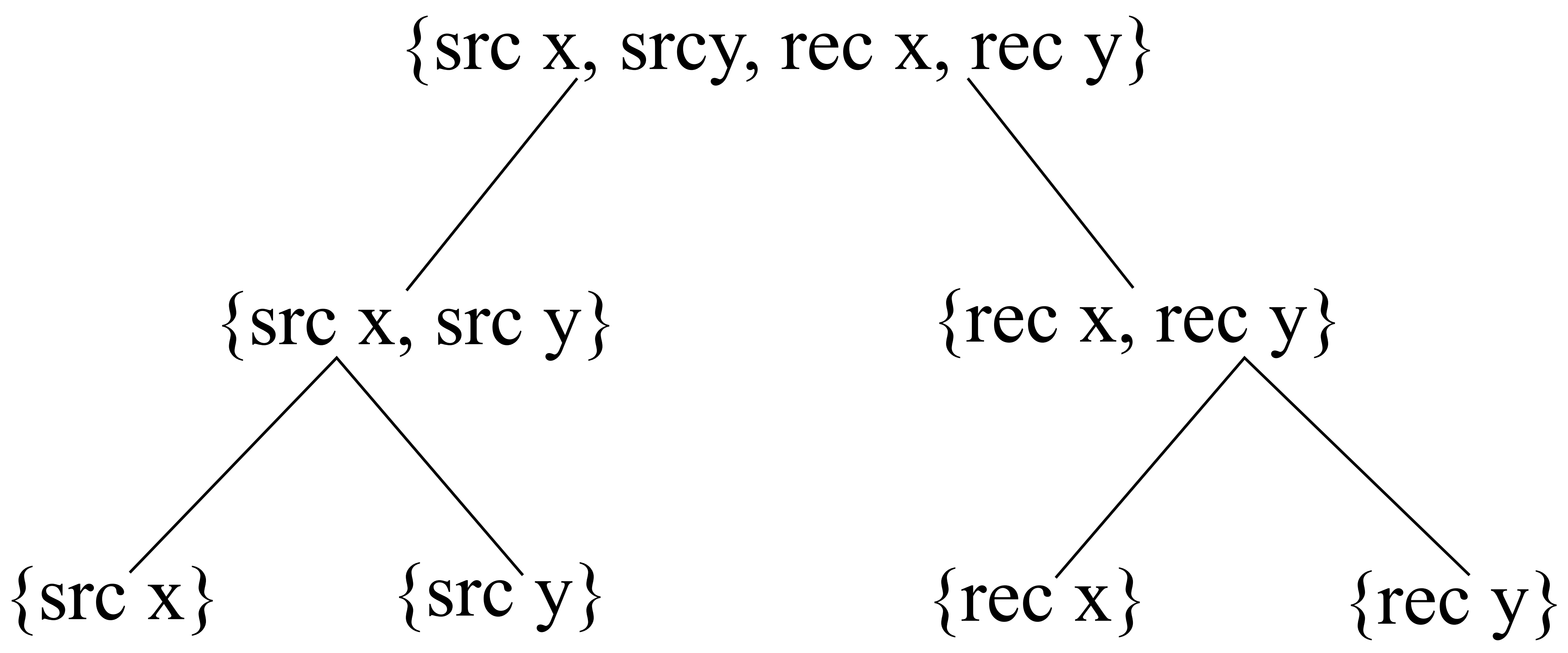
We consider a 3D seismic survey with coordinates
(src x, src y, rec x, rec y, time)

We take a Fourier transform in time and restrict ourselves to a single frequency slice

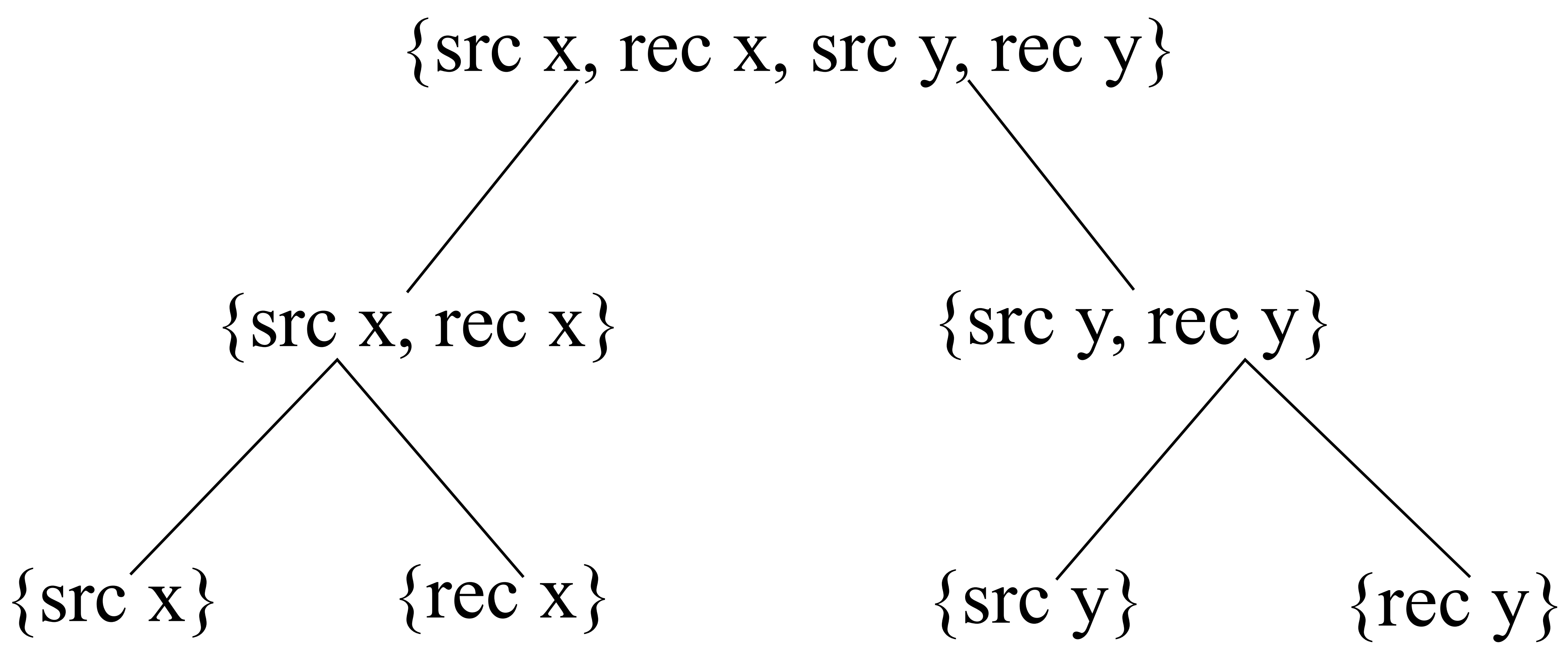
Seismic Hierarchical Tucker

For a frequency slice with coordinates (src x, src y, rec x, rec y),

there are essentially two choices of dimension splitting (by reciprocity)

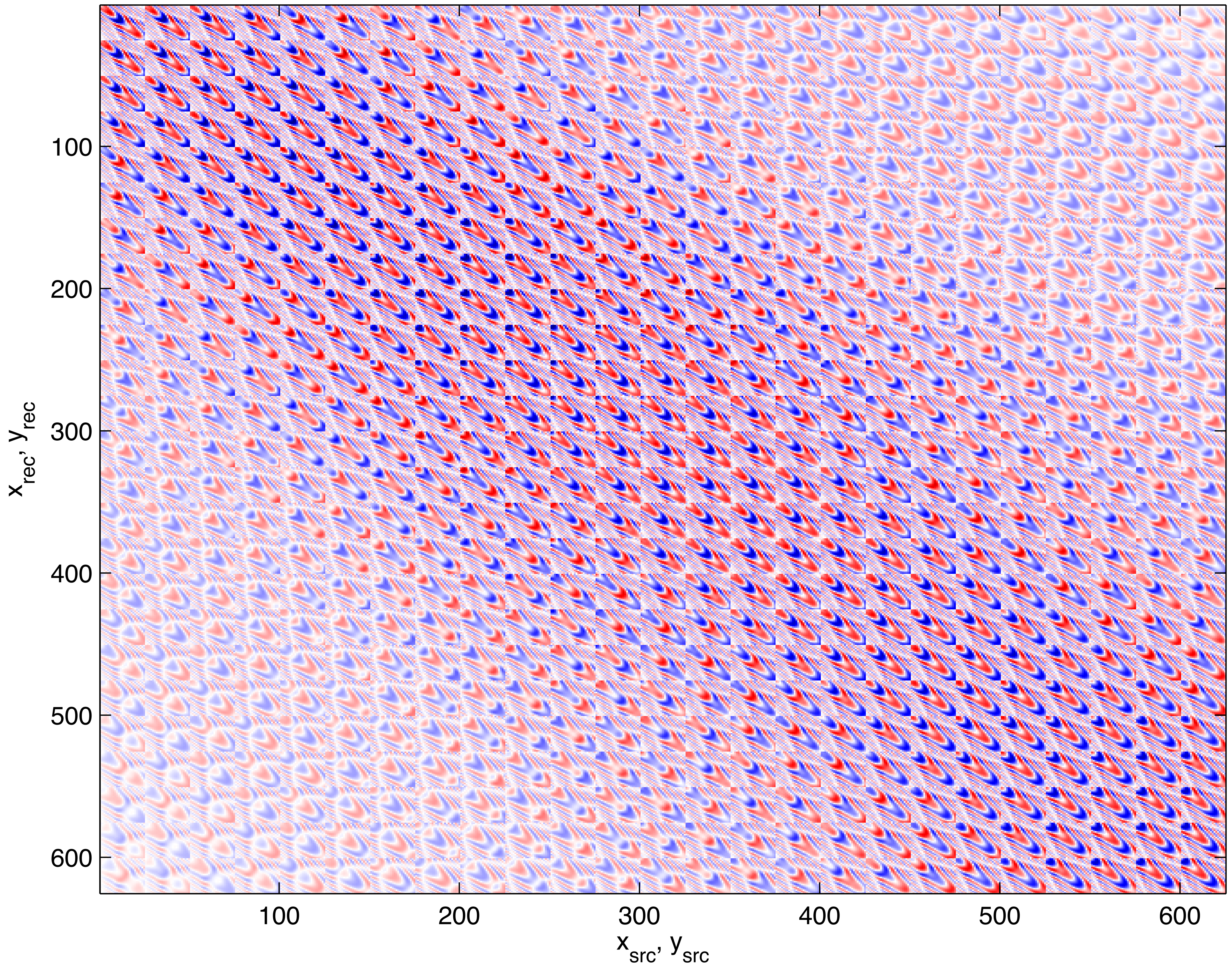


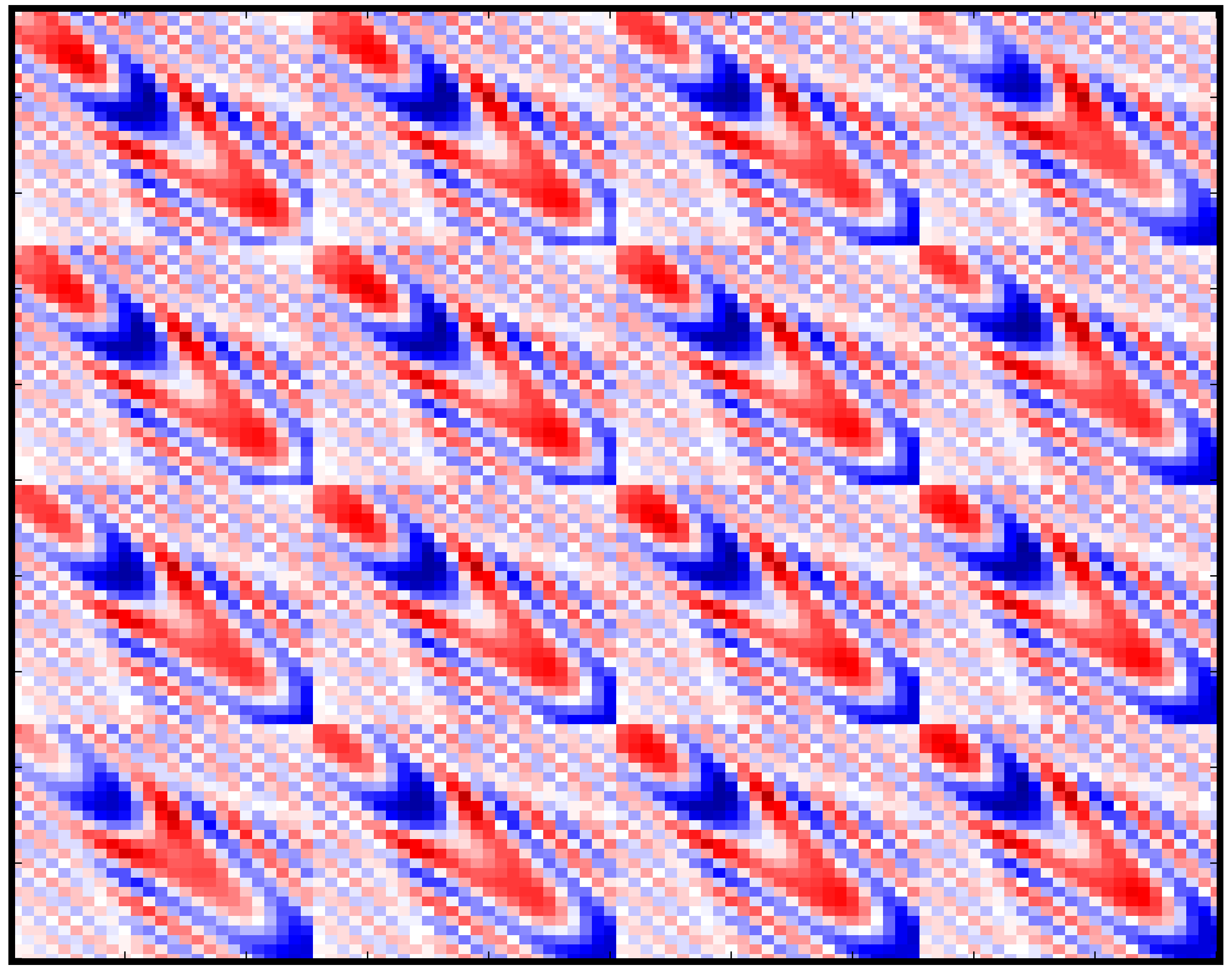
Canonical Decomposition

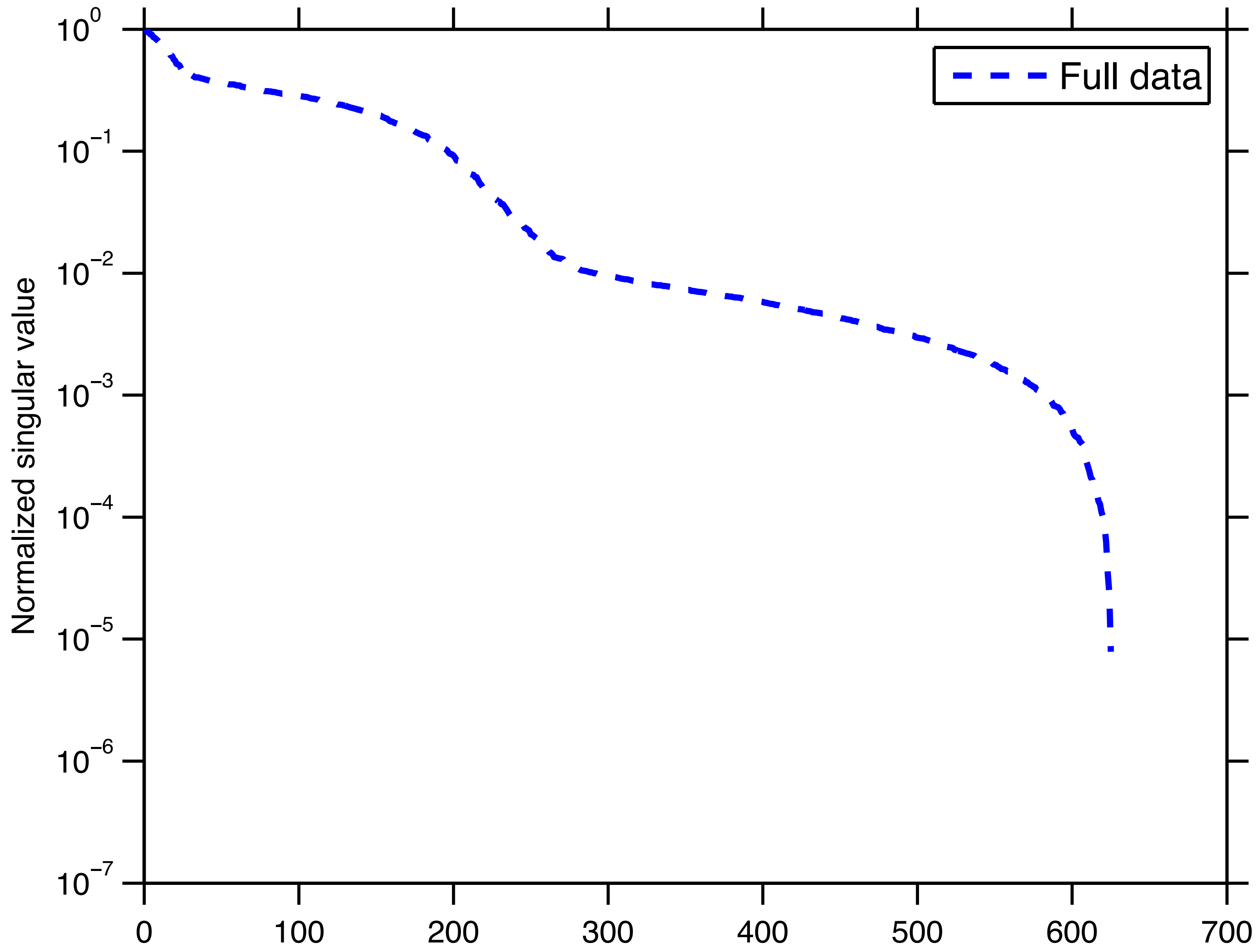


Non-canonical Decomposition

Matricizations

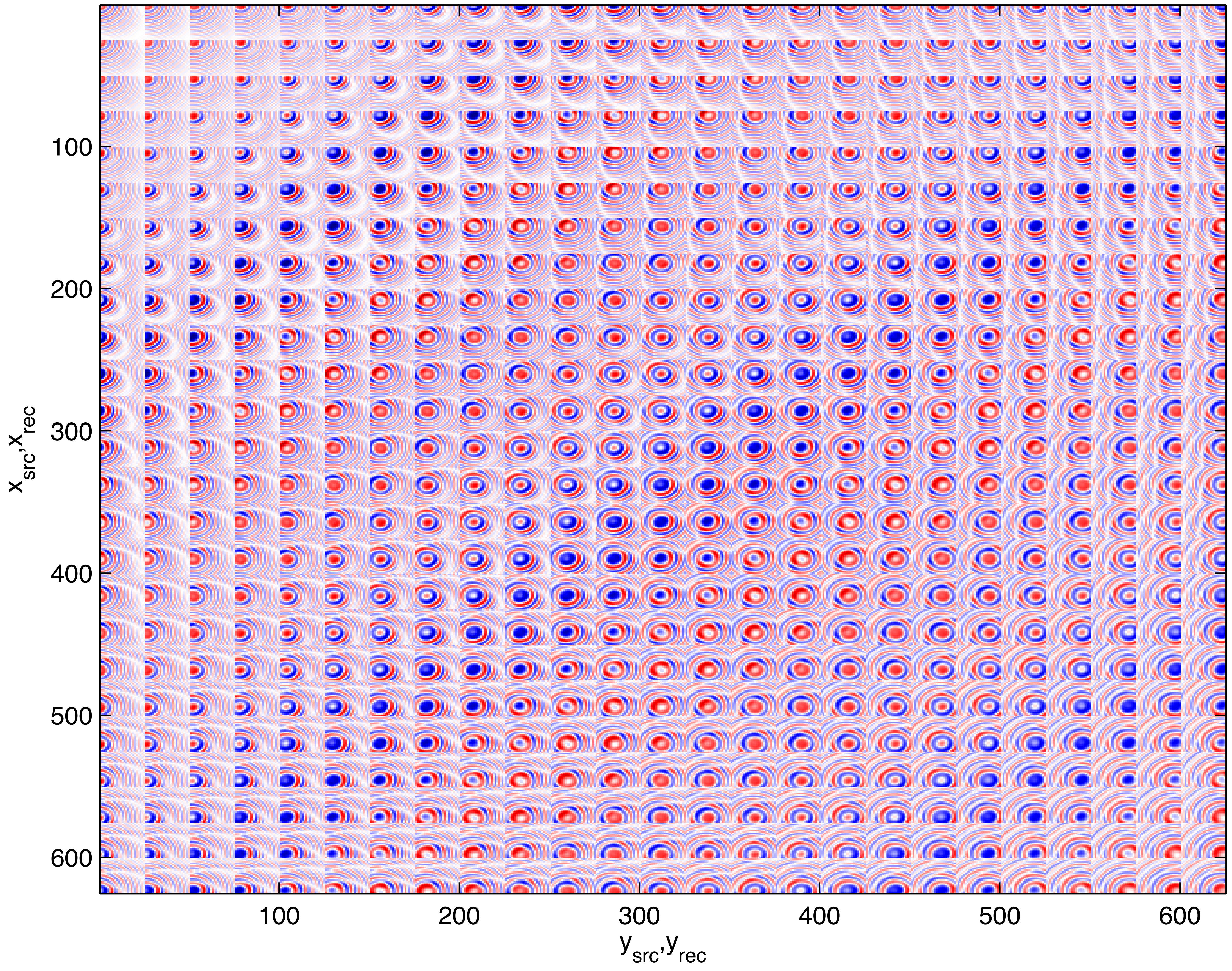


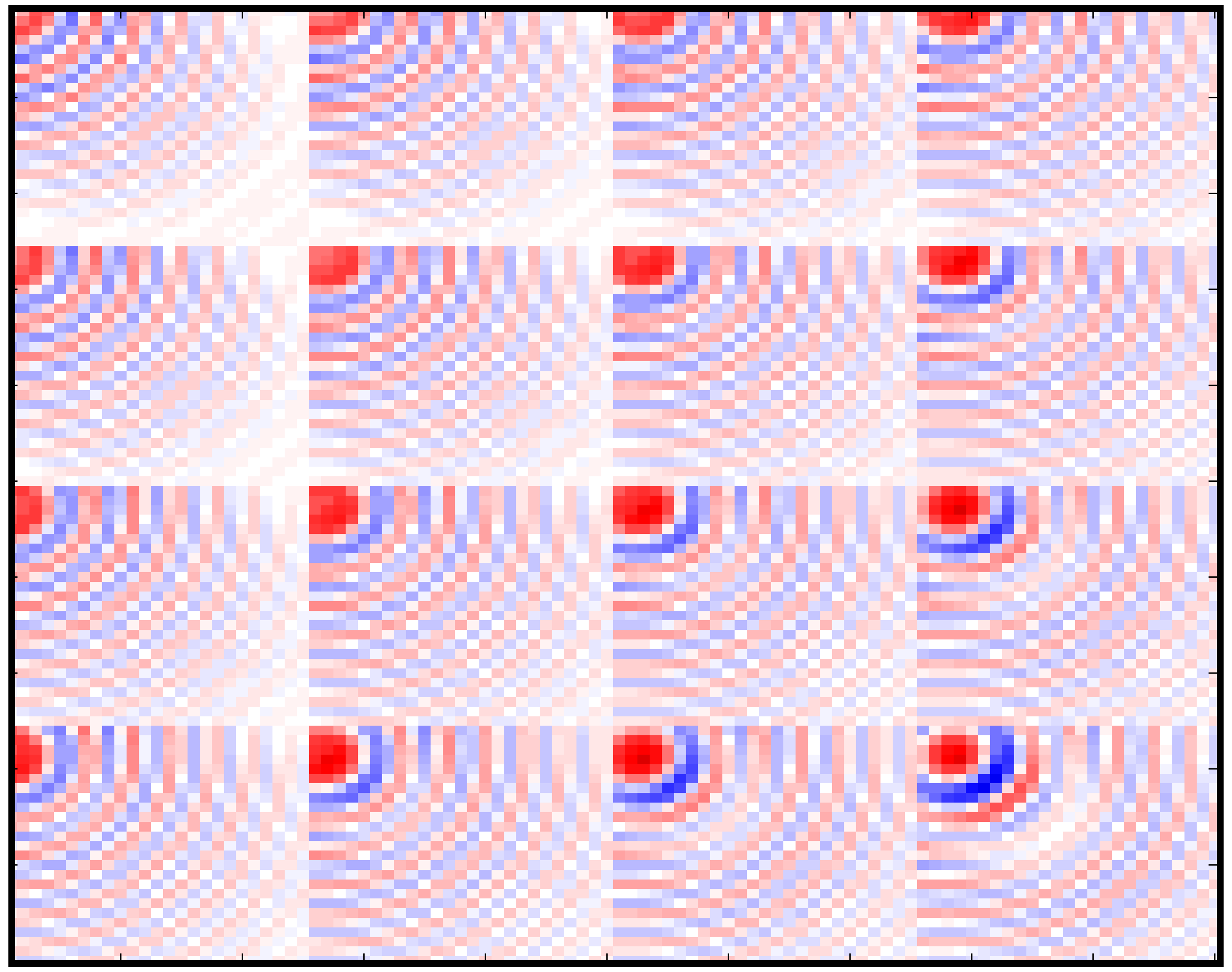


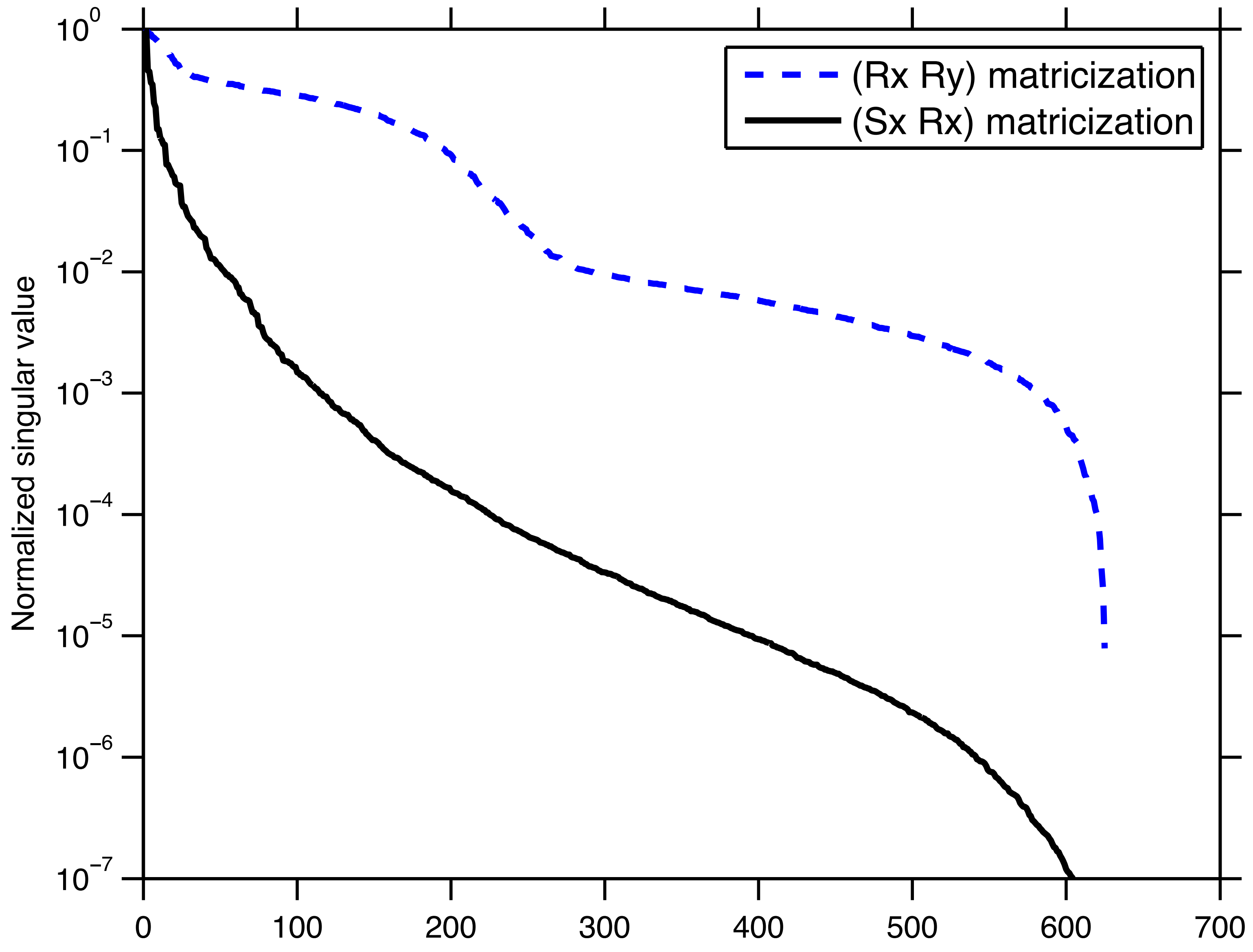


(Rec x, Rec y) matricization - Canonical ordering

Matricizations







(Src x, Rec x) matricization - Noncanonical ordering

Multidimensional interpolation

with Hierarchical Tucker

Successful reconstruction scheme

Signal structure

- Hierarchical Tucker

Sampling

- ***subsampling increases hierarchical rank***

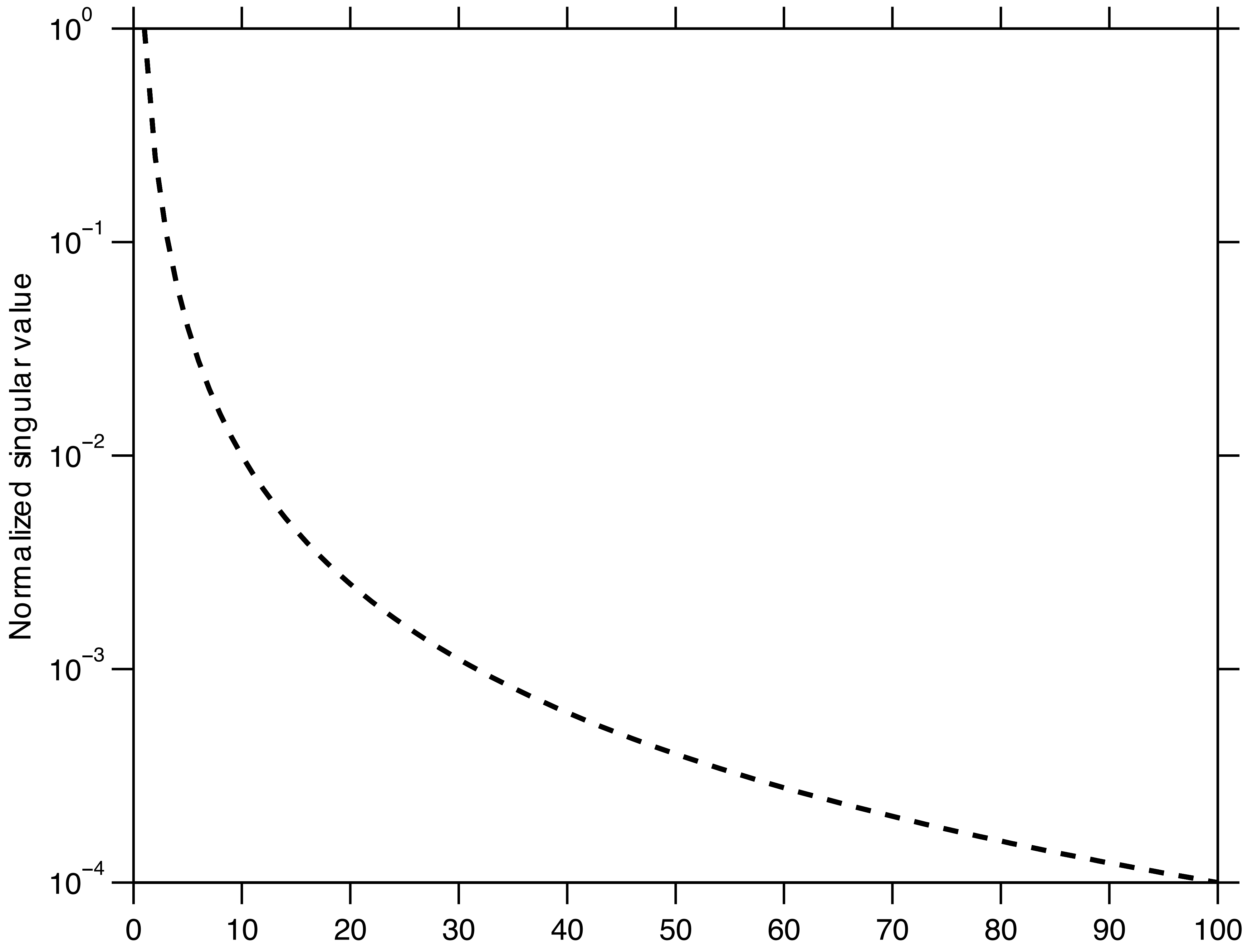
Optimization

- fit data in the Hierarchical Tucker format

Matrix Completion

$$\mathbf{X}$$

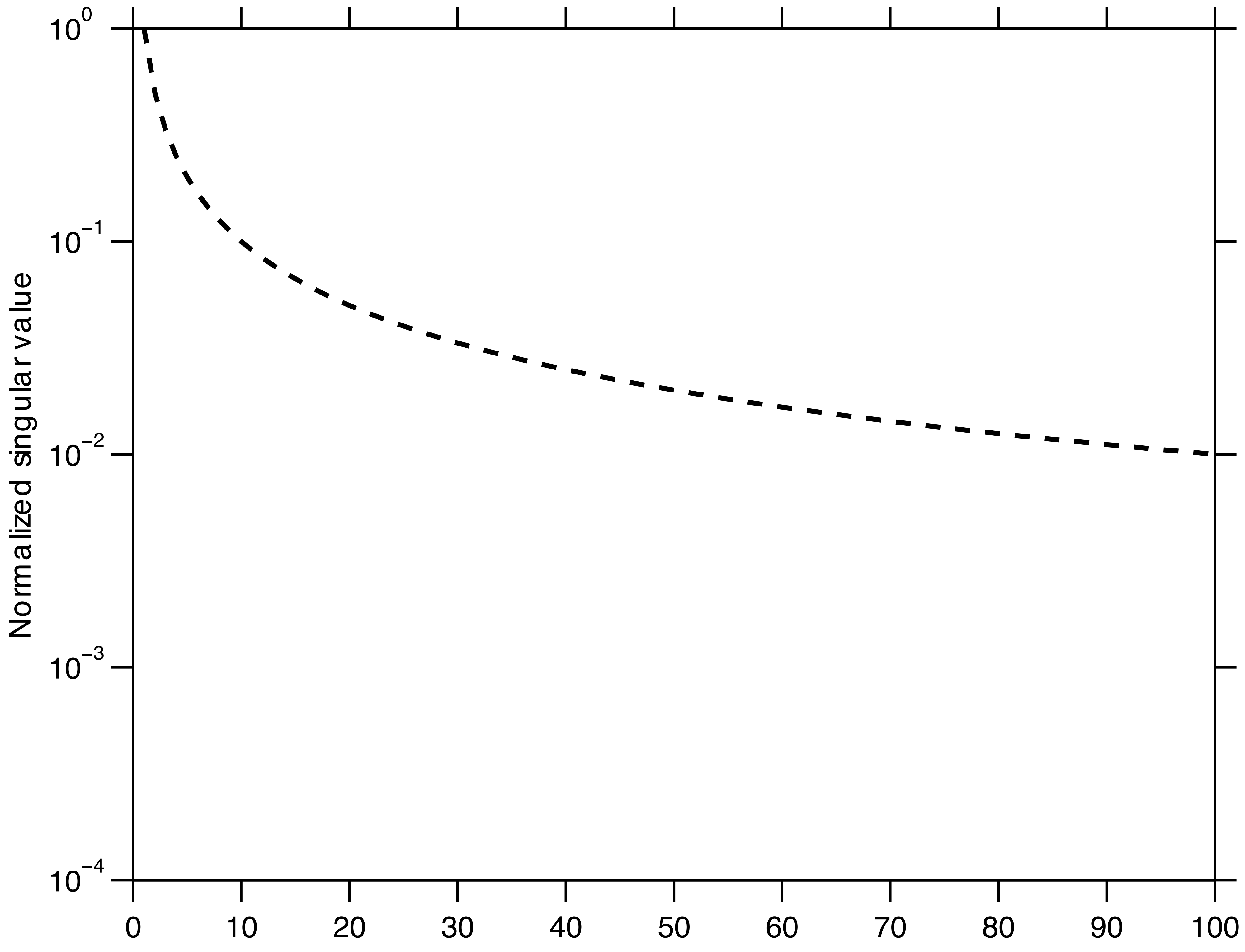
$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$



Matrix Completion

$$\mathcal{A}(\mathbf{X})$$

$$\begin{bmatrix} * & * & * & 0 & * \\ * & 0 & 0 & * & 0 \\ * & * & * & * & * \\ * & * & 0 & * & * \\ 0 & * & * & * & 0 \end{bmatrix}$$



Tensor Completion

Structure - recover a tensor X which has low hierarchical rank

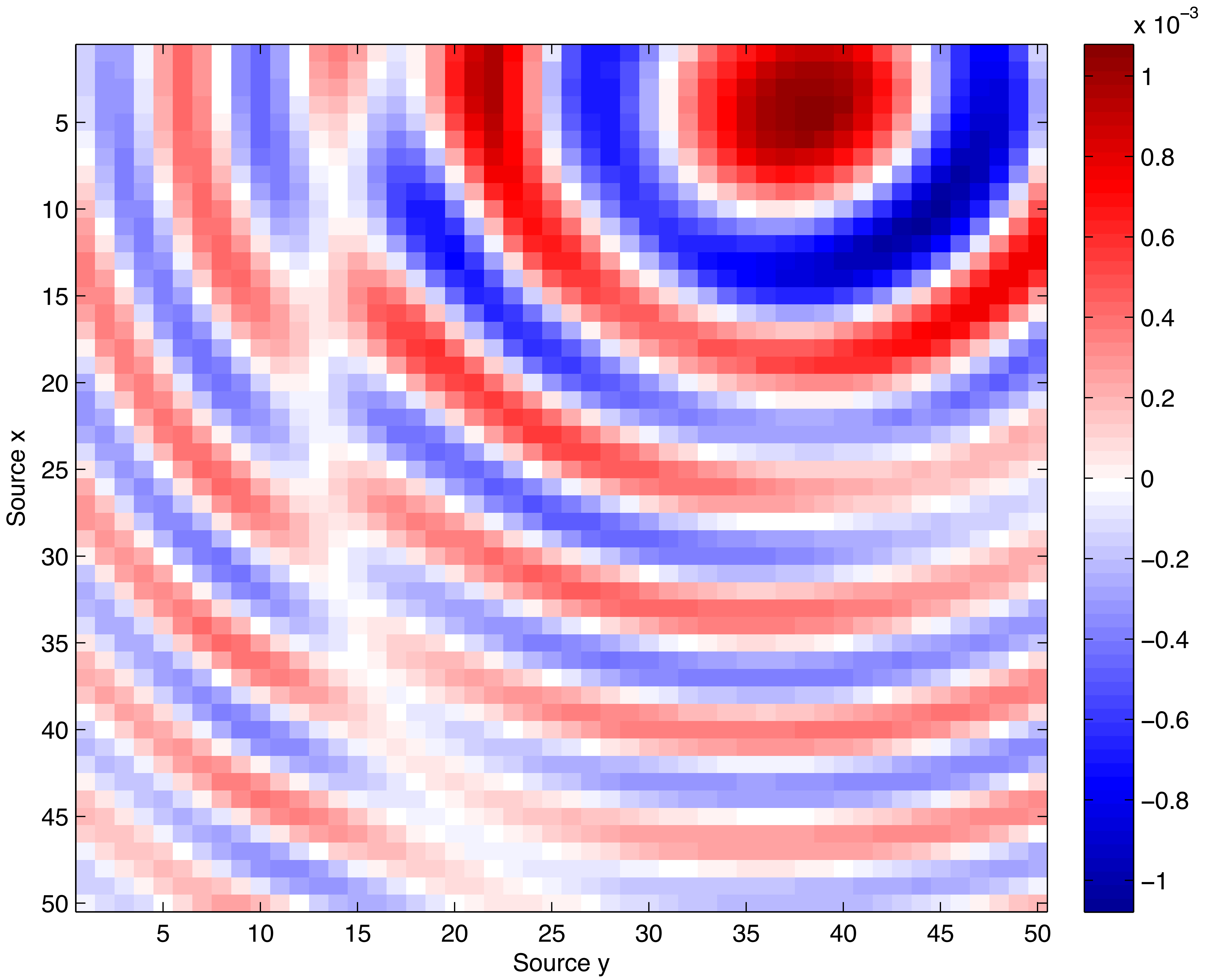
- Well represented in HT

Sampling - random removal of points increases rank

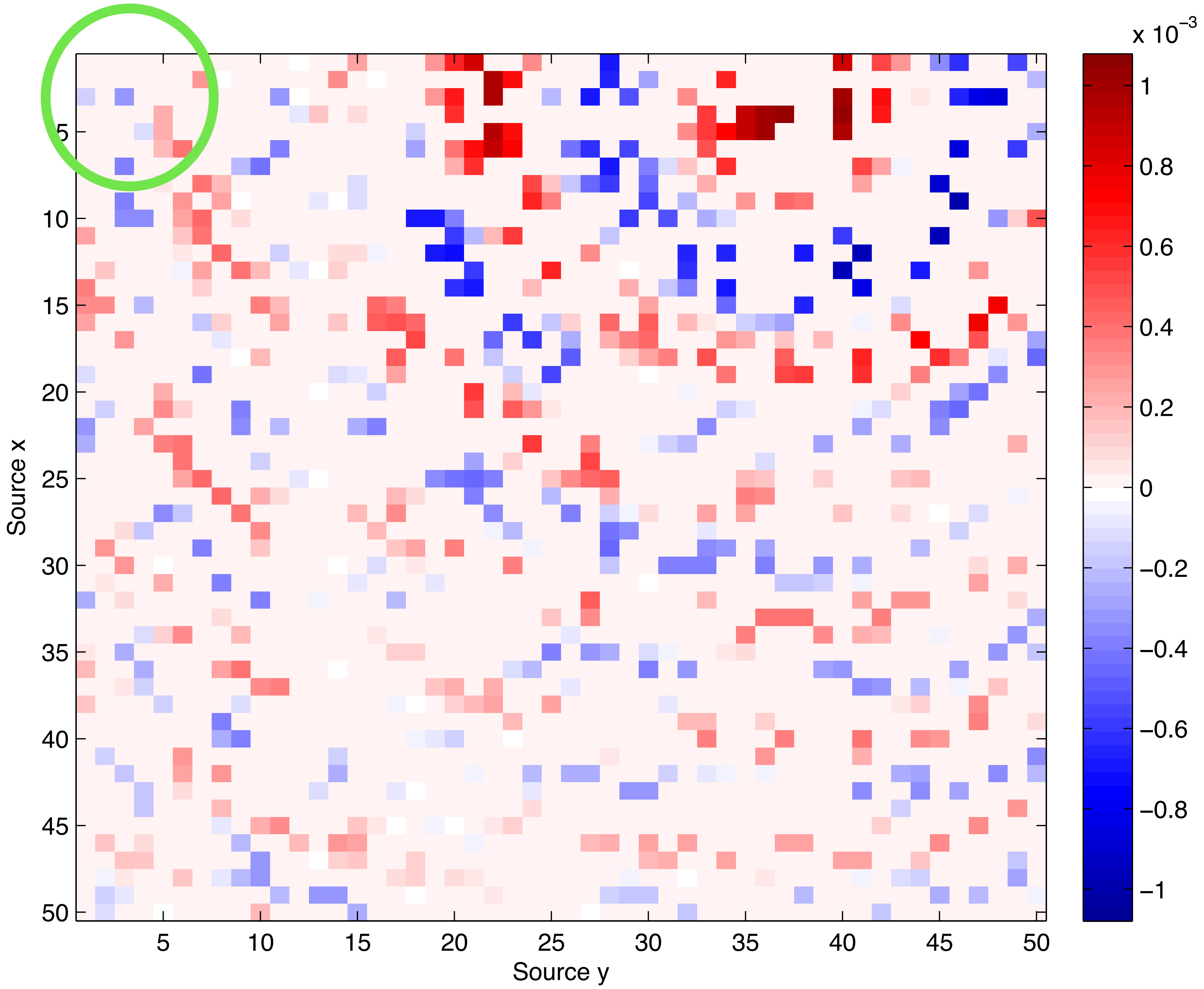
- Poorly represented in HT
- Idealized sampling

Idealized recovery

75% random entries removed
Common receiver gather



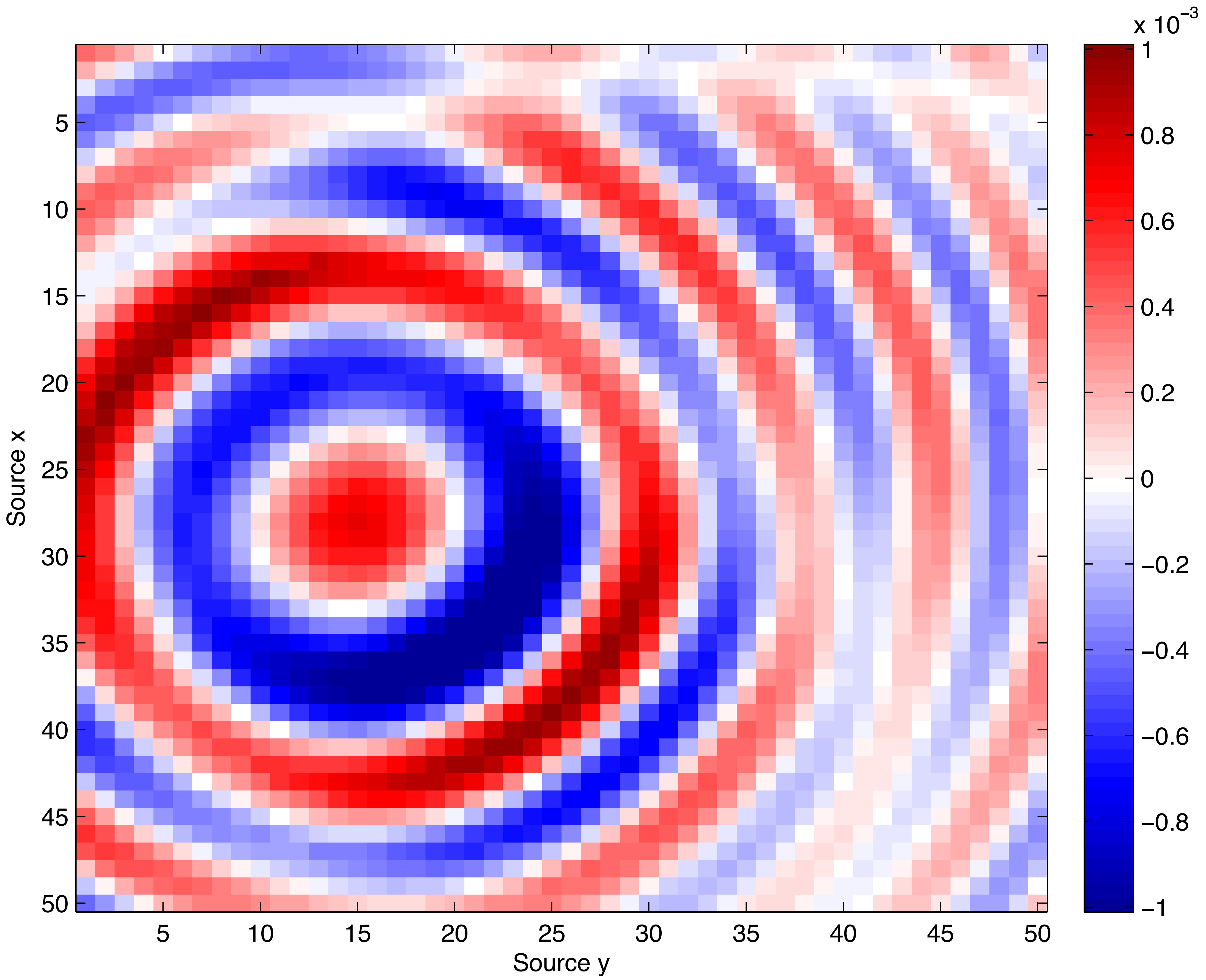
True data



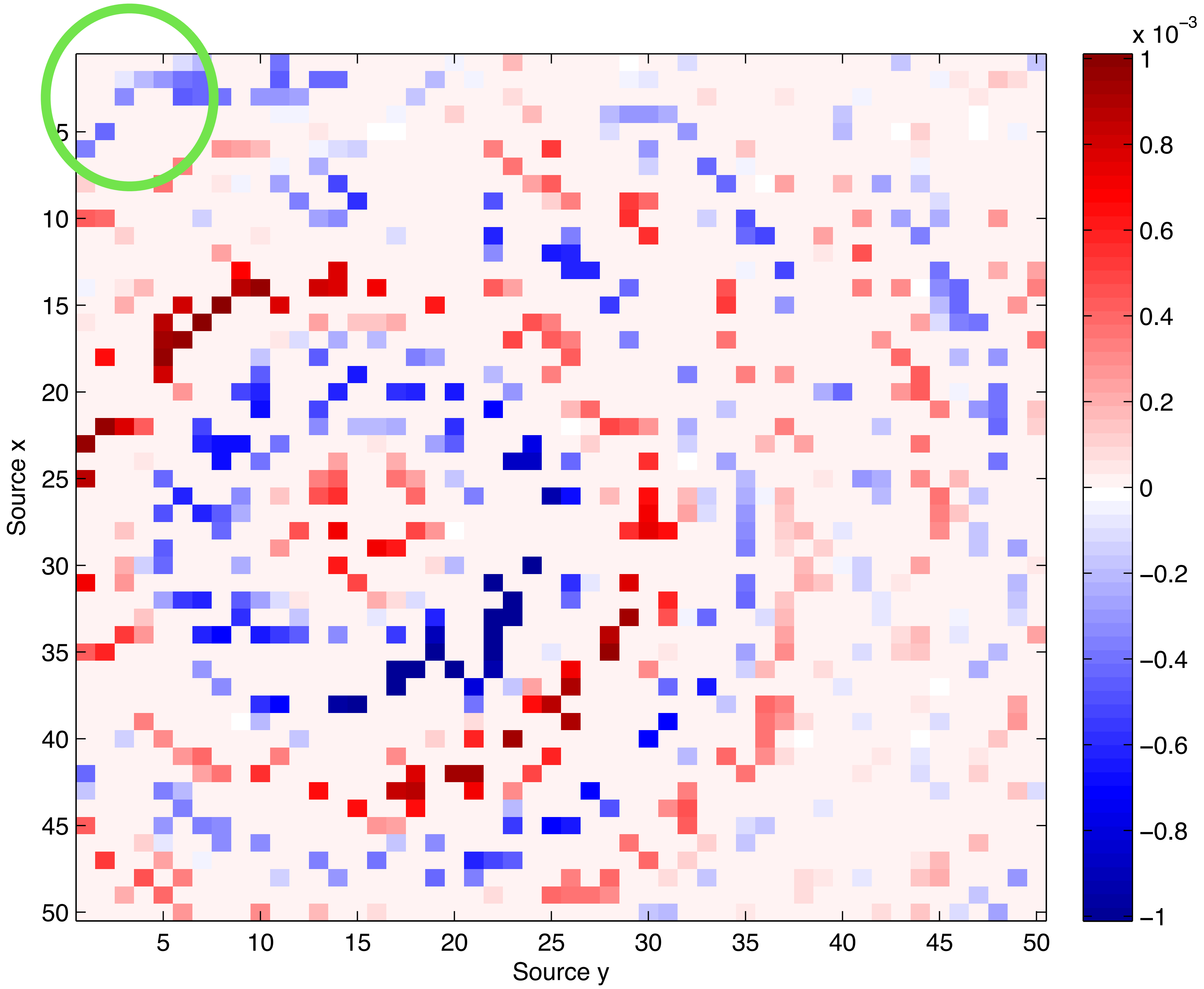
Subsampled data

Idealized recovery

75% random entries removed
Common receiver gather



True data



Subsampled data

Idealized recovery

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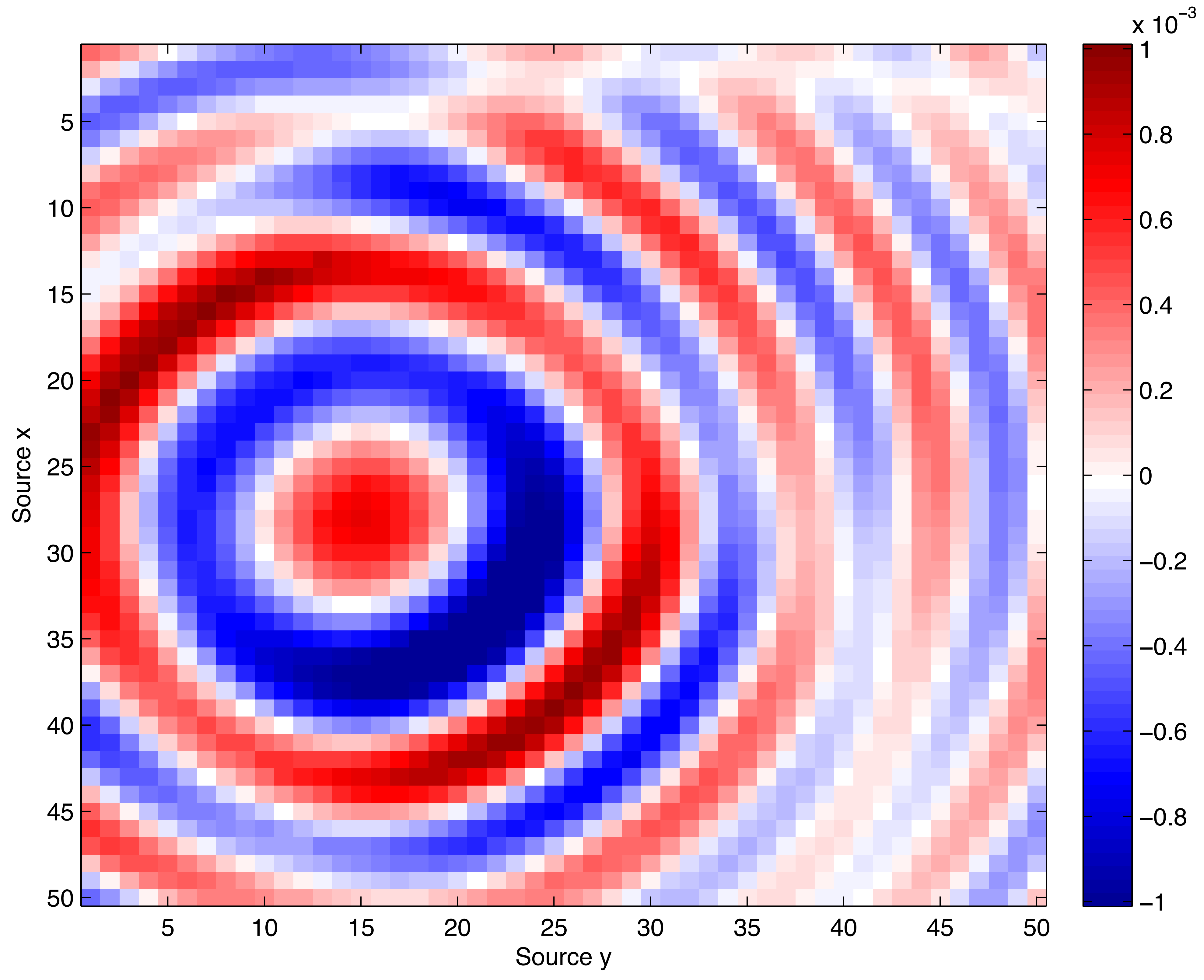
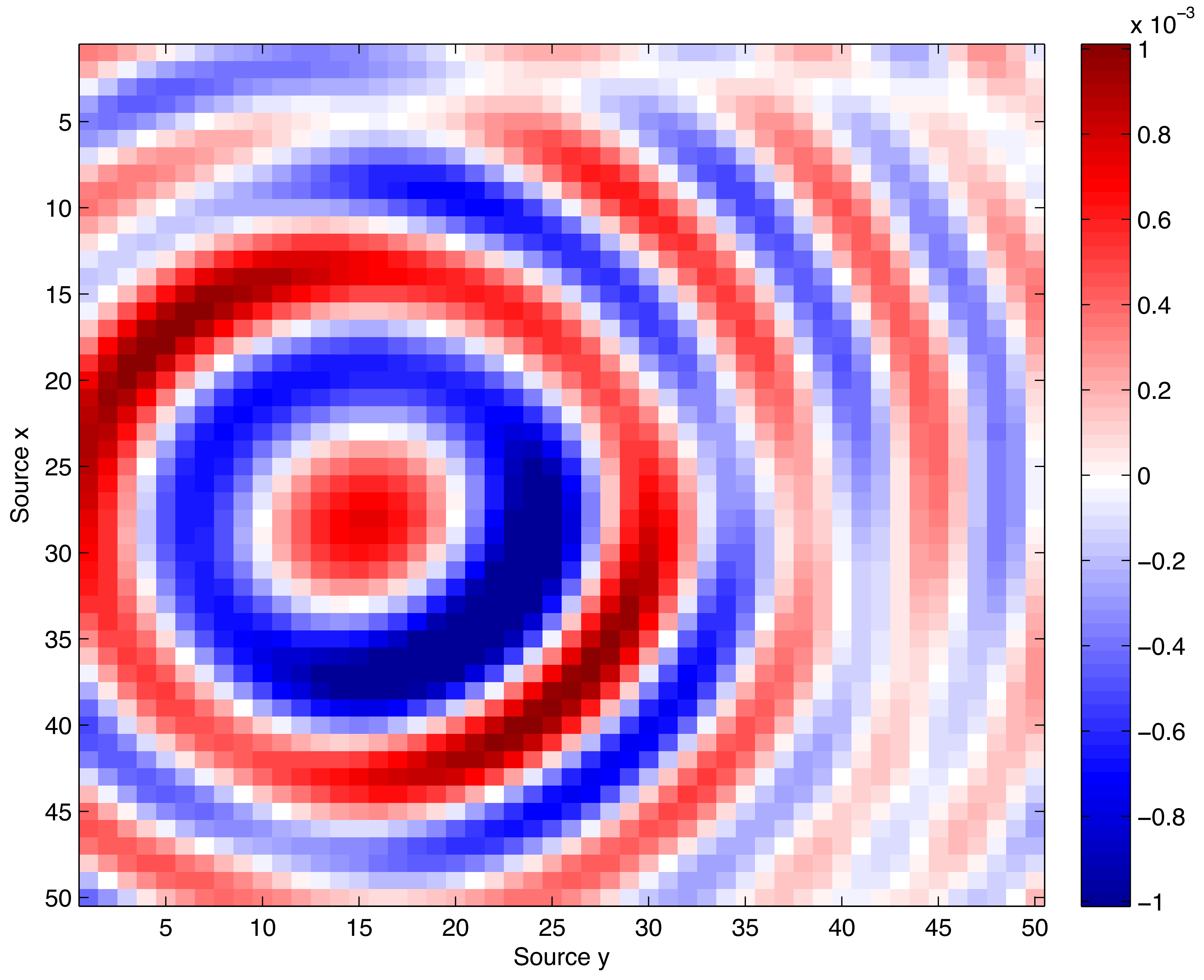
Common receiver gather



True data



Recovered data - SNR 19.3 dB

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Sampling

Sampling $(x_{\mathrm{src}}, y_{\mathrm{src}}, x_{\mathrm{rec}}, y_{\mathrm{rec}})$ points from the data

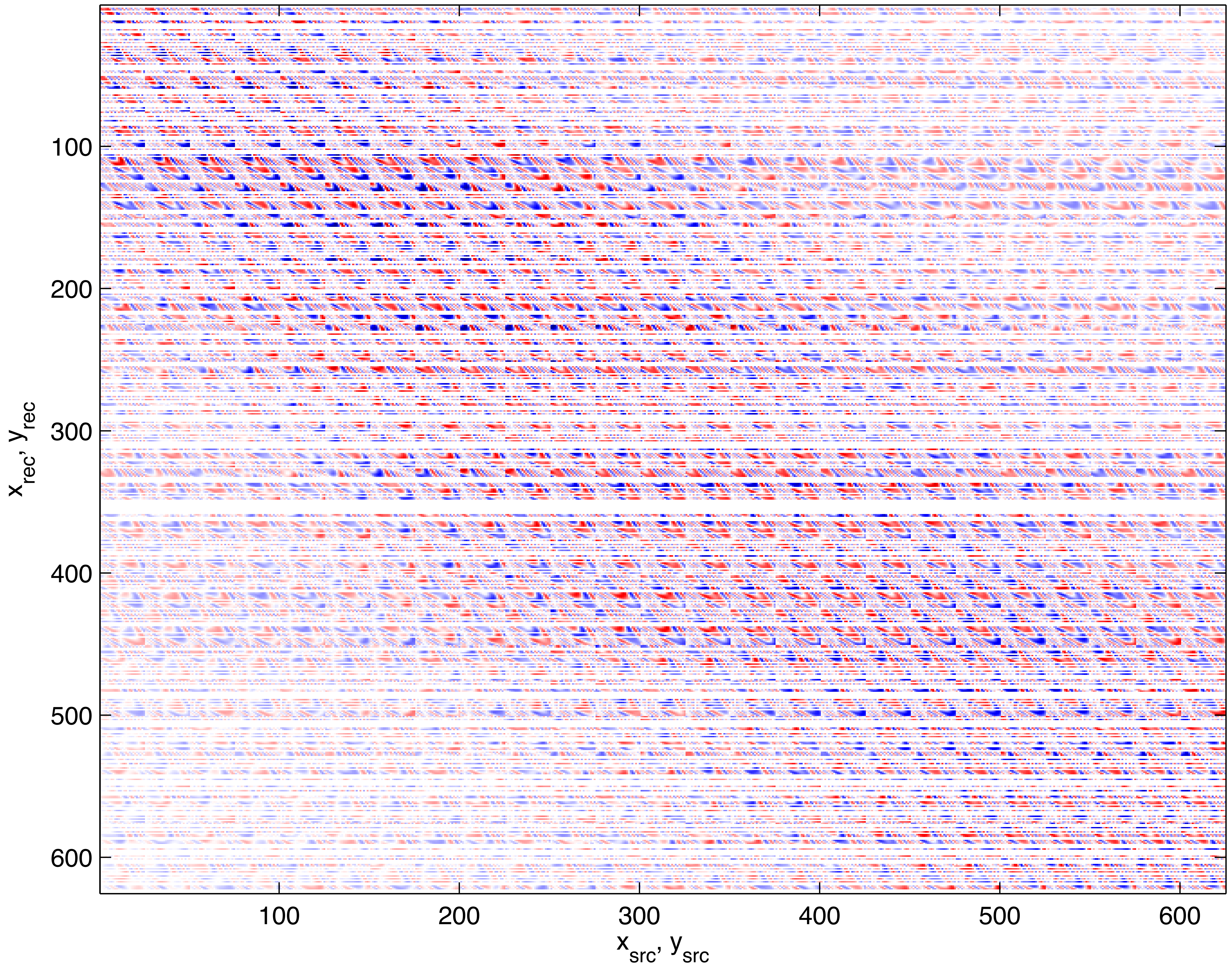
- idealized recovery
- impossible to physically implement

Sampling $(x_{\mathrm{rec}}, y_{\mathrm{rec}})$ points from the data

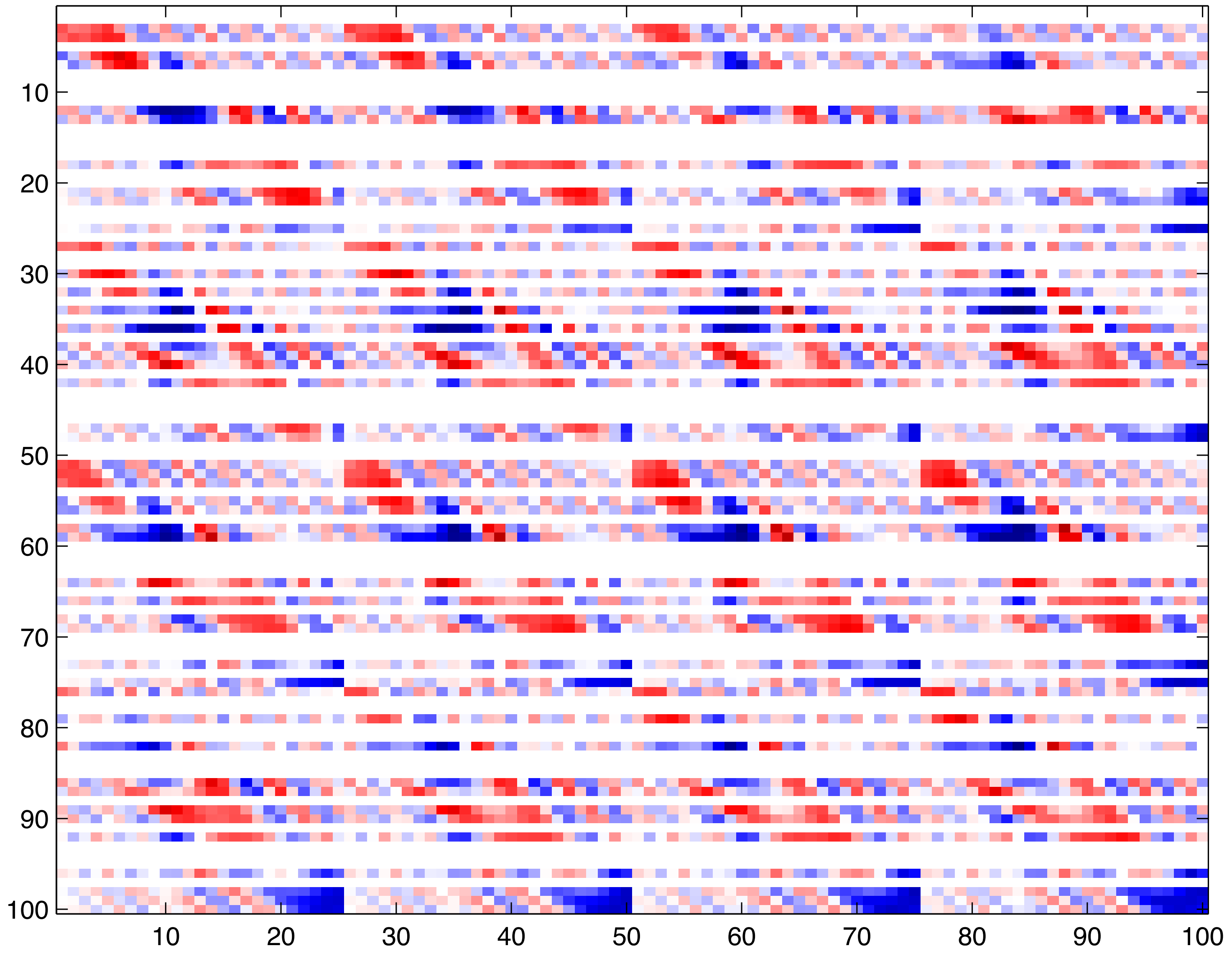
- less idealized
- possible to acquire data - e.g. ocean bottom nodes

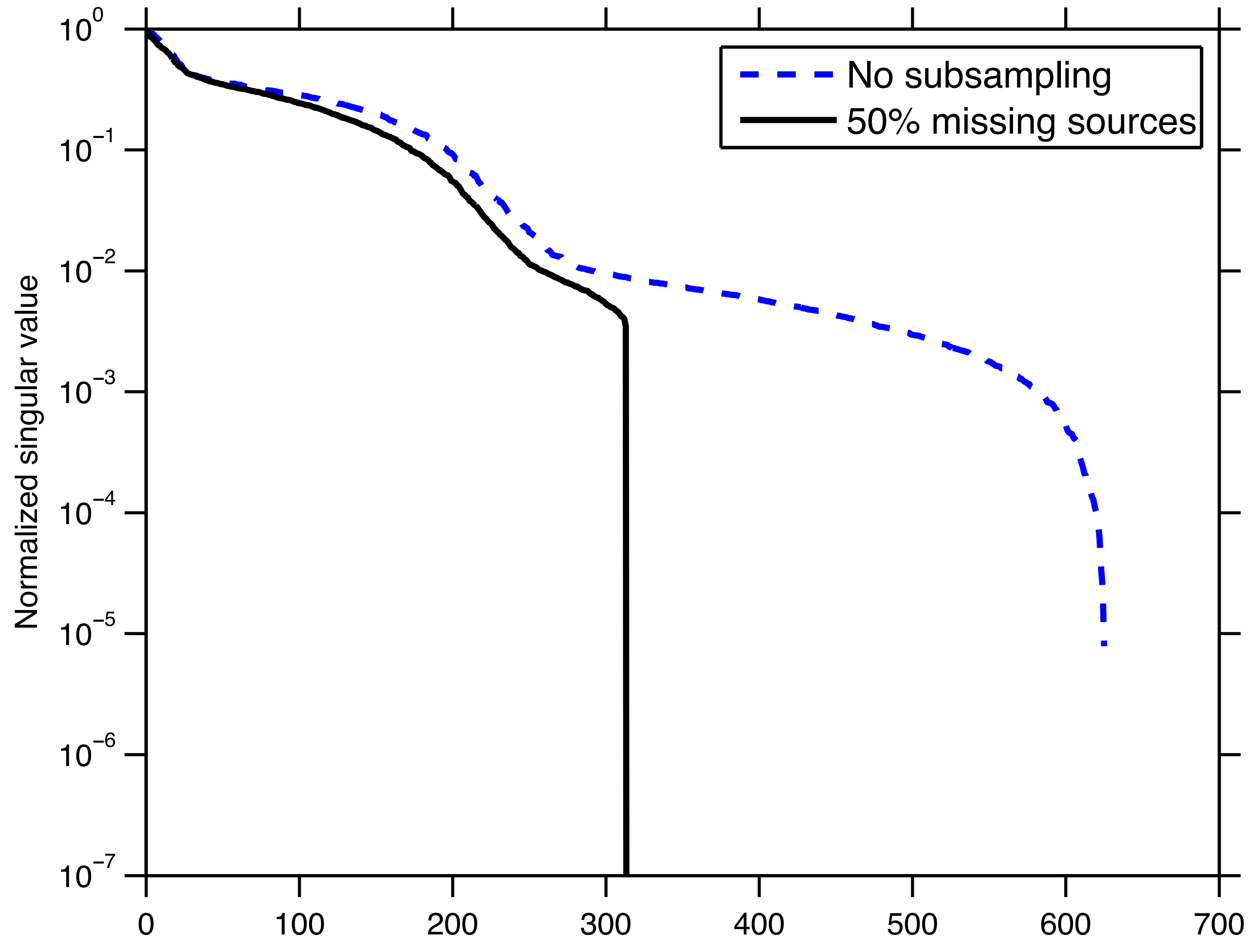
Realistic recovery

50% random receivers removed



(Rec x, Rec y) matricization - Canonical ordering

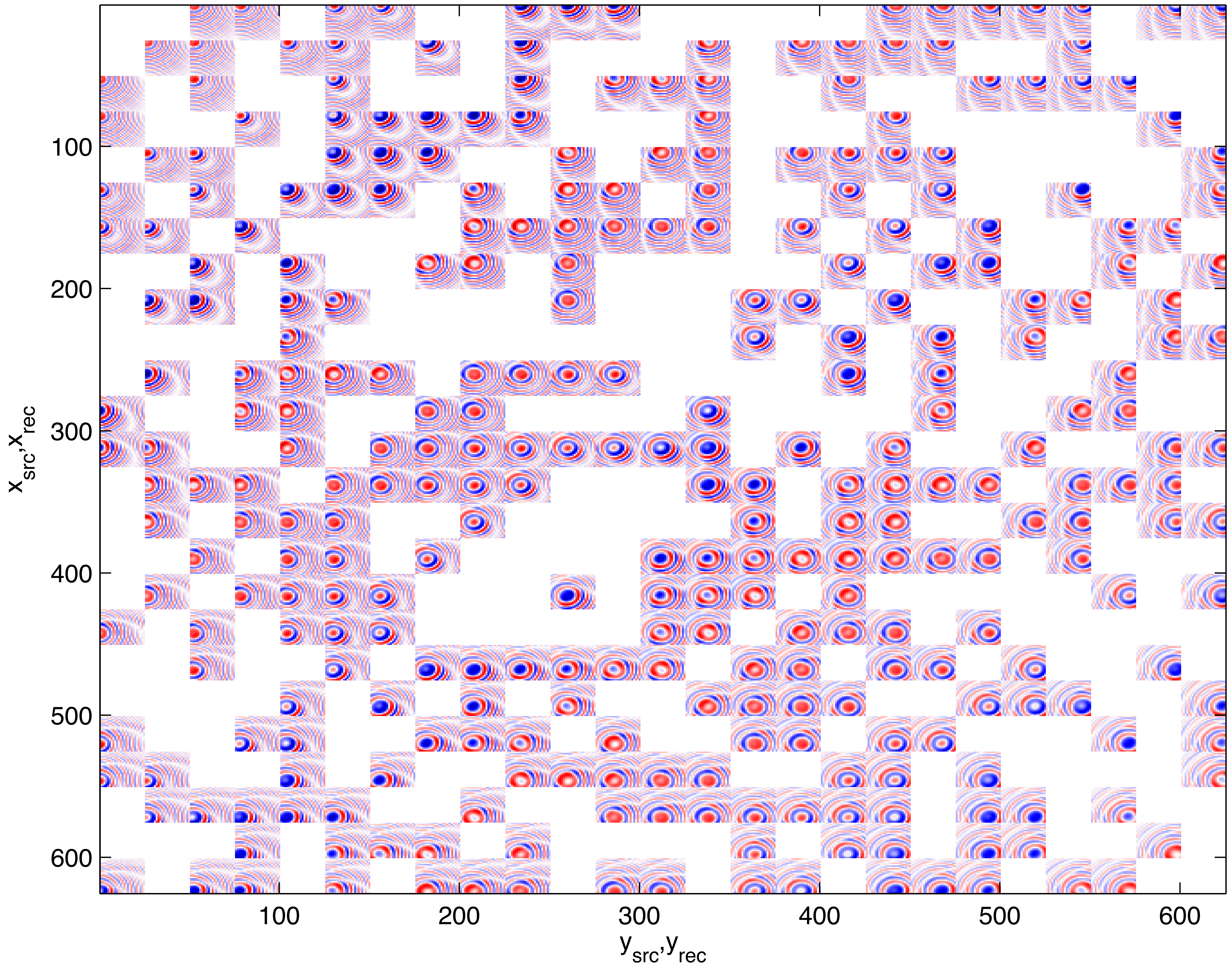


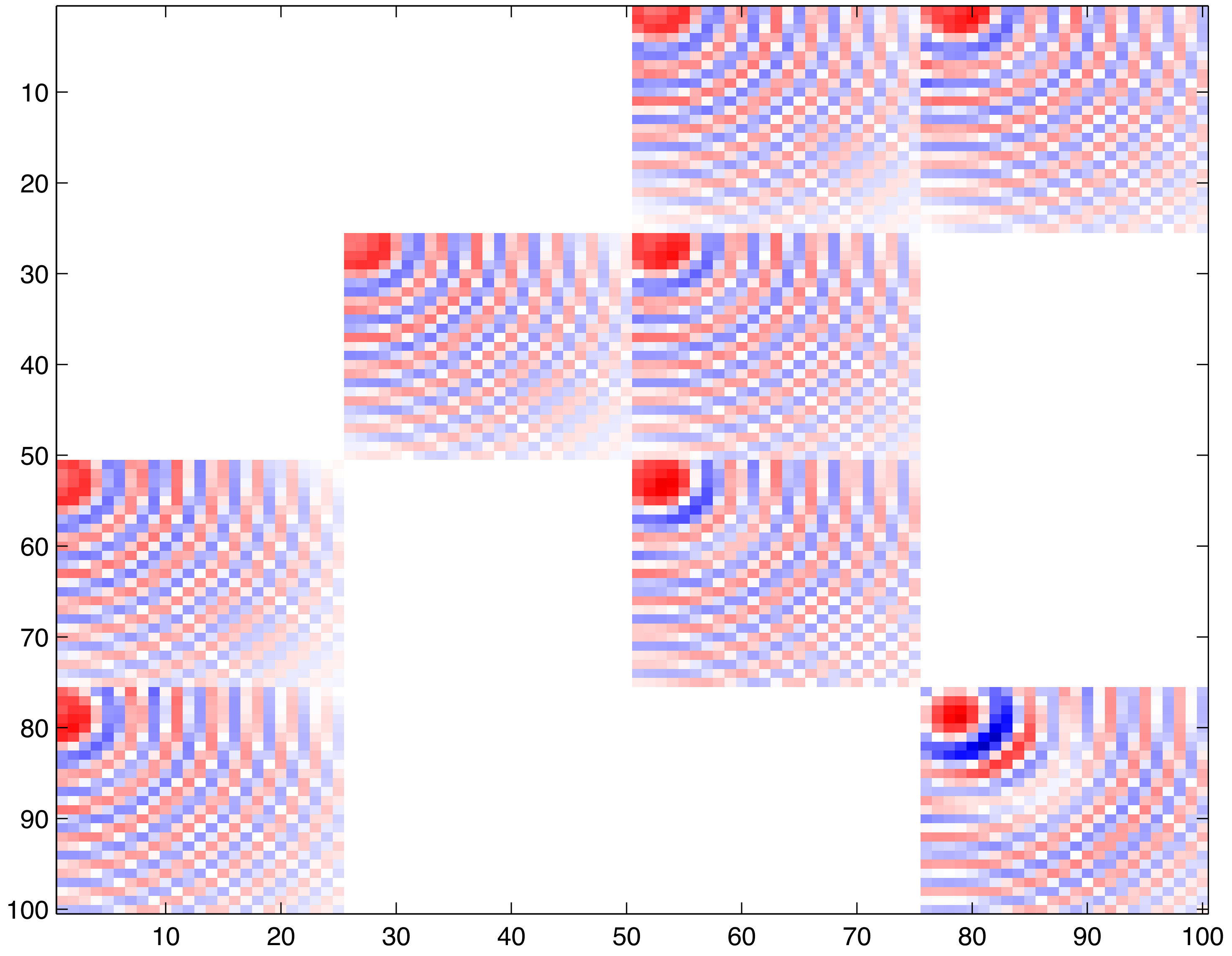


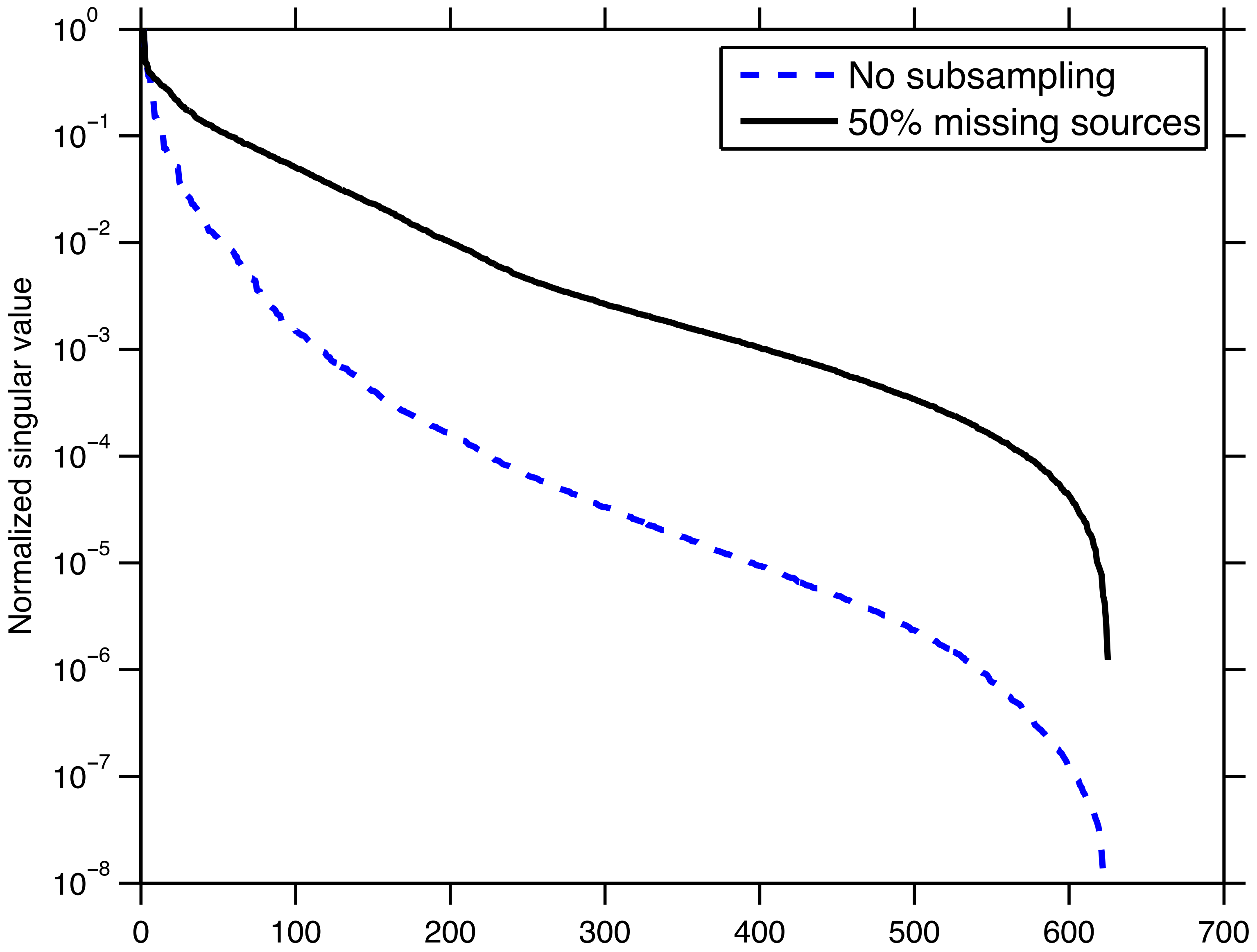
SLIM

Realistic recovery

50% random receivers removed







(Src x, Rec x) matricization - Noncanonical ordering

Data organization

In summary:

(rec x, rec y) organization

- High rank
- Missing sources operator - removes columns
- Poor recovery scenario

(src x, rec x) organization

- Low rank
- Missing sources operator - removes blocks
- Closer to ideal recovery scenario

Multidimensional interpolation
with Hierarchical Tucker

Successful reconstruction scheme

Signal structure

- Hierarchical Tucker

Sampling

- subsampling increases hierarchical rank

Optimization

- ***fit data in the Hierarchical Tucker format***

Optimization

Given data b with missing sources and/or receivers, subsampling operator A, full tensor expansion operator

$$\phi : (U_t, B_t) \rightarrow \mathbb{C}^{n_1 \times \ldots n_d}$$

solve

$$\min_{x=(U_t, B_t)} \frac{1}{2} \|A\phi(x) - b\|_2^2$$

A. Uschmajew, B. Vandereycken. *The geometry of algorithms using hierarchical tensors*. Linear algebra and its applications, 2013
C. Da Silva and F. J. Herrmann, *Optimization on the Hierarchical Tucker manifold - applications to tensor completion*, 2013

Differential geometry

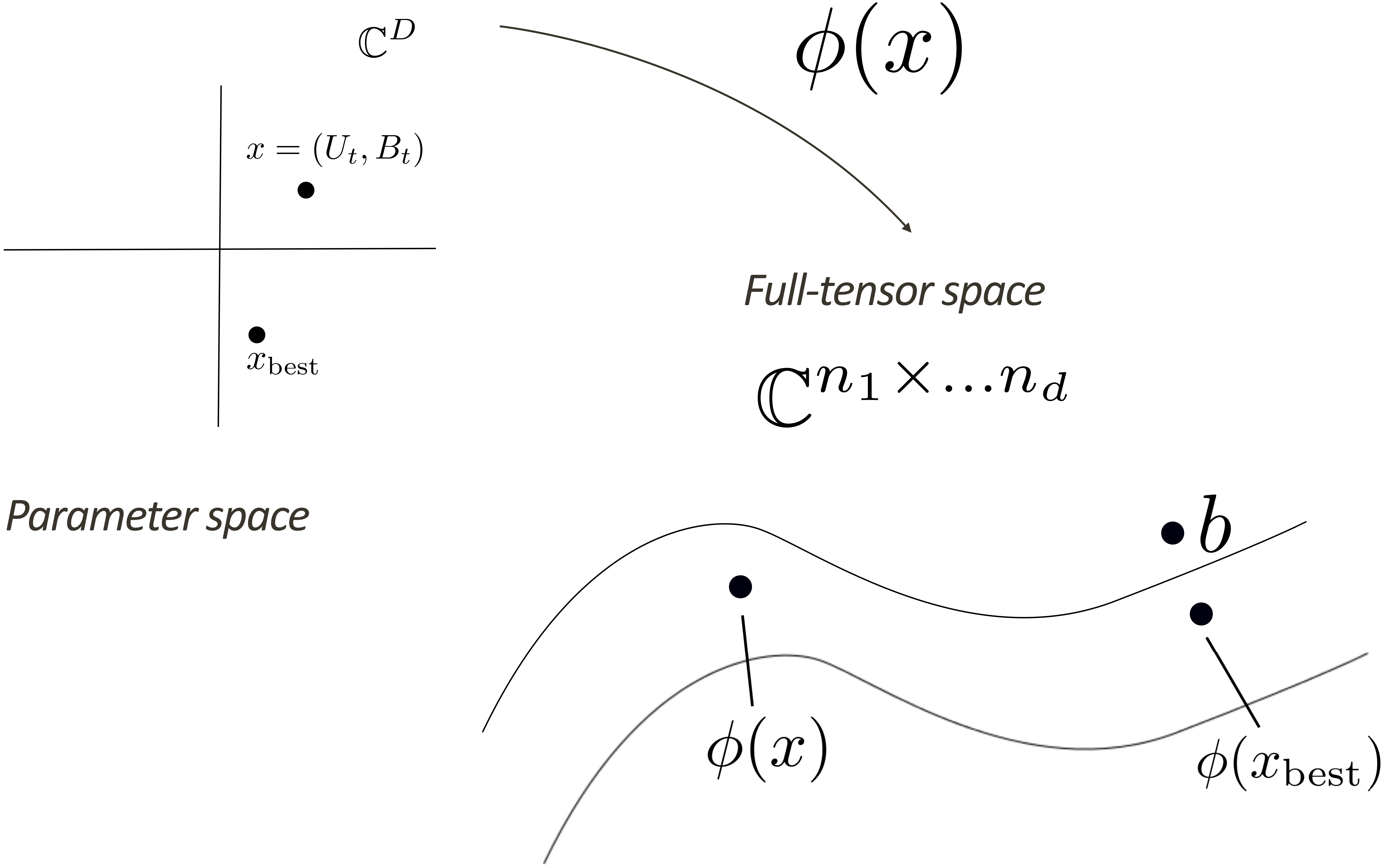
HT tensors parametrize a submanifold of full tensor space $\mathbb{C}^{n_1 \times \dots n_d}$

- Nonlinear, nonconvex space
- Generalization of curved surfaces

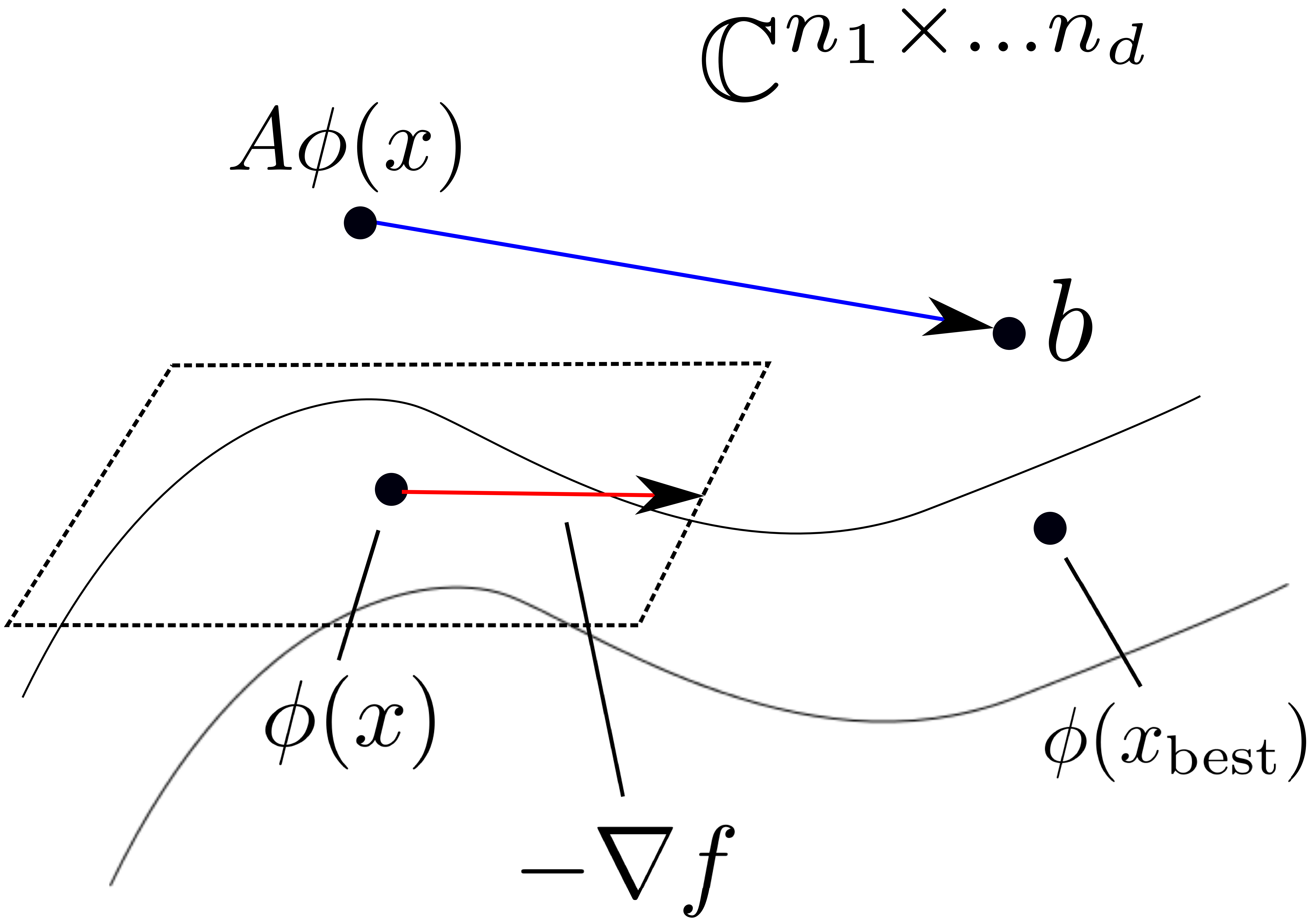
Steepest Descent, Conjugate gradient, Gauss-Newton

- ***without*** SVDs in the full tensor space

Optimization program



Optimization program



Derivatives

Derivatives of a particular node with respect to its children can be computed efficiently, i.e. via

$$(I - U_{t_l} U_{t_l}^H) \langle U_{t_r}^T \circ_2 Z, B_t \rangle_{(2,3),(2,3)}$$
$$(I - U_{t_r} U_{t_r}^H) \langle U_{t_l}^T \circ_1 Z, B_t \rangle_{(1,3),(1,3)}$$

$P \circ_i Q$ multiplies Q by P in dimension i, $\langle X, Y \rangle_{(1,3),(1,3)} = \sum_{i_1, i_3} \overline{X}_{i_1,:,i_3} Y_{i_1,:,i_3}$

The chain rule gives the gradient of the function ϕ

Derivatives

Only involves matrix-matrix multiplications of small matrices compared to the full-tensor space

Parallelizable - multilinear product can be done in parallel

SVD-free - no large-scale SVDs, unlike nuclear norm-based methods

Results

Synthetic BG Group data

Unknown model

- 68 x 68 sources with 401 x 401 receivers, data at 7.34 Hz

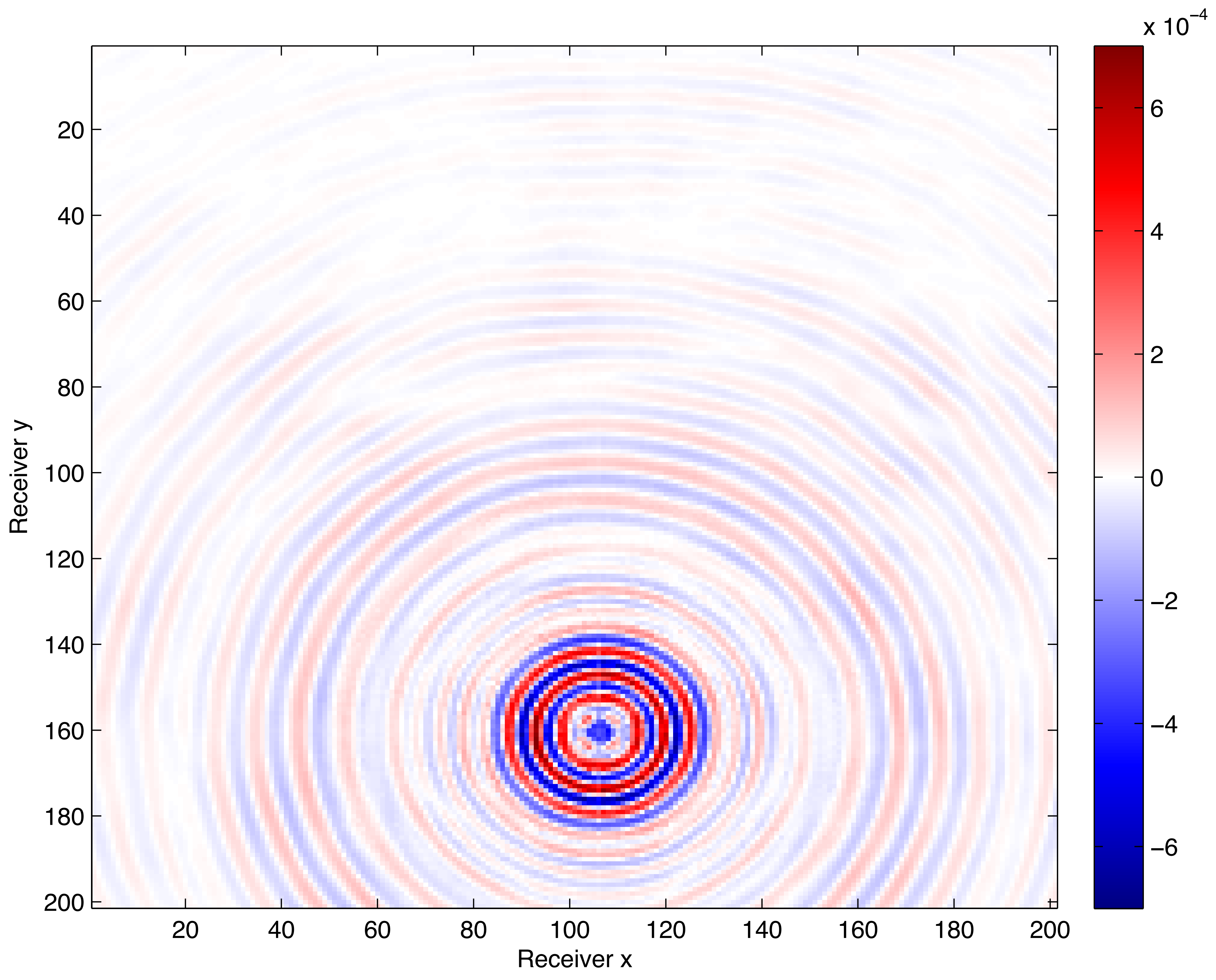
Receivers subsampled to 201 x 201

Recovered with Gauss-Newton

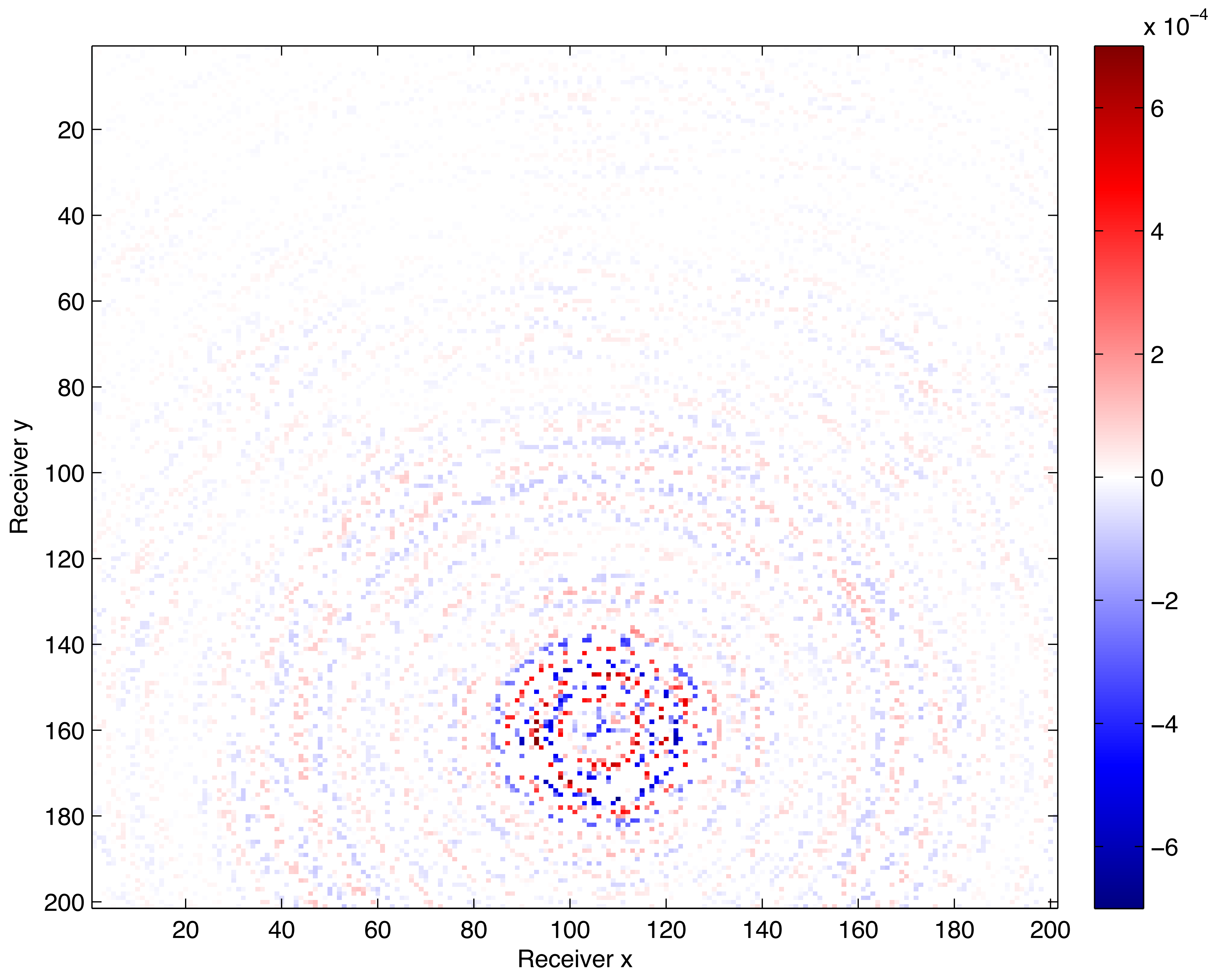
- code available at https://www.slim.eos.ubc.ca - HTOpt toolbox
- ~ 1 hour serial running time

7.34 Hz - 75% missing receivers

Common source gather



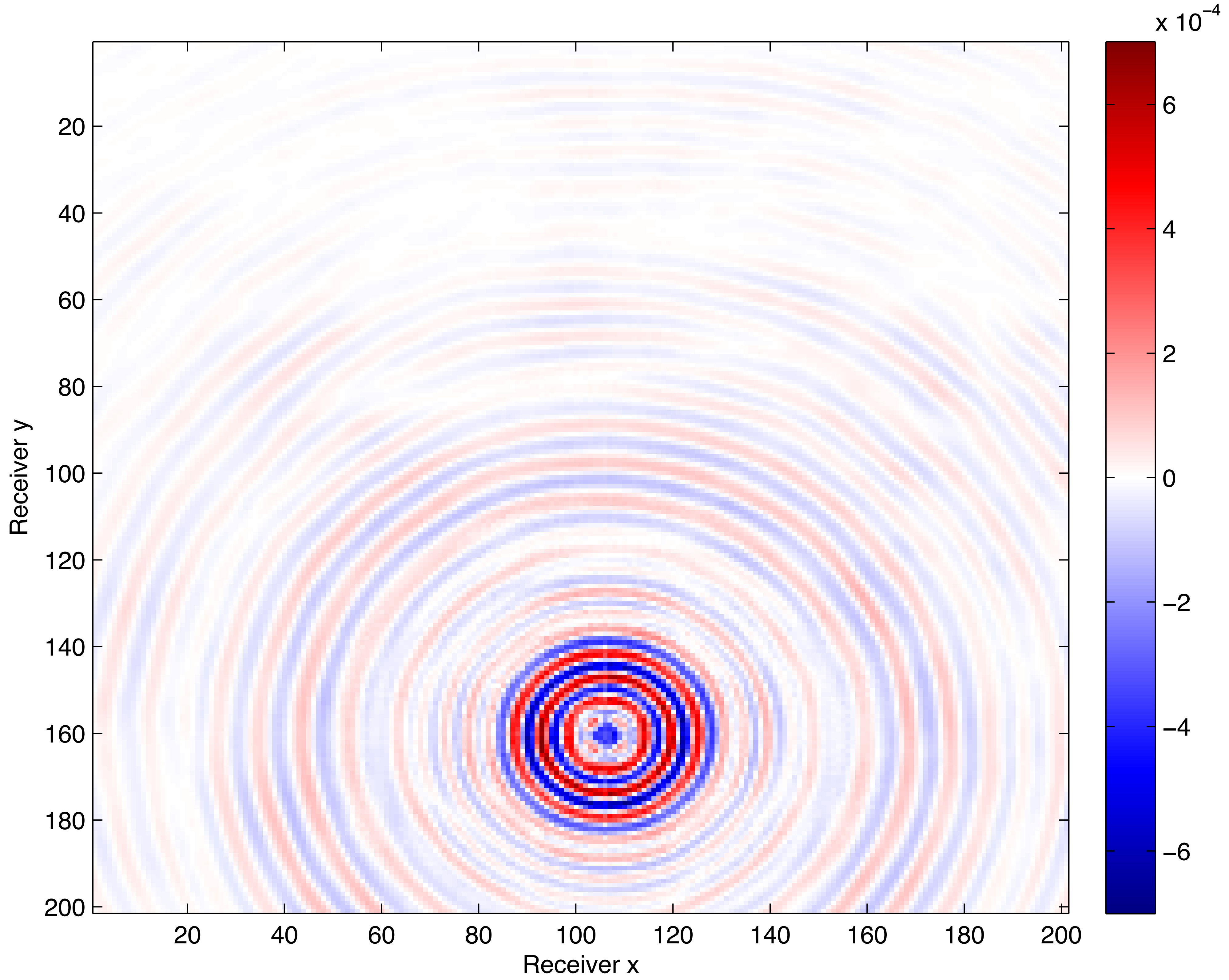
True data



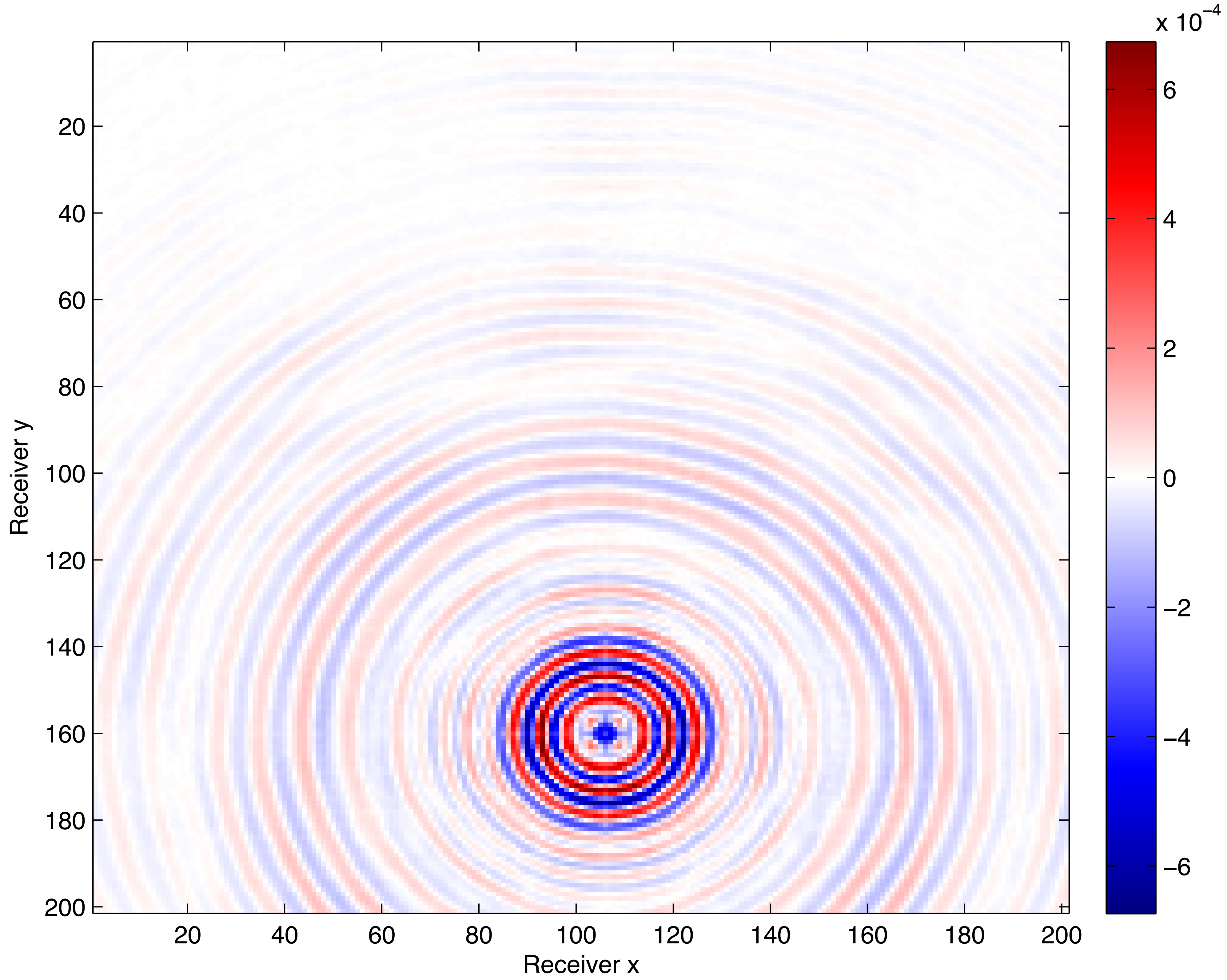
Subsampled data

7.34 Hz - 75% missing receivers

Common source gather



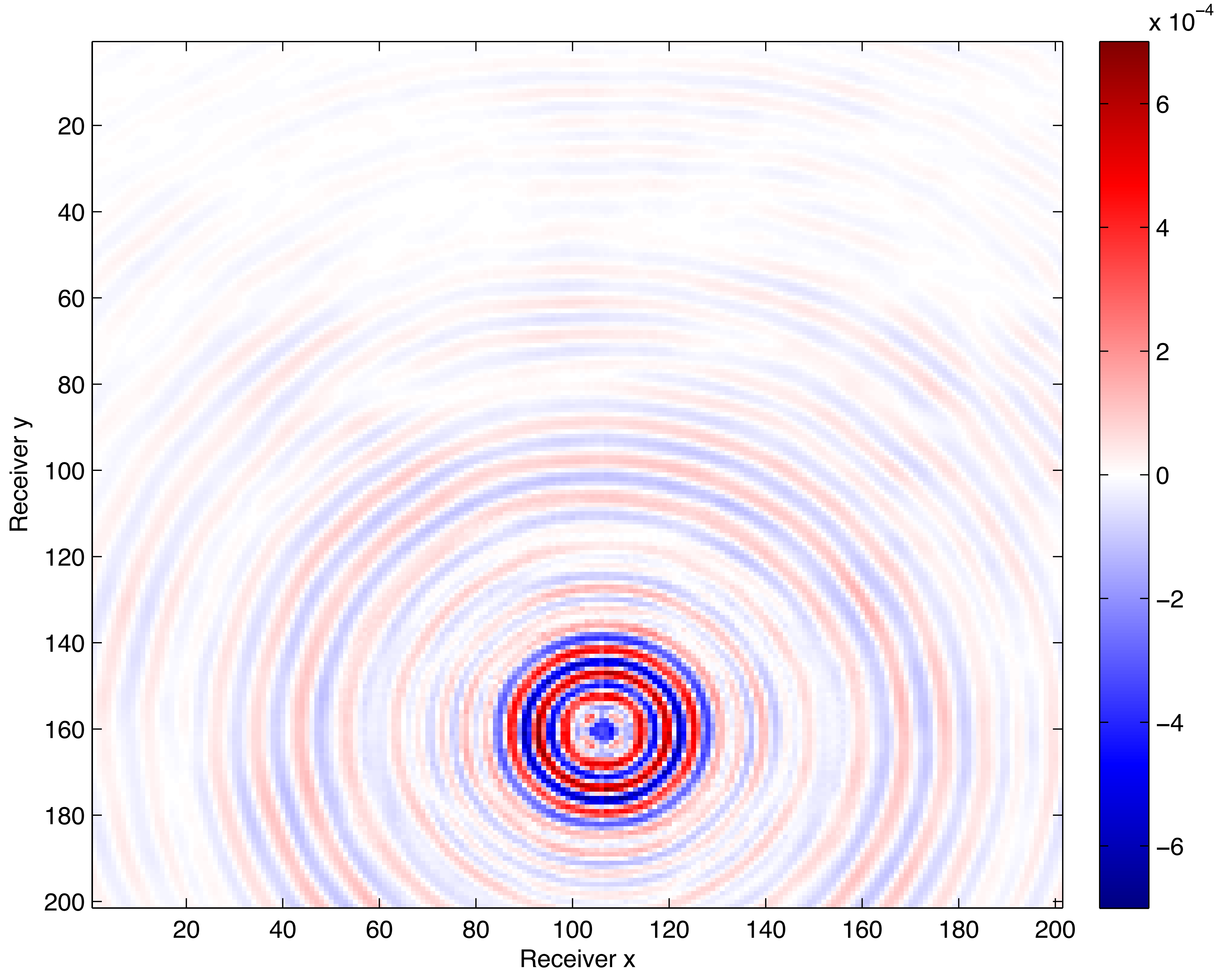
True data



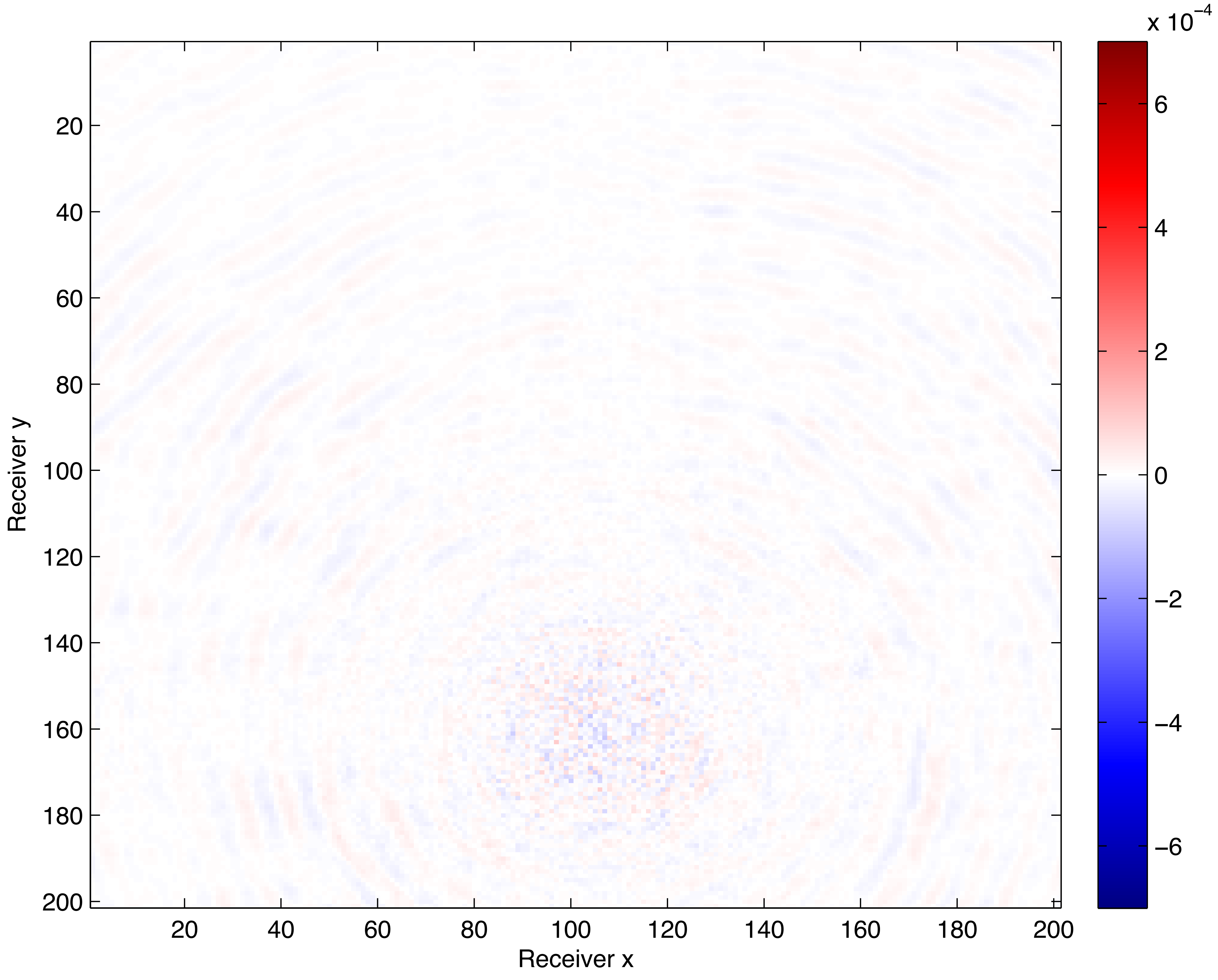
Recovered data - SNR 17.6 dB

7.34 Hz - 75% missing receivers

Common source gather



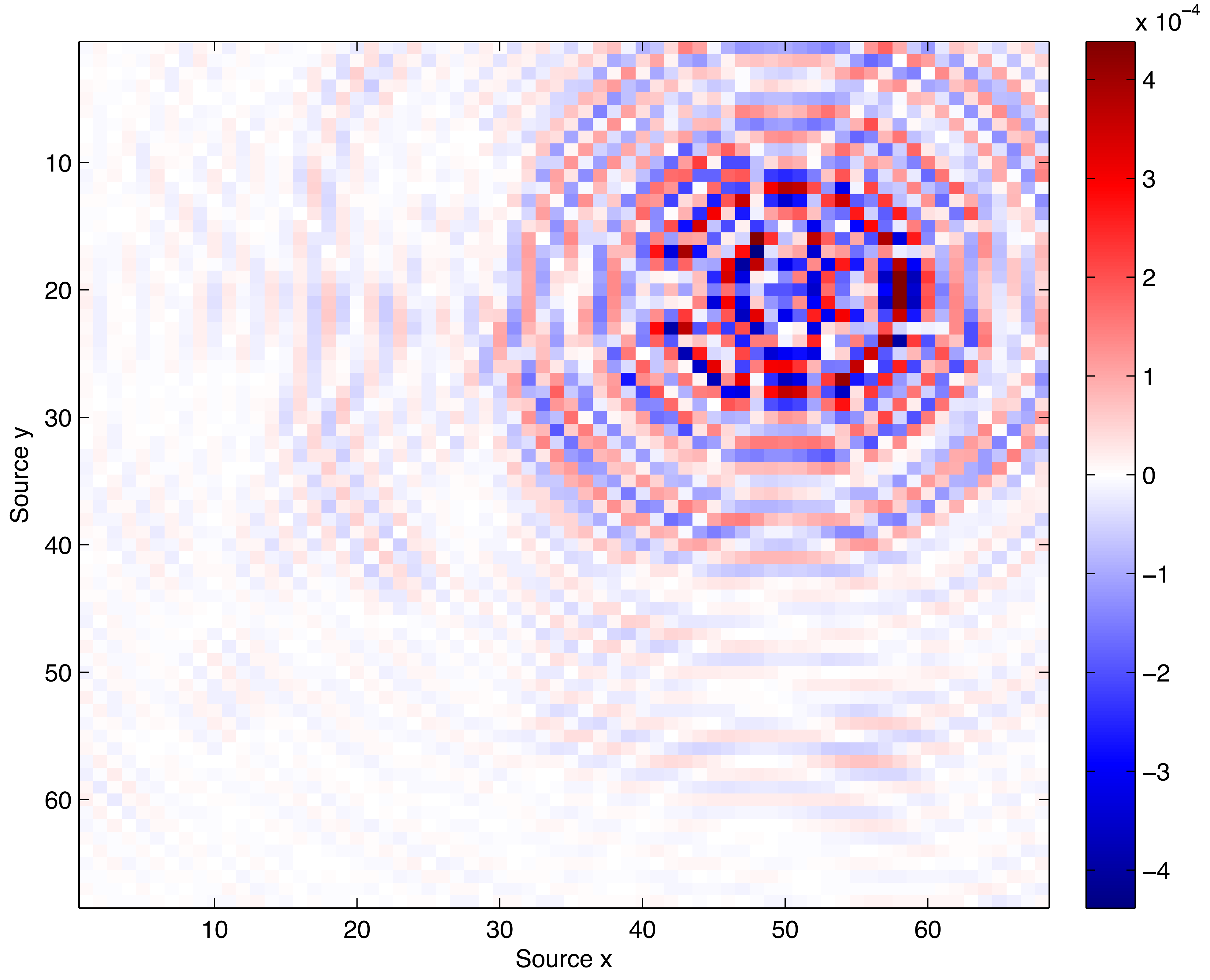
True data



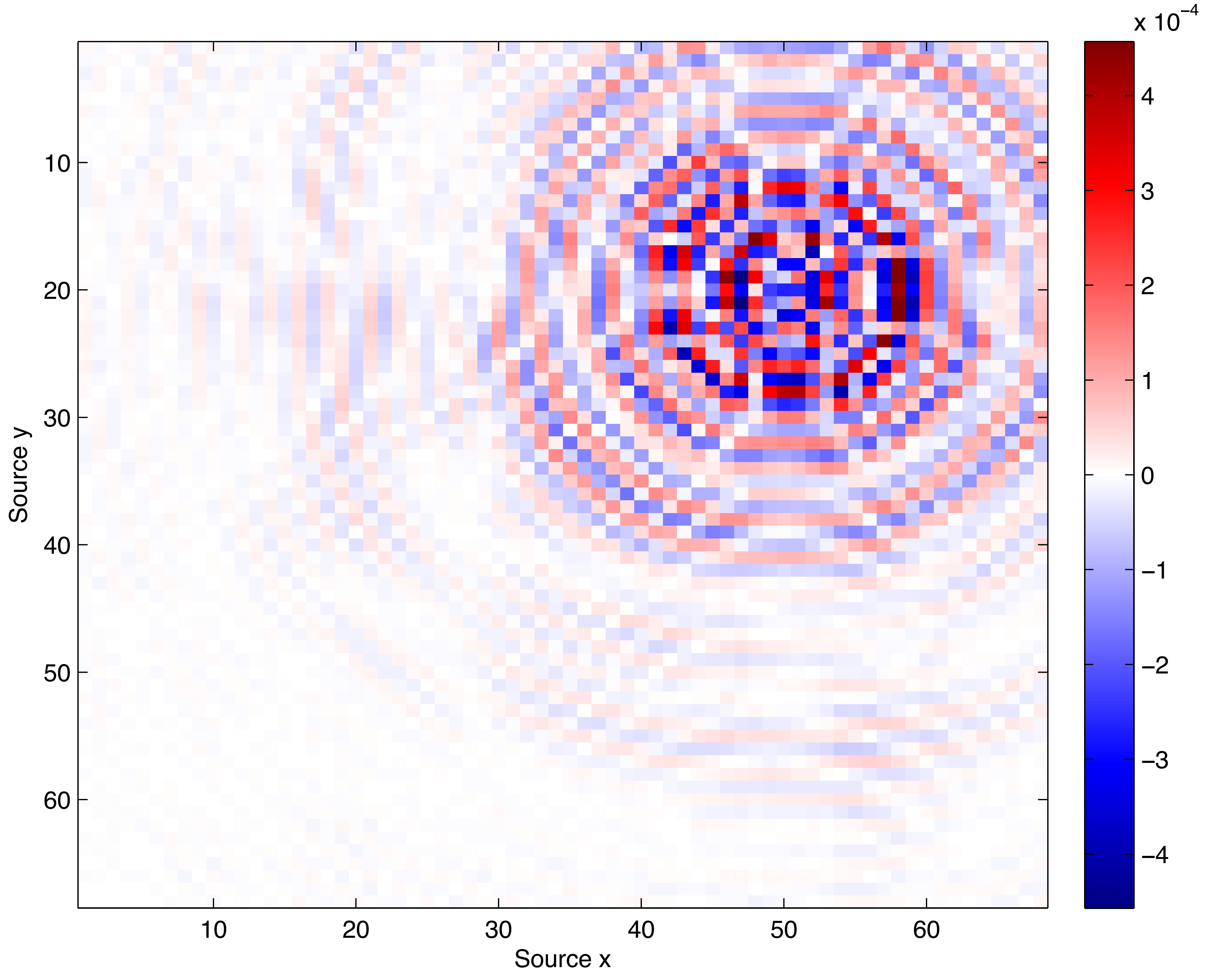
Difference

7.34 Hz - 75% missing receivers

Common receiver gather - no data initially



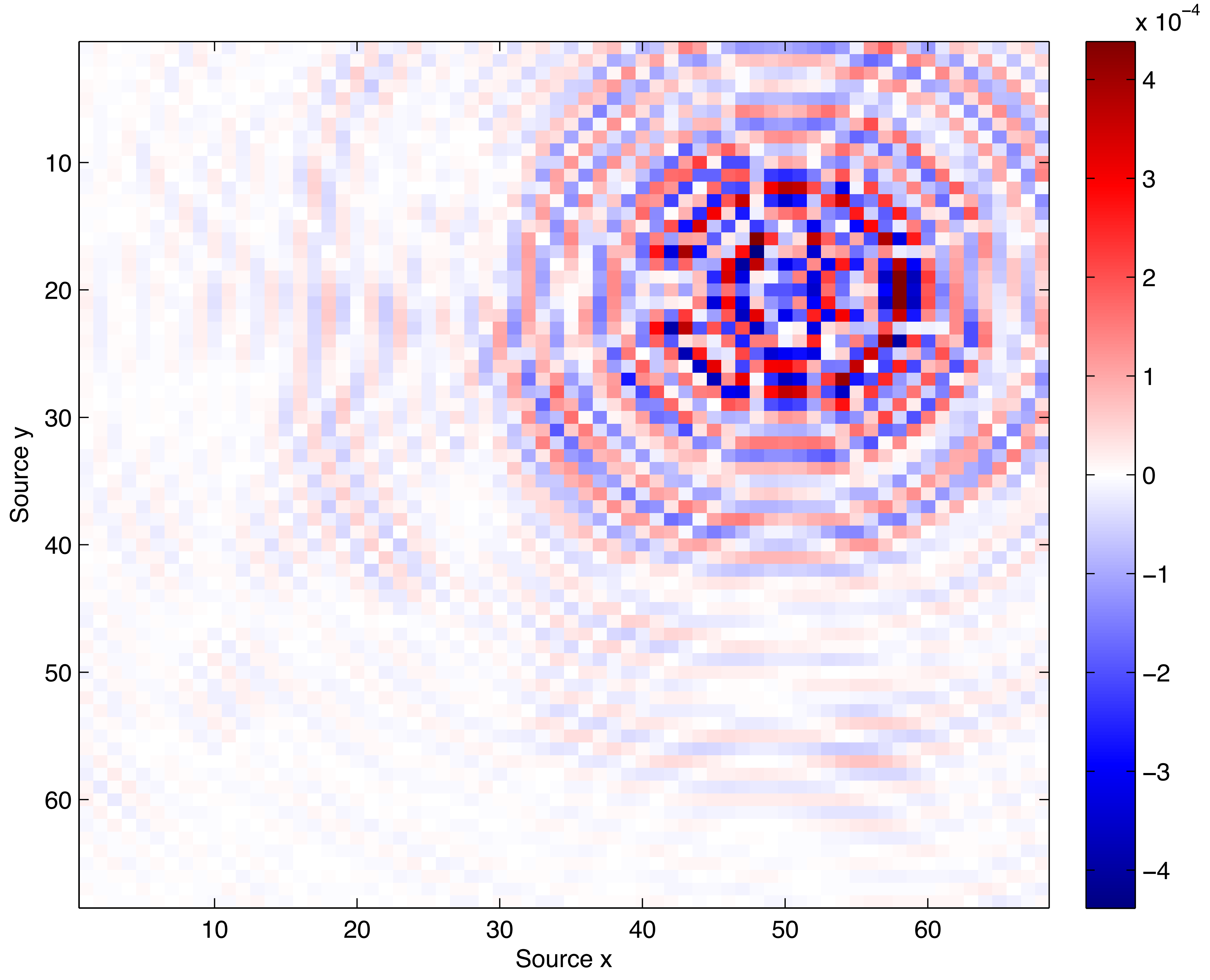
True data



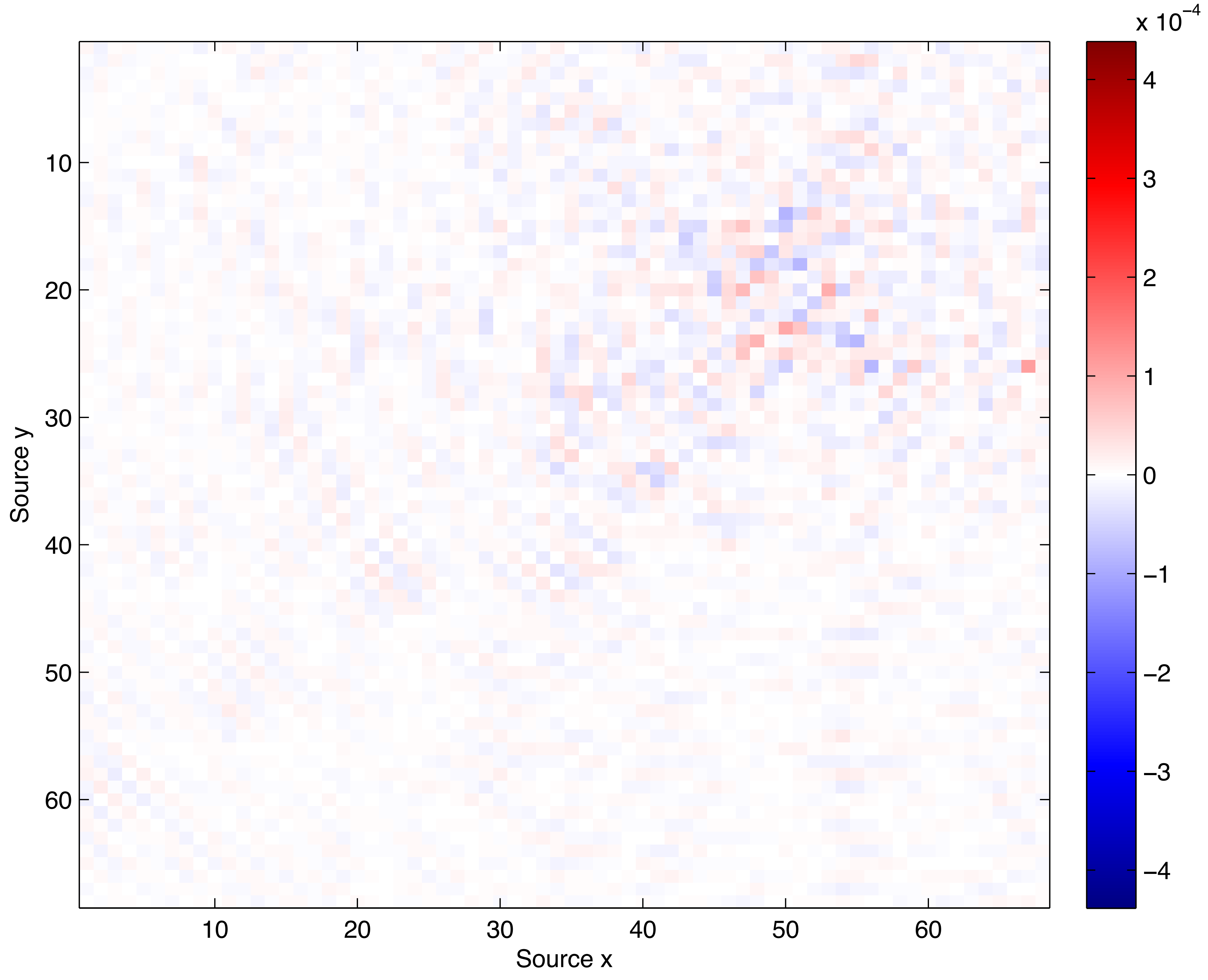
Recovered data - SNR 16.2 dB

7.34 Hz - 75% missing receivers

Common receiver gather - no data initially



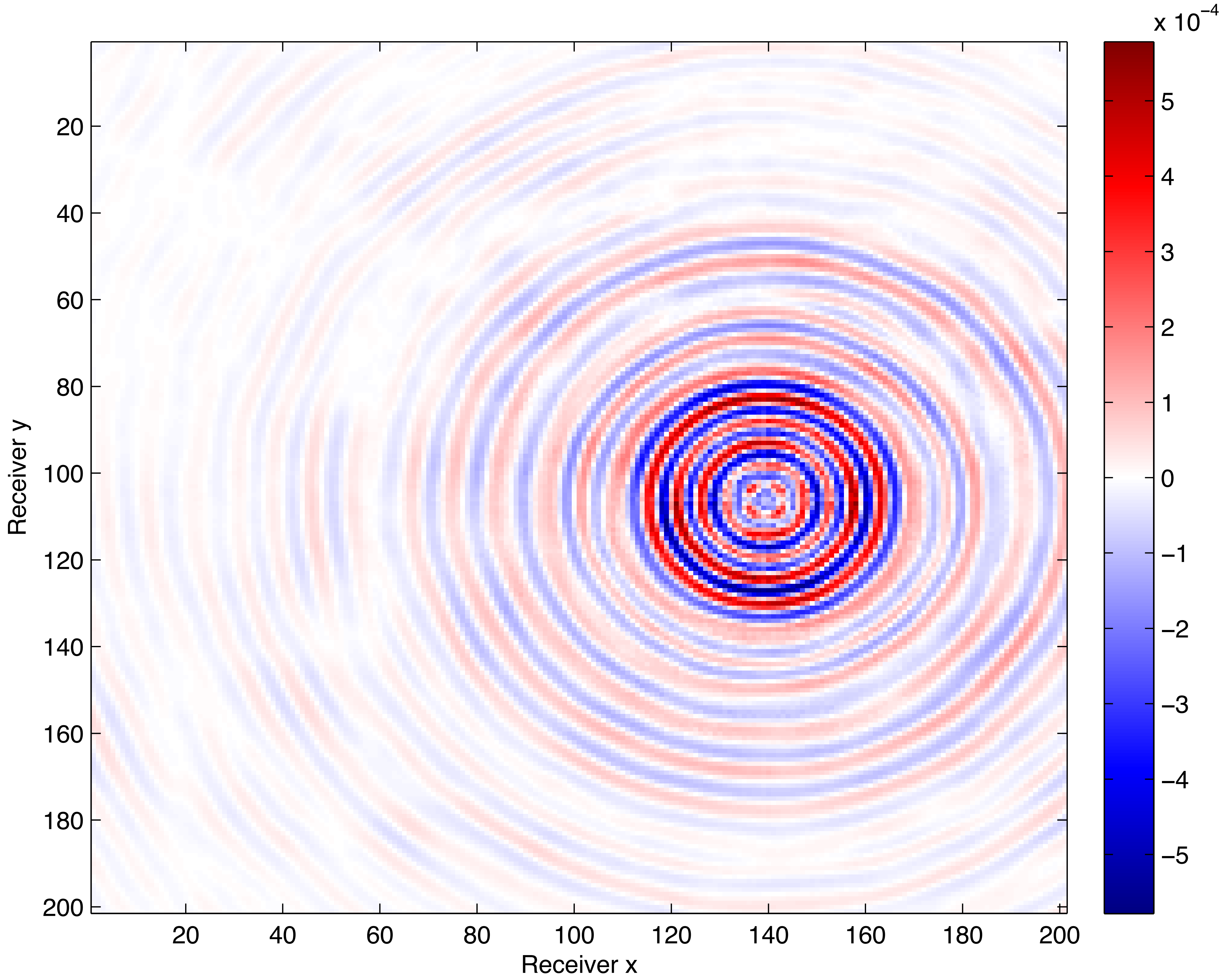
True data



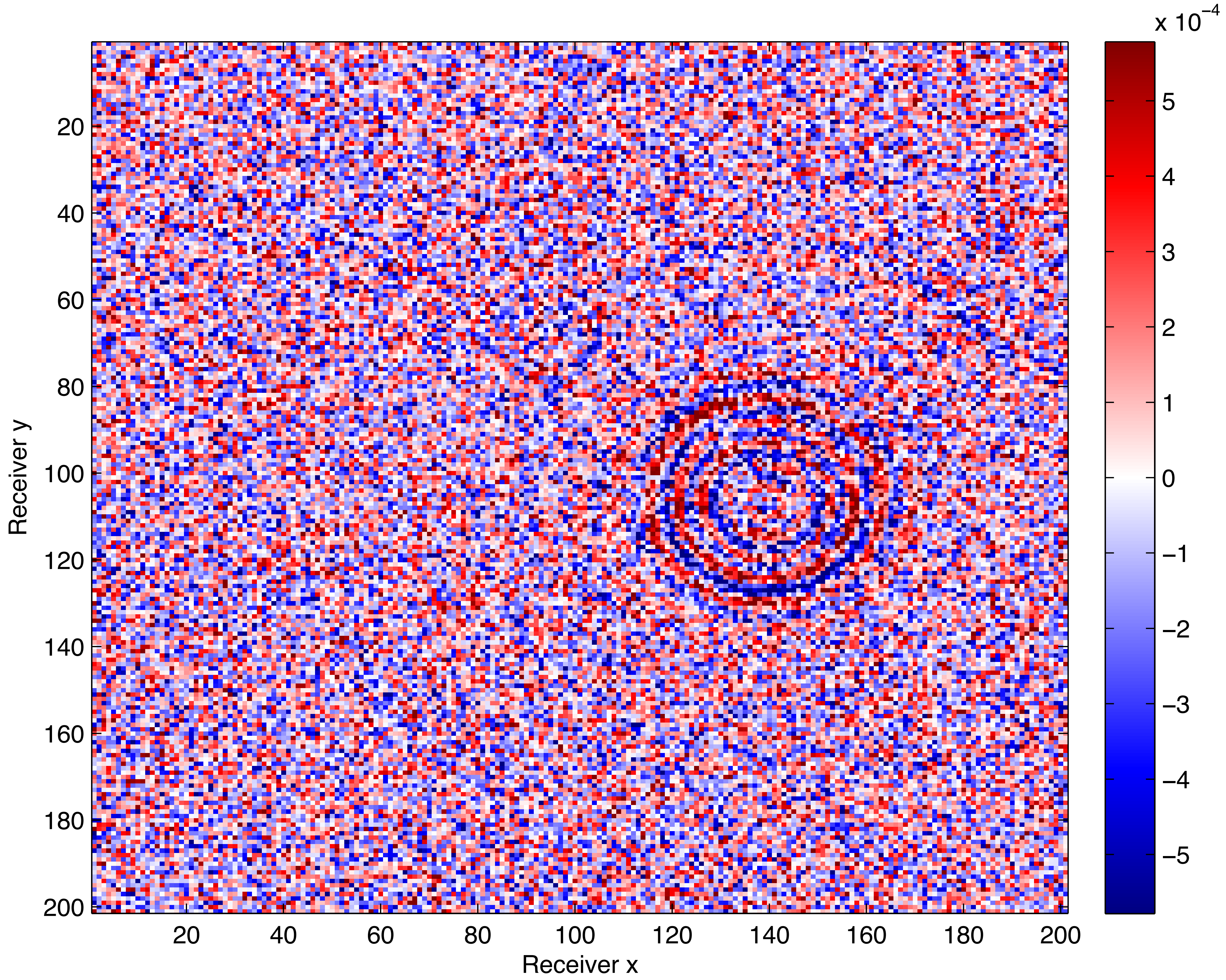
Difference

7.34 Hz - simultaneous receivers - 90% data reduction

Common source gather



True data

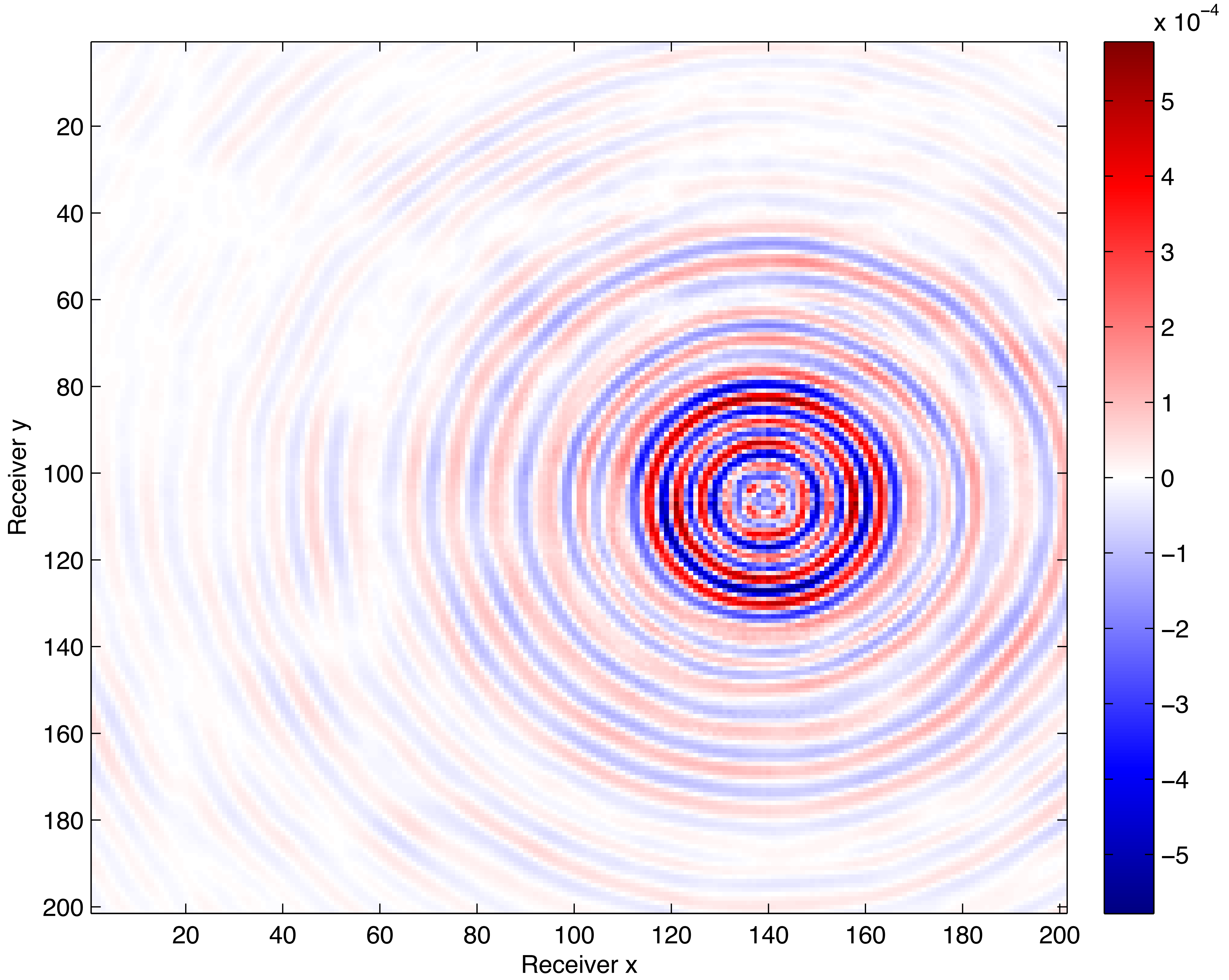


Input data - $A^T Ab$

A - subsampling operator b - full data

7.34 Hz - simultaneous receivers - 90% data reduction

Common source gather



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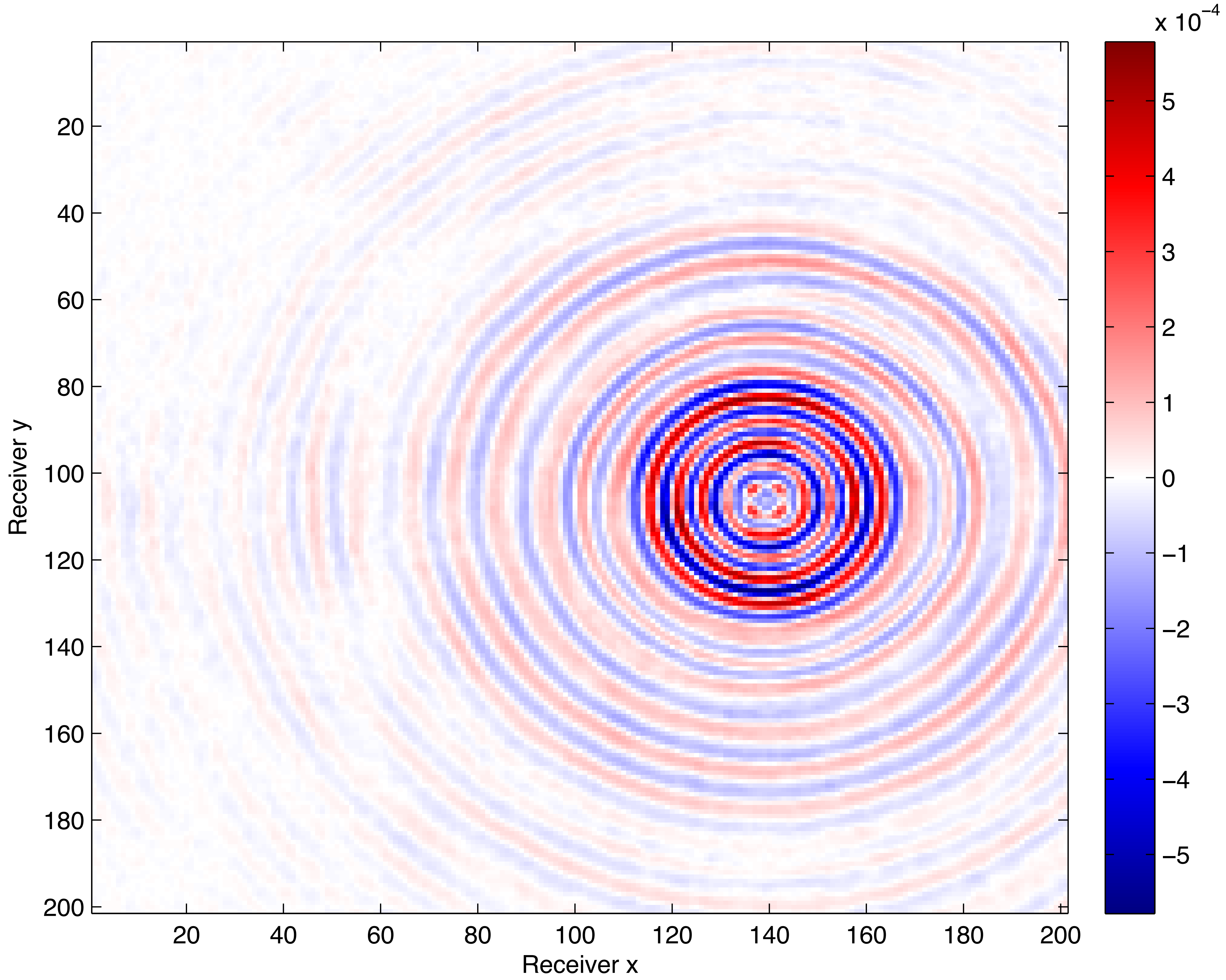
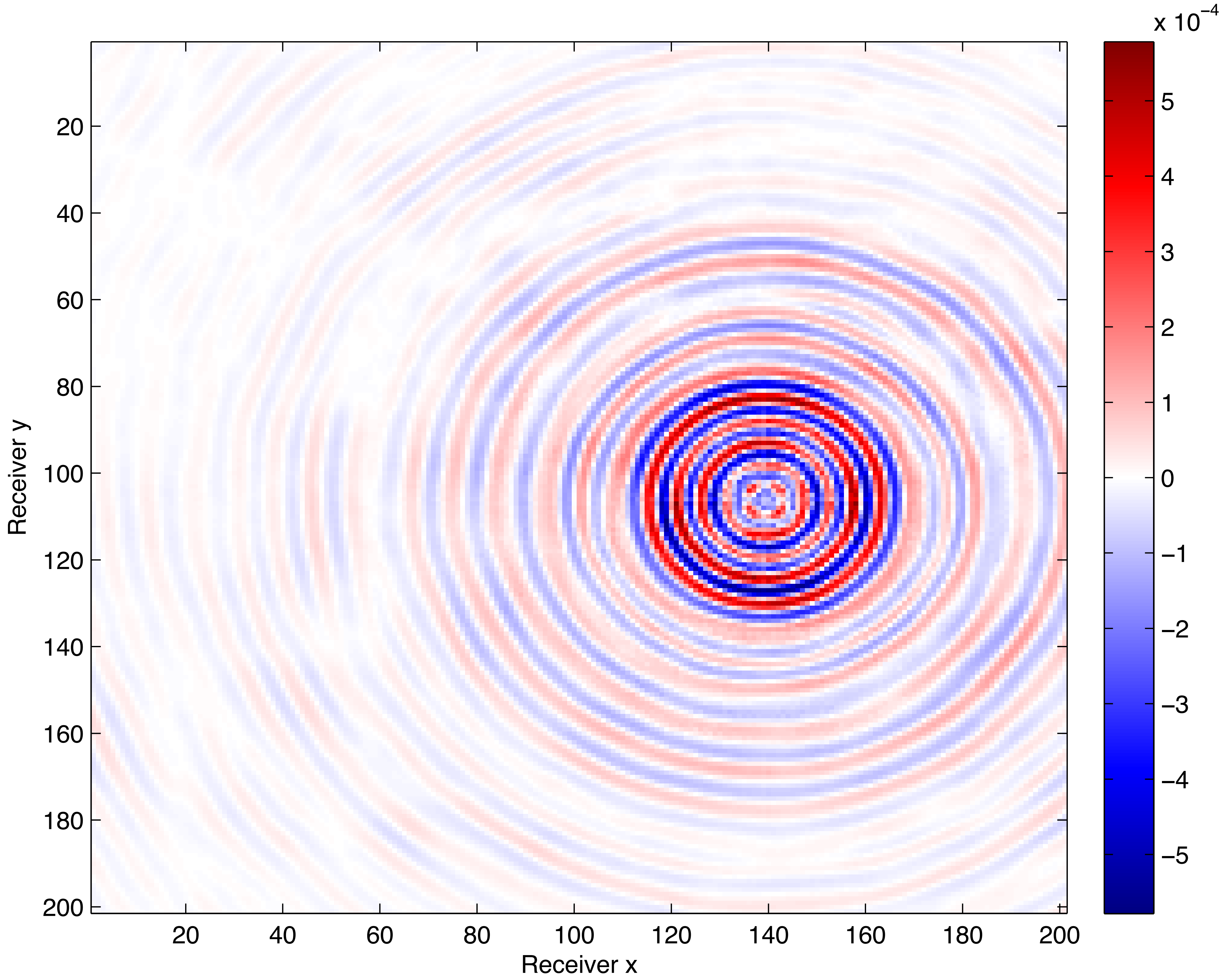


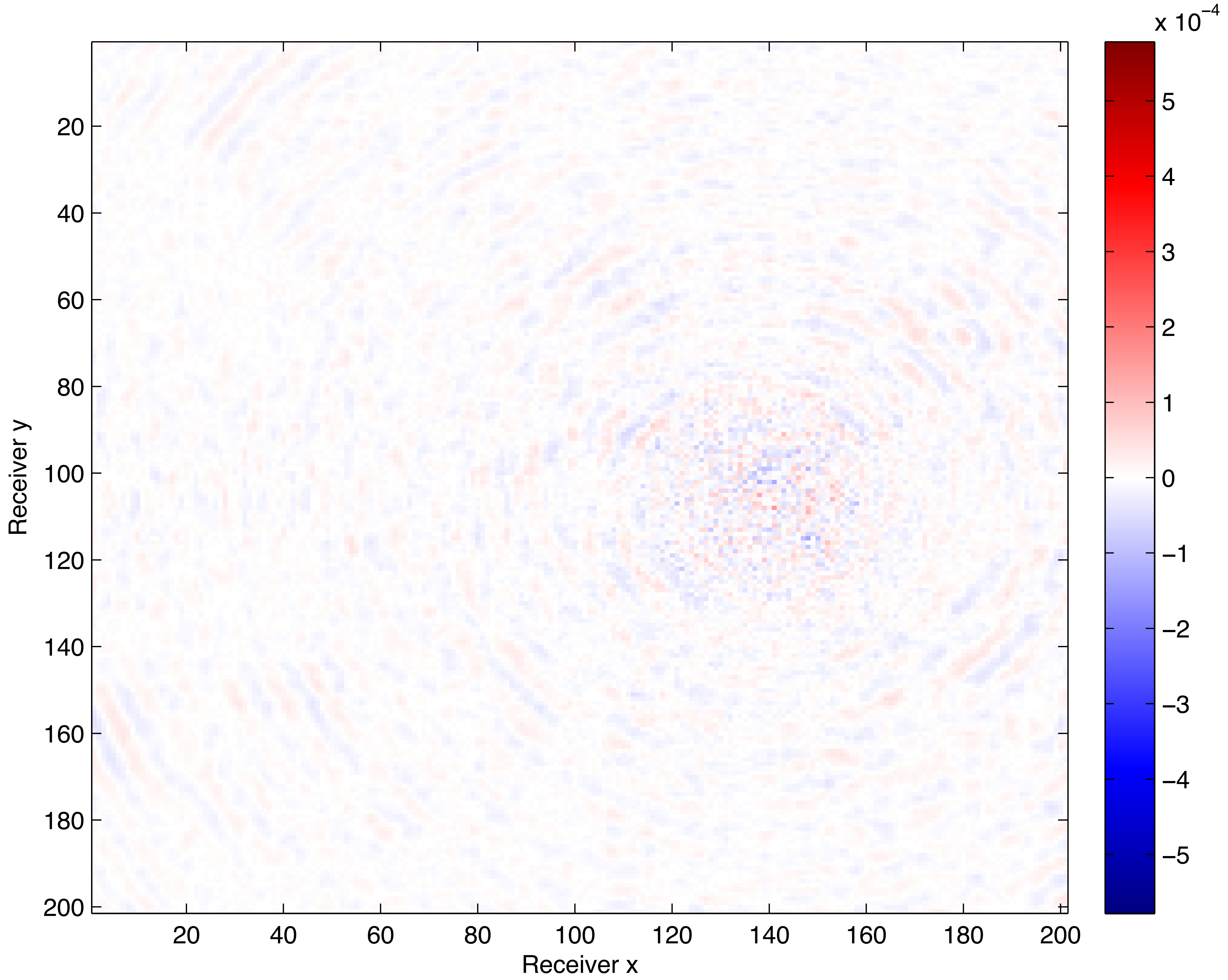
Recovered data - SNR 16.4 dB

7.34 Hz - simultaneous receivers - 90% data reduction

Common source gather



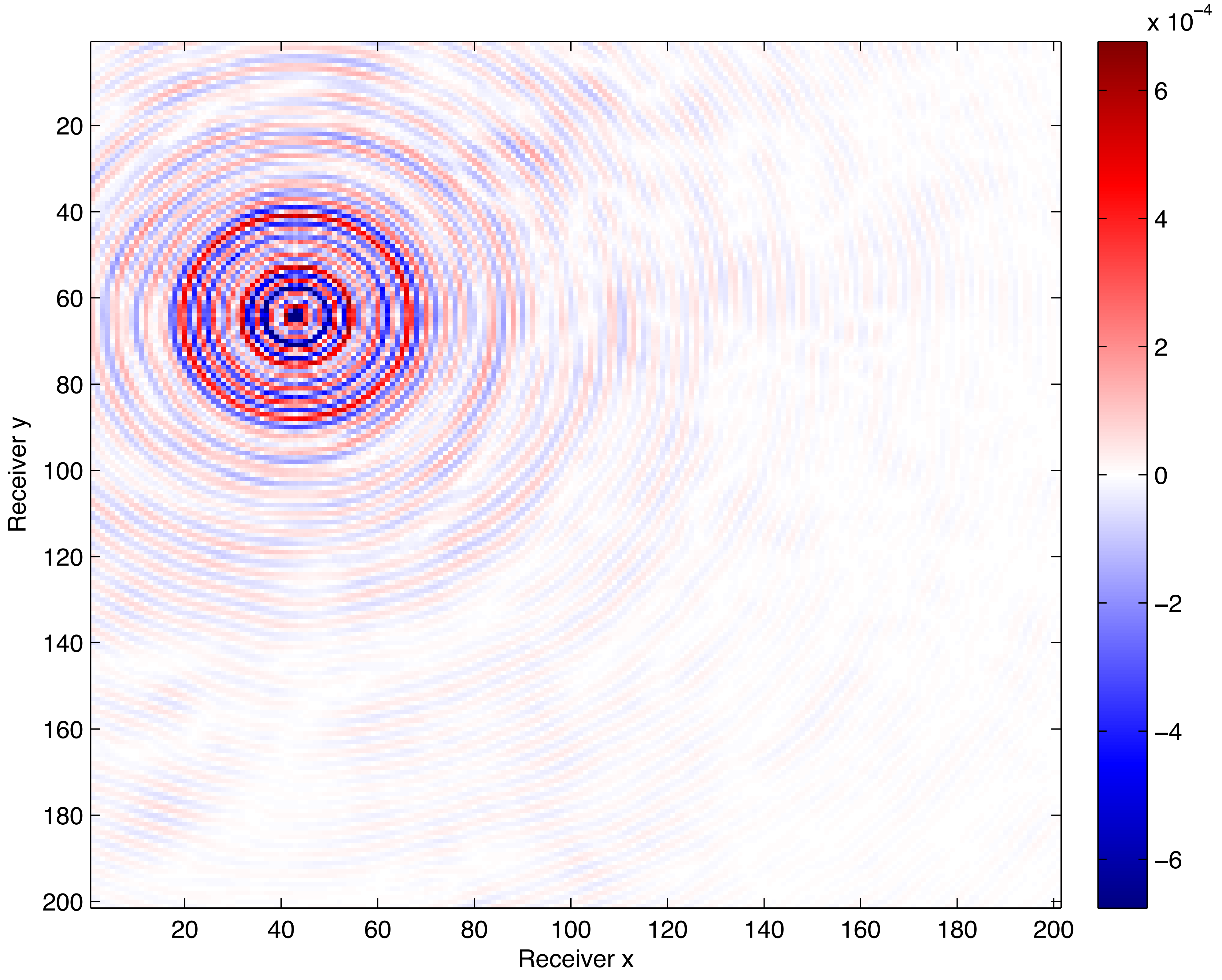
True data



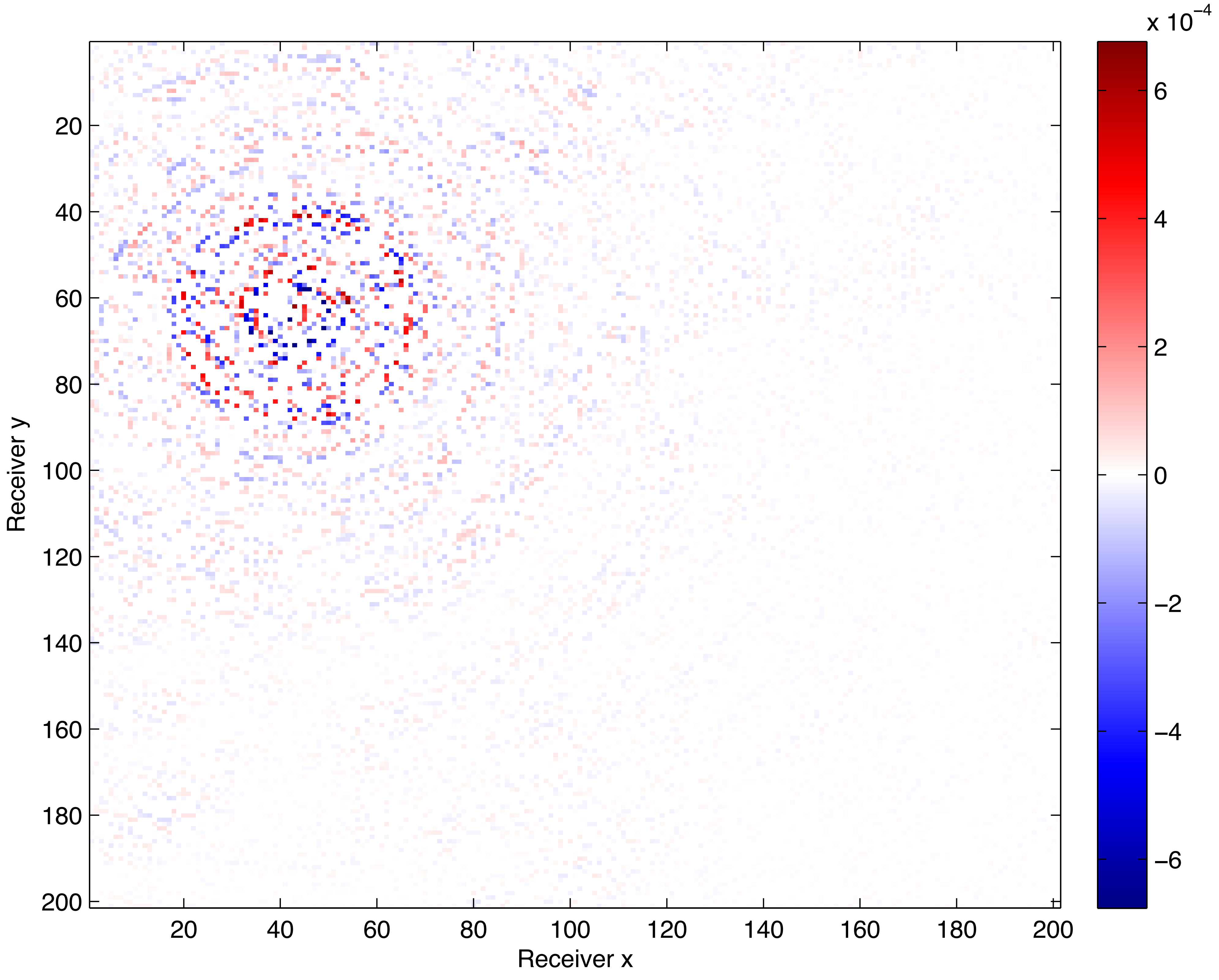
Difference

12.3 Hz - 75% missing receivers

Common source gather



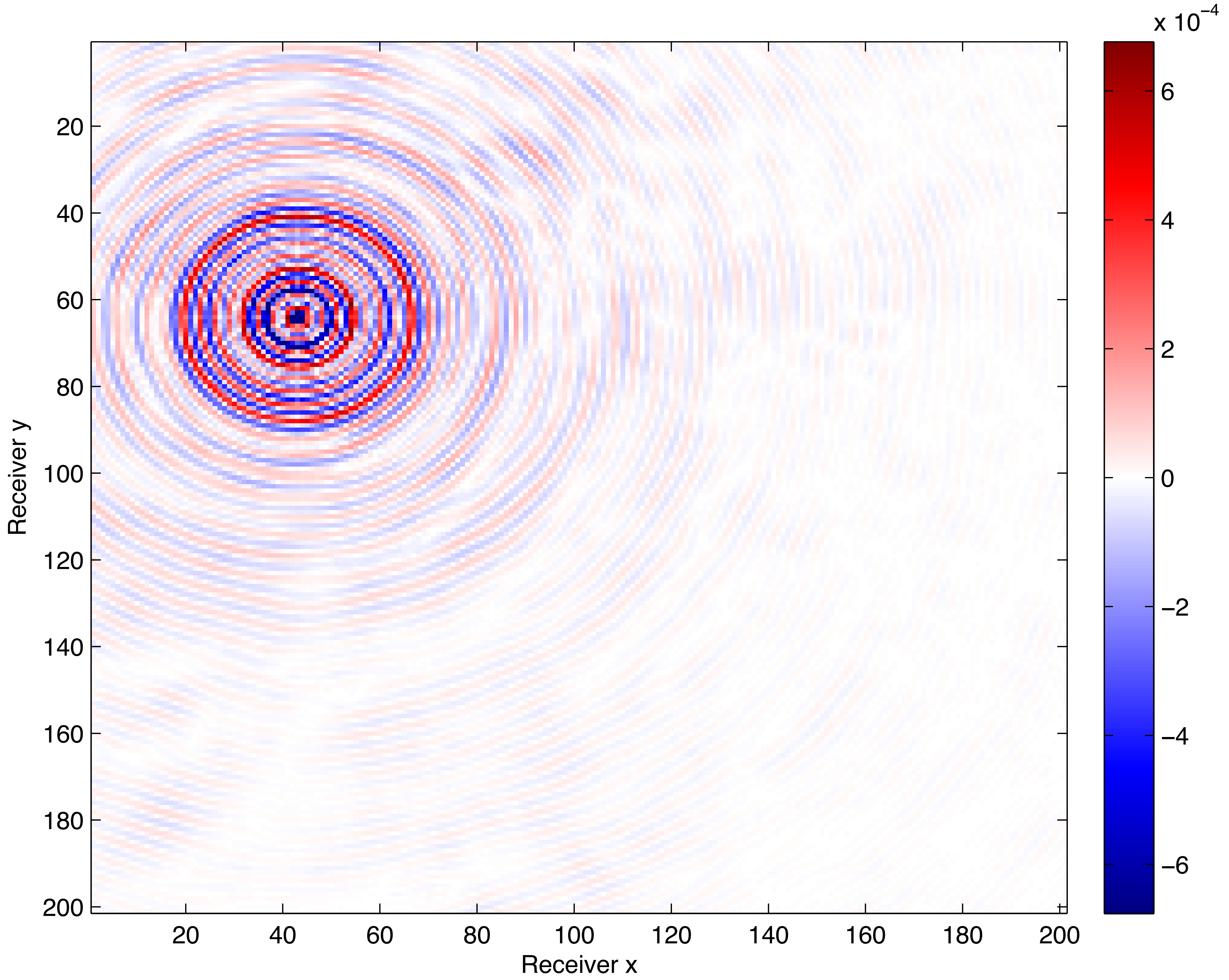
True data



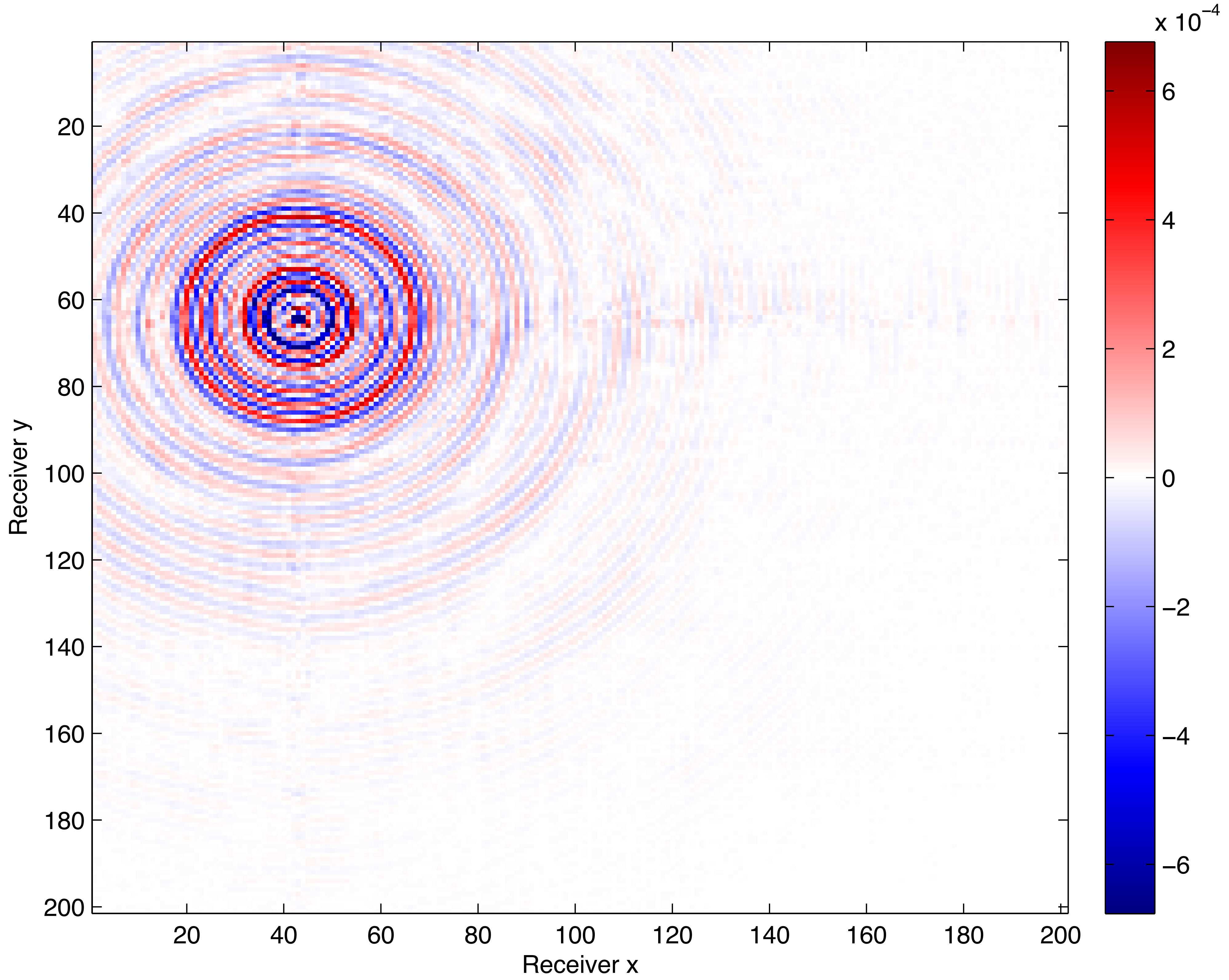
Subsampled data

12.3 Hz - 75% missing receivers

Common source gather



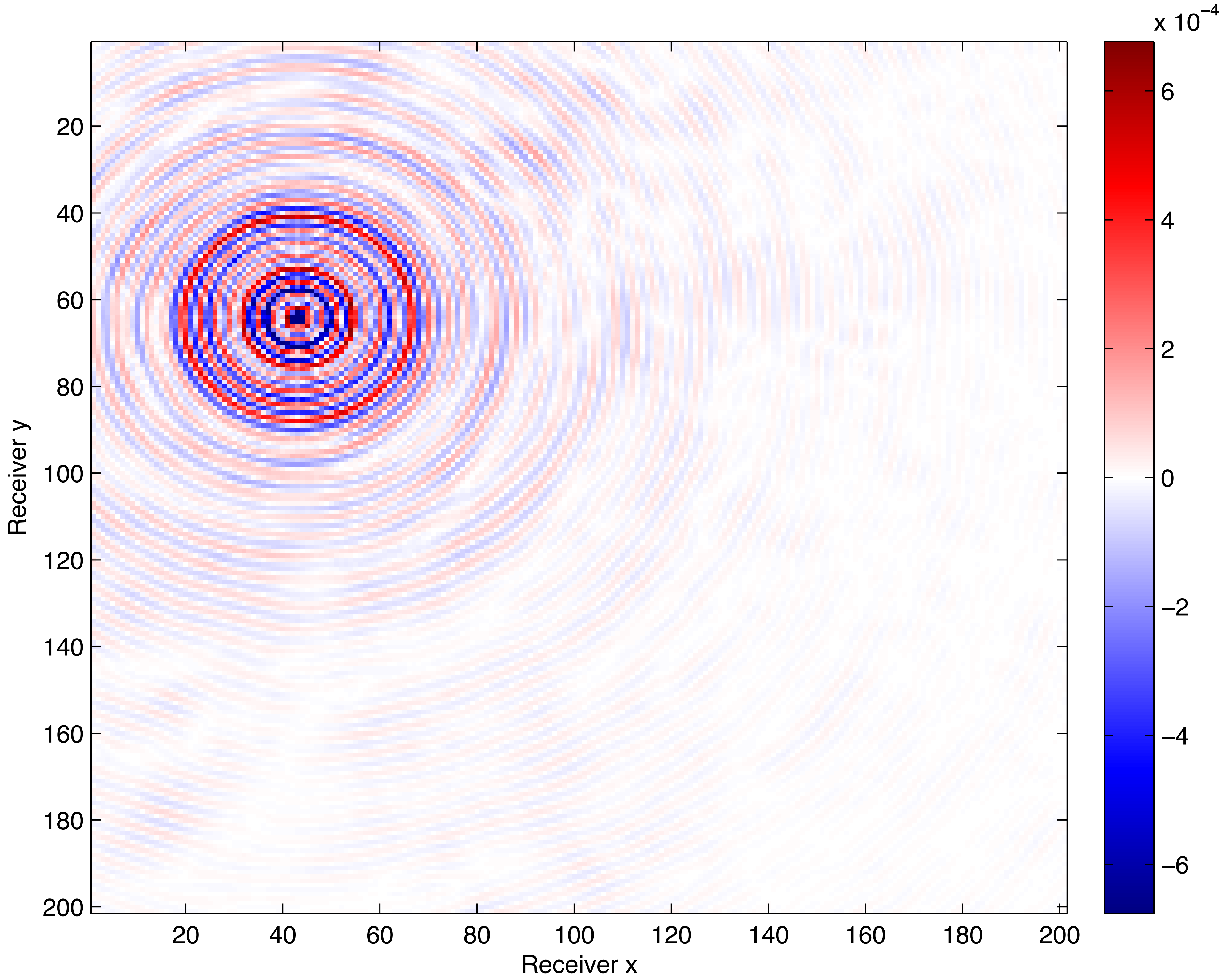
True data



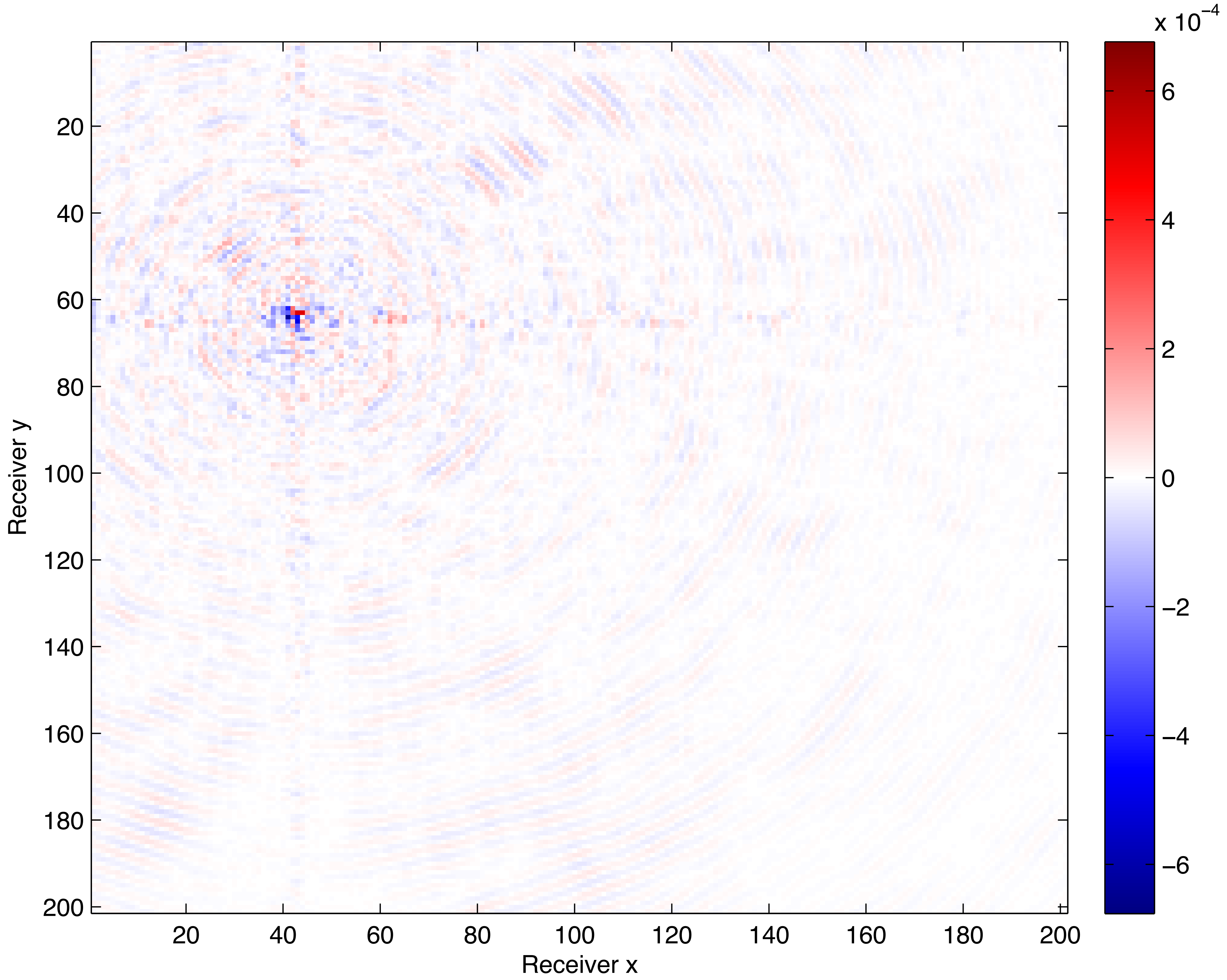
Recovered data - SNR 11.9 dB

12.3 Hz - 75% missing receivers

Common source gather



True data



Difference

Conclusion

3D seismic data has an underlying structure that we can exploit for interpolation (Hierarchical Tucker format)

Different schemes for organizing data - important for recovery

Conclusion

We can interpolate HT tensors with missing entries using the Riemannian manifold structure of the HT format

Achieve good results from largely subsampled data (75% missing receivers, 90% reduction in simultaneous receivers)

Can use this method to create full volumes from subsampled data

- Migration, multiple removal, etc.

What next?

Simulating waves in random media

$m(x,y)$ - earth medium as a function of space, x, random variable, y

$u(x,y)$ - wavefield depending on x, y

$$\left(\frac{1}{m(x,y)^2}\frac{\partial^2}{\partial t^2} + \Delta\right) u(x,y) = f(x)$$

SLIM

Why are we interested?

Full Waveform Inversion with *uncertainty quantification*

- we are interested in computing

$$\tilde{m}(x) = \mathbb{E}_y(m(x,y)|d)$$

$$\begin{aligned} v(x) &= \mathrm{var}(m|d) \\ &= \mathbb{E}_y((m(x,y) - \tilde{m}(x))^2|d) \end{aligned}$$

Challenges

$m(x, y)$

- what's an appropriate random model?

Typically $u(x, y_1, y_2, \ldots, y_N)$ for some finite random variables y_i

- naive approach - solve wave equation for *each* fixed $y_1, y_2, \ldots, y_N$
  - curse of dimensionality
- exploit low-rank hierarchical tucker structure?

Random models

Write $m(x, y)$ as

$$m(x, y) = \hat{m}(x) + \sum_{i=1}^{N} y_i \psi_i(x)$$

$\hat{m}(x)$ is the mean, $\psi_i(x)$ are basis functions, y_i are random variables

Random layered models

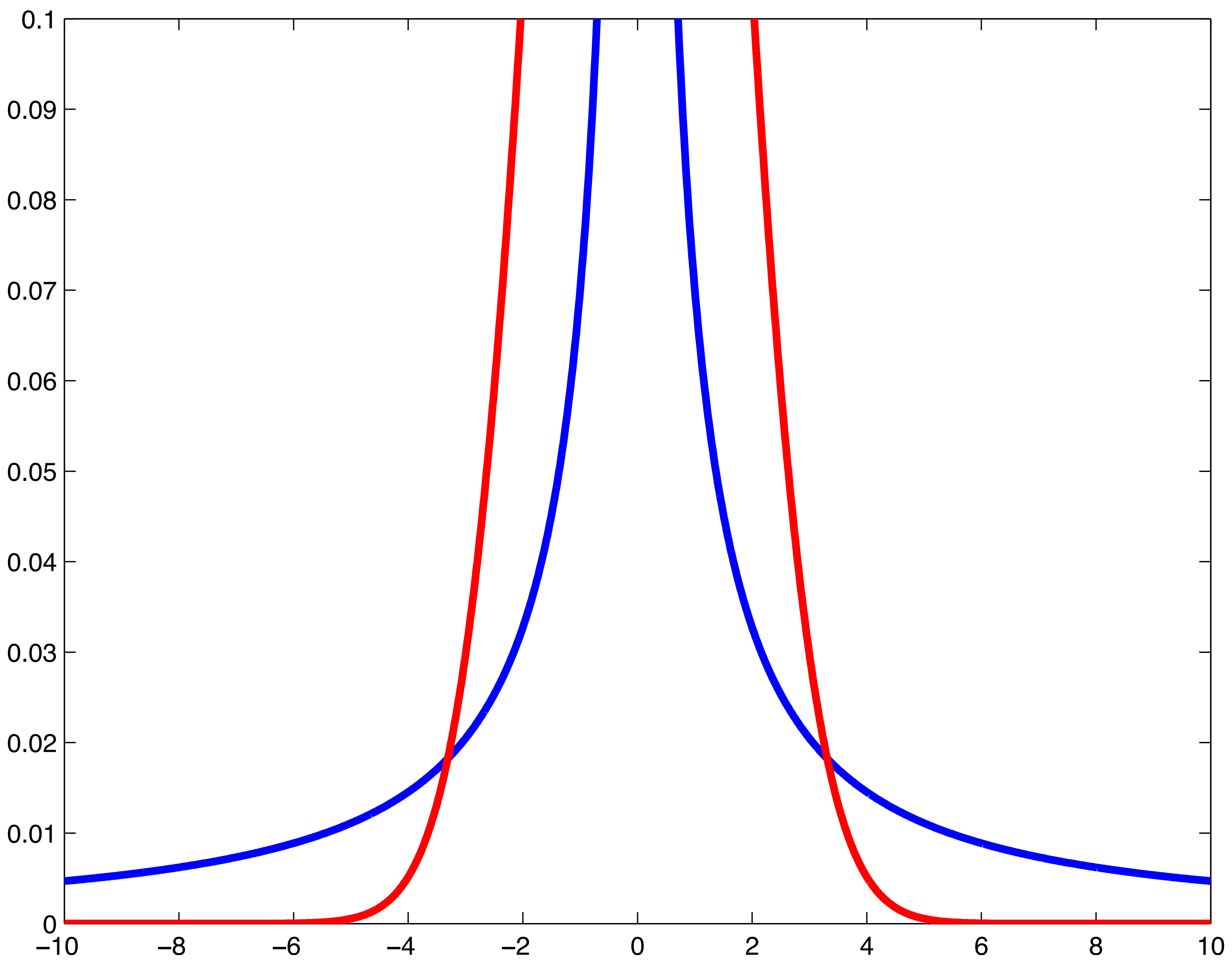
Simple model

- $\psi_i(x)$ - Haar wavelet basis
- y_i - drawn from an α-stable distribution

α-stable distributions

Probability density

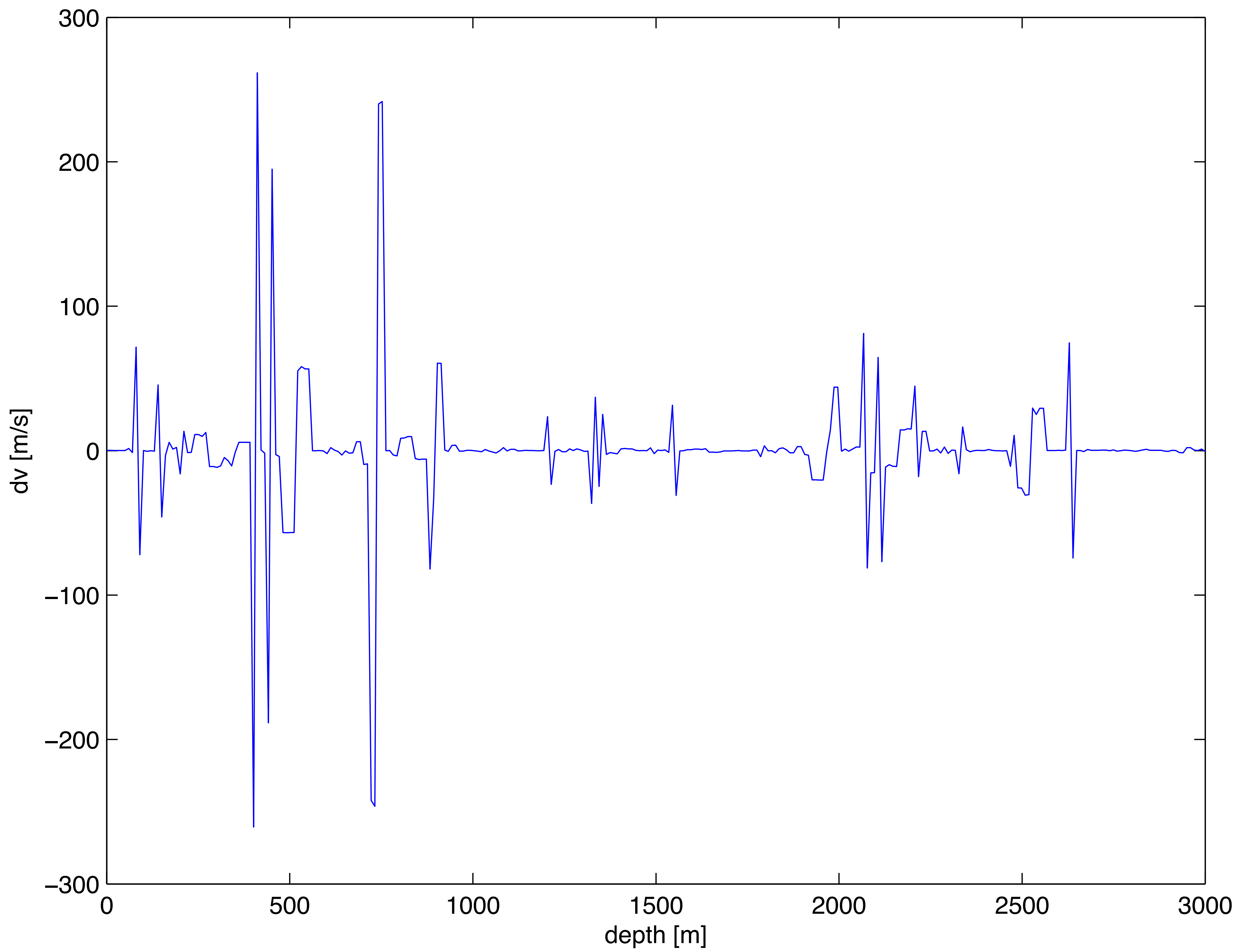
- Blue - α-stable
- Red - Gaussian

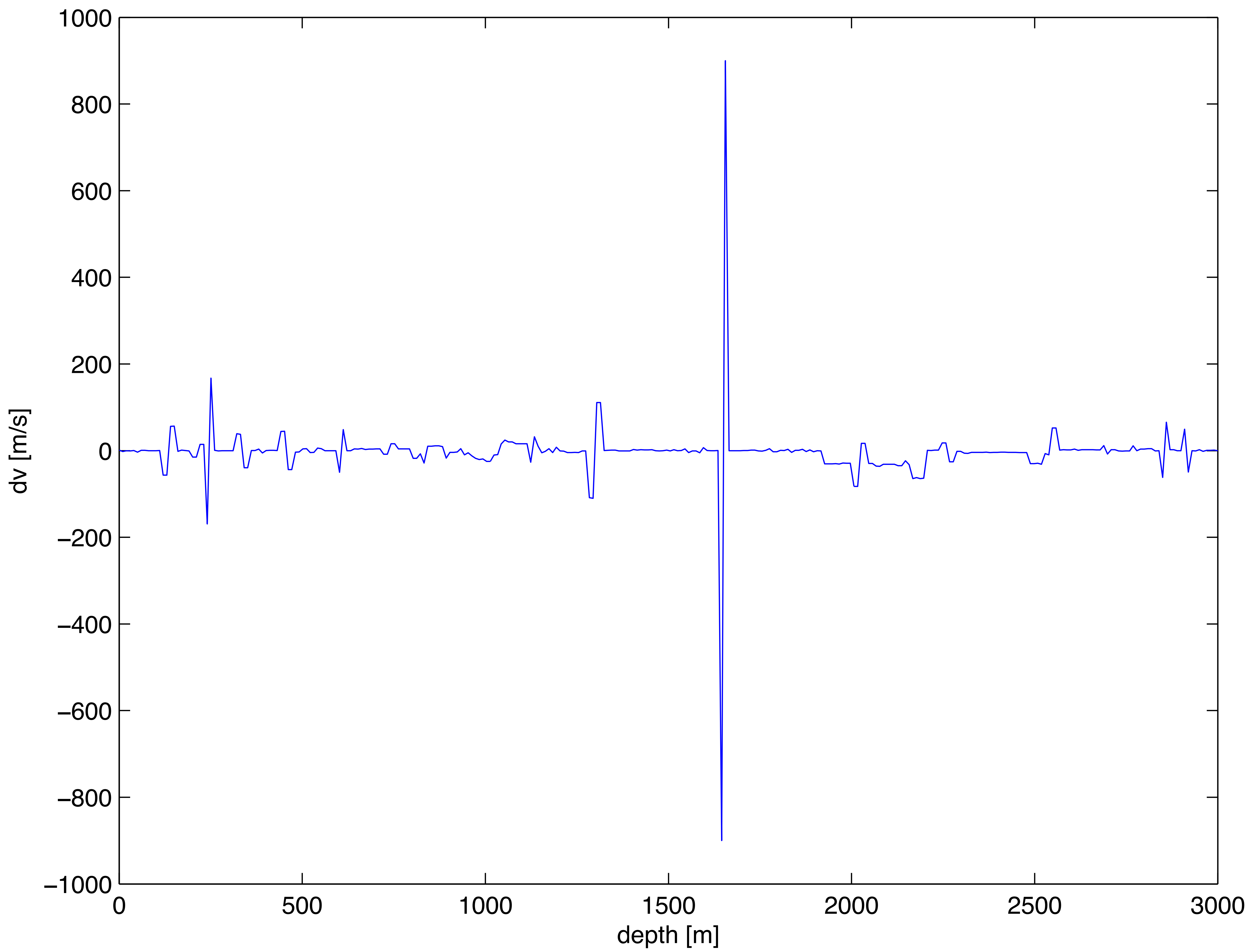


α-stable distributions

- allow *sparse* noise
- allow *large outliers*

Random model perturbations





Future work

Scale-dependent probability - wavelets

- if a coefficient is zero at scale j, it should probably be zero at scale j+1

2D random models

- curvelets + sparse distribution

Efficient implementation for computing random wavefields

Fullwaveform inversion with uncertainty quantification

Acknowledgements

Thank you for your attention

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