3D frequency domain FWI

using a row-projected Helmholtz solver

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joint work with:

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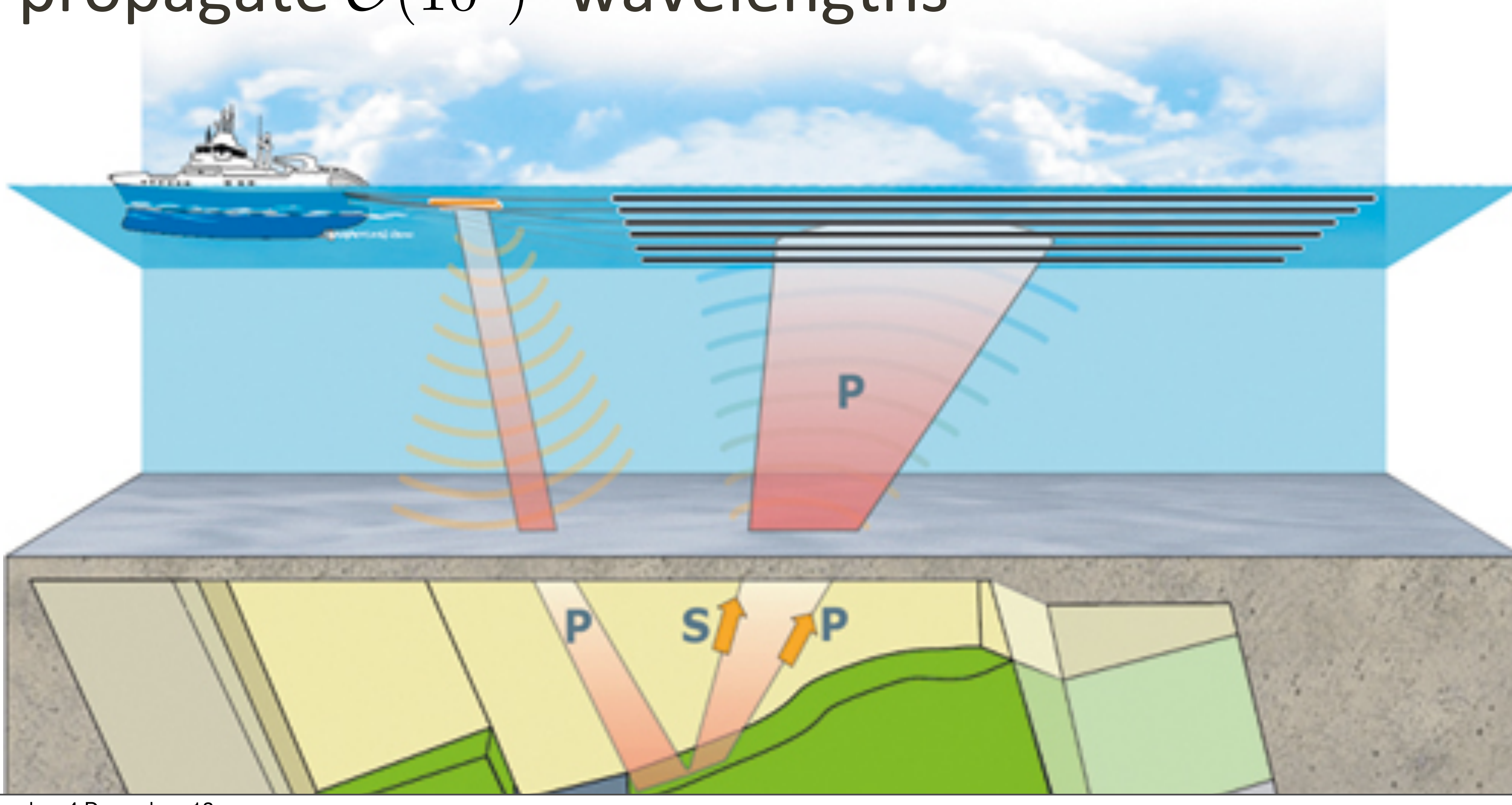
SLIM
University of British Columbia

Infer 3D velocity model from *multi-experiment* data:

- $\mathcal{O}(10^9)$ unknowns
- $\mathcal{O}(10^{15})$ datapoints
- propagate $\mathcal{O}(10^2)$ wavelengths



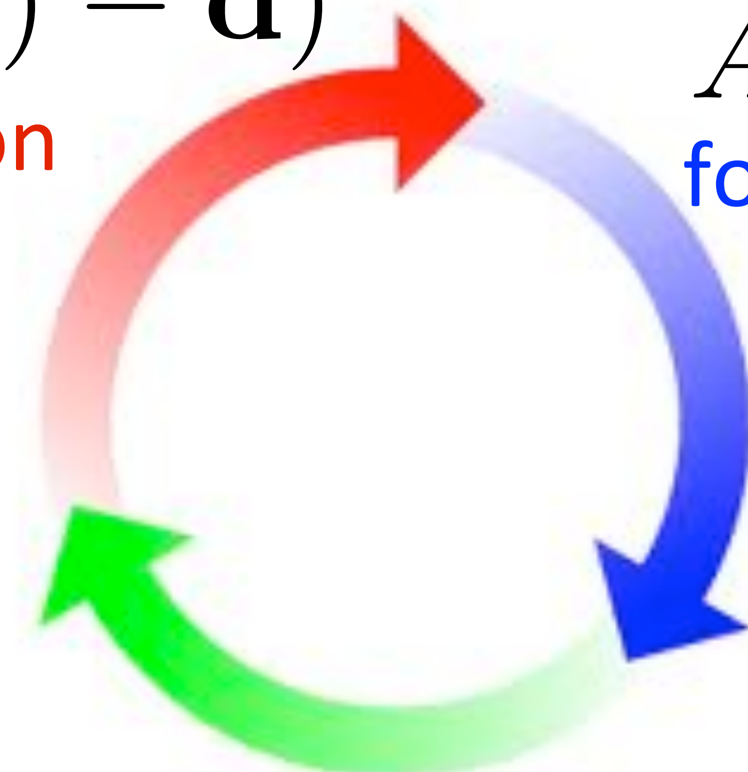
SLIM

$$\min_{\mathbf{m}} \rho(F(\mathbf{m}) - \mathbf{d})$$

formulation

$$A(\mathbf{m})\mathbf{u} = \mathbf{q}$$

forward modelling



$$\mathbf{m}_{k+1} = \mathbf{m}_k + \alpha_k \mathbf{s}_k$$

optimization strategies

computational framework

Formulation

non-linear least-squares problem:

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) = \sum_{i=1}^{M} ||\mathbf{d}_i - P_i \mathbf{u}_i||_2^2$$

gradient:

$$\frac{\partial \Phi}{\partial m_k} = \sum_{i=1}^{M} \mathbf{u}_i^H \left(\frac{\partial A(\mathbf{m})}{\partial m_k} \right)^H \mathbf{v}_i$$

where:

$$A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

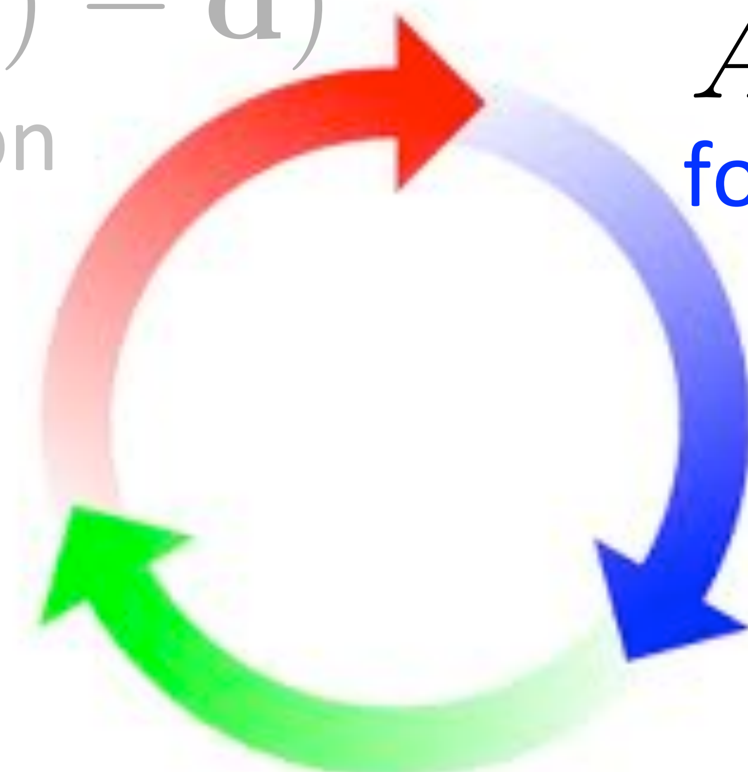
$$A(\mathbf{m})^H \mathbf{v}_i = P_i^T (\mathbf{d}_i - P_i \mathbf{u}_i)$$

$$\min_{\mathbf{m}} \rho(F(\mathbf{m}) - \mathbf{d})$$

formulation

$$A(\mathbf{m})\mathbf{u} = \mathbf{q}$$

forward modelling



$$\mathbf{m}_{k+1} = \mathbf{m}_k + \alpha_k \mathbf{s}_k$$

optimization strategies

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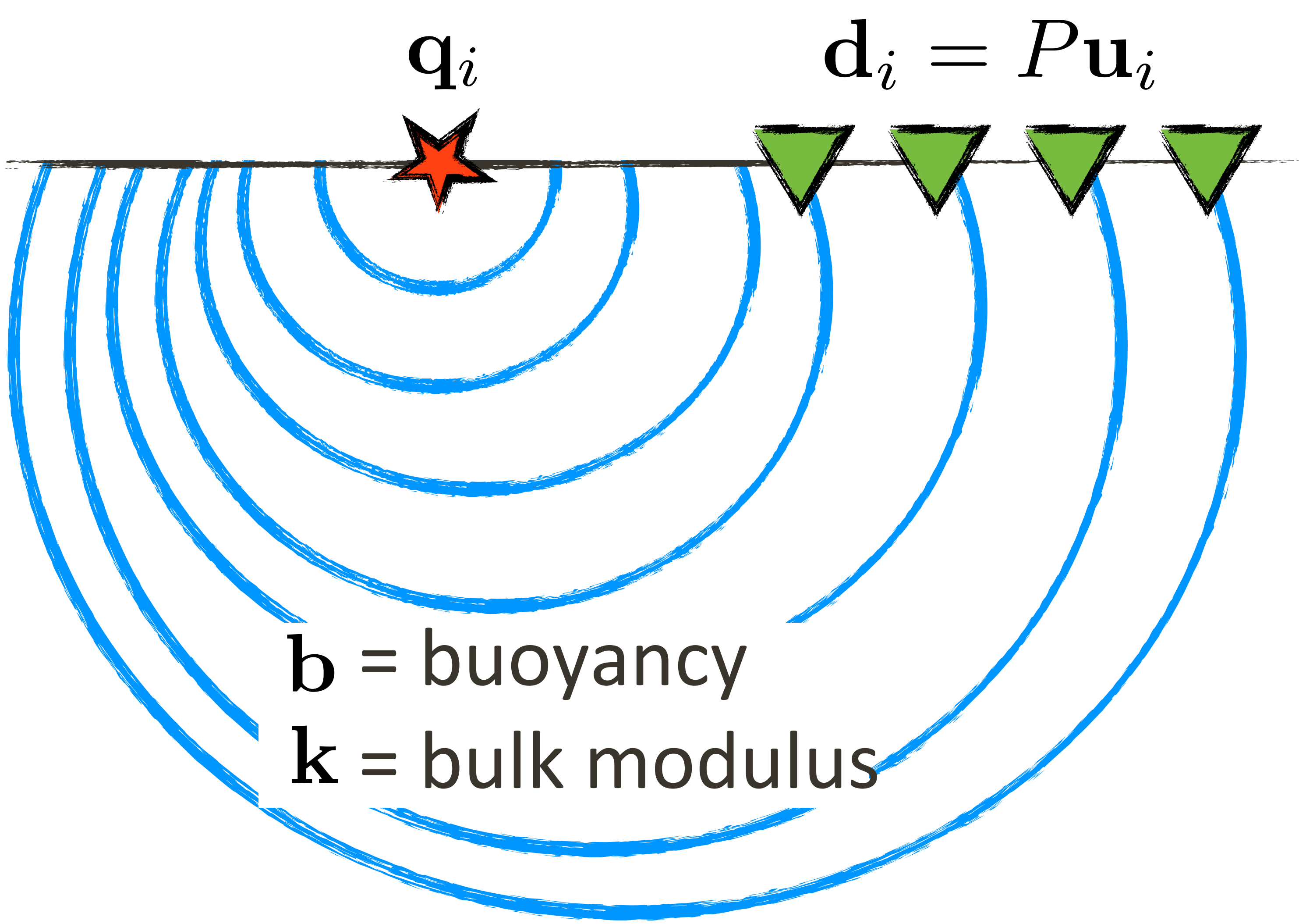
Forward modelling

We model the data in the *acoustic* approximation

$$\big(\omega^2 + \mathbf{k}\nabla \cdot (\mathbf{b}\nabla)\big)\mathbf{u}_i = \mathbf{q}_i$$



Preconditioning

Replace original system $A\mathbf{x} = \mathbf{b}$ by

$$M^{-1}A\mathbf{x} = M^{-1}\mathbf{b}$$

where $M^{-1} \approx A^{-1}$ *and* can be applied cheaply.

Popular approaches:

- *Asymptotic/one-way approximations*
- *Multi-grid on damped equation*

Our approach:

- *Generic, simple and robust*
- *Easily parallelized and optimized*

[Ernst & Martin, 2012]

Kaczmarz

The Kaczmarz method solves a system

$$A\mathbf{x} = \mathbf{b}$$

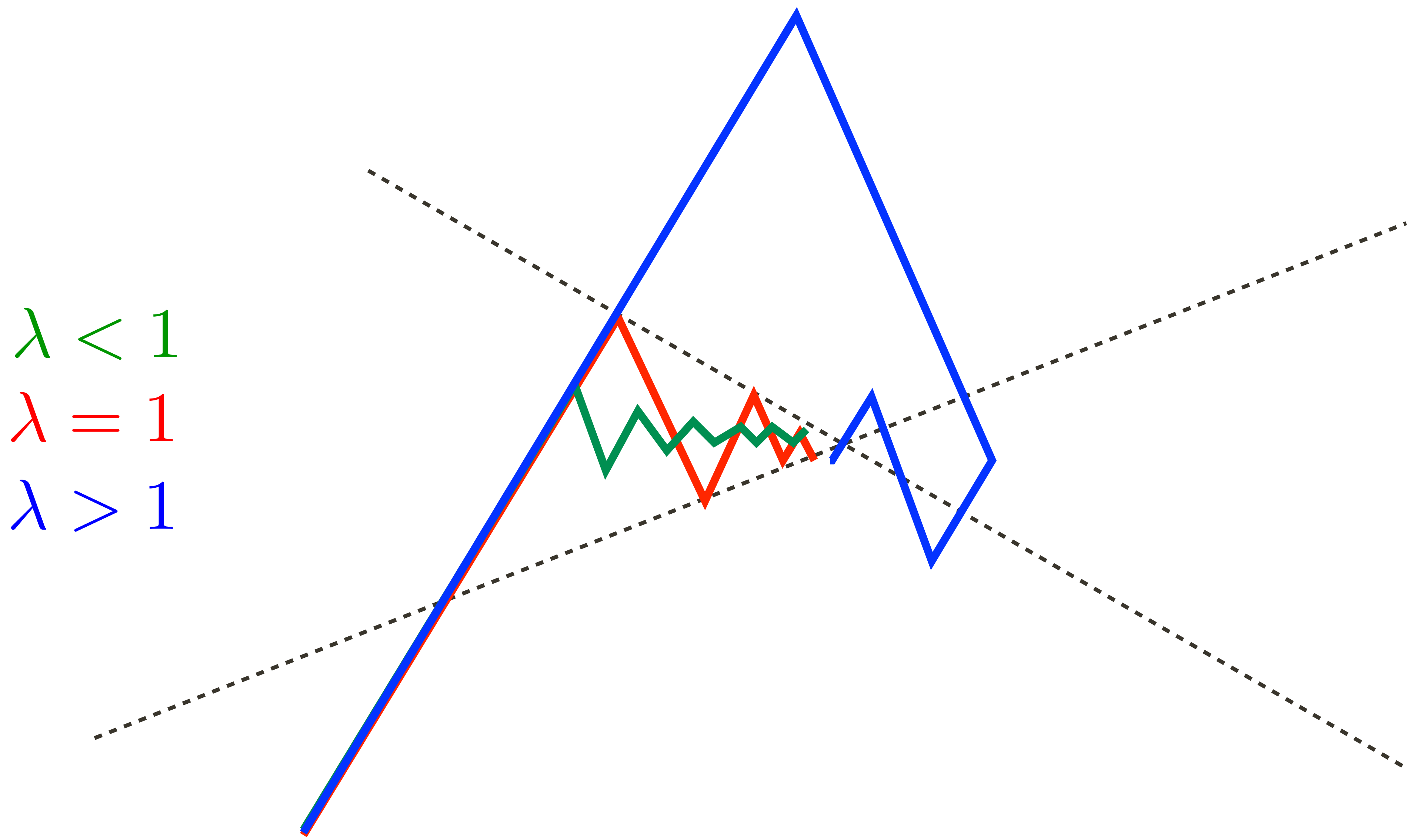
by successive row projections

$$\mathbf{x} := \mathbf{x} + \frac{\lambda}{||\mathbf{a}_i||_2^2}\left(b_i - \mathbf{a}_i^T\mathbf{x}\right)\mathbf{a}_i,$$

with relaxation parameter $0 < \lambda < 2$

[Kaczmarz, ’37]

Kaczmarz



Kaczmarz

Split matrix: $AA^T = D + L + L^T$

The corresponding preconditioner is given by

$$M^{-1} = A^T H$$

where

$$H = \lambda(2-\lambda)(D+\lambda L^T)^{-1} D (D+\lambda L)^{-1}$$

The matrix $M^{-1}A$ is *symmetric* and *positive semidefinite*; we can use CG.

Error propagation

analyze error propagation:

$$\mathbf{e}_{k+1} = Q(\lambda)\mathbf{e}_k$$

with $\quad Q(\lambda) = I - A_\lambda^T H_\lambda A_\lambda$

Fourier analysis: $\quad e_j(\theta) = \exp(\imath j\theta)$

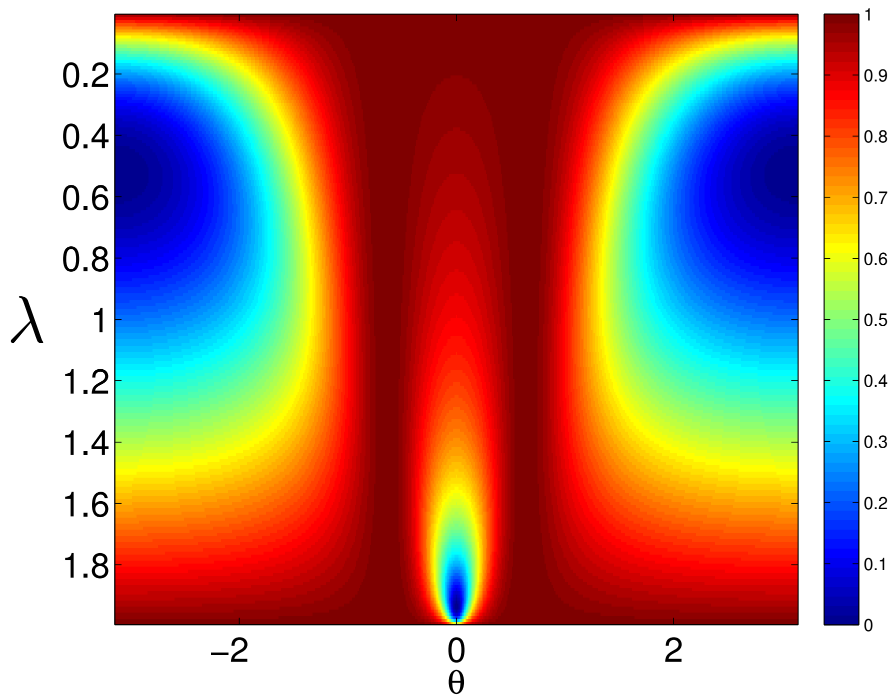
$$a(\lambda,\theta) = |Q(\lambda)\mathbf{e}(\theta)|/|\mathbf{e}(\theta)|$$

Error propagation

study the amplitude as a function of $n_g = 2\pi/(kh)$

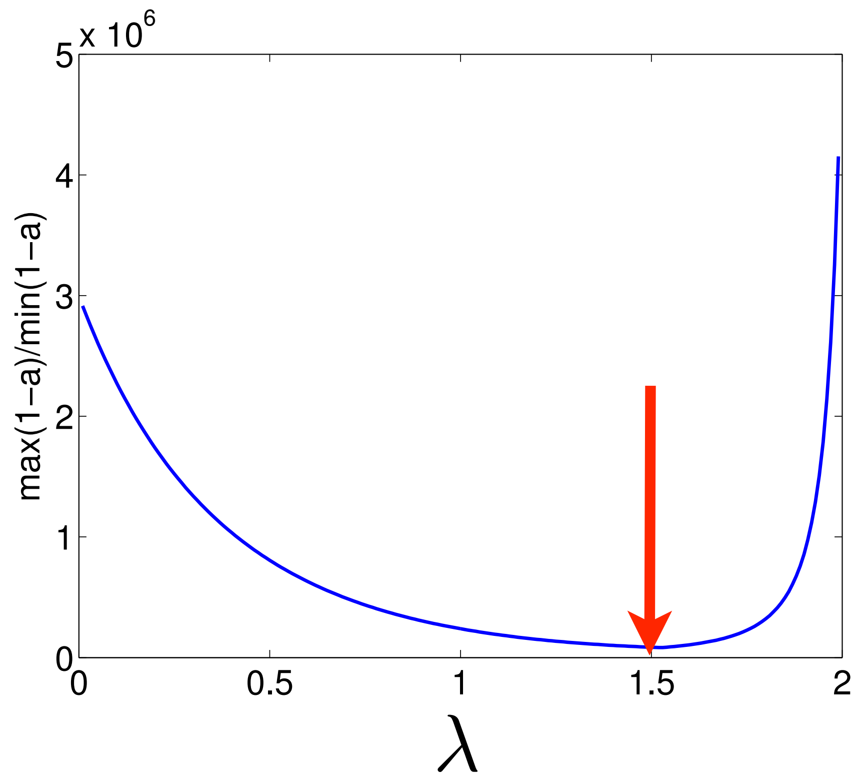
$n_g = 10$

$a(\lambda, \theta)$

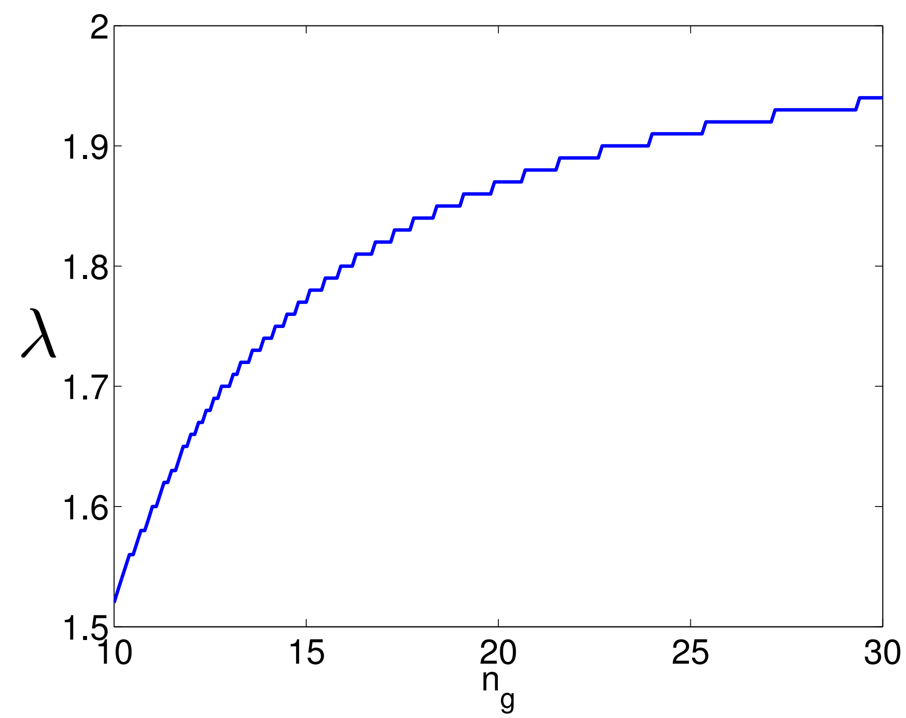


$n_g = 10$

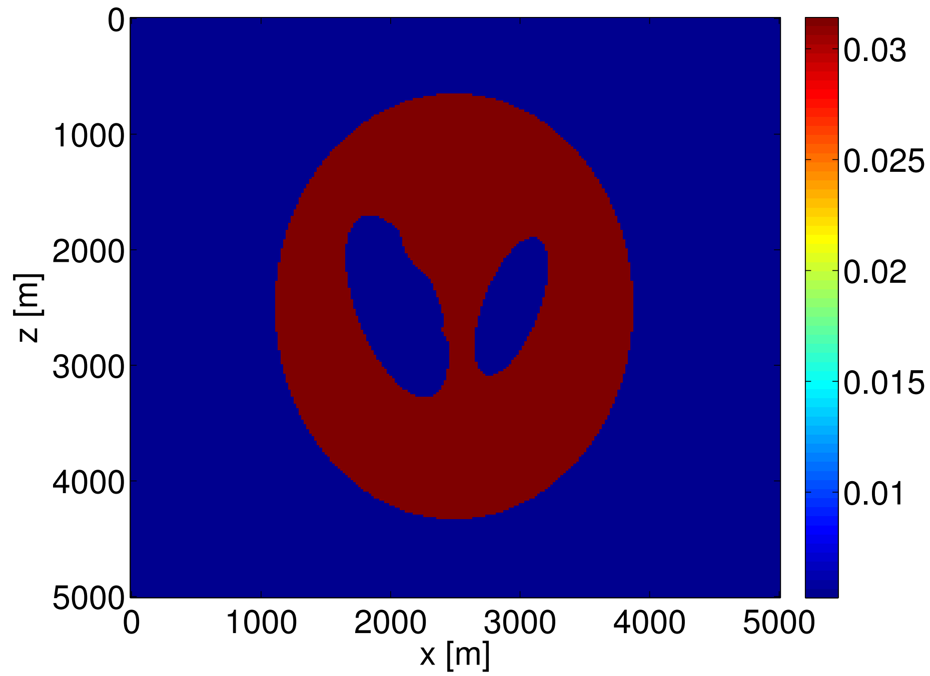
`condition number'

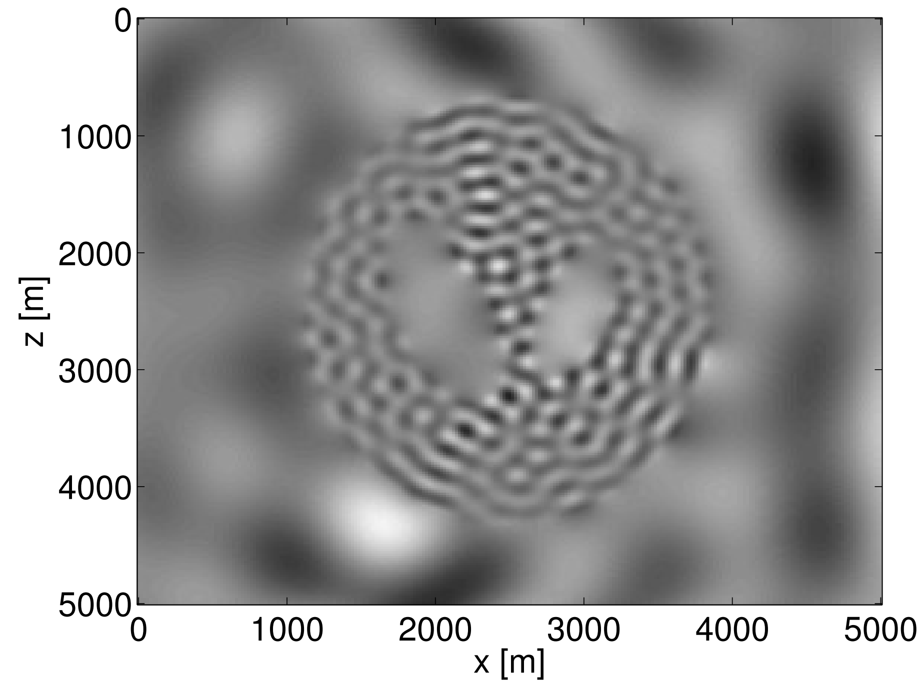


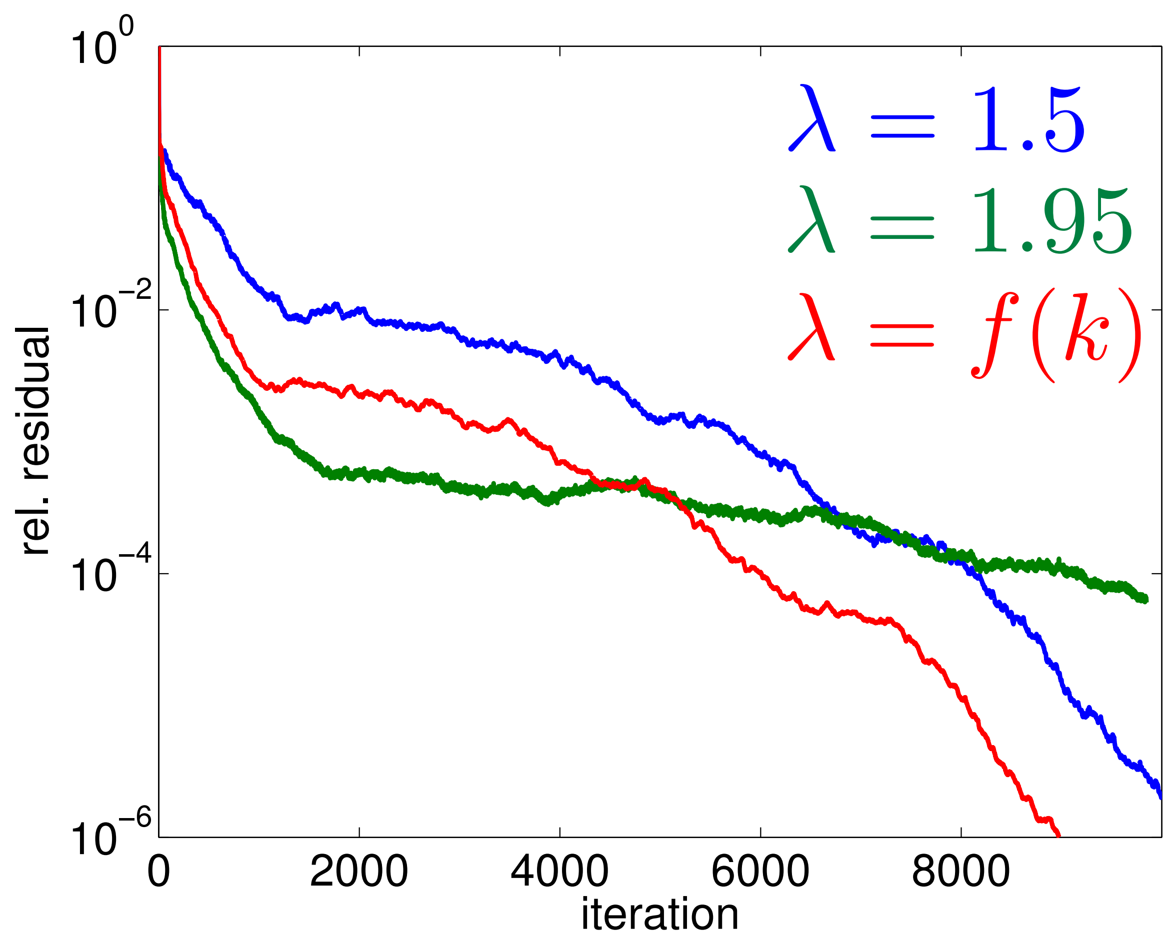
optimal λ



Error propagation







Kaczmarz

We never form the matrix explicitly, but compute its action:

Algorithm 1 $\mathsf{DKSWP}(A, \mathbf{r}, \lambda) = M^{-1}\mathbf{r}$

$\mathbf{x} = 0$
forward sweep
for $i = 1$ to n **do**
  $\mathbf{x} := \mathbf{x} + \lambda(r_i - \mathbf{a}_i^T\mathbf{x})\mathbf{a}_i / ||\mathbf{a}_i||_2^2$
end for
backward sweep
for $i = n$ to 1 **do**
  $\mathbf{x} := \mathbf{x} + \lambda(r_i - \mathbf{a}_i^T\mathbf{x})\mathbf{a}_i / ||\mathbf{a}_i||_2^2$
end for
return x

CARP CG

Algorithm 1 $\mathsf{CARP}-\mathsf{BCG}(A, U_0, S, \gamma)$

$P_0 = R_0 = \mathsf{DKSWP}(A, U_0, B, \gamma) - U_0$

while not converged **do**

$Q_k = P_k - \mathsf{DKSWP}(A, P_k, 0, \gamma)$

$\alpha_k = ({P_k}^* Q_k)^{-1}({R_k}^* R_k)$

$X_{k+1} = X_k + P_k \alpha_k$

$R_{k+1} = R_k - Q_k \alpha_k$

$\beta_k = ({R_k}^* R_k)^{-1}({R_{k+1}}^* R_{k+1})$

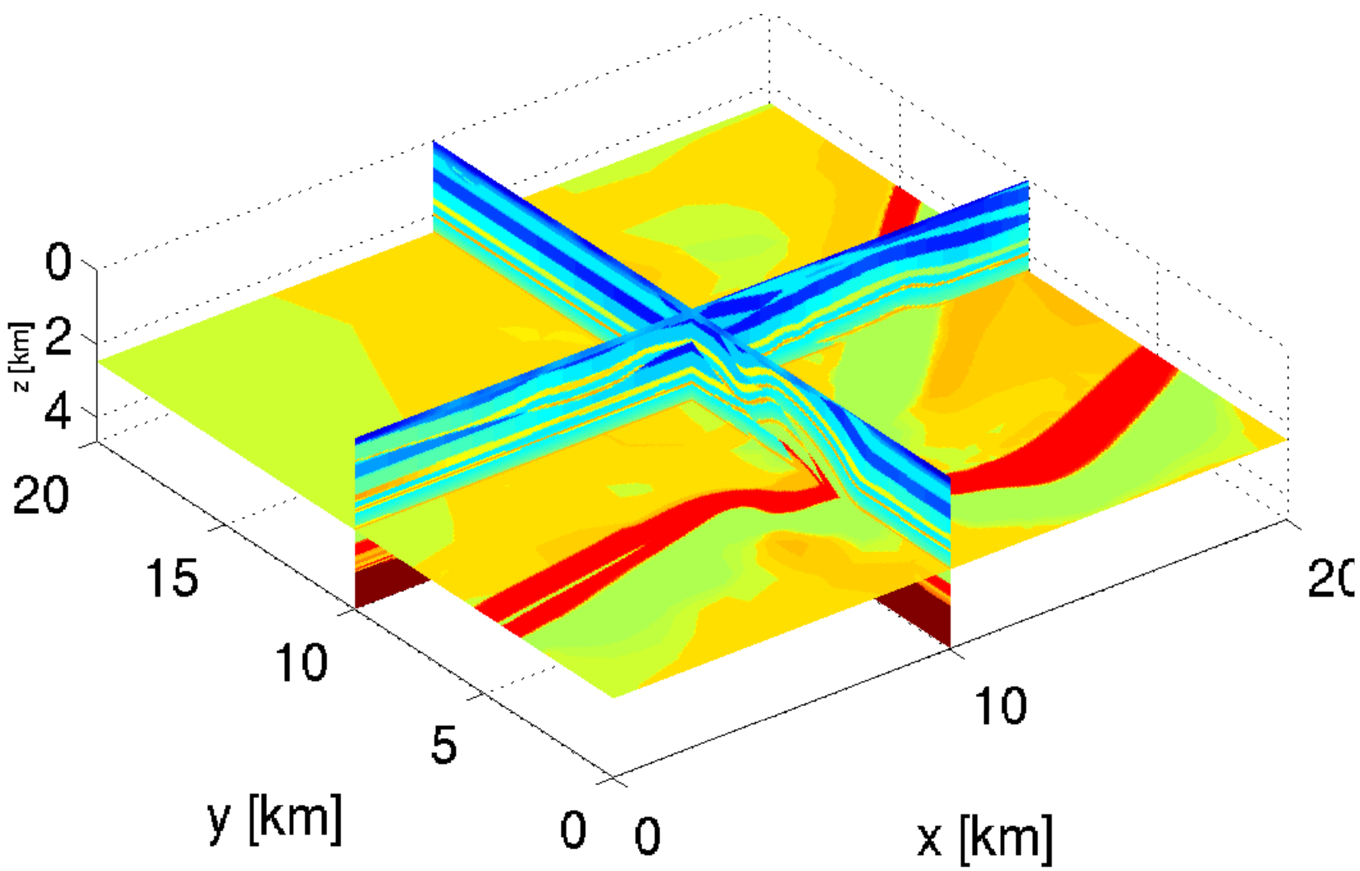
$P_{k+1} = R_k + P_k \beta_k$

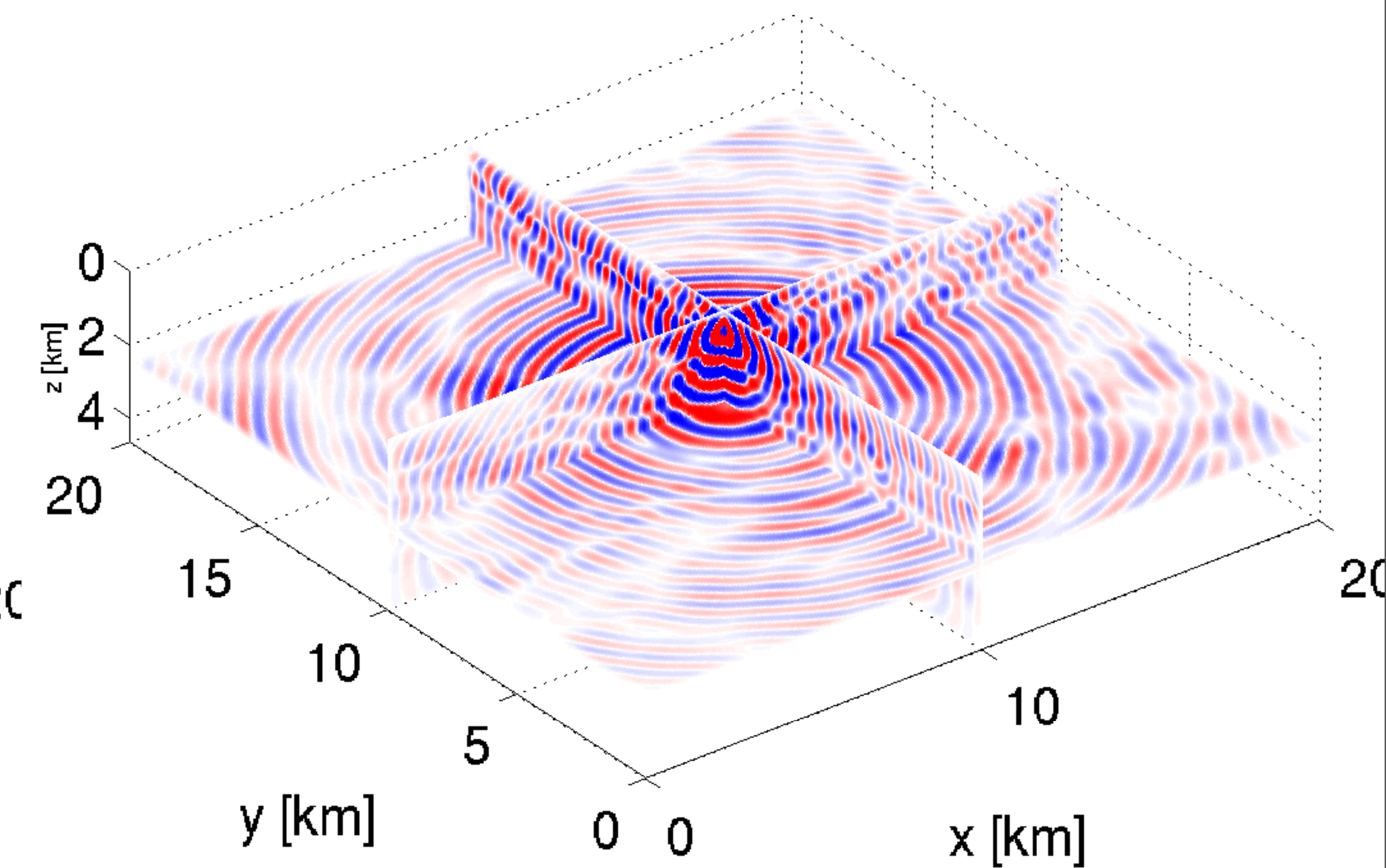
$k = k + 1$

end while

Example

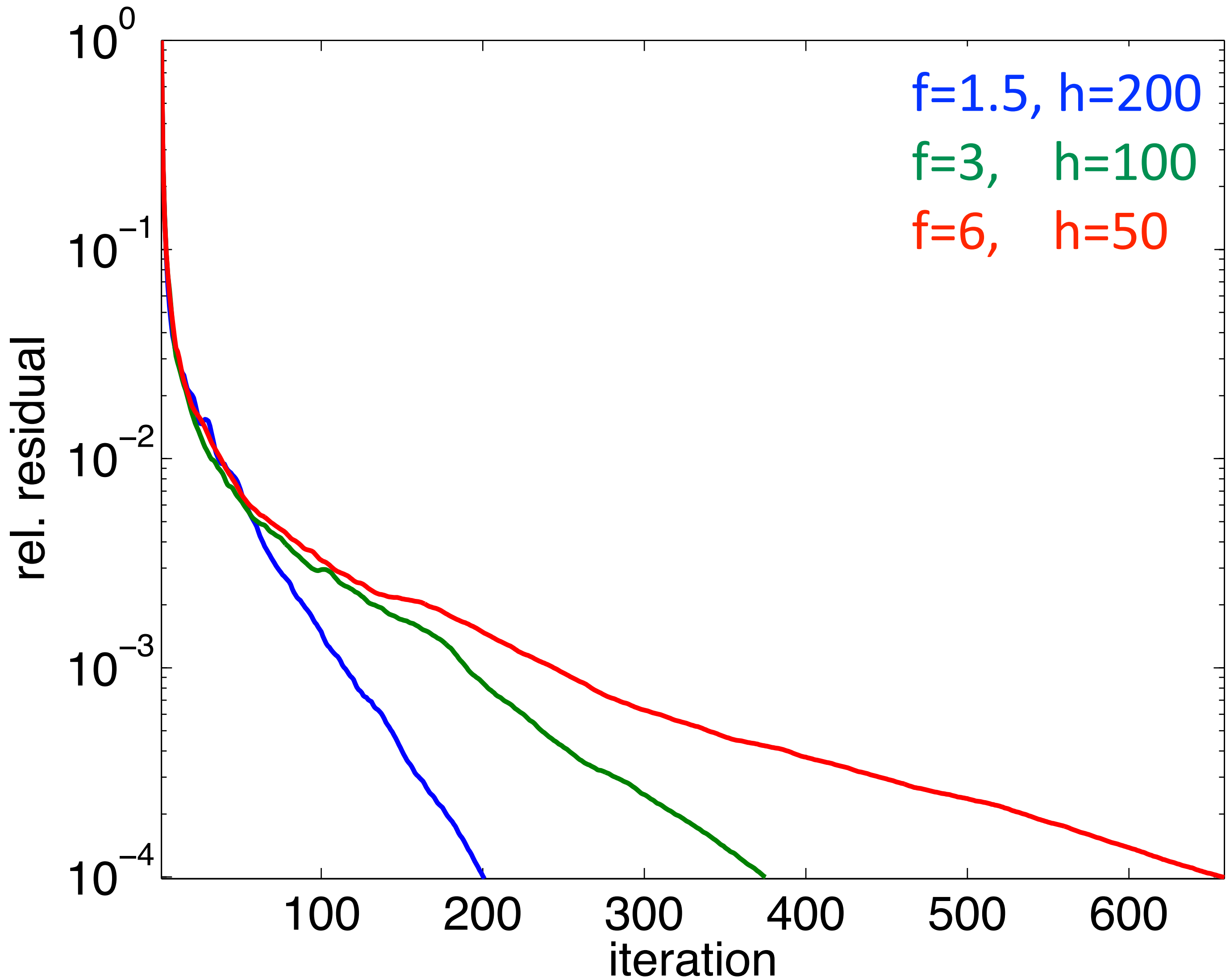
27 point stencil (2nd order), PML





Example

one r.h.s.



on 1 processor

Example

Parallelization via domain-decomposition:

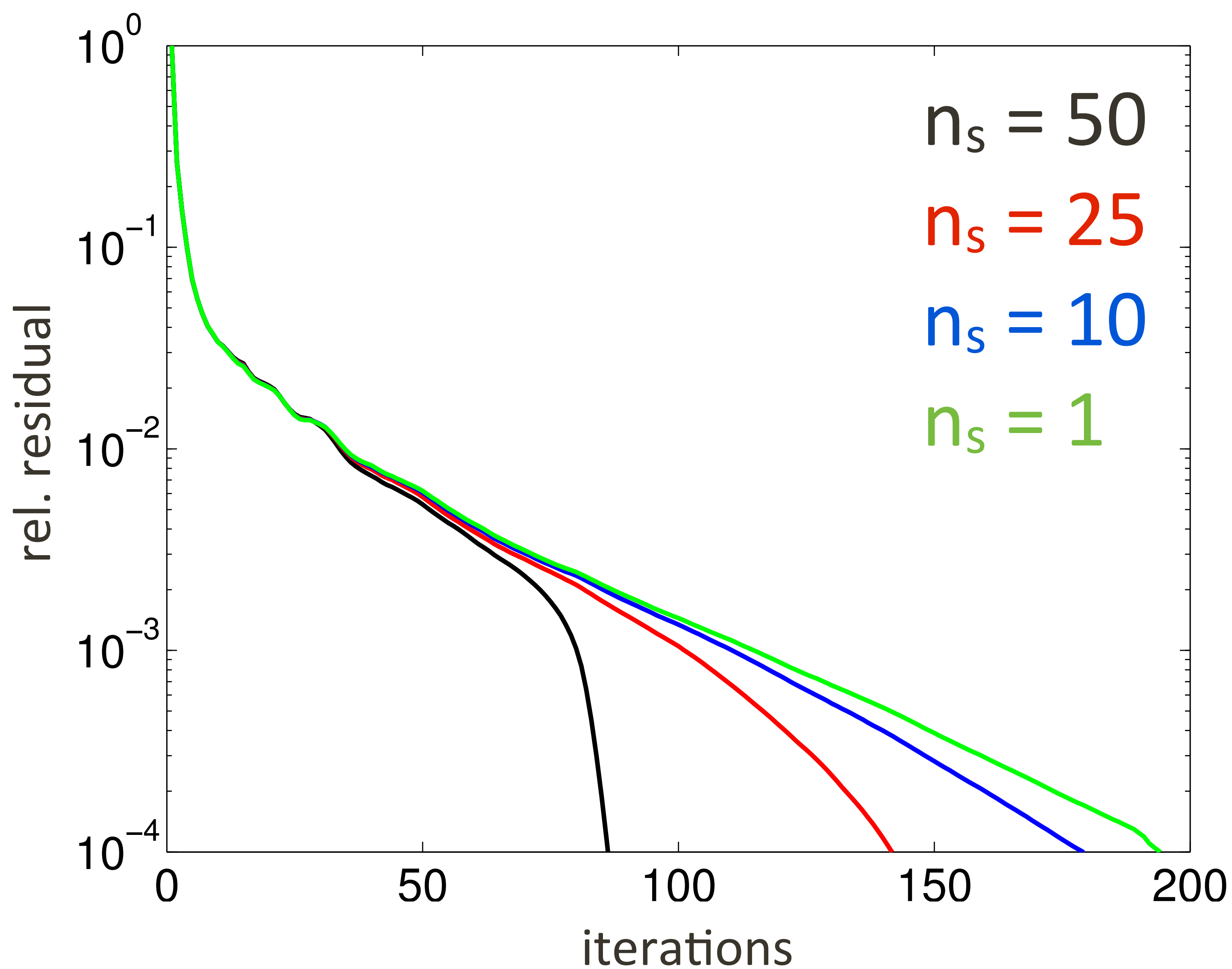
- sweeps applied on subdomains
- `additive Schwarz'
- convergence guaranteed

np	iter	time	efficiency
1	659	20785.40	1.00
2	657	11306.90	0.92
4	596	4882.50	0.96
8	603	3960.10	0.60

grid: 47x201x201, h=100, f=3 Hz, $\epsilon = 10^{-4}$

Example

multiple r.h.s.

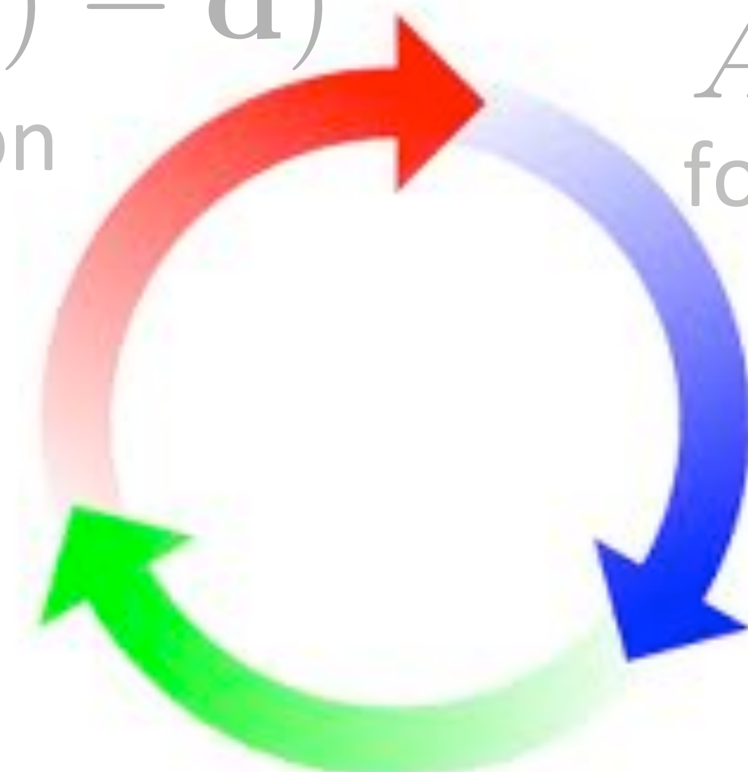


$$\min_{\mathbf{m}} \rho(F(\mathbf{m}) - \mathbf{d})$$

formulation

$$A(\mathbf{m})\mathbf{u} = \mathbf{q}$$

forward modelling



$$\mathbf{m}_{k+1} = \mathbf{m}_k + \alpha_k \mathbf{s}_k$$

optimization strategies

computational framework

Fast optimization

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) = \frac{1}{M}\sum_{i=1}^{M} \phi_i(\mathbf{m})$$

steepest descent

$$\mathbf{m}_{k+1} = \mathbf{m}_k - \lambda_k \nabla\Phi(\mathbf{m}_k)$$

But: evaluation of *full* misfit and gradient is very expensive.

Fast optimization

Gradient descent with errors:

$$\nabla\widetilde{\Phi}_k = \nabla\Phi_k + \mathbf{e}_k$$

stochastic/incremental gradient require an *unbiased* estimate*:*

$$\mathbb{E}(\mathbf{e}_k) = 0$$

and have *sublinear* convergence rate

[Bertsekas ’96,’08; Nemirovski ’00]

Fast optimization

Instead, require that the error is *bounded*

$||\mathbf{e}_k|| \leq B_k$ and $\lim_{k\to\infty} B_{k+1}/B_k \leq 1$

Then

$$\Phi(\mathbf{m}_k) - \Phi(\mathbf{m}^*) = c^k(\Phi(\mathbf{m}_k) - \Phi(\mathbf{m}_0)) + \mathcal{O}(\max\{B_k, c^k\})$$

where $0 < c < 1$

Linear convergence rate if $B_k = \mathcal{O}(\gamma^k)$

[Friedlander & Schmidt '12]

SLIM

Fast optimization

The gradient is an *average*

$$\nabla \Phi = \frac{1}{M} \sum_{i=1}^{M} \nabla \phi_i$$

which we can approximate by

$$\nabla \Phi \approx \nabla \widetilde{\Phi} = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \nabla \phi_i$$

Express the error as

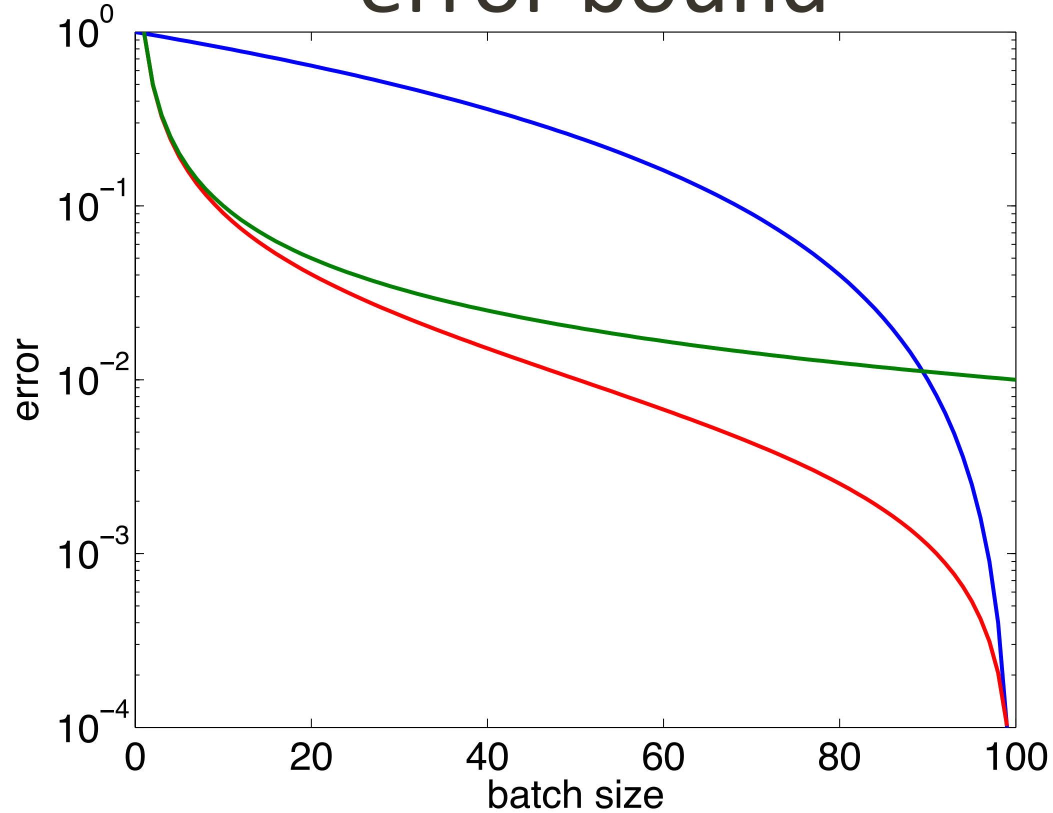
$$\mathbf{e}_k = \frac{M - |\mathcal{I}_k|}{M|\mathcal{I}_k|} \sum_{i \in \mathcal{I}_k} \nabla \phi_i + \frac{1}{M} \sum_{i \notin |\mathcal{I}|} \nabla \phi_i$$

Fast optimization

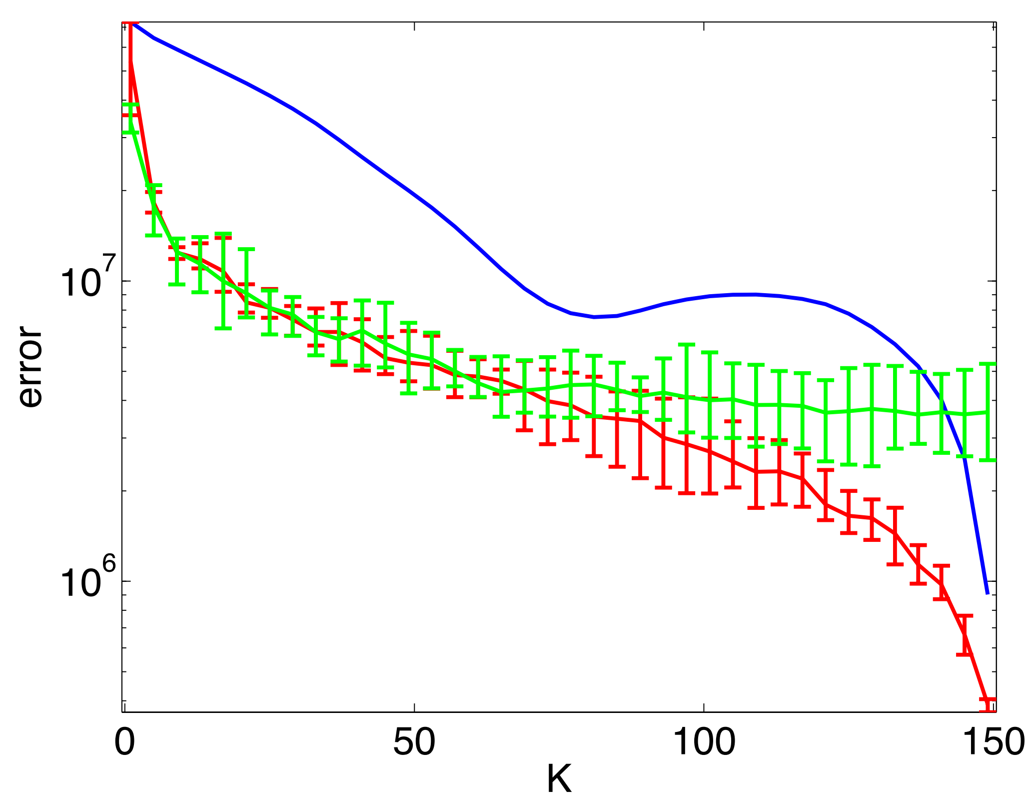
Decrease the error by adding elements to the batch

- in a pre-scribed order
- random *without* replacement
- random *with* replacement

error bound



numerical result

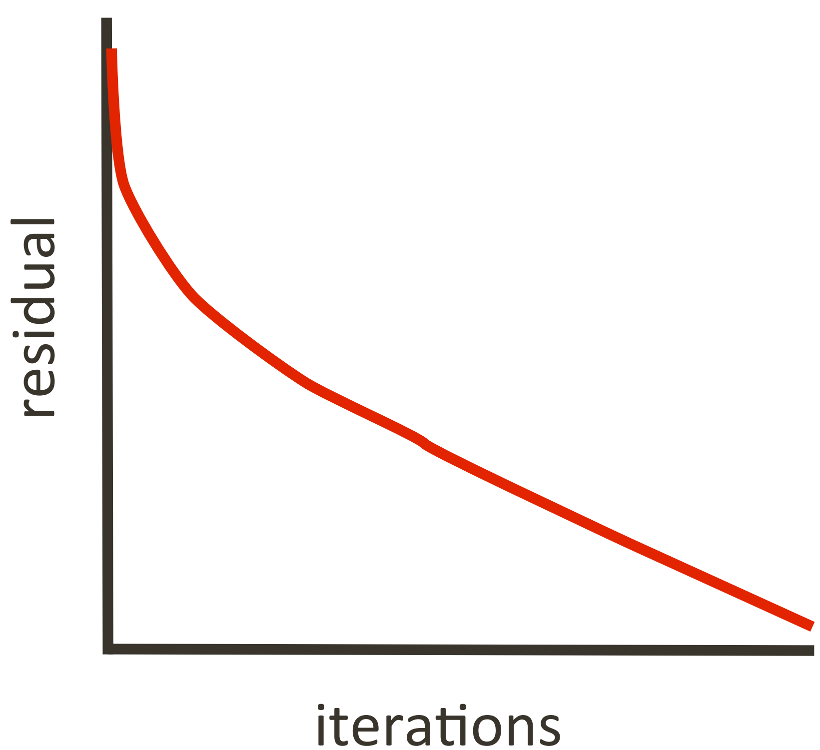


Fast optimization

use approximate PDE solves:

$$A(\mathbf{m})\mathbf{u}_i \approx \mathbf{q}_i$$
$$A(\mathbf{m})^H\mathbf{v}_i \approx P_i^T(\mathbf{d}_i - P_i\mathbf{u}_i)$$

control the accuracy by the number of iterations



Examples

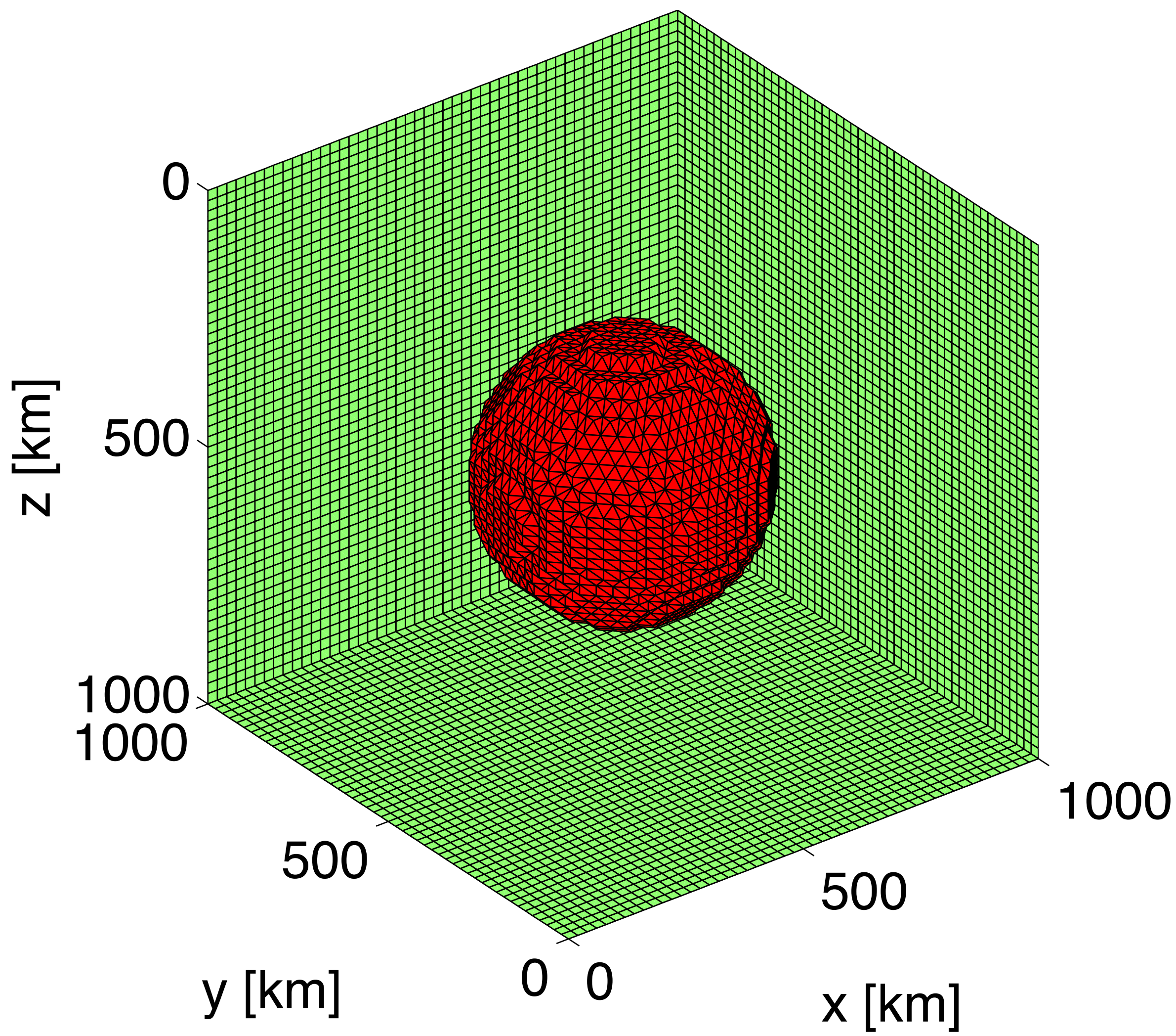
Invert multi-source, multi-frequency data:

$$\min_{\mathbf{m},\mathbf{w}} \sum_{i=1}^{M} ||w_i F_i(\mathbf{m}) - \mathbf{d}_i||_2^2$$

- source calibration via variable projection
- increase sources as: $b_{k+1} = \min\{b_k + \beta, b_{\max}\}$
- increase accuracy as: $\epsilon_{k+1} = \max\{\alpha\epsilon_k, \epsilon_{\min}\}$
- L-BFGS with 5 vectors

Edam model

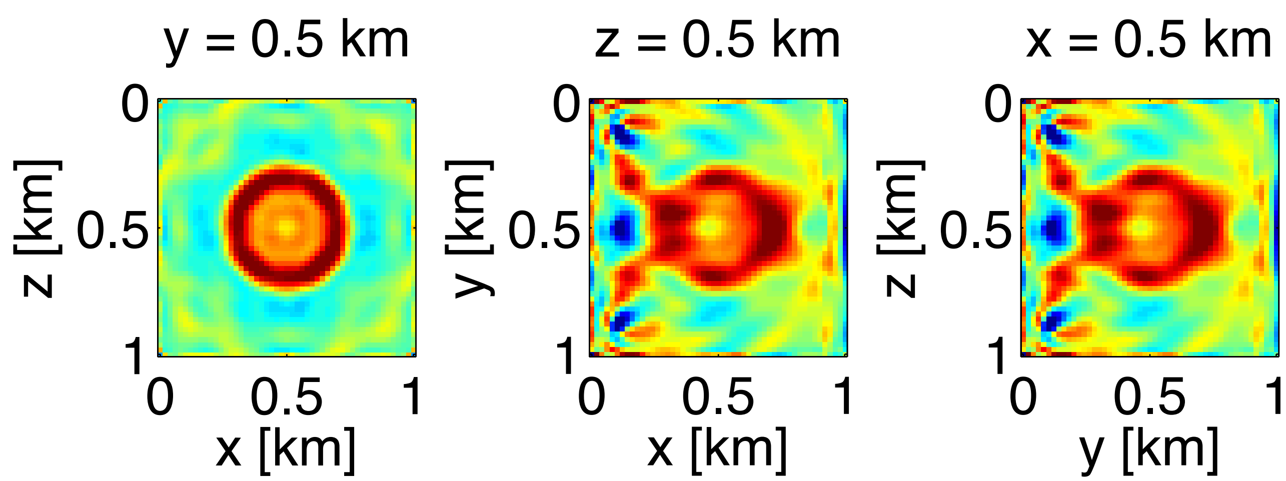
51^3 gridpoints, f = [5,10,15]

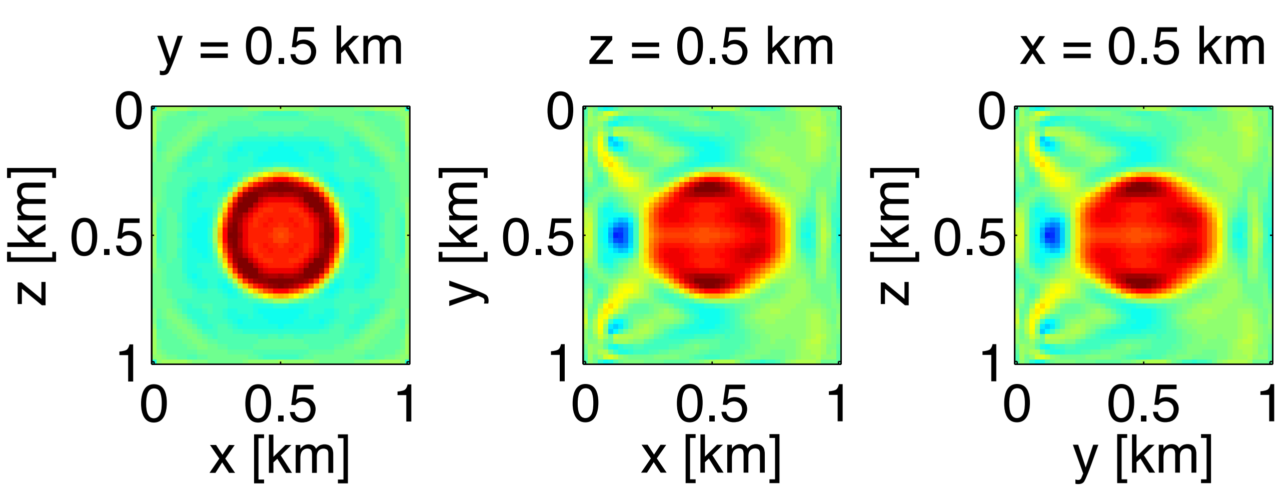


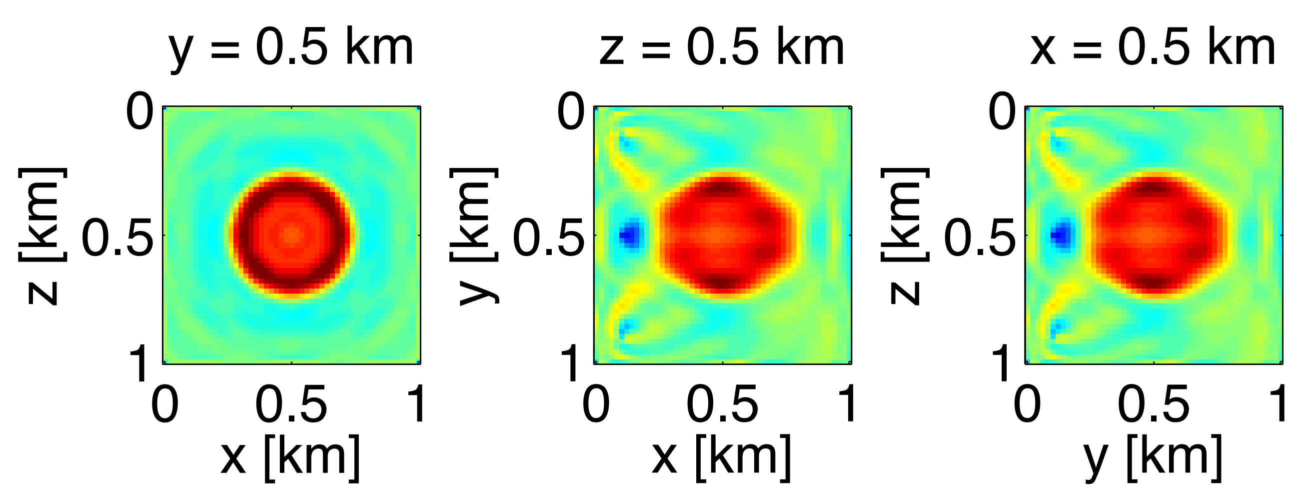
Edam model

$$b_0 = 9, \beta = 0, \epsilon_{\min} = 10^{-6}, n_{\text{iter}} = 10$$

$$\epsilon_0 = 10^{-3}, \alpha = 1,$$



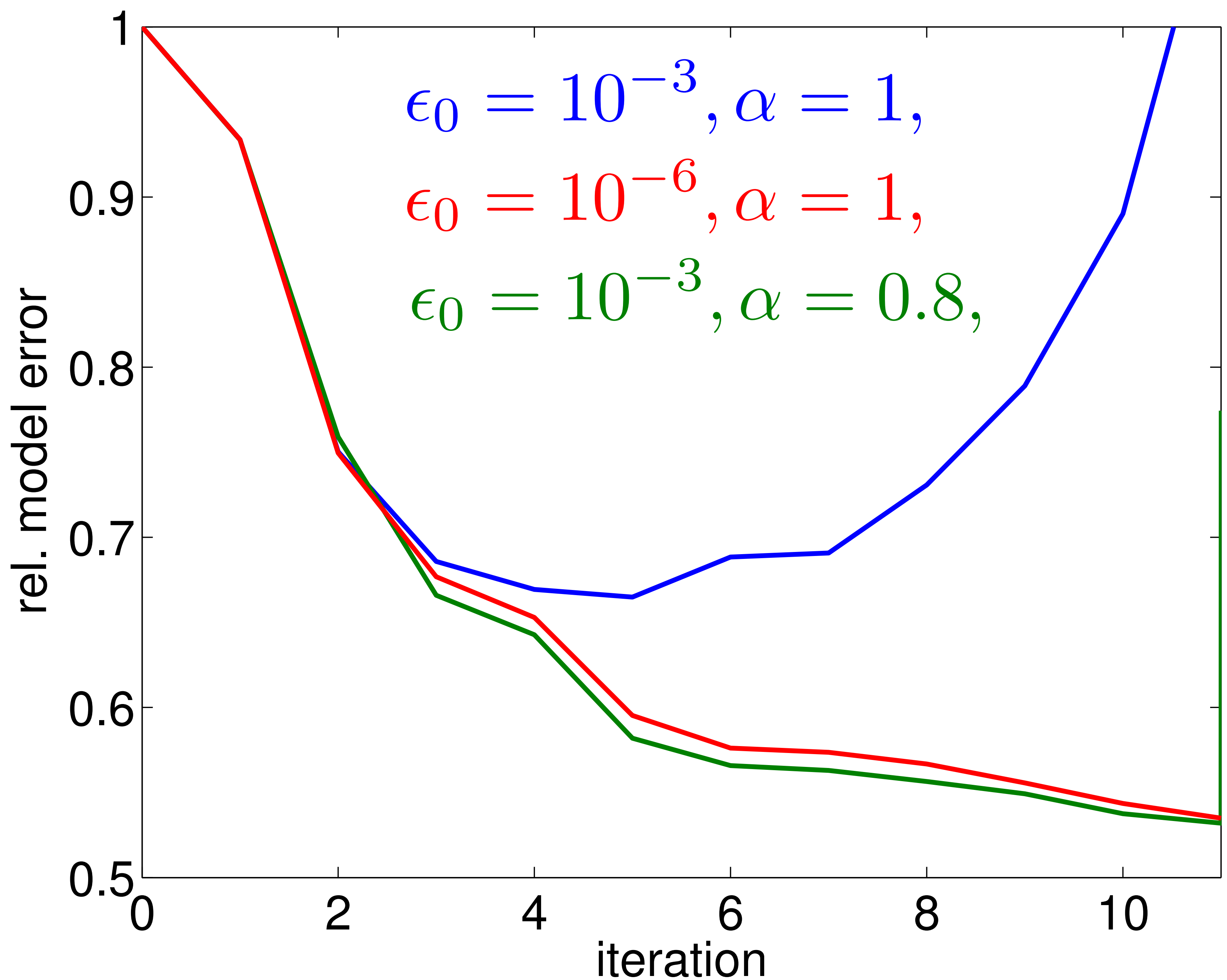




$$\epsilon_0 = 10^{-6}, \alpha = 1,$$

$$\epsilon_0 = 10^{-3}, \alpha = 0.8,$$

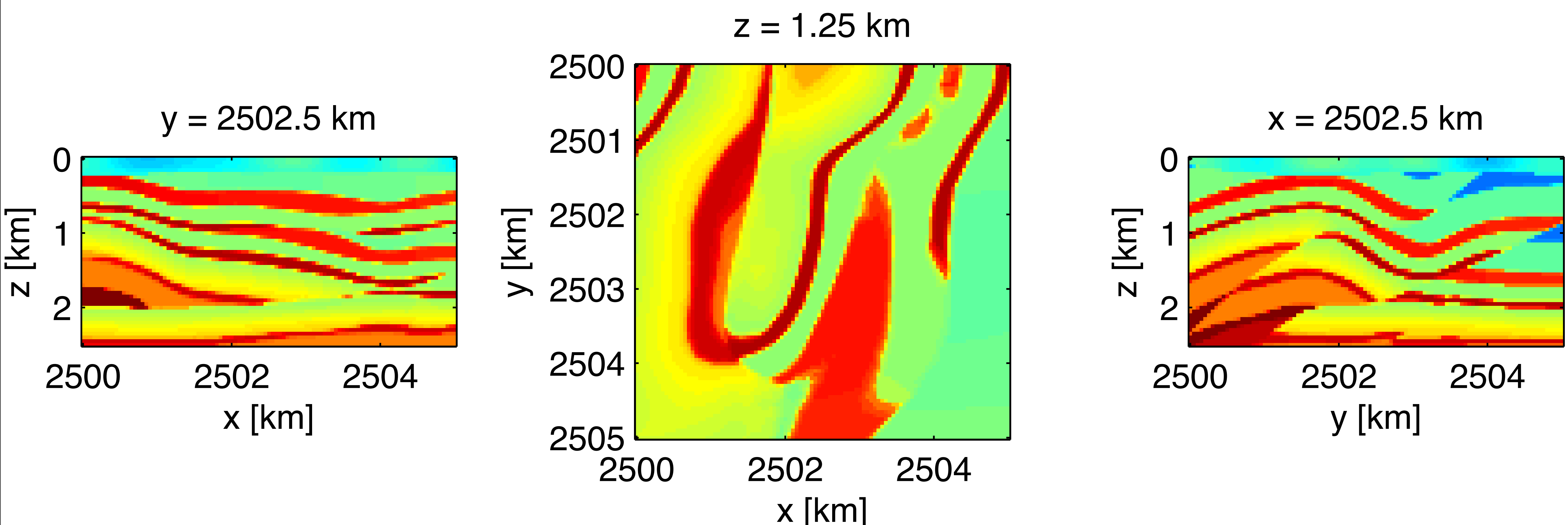
Edam model



$\epsilon_0 = 10^{-3}, \alpha = 1,$
$\epsilon_0 = 10^{-6}, \alpha = 1,$
$\epsilon_0 = 10^{-3}, \alpha = 0.8,$
rel. model error
iteration
1
0.9
0.8
0.7
0.6
0.5
0
2
4
6
8
10

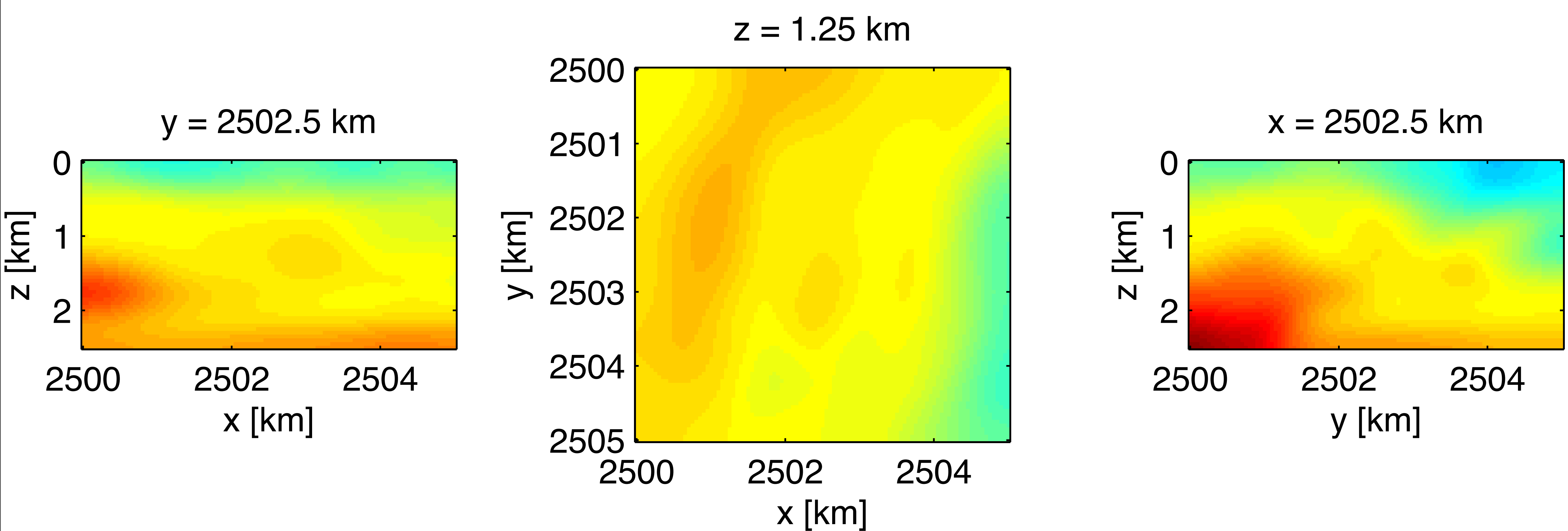
Overthrust model

101x101x51 gridpoints, f = 4



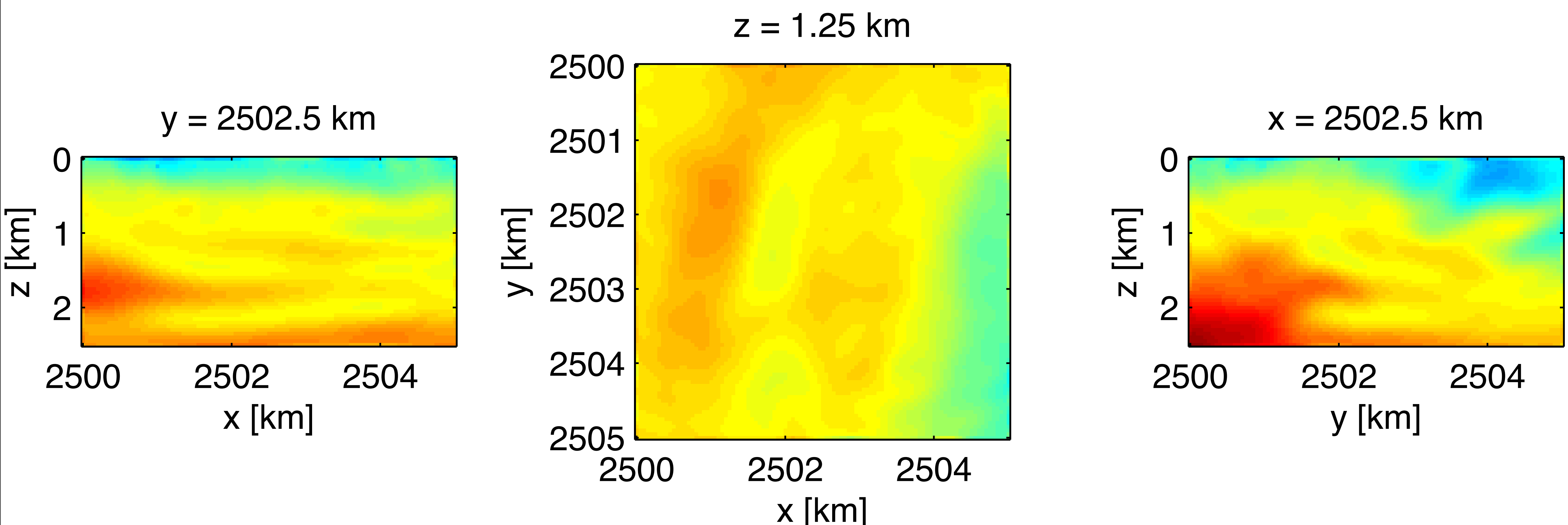
Overthrust model

initial model



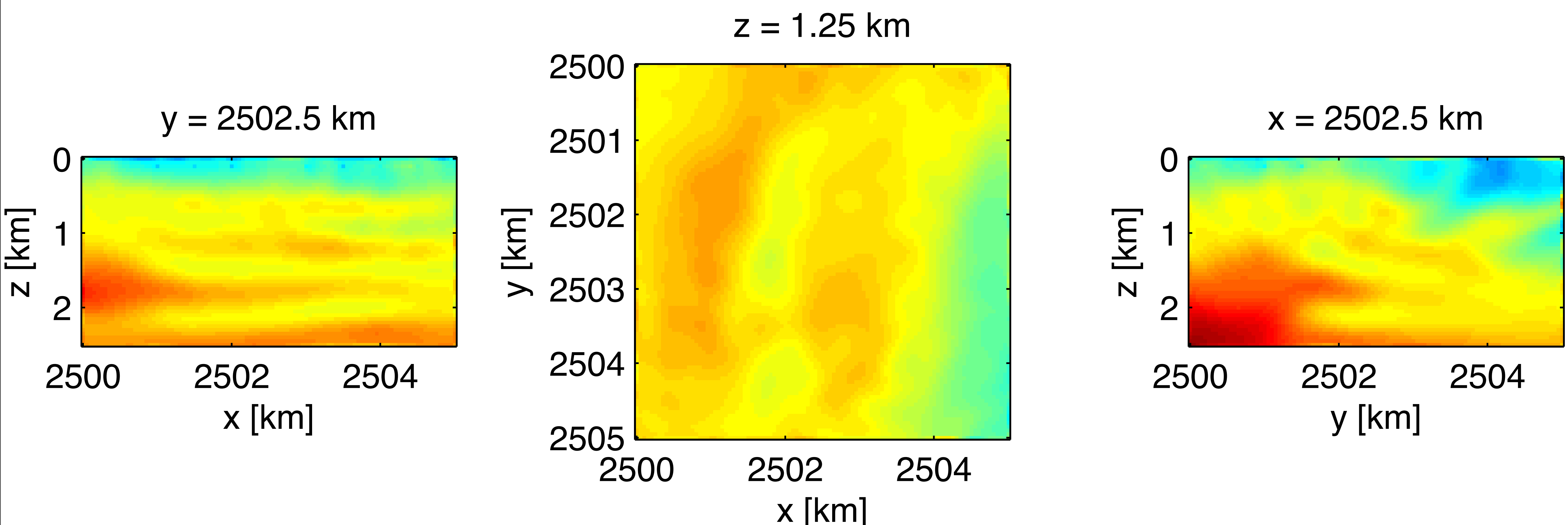
Overthrust model

$$b_0 = 1, \beta = 0, \epsilon_0 = 10^{-3}, \epsilon_{\min} = 10^{-6}, \alpha = 1, n_{\text{iter}} = 100$$



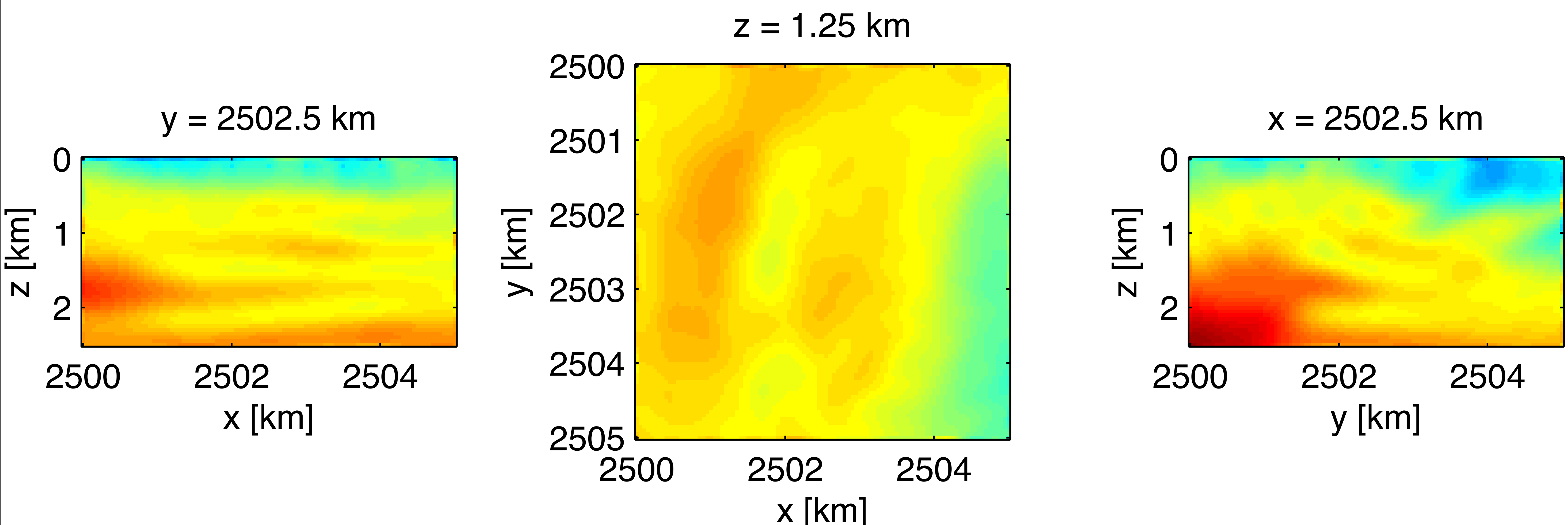
Overthrust model

$$b_0 = 5, \beta = 0, \epsilon_0 = 10^{-3}, \epsilon_{\min} = 10^{-6}, \alpha = 1, n_{\text{iter}} = 20$$

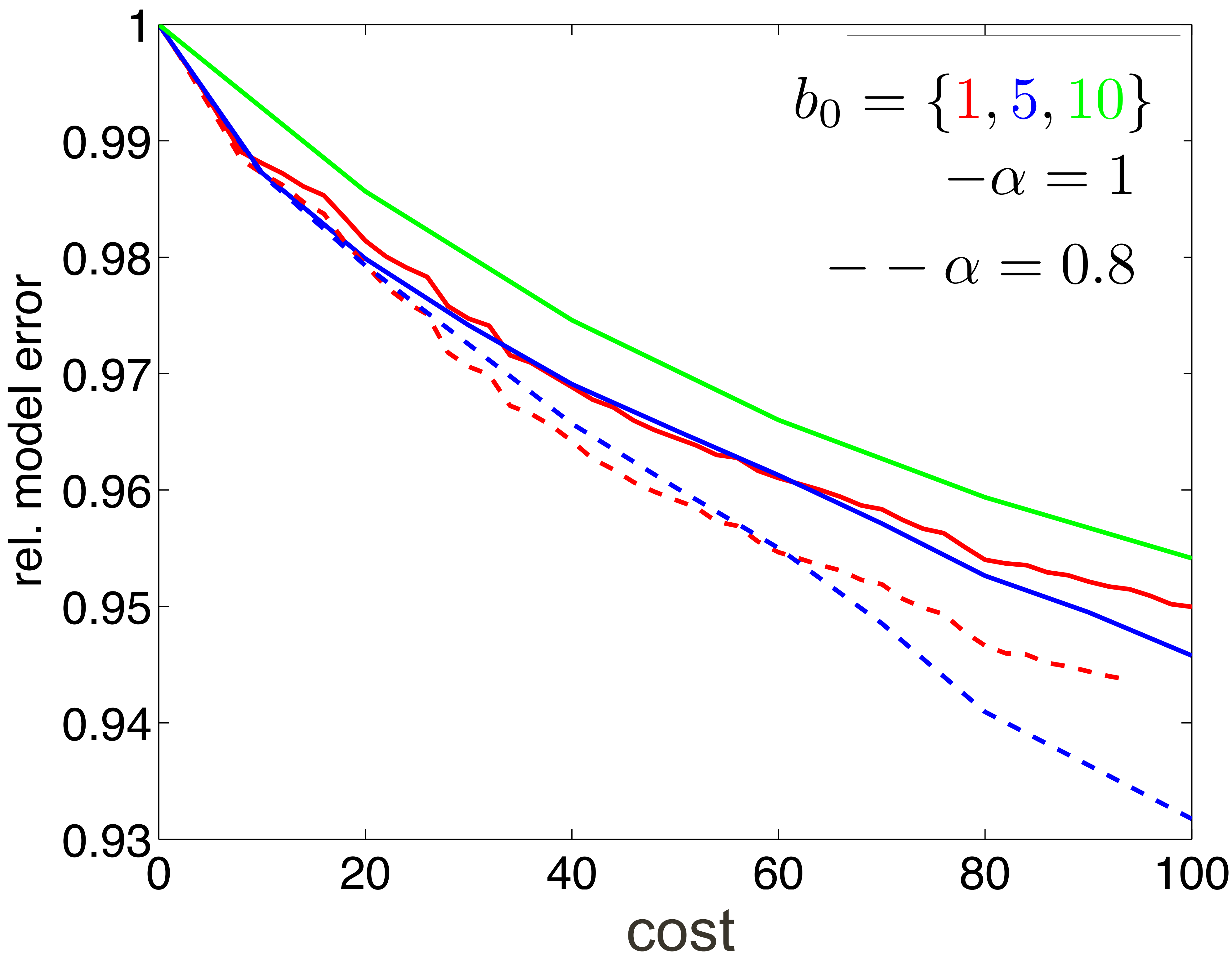


Overthrust model

$$b_0 = 10, \beta = 0, \epsilon_0 = 10^{-3}, \epsilon_{\min} = 10^{-6}, \alpha = 1, n_{\text{iter}} = 10$$



Overthrust model



Summary

- Generic preconditioner for Helmholtz with controllable accuracy
- efficient optimization strategy that can exploit inaccurate gradients
- good results by working on small (random) subsets of the data

- further computational aspects discussed by Art Petrenko (now) and me (tomorrow)
- Extension to elastic discussed by Bas Peters

SLIM

Acknowledgements

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Further reading

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