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An introduction to **cosparse** signal **reconstruction**

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Analysis vs Synthesis

(assume **x** is sparse)

$$\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

"Traditional" basis-pursuit compressive sensing with sparse signal

Analysis vs Synthesis

$\mathbf{x} = \mathbf{D}\mathbf{z}$ (assume **x** is not sparse, but **z** is)

$$\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

“Traditional” basis-pursuit compressive sensing with sparse signal

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$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \, ||\mathbf{z}||_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

"Synthesis"-based sparse signal reconstruction

Analysis vs Synthesis

(assume **x** is not sparse, but $\mathbf{D}^\dagger \mathbf{x}$ is)

$$\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \, ||\mathbf{z}||_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{D}\mathbf{z}$$

"Synthesis"-based sparse signal reconstruction

Analysis vs Synthesis

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$$\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

"Analysis"-based sparse signal reconstruction

Equivalence?

Synthesis $\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{ADz}$

"Synthesizes" the signal using sparse sets of columns of **D**

Analysis $\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{Ax}$

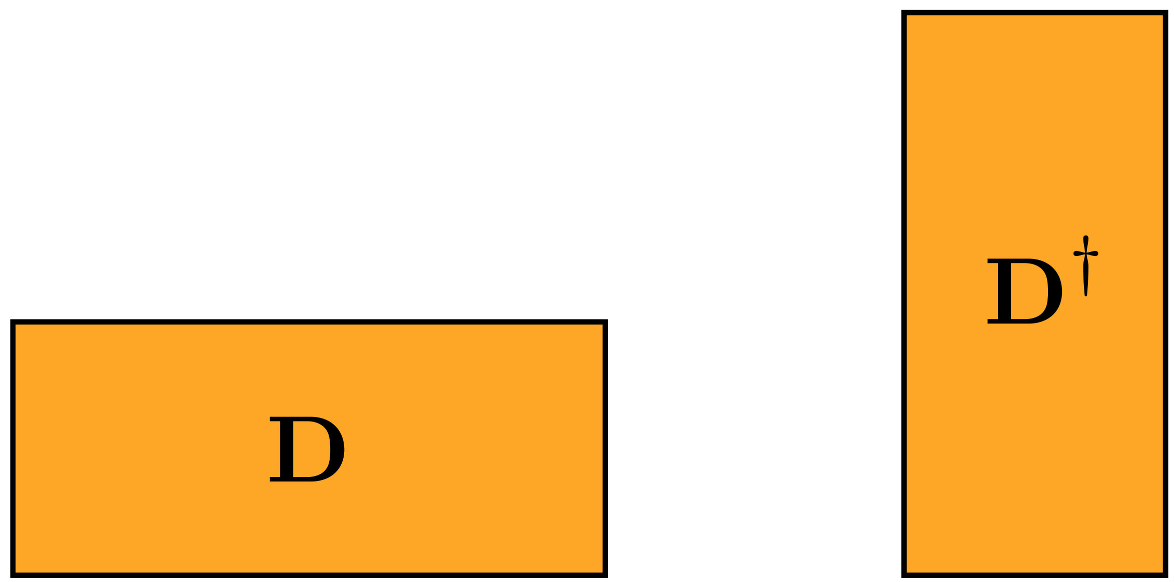
"Analyses" the sparsity of the signal under an operator

Equivalence?

$\mathbf{D}$ $\mathbf{D}^{-1}$

If **D** is square and invertible, then synthesis = analysis

Equivalence?



If **D** is “flat” and redundant, then *not equal*

Equivalence?

- Many ways to choose

$$\mathbf{z} \text{ s.t. } \mathbf{x} = \mathbf{D}\mathbf{z}$$

- But there is only one

$$\mathbf{D}^{\dagger}\mathbf{x}$$

Equivalence?

sparsity in $\mathbf{D}^{\dagger}\mathbf{x}$ means that $\mathbf{x}$ is orthogonal to most rows of $\mathbf{D}^{\dagger}$

Equivalence?

Synthesis $\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{ADz}$

Analysis $\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{z}\|_1 \text{ subject to } \mathbf{y} = \mathbf{Ax}$

Equivalence

Synthesis* $\tilde{\mathbf{x}} = \mathbf{D} \cdot \underset{\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1$ subject to $\mathbf{y} = \mathbf{ADz}$

$$\mathbf{z} = \mathbf{D}^\dagger \mathbf{D} \mathbf{z}$$

⇕

Analysis $\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{D}^\dagger \mathbf{z}\|_1$ subject to $\mathbf{y} = \mathbf{Ax}$

Analysis-sparsity is a **stronger** condition than Synthesis-sparsity

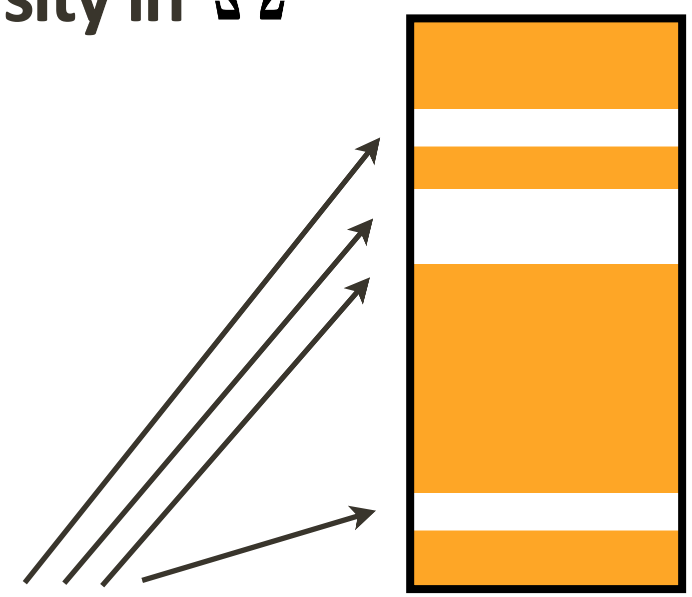
Caveats for L1

- Resulting Analysis *coefficients* generally *less* sparse than Synthesis
- Says very little about actual recovery of the underlying signal
  (doesn't give tighter recovery bounds compared to synthesis for most operators)

S. Nam et al., The cosparse analysis model and algorithms, 2012
Candes et al., Compressed sensing with coherent and redundant dictionaries, 2011

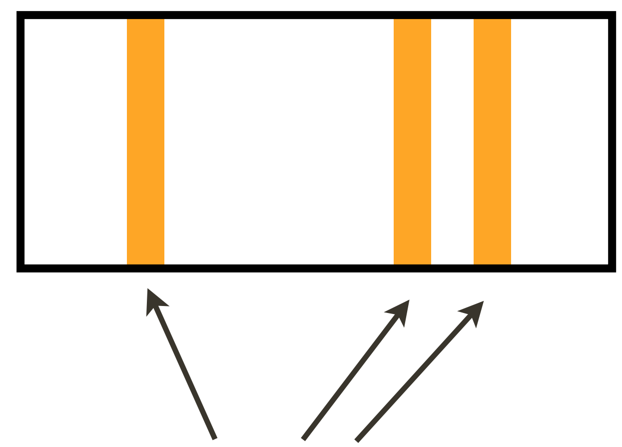
Cosparsity

Cosparsity in Ω



Constrains signal to be orthogonal to some number of rows

Sparsity in $\mathbf{D}$

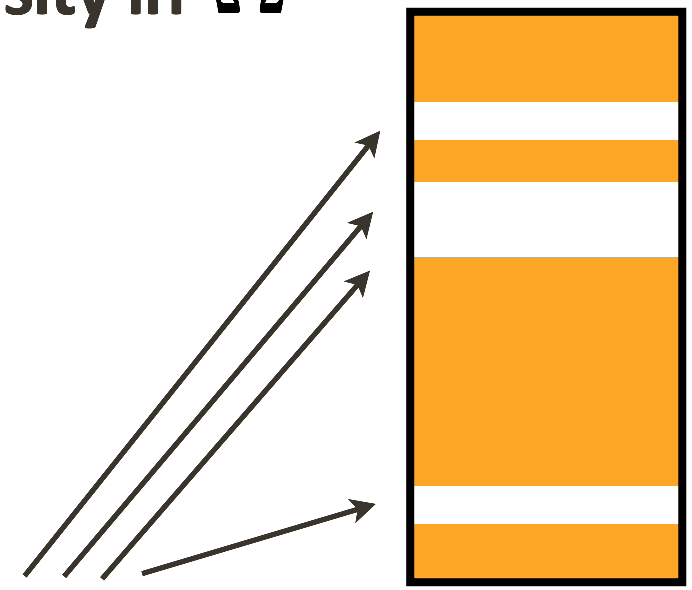


Constrains signal to lie on the support of a few number of columns

S. Nam et al., The cosparse analysis model and algorithms, 2012

Cosparsity

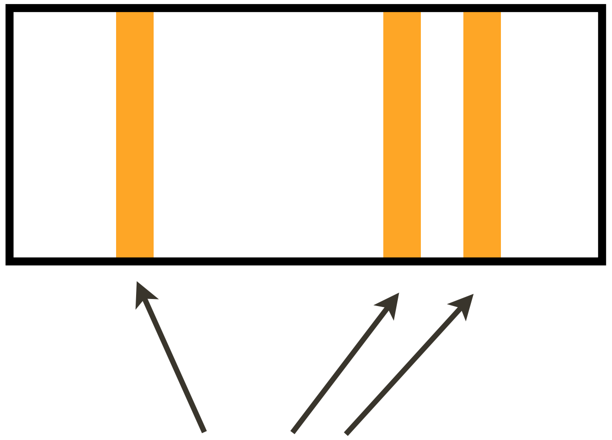
Cosparsity in $\mathbf{\Omega}$



Constrains signal to be orthogonal to some number of rows

cosparsity ℓ

Sparsity in $\mathbf{D}$



Constrains signal to lie on the support of a few number of columns

sparsity k

Constraining signal

Cosparsity +1 :

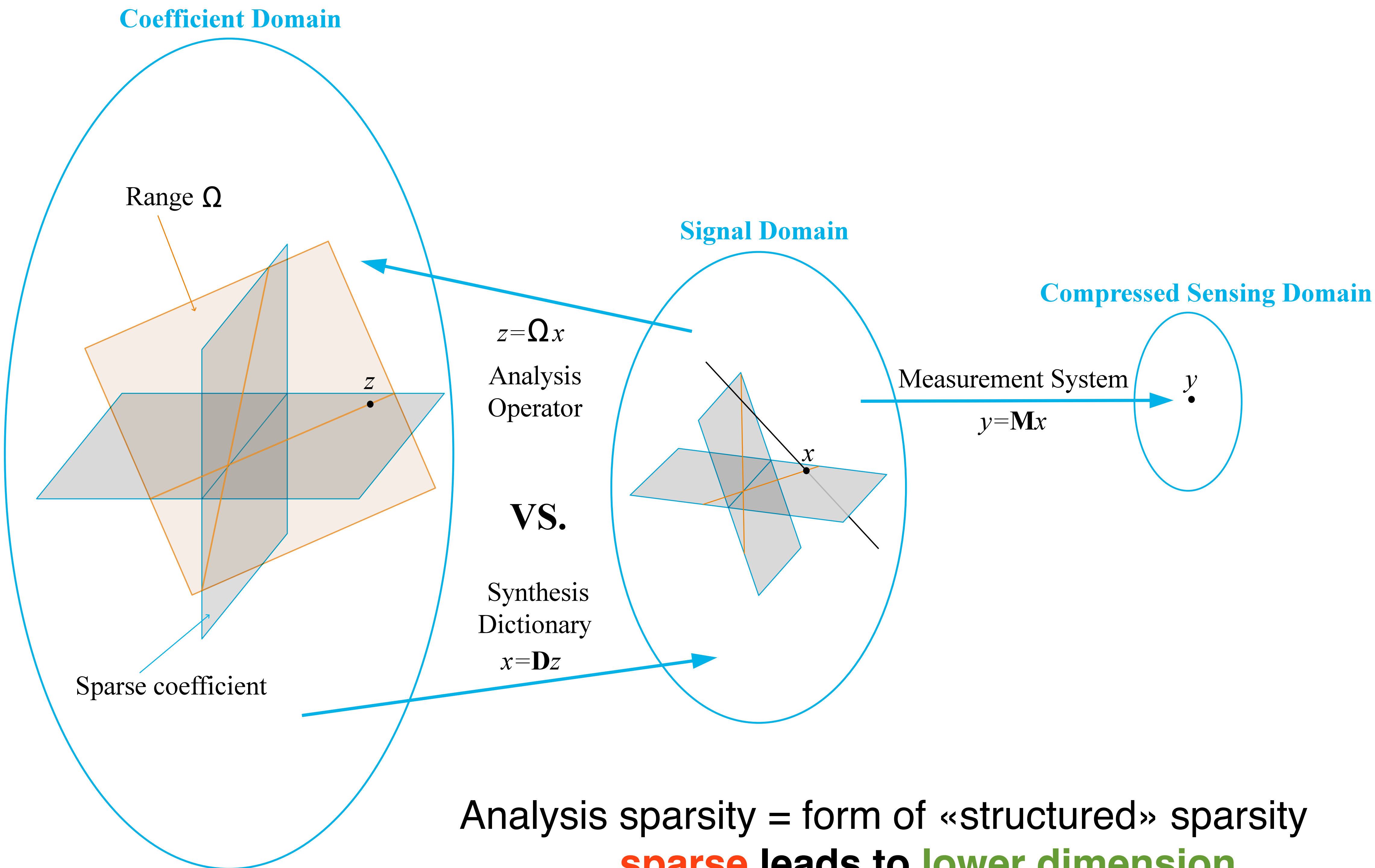
Coefficient subspace dimension -1

Signal orthogonal to +1 more row of Ω

Sparsity -1 :

Coefficient subspace dimension -1

Cosparsity can be much lower than $n-k$ to constrain signal "just as much as" specifying sparsity level k



Analysis sparsity = form of «structured» sparsity

sparse leads to lower dimension

from S. Nam et al., The cosparse analysis model and algorithms, 2012

SLIM

Algorithm implication

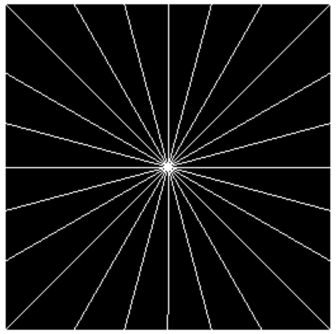
Goal is to pick out as many rows of $\mathbf{\Omega}$ that the signal is orthogonal to as possible

e.g., *Greedy Analysis Pursuit* (GAP)

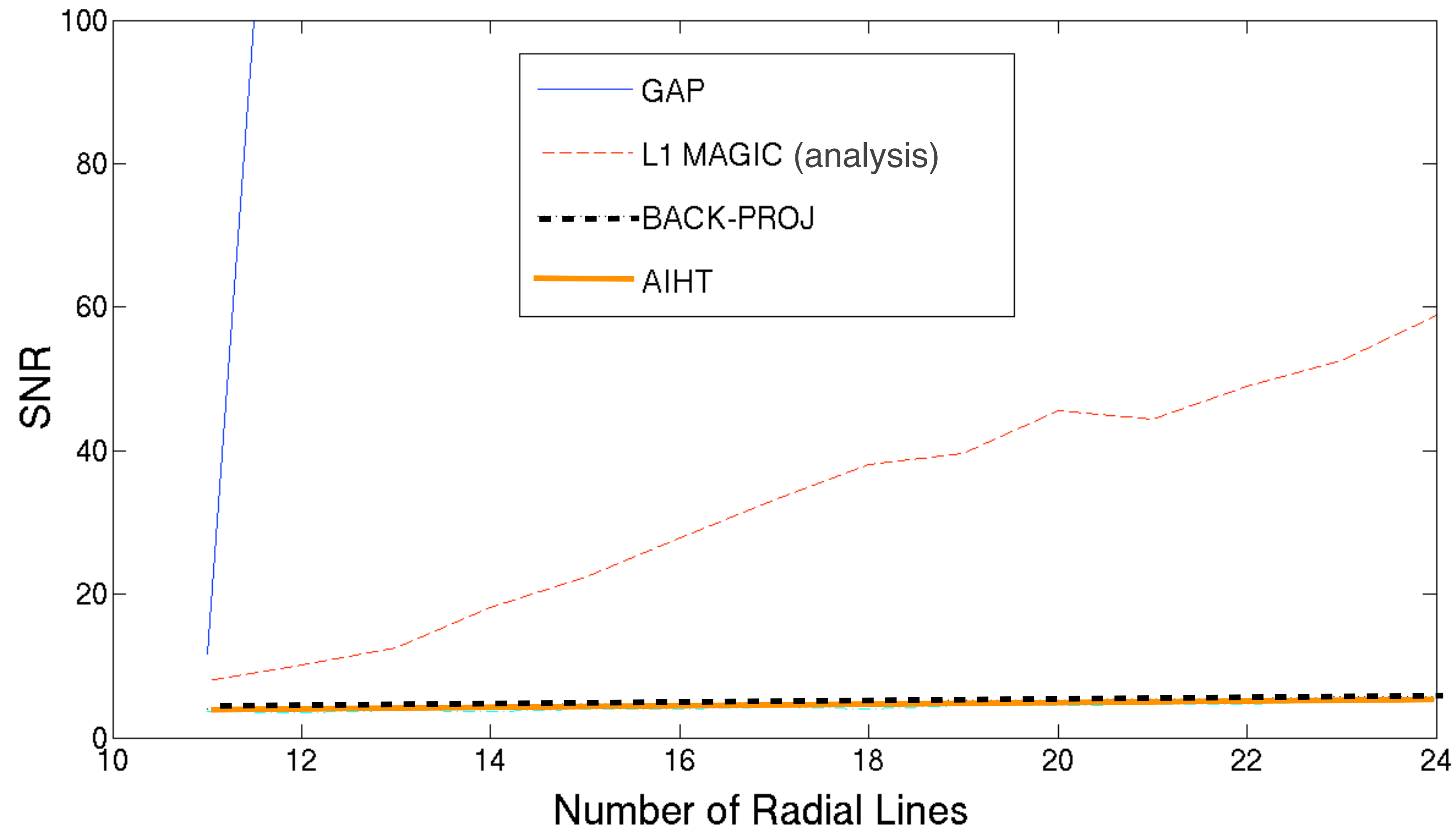
S. Nam et al., The cosparse analysis model and algorithms, 2012

GAP Performance: MRI

Observe radial lines in spatial frequency domain



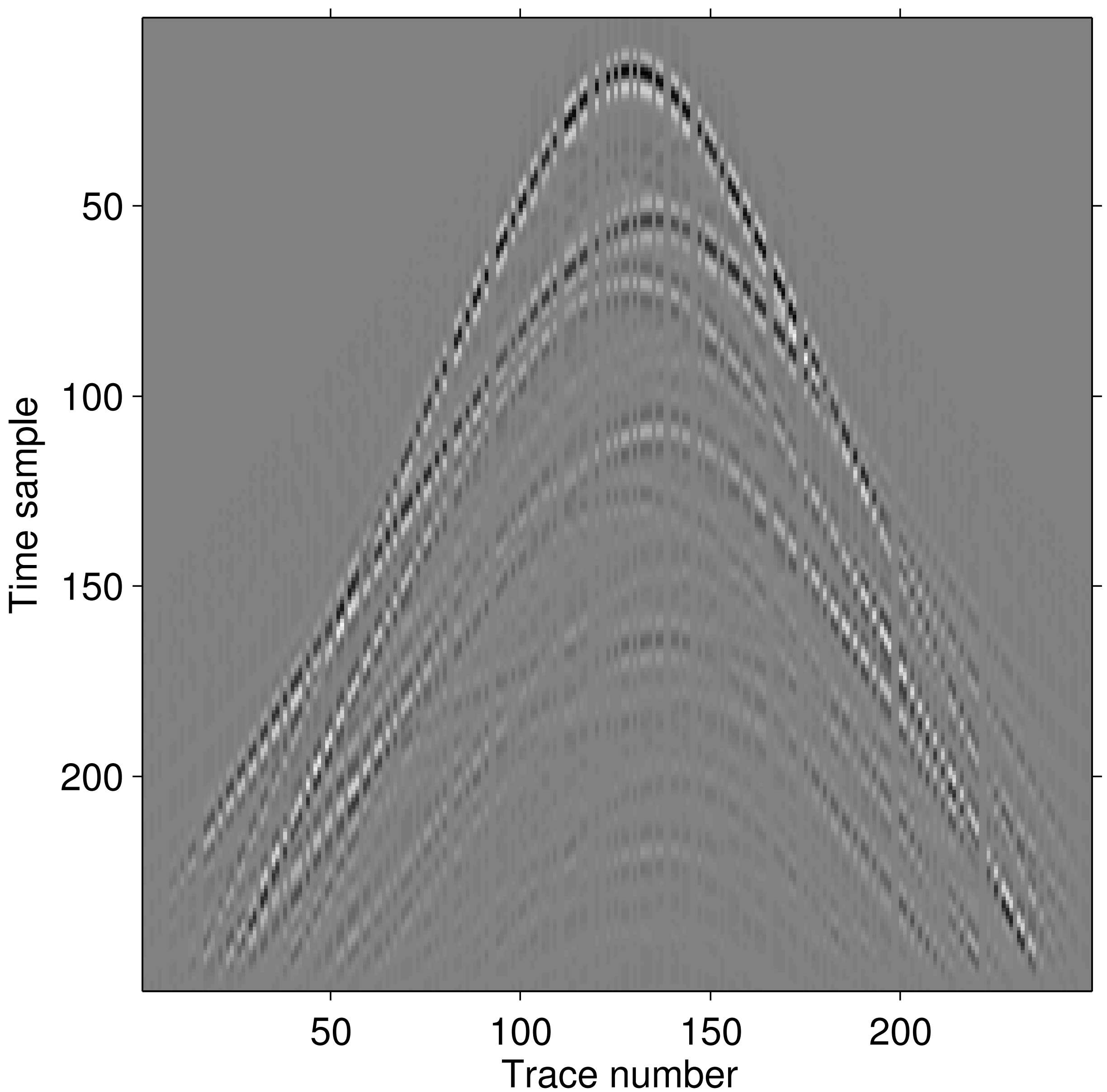
Recover piecewise-constant image using "TV"



from S. Nam et al., The cosparse analysis model and algorithms, 2012

GAP Performance: CRSI

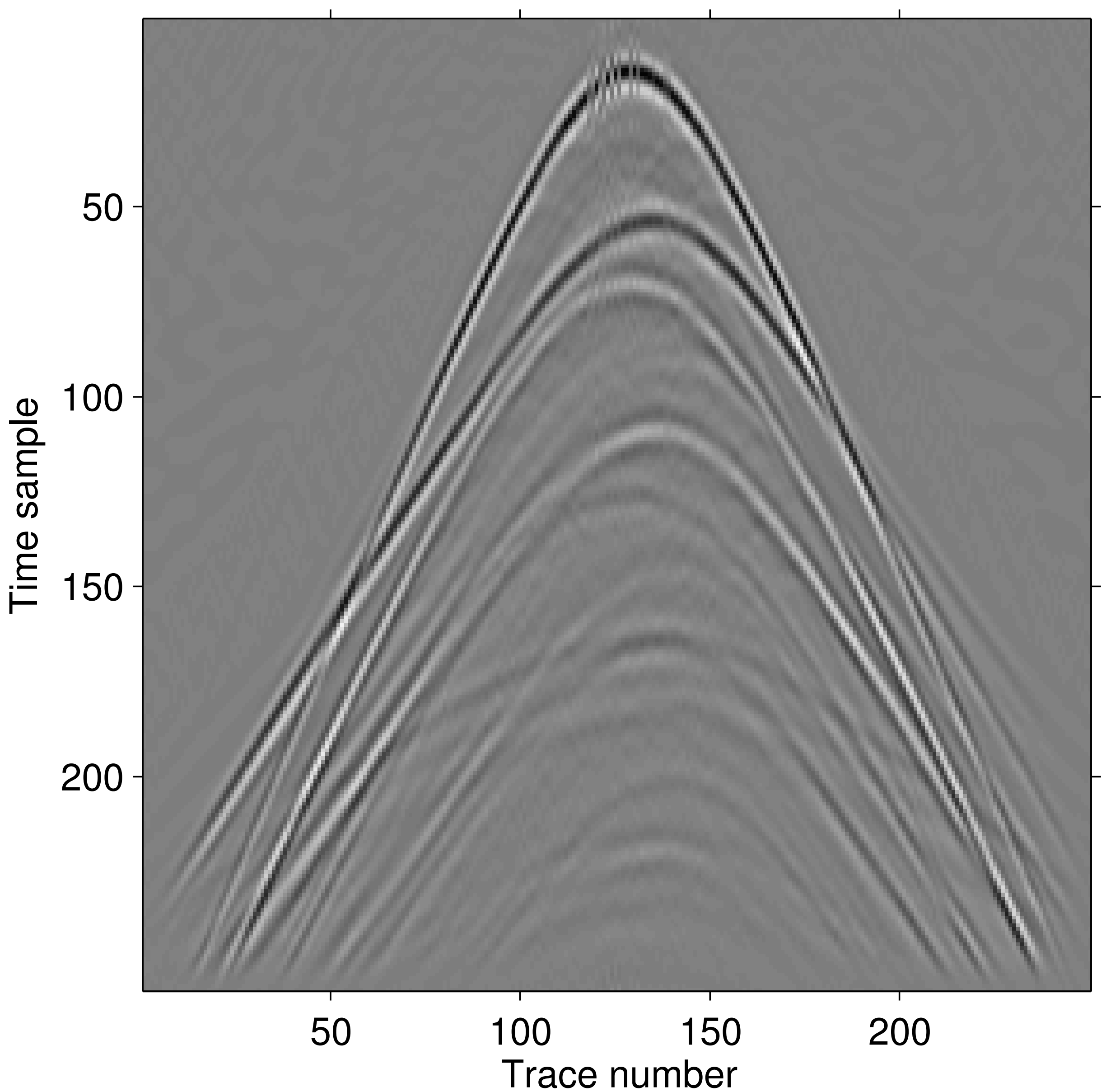
40% missing traces



data

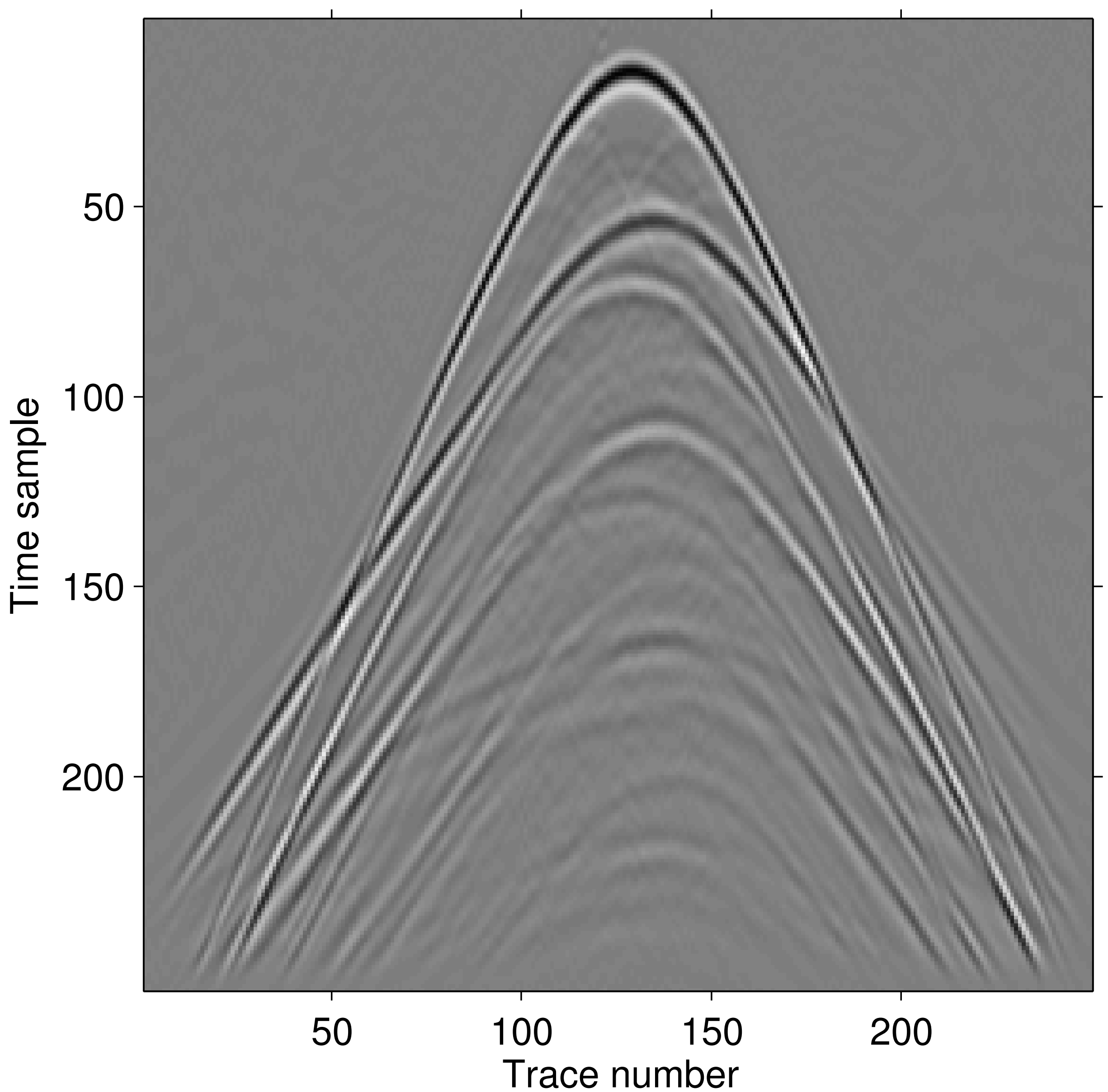
Recover complete shot-record in terms of Curvelet coefficients

GAP Performance: CRSI



L1-Analy (NESTA)

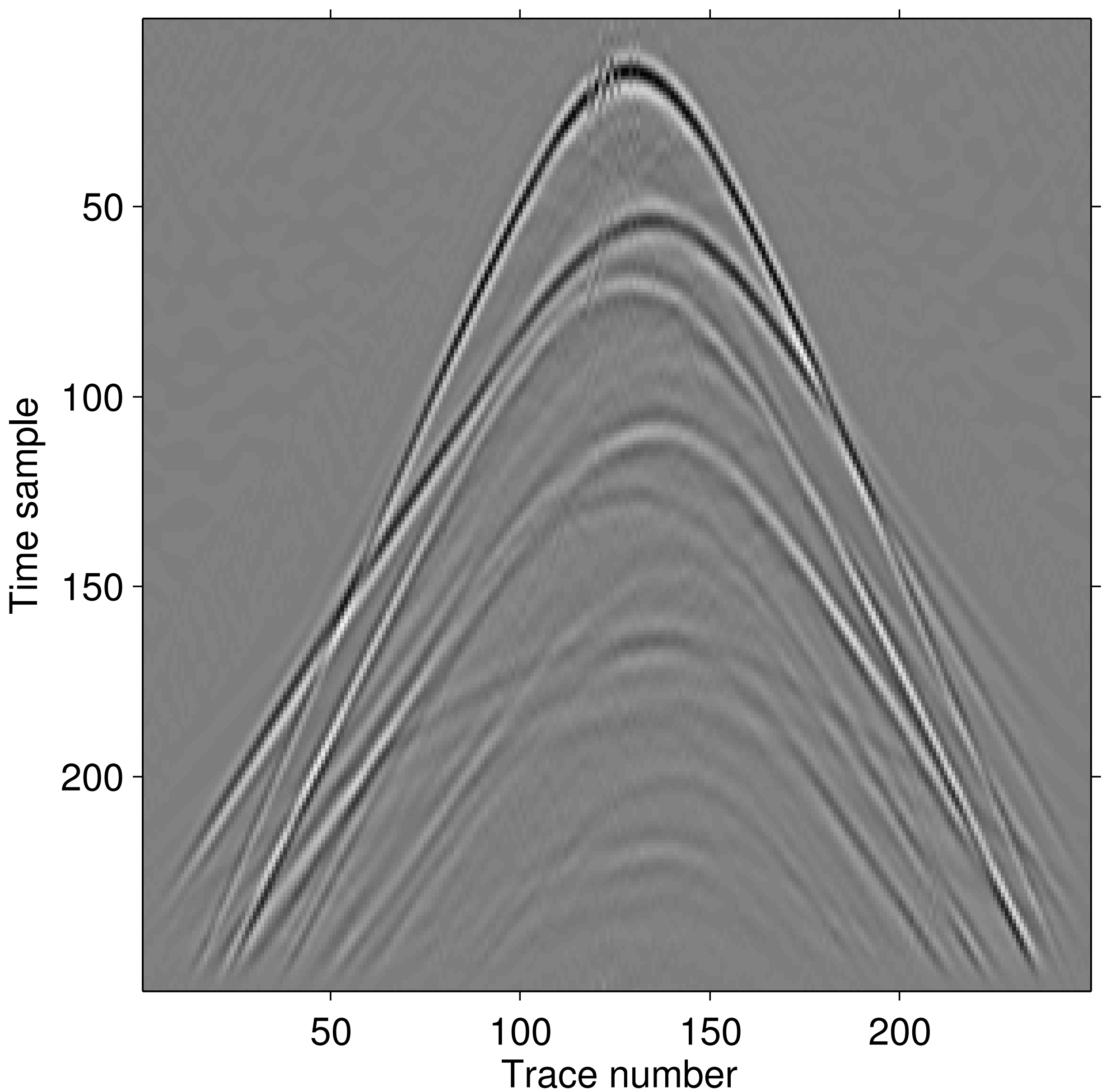
Rel error: 1.7E-01



GAP

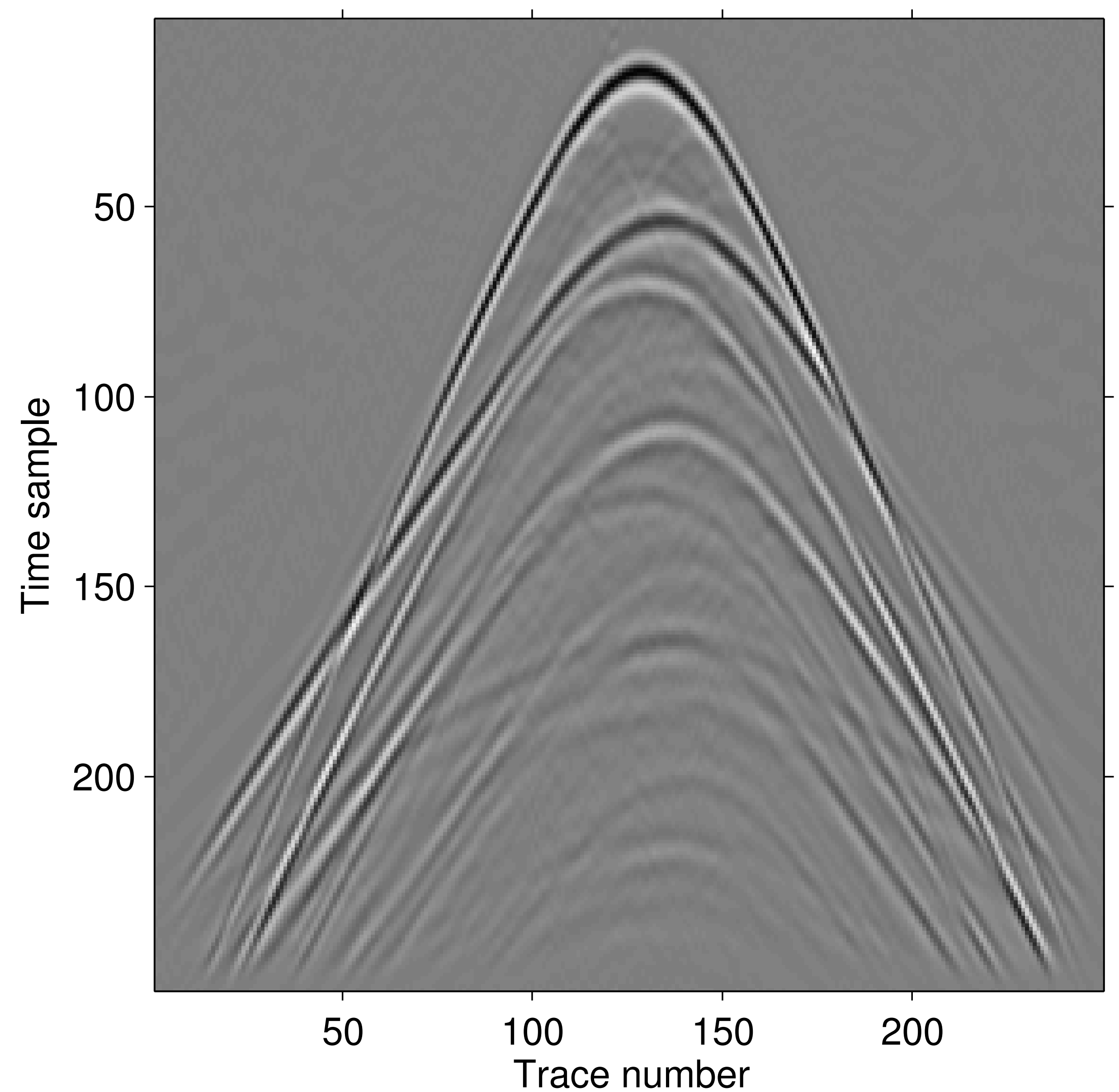
Rel error: 5.4E-02

GAP Performance: CRSI



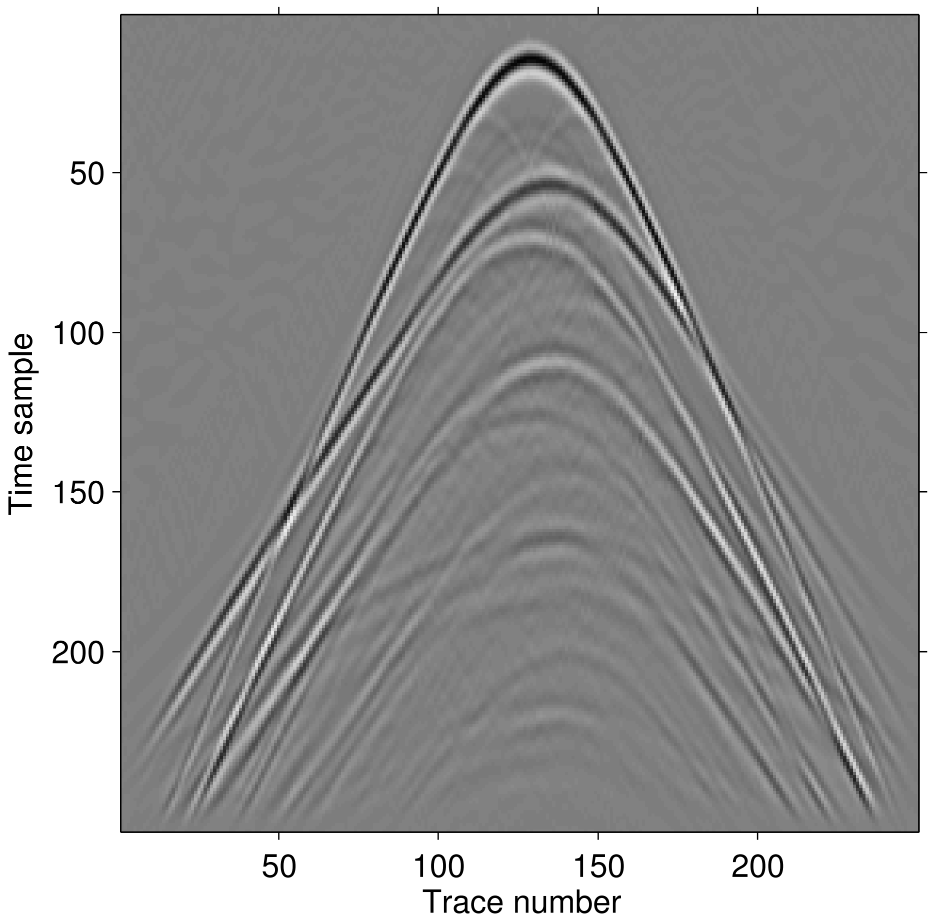
L1-Synth (NESTA)

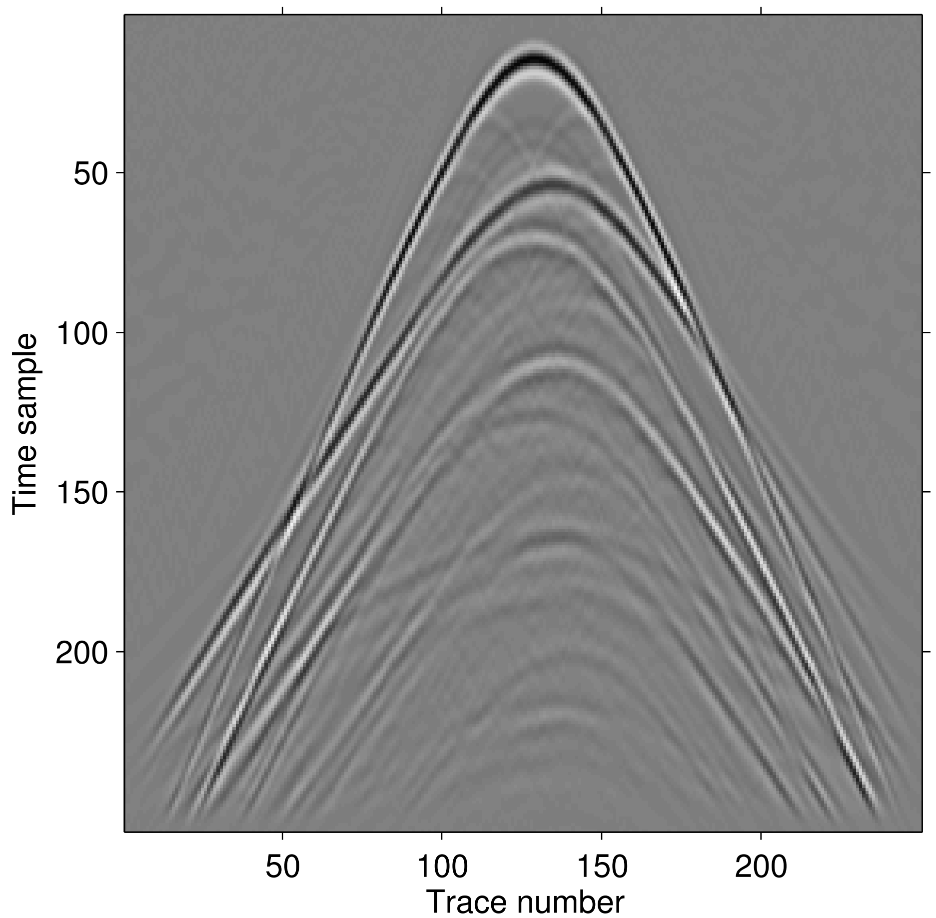
Rel error: 1.6E-01



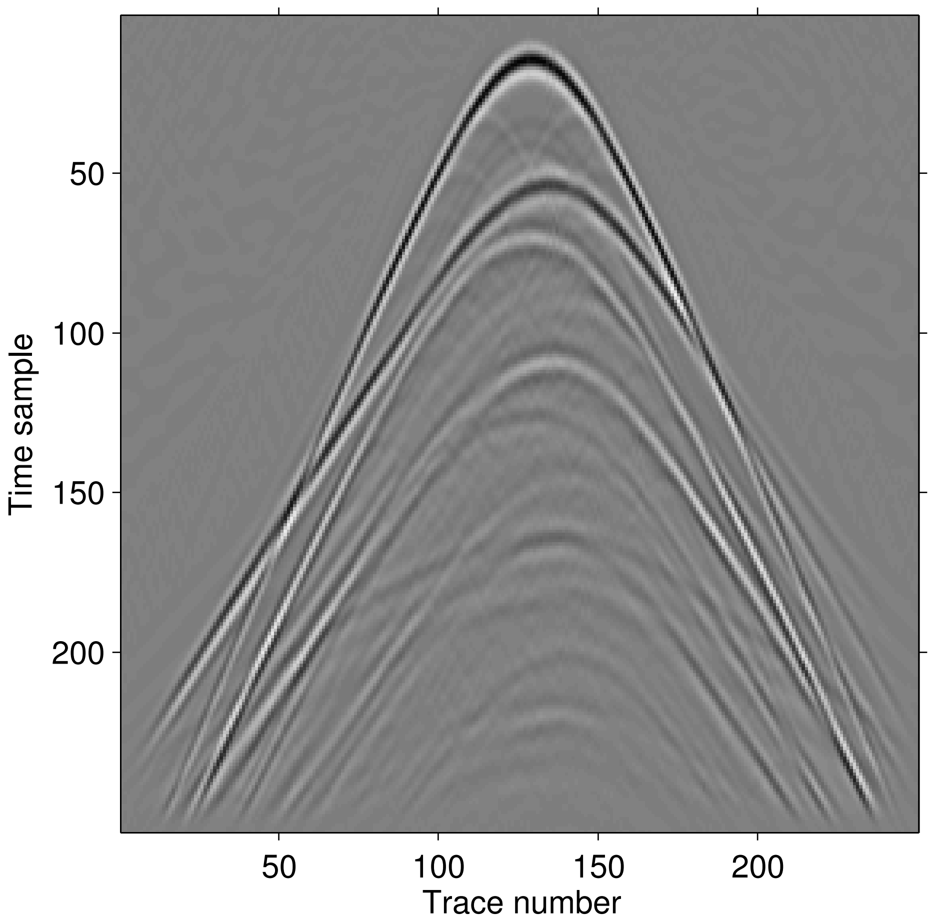
GAP

Rel error: 5.4E-02

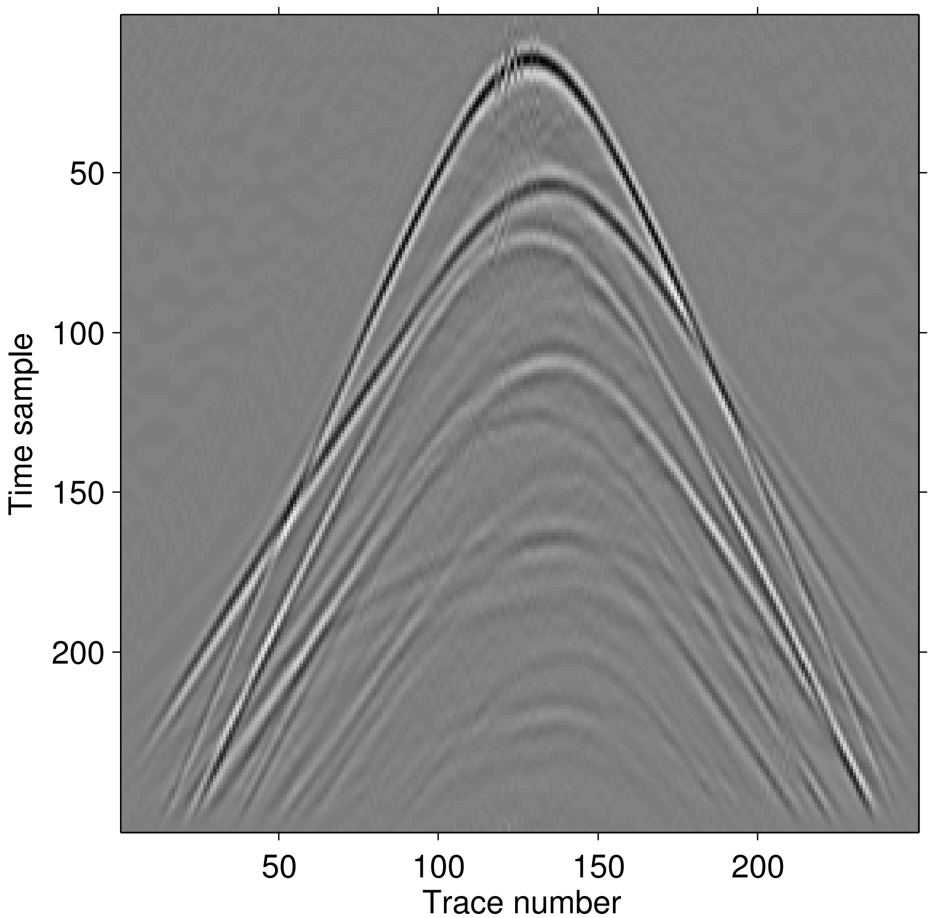


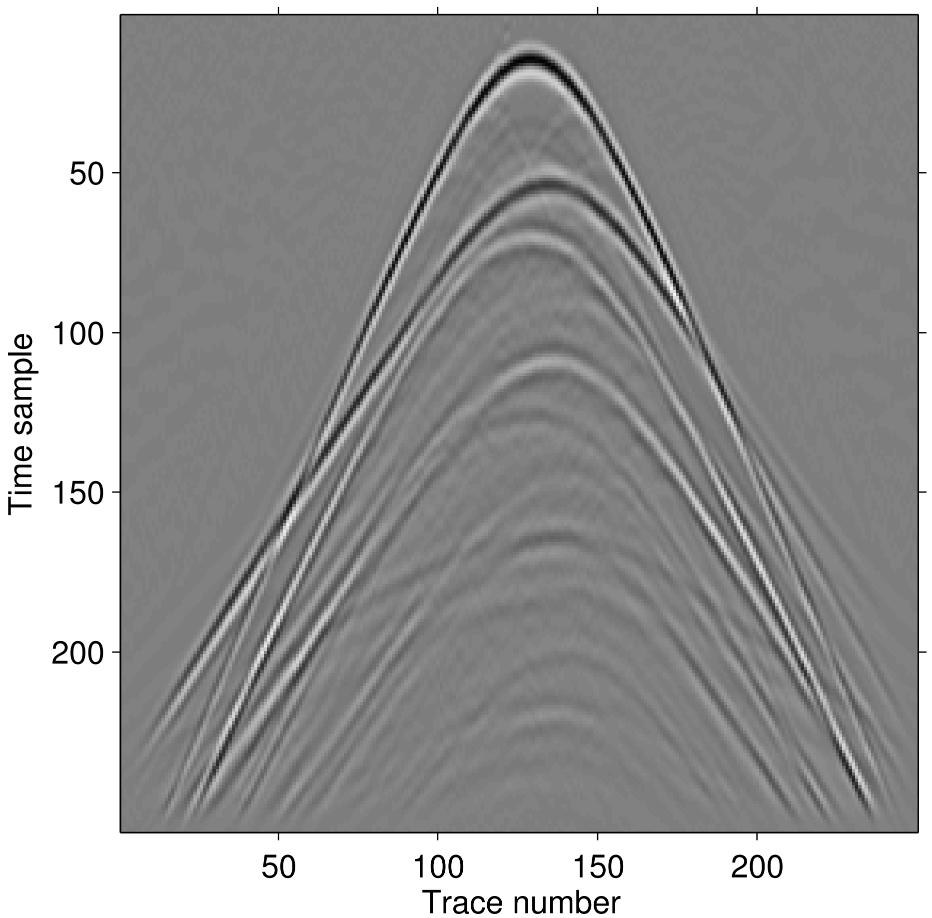


25% missing

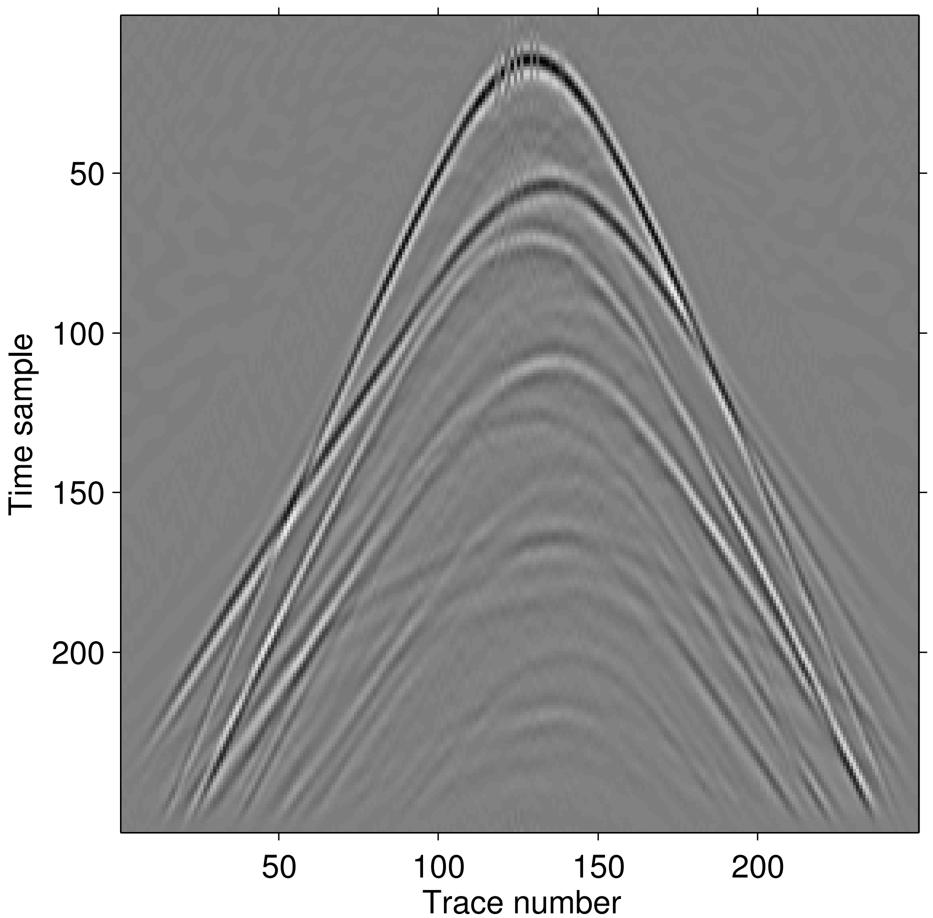


method	rel. model residual
L1-Synth	5.20E-02
L1-Analy	4.30E-02
GAP	2.60E-02



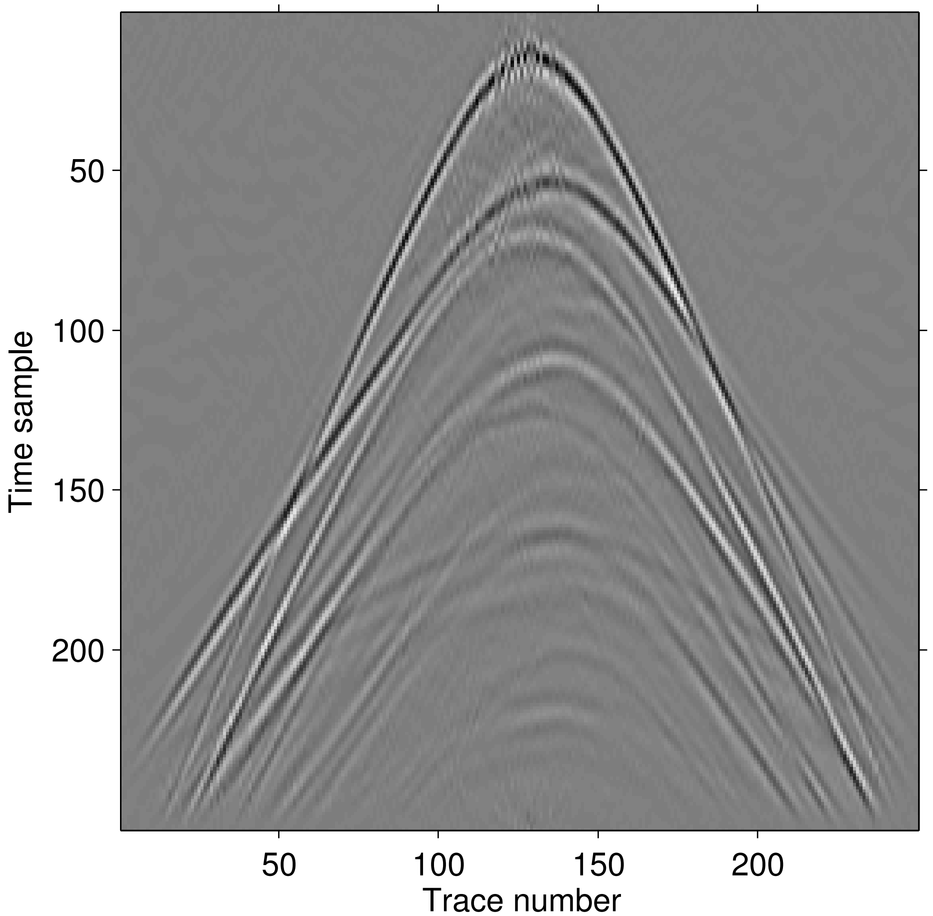


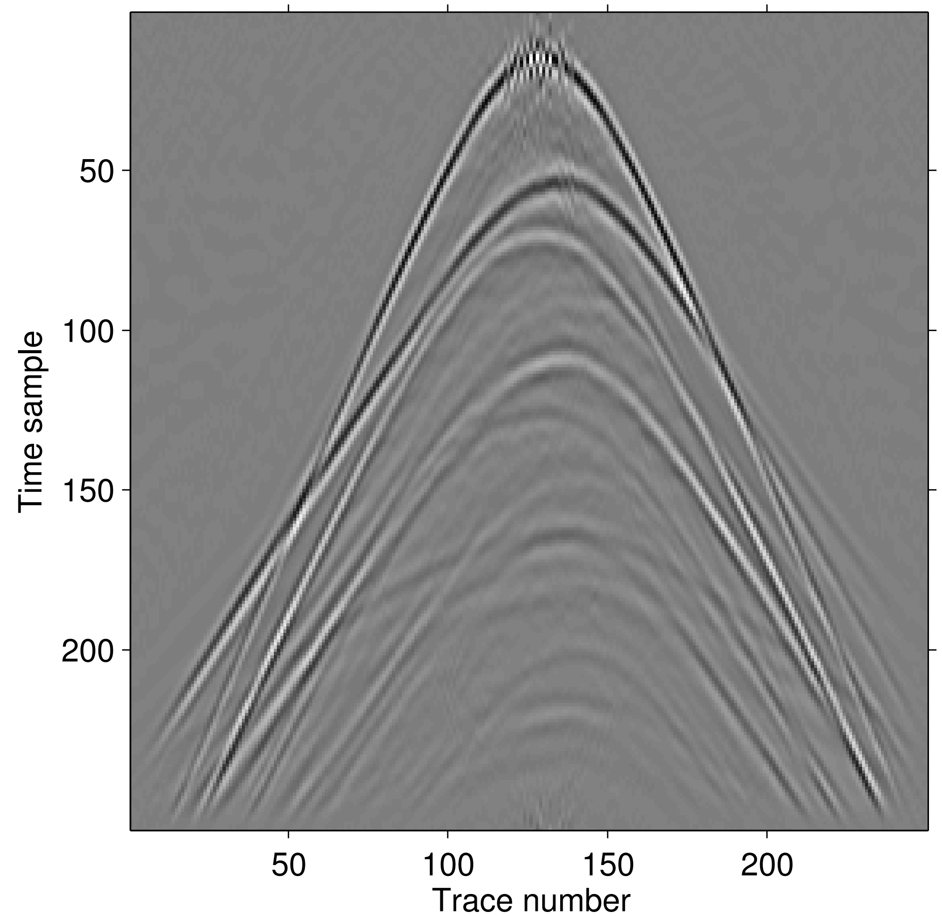
40% missing

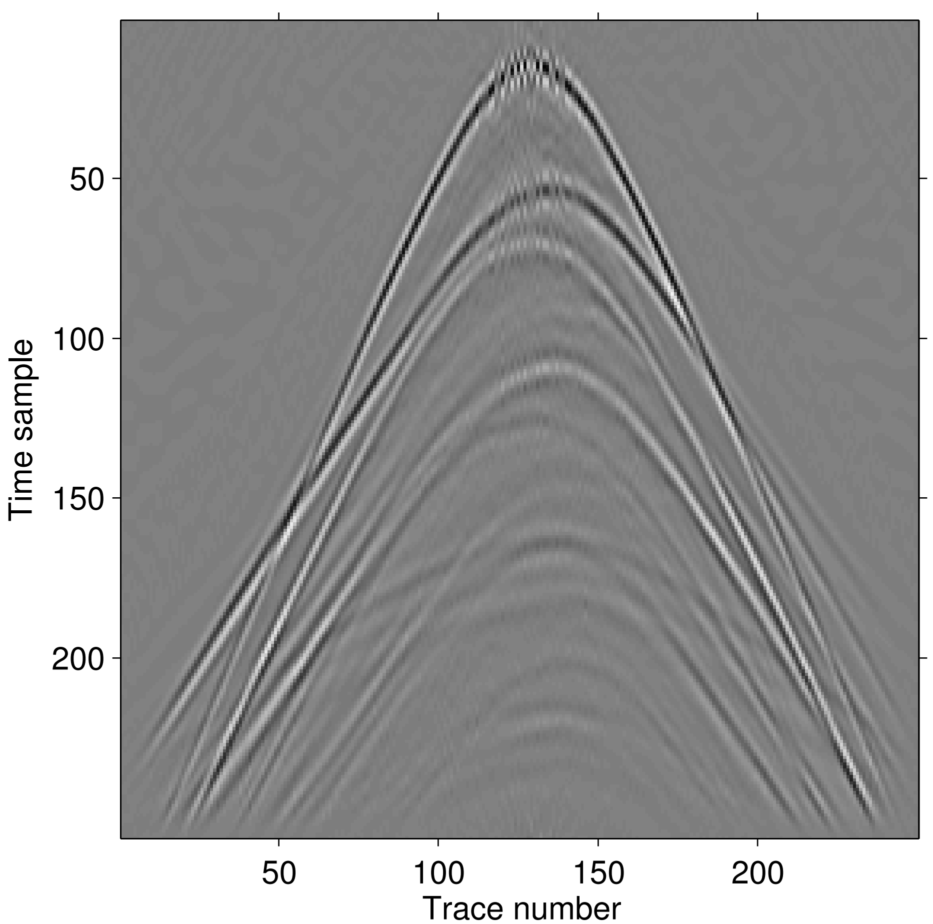


method	**rel. model residual**
L1-Synth	1.62E-01
L1-Analy	1.70E-01
GAP	5.40E-02

55% missing







method	rel. model residual
L1-Synth	3.00E-01
L1-Analy	3.20E-01
GAP	3.10E-01

GAP basic outline

Start with full index set of rows of $\Omega \in \mathbb{C}^{n \times d}$

$$\Lambda = \{1, 2, 3, \ldots, n\}$$

1. **Projection:** compute $\alpha = \Omega \mathbf{x}_k$
2. **Find largest** element(s) of α
3. **Remove** the corresponding row(s) from Λ
4. **Update solution** estimate

$$\tilde{\mathbf{x}}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\Omega_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

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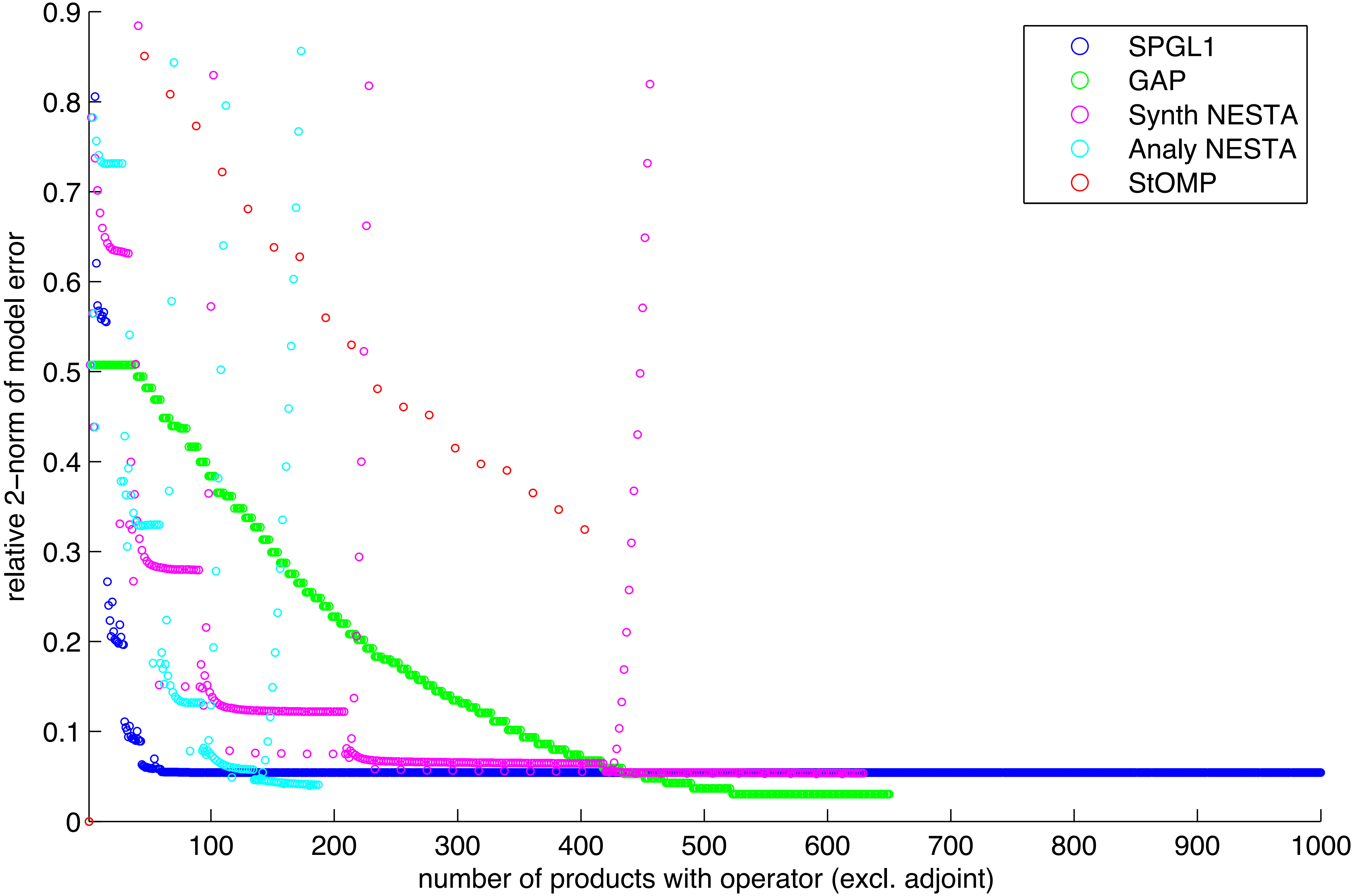
Stop at convergence of $\Delta \mathbf{x}$, or small $\|\Omega_\Lambda \mathbf{x}\|_\infty$

Model error vs computation cost
25% missing traces



Model error vs computation cost
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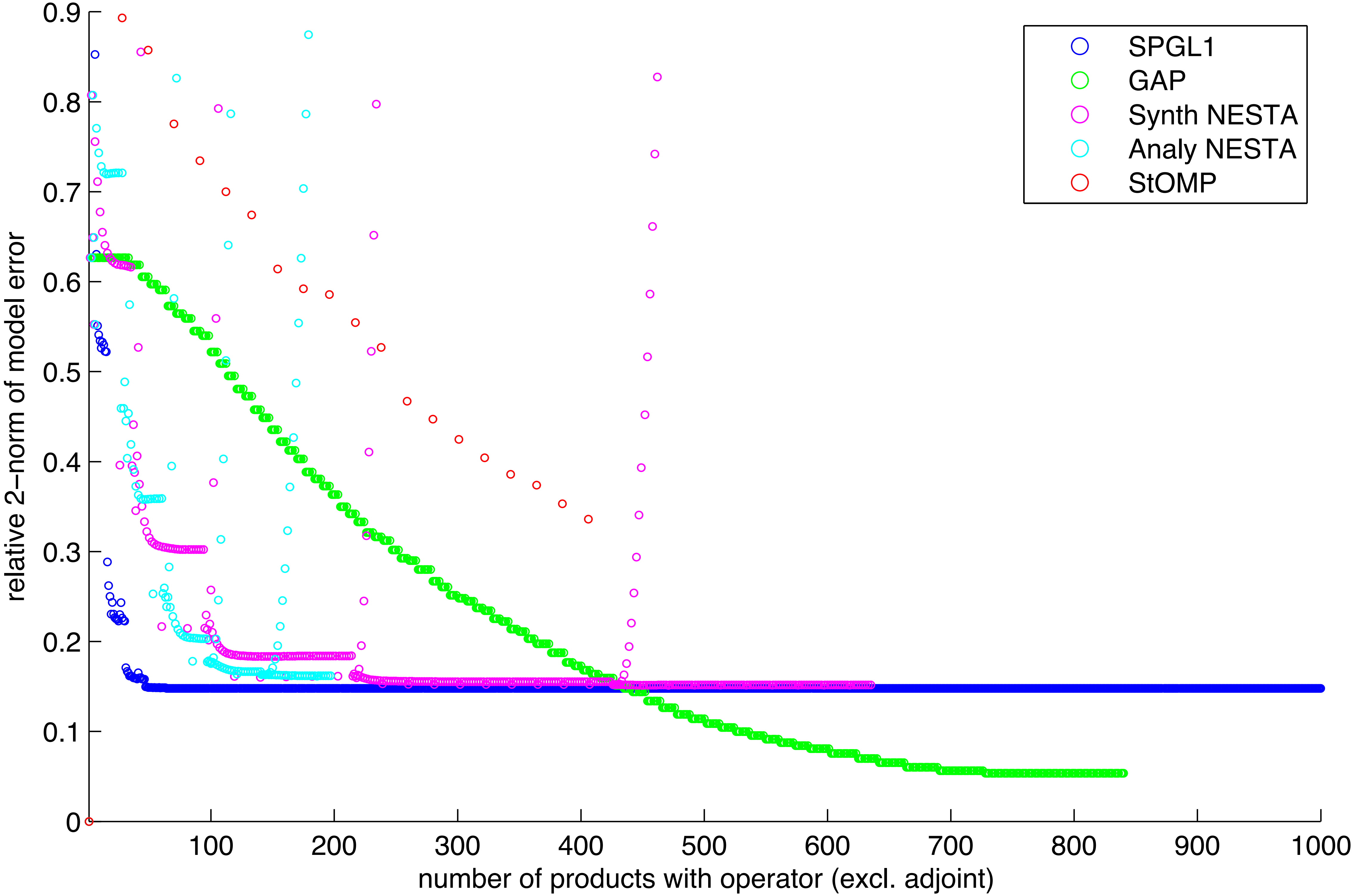

Model error vs computation cost
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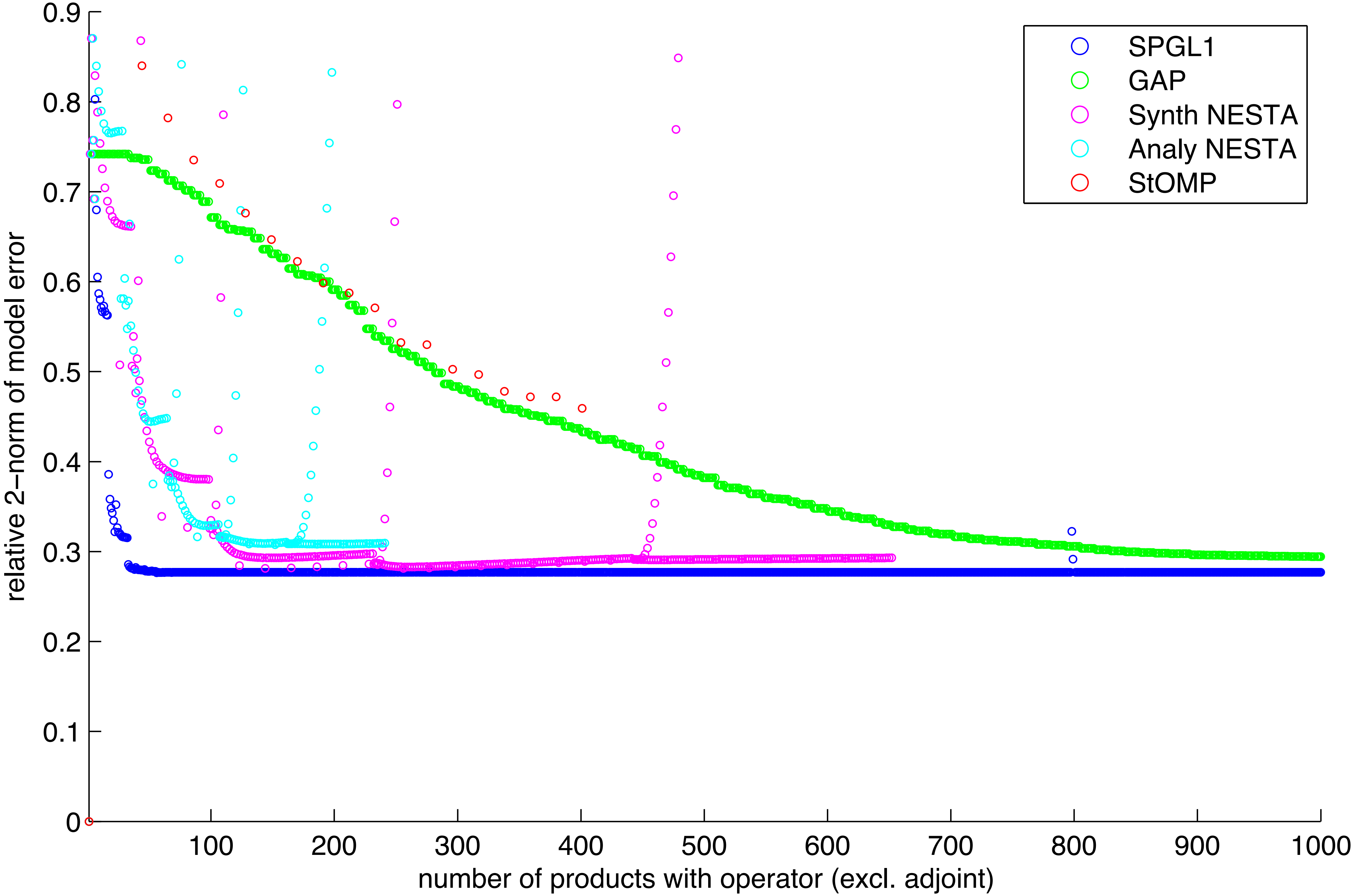
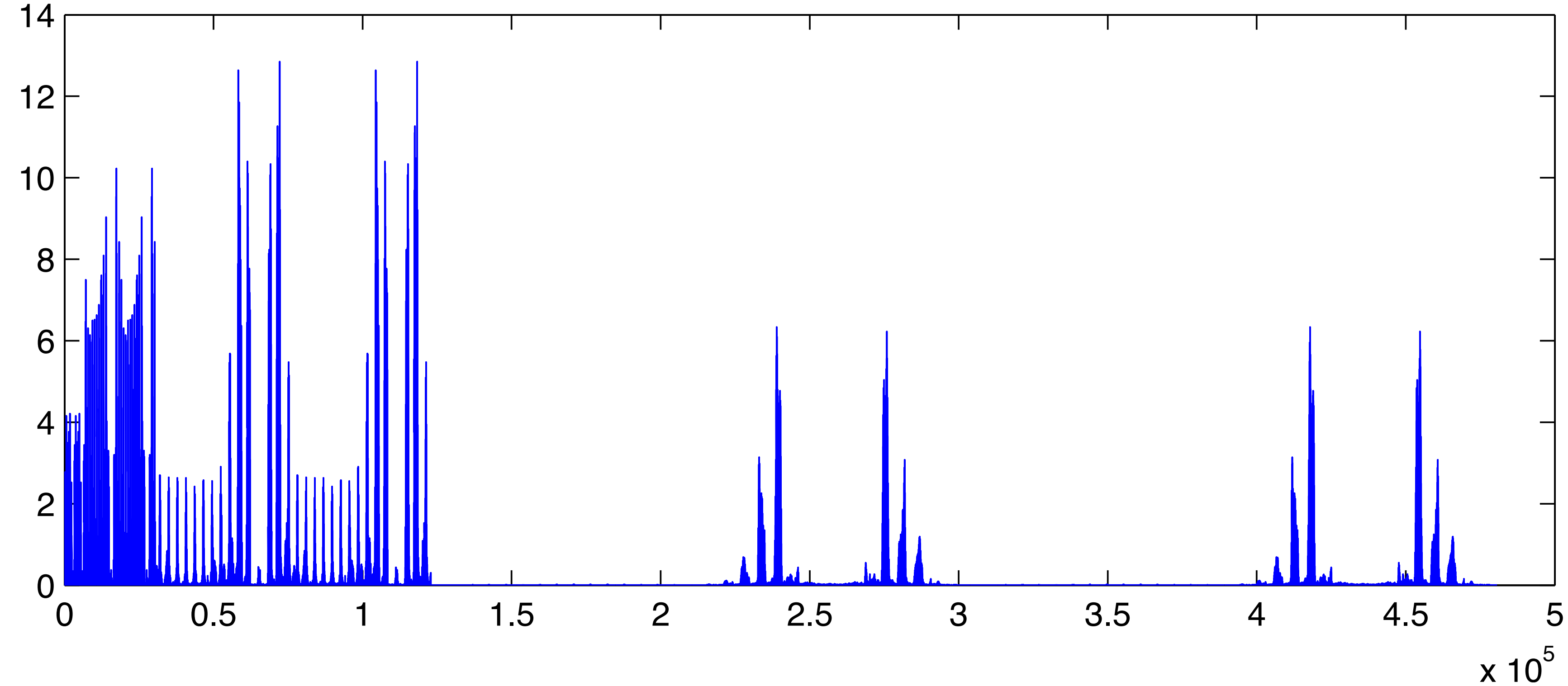


GAP does not detect "support"

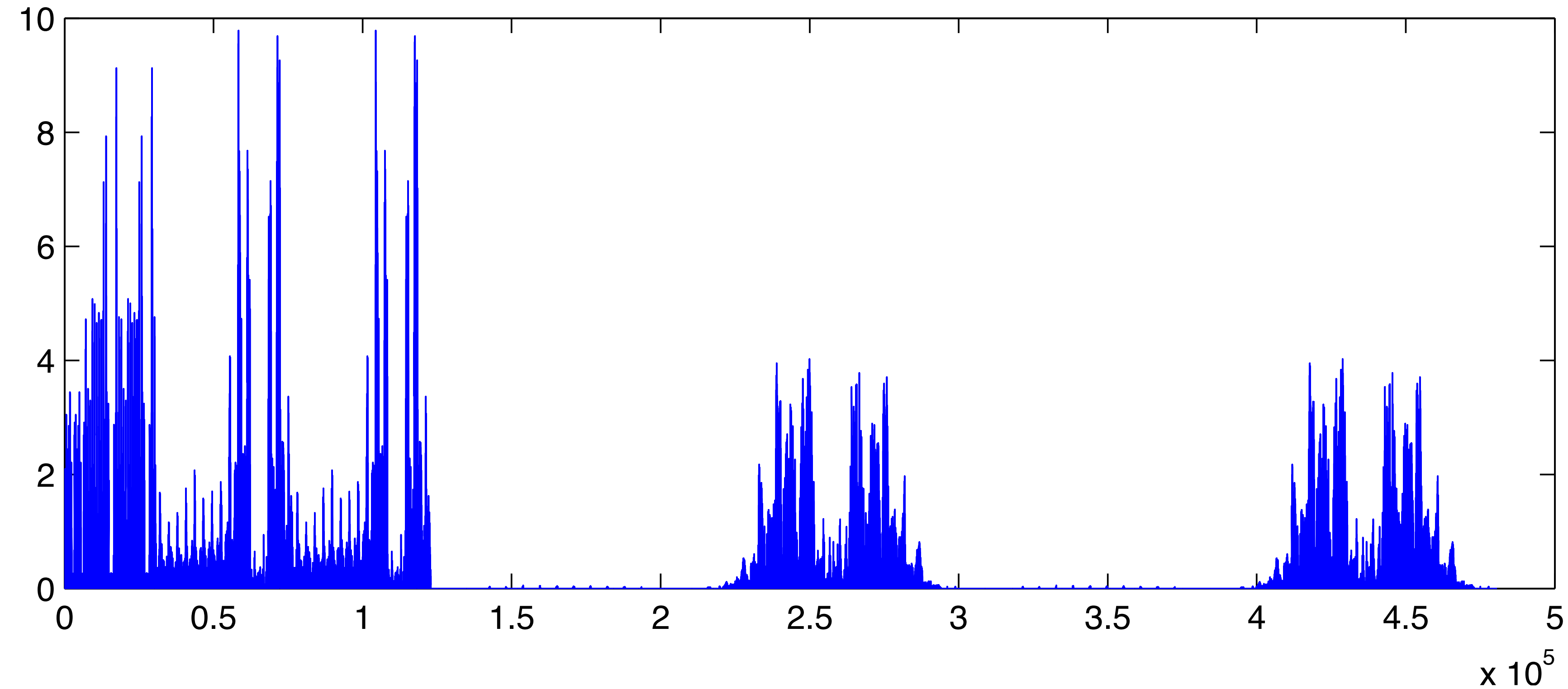
Out of 480617 coefficients:

GAP kept 150951 **(31.4%)**
L1-Synthesis kept 49519 **(10.3%)**

(signal size > 0.5% of largest)

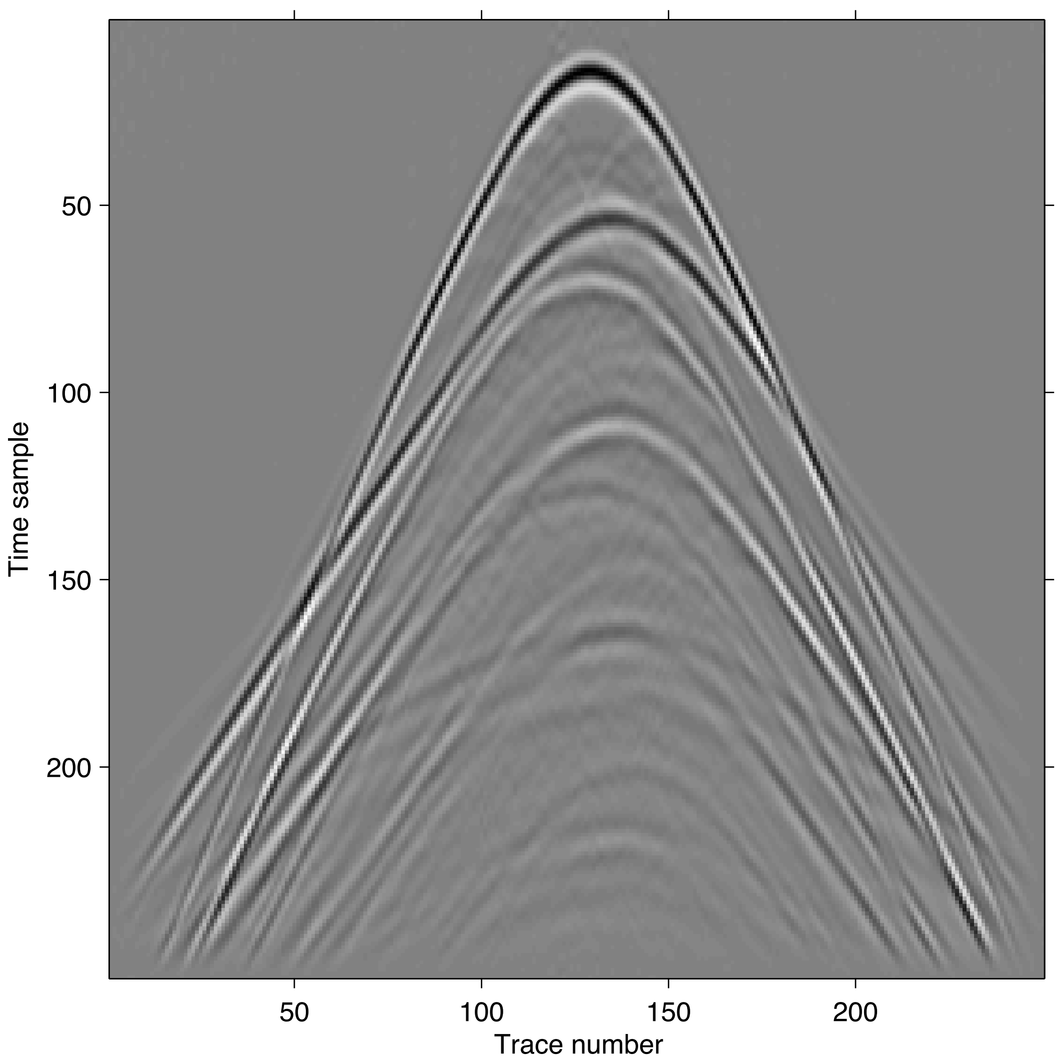


$$\min_{\mathbf{x}} \|\mathbf{C}_\Lambda \mathbf{x}\|_2 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$ GAP solution

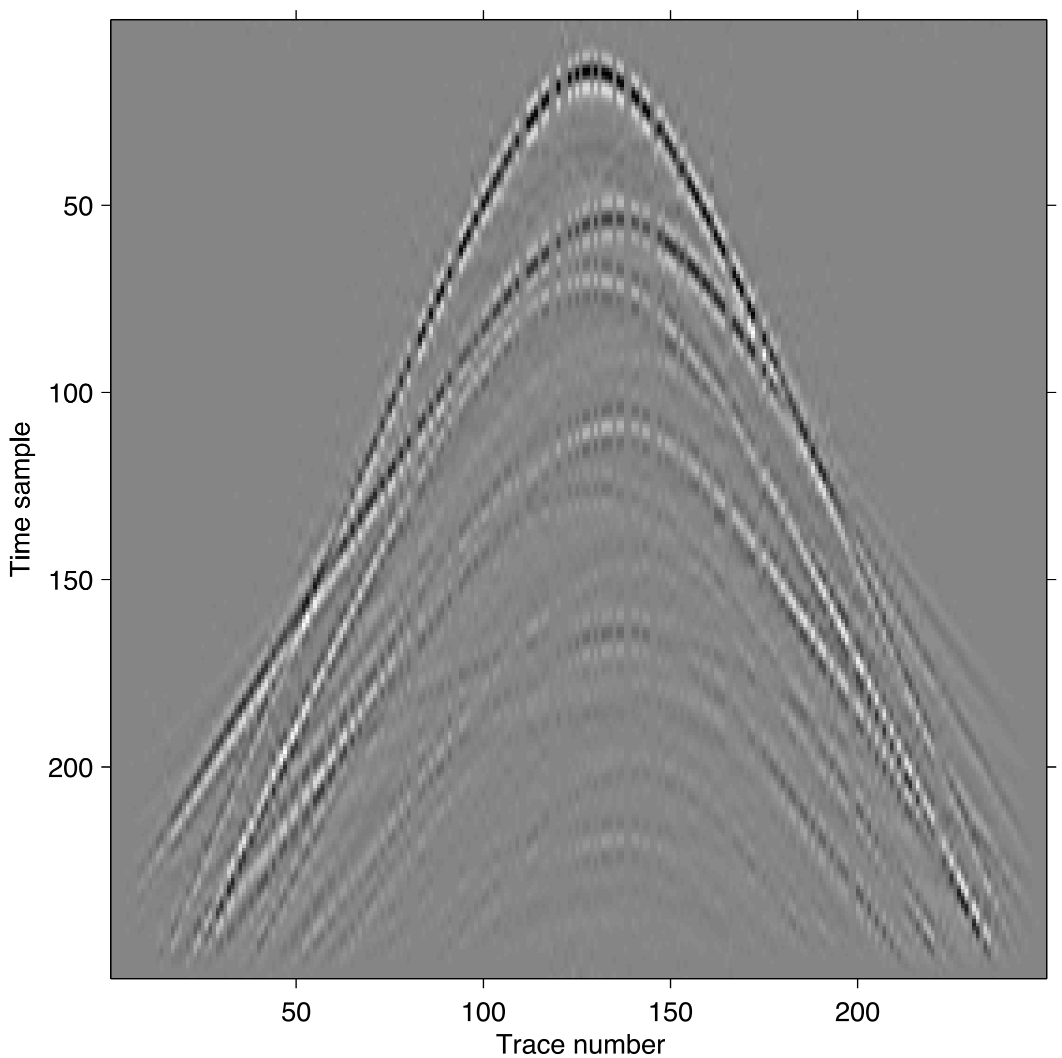


$$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{C}^H_{\Lambda^c}\mathbf{x}\|_2$$ GAP detected “support” solution

GAP does not detect "support"

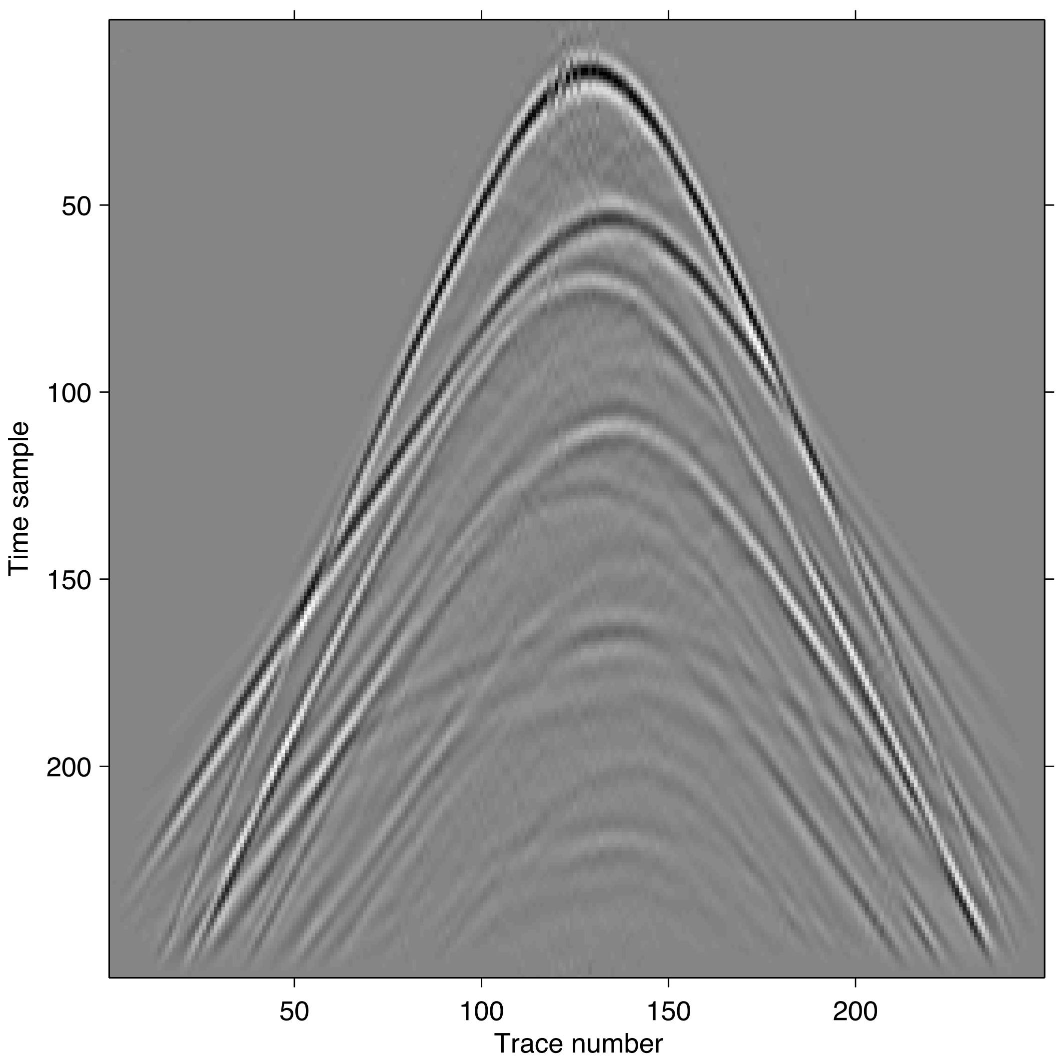


GAP solution

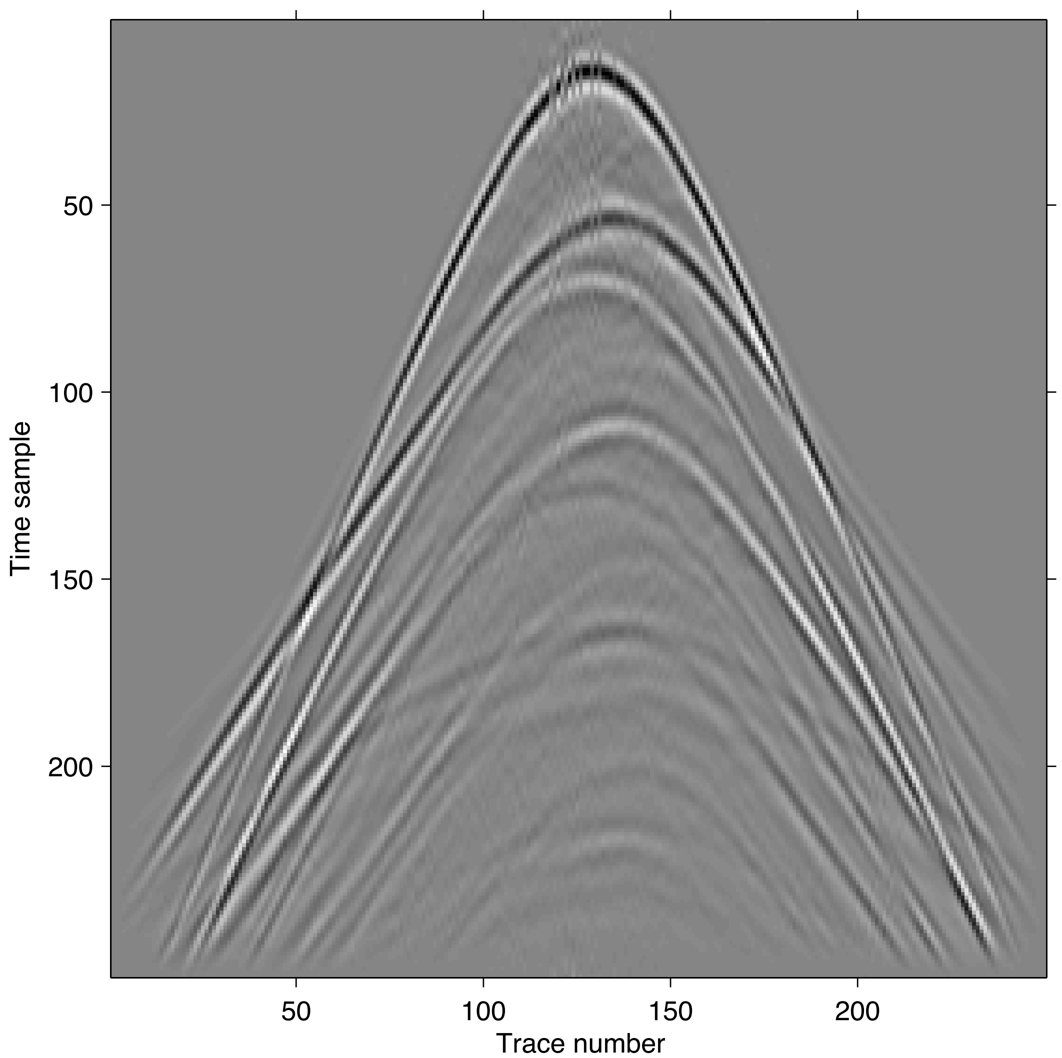


GAP detected "support" solution

GAP does not detect “support”



L1-Synth solution



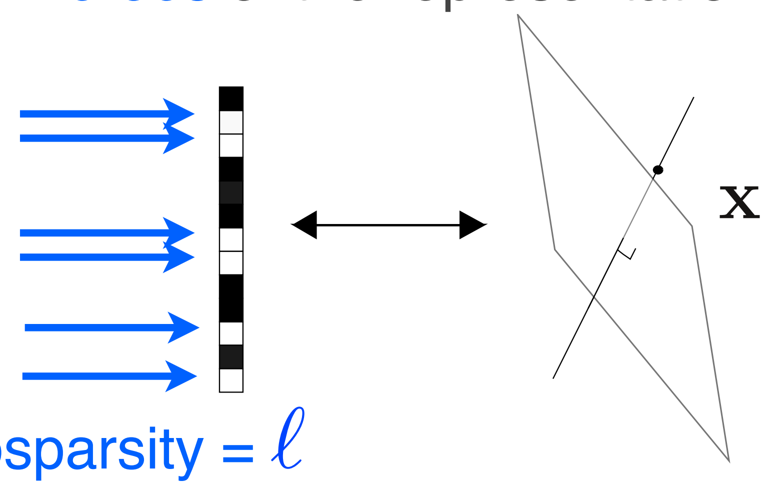
L1-Synth “debiased” solution

Cosparsity results

- uniqueness of solution when recovering from undersampled cosparse signals
- sufficient condition ("ERC-like") for success of L1-Analysis and GAP in reconstructing the above

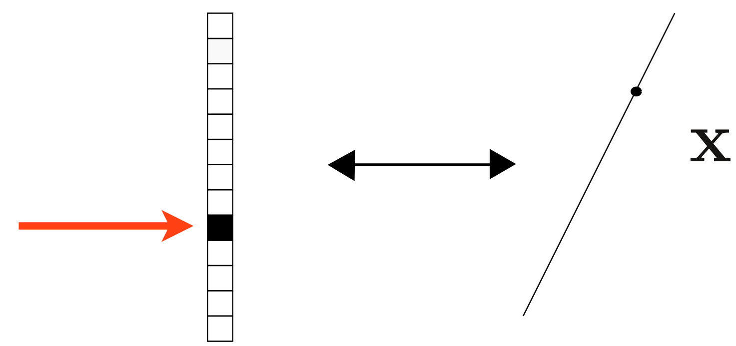
S. Nam et al., The cosparse analysis model and algorithms, 2012

- **Cosparse analysis model**
  - ✓ Analysis operator $\mathbf{\Omega}$
  - ✓ Representation $\mathbf{\Omega x}$
  - ✓ Zeroes of the representation

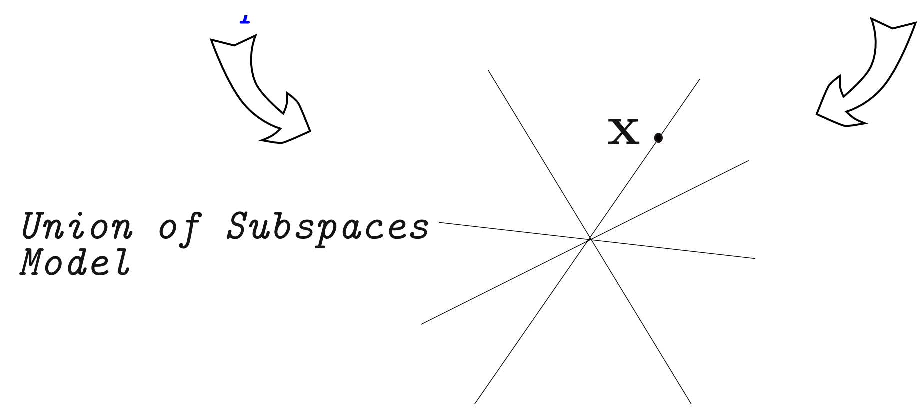


cosparsity = ℓ
= codimension of subspace

- **Sparse synthesis model**
  - ✓ Synthesis dictionary $\mathbf{D}$
  - ✓ Representation $\mathbf{z}$ s.t. $\mathbf{x} = \mathbf{Dz}$
  - ✓ Nonzeroes of the representation



sparsity = *k*
= dimension of subspace



from S. Nam et al., The cosparse analysis model and algorithms, 2012

Constructing signals with...

Sparsity

Cosparsity

Acknowledgements

- All cosparsity theoretical results, along with the GAP algorithm (and codes) from S.Nam, M. Davies, M. Elad, and R. Gribonval

SINBAD

NSERC CRSNG

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