Wireless Sensor Network Localization

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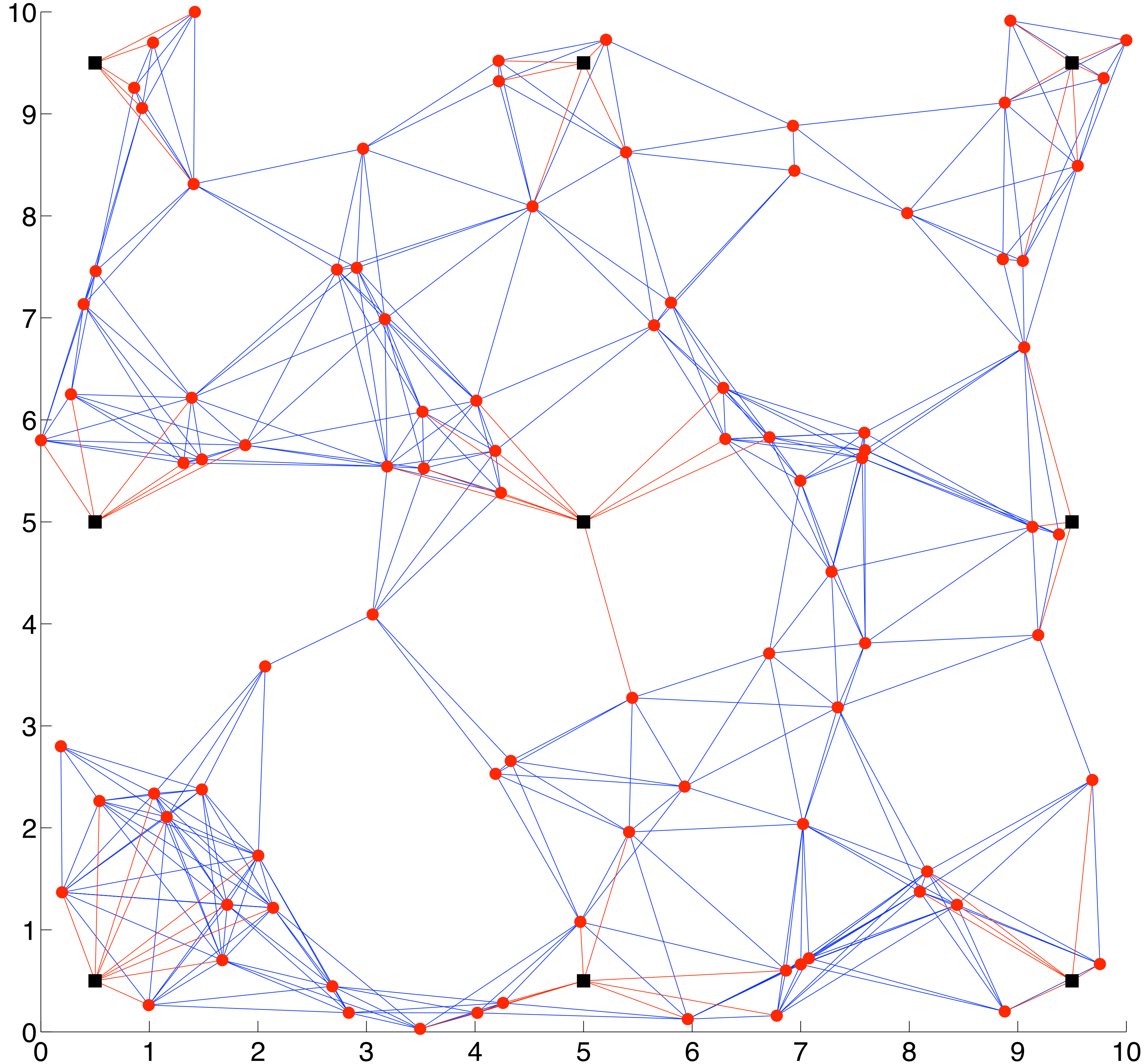
Sensor Network Localization



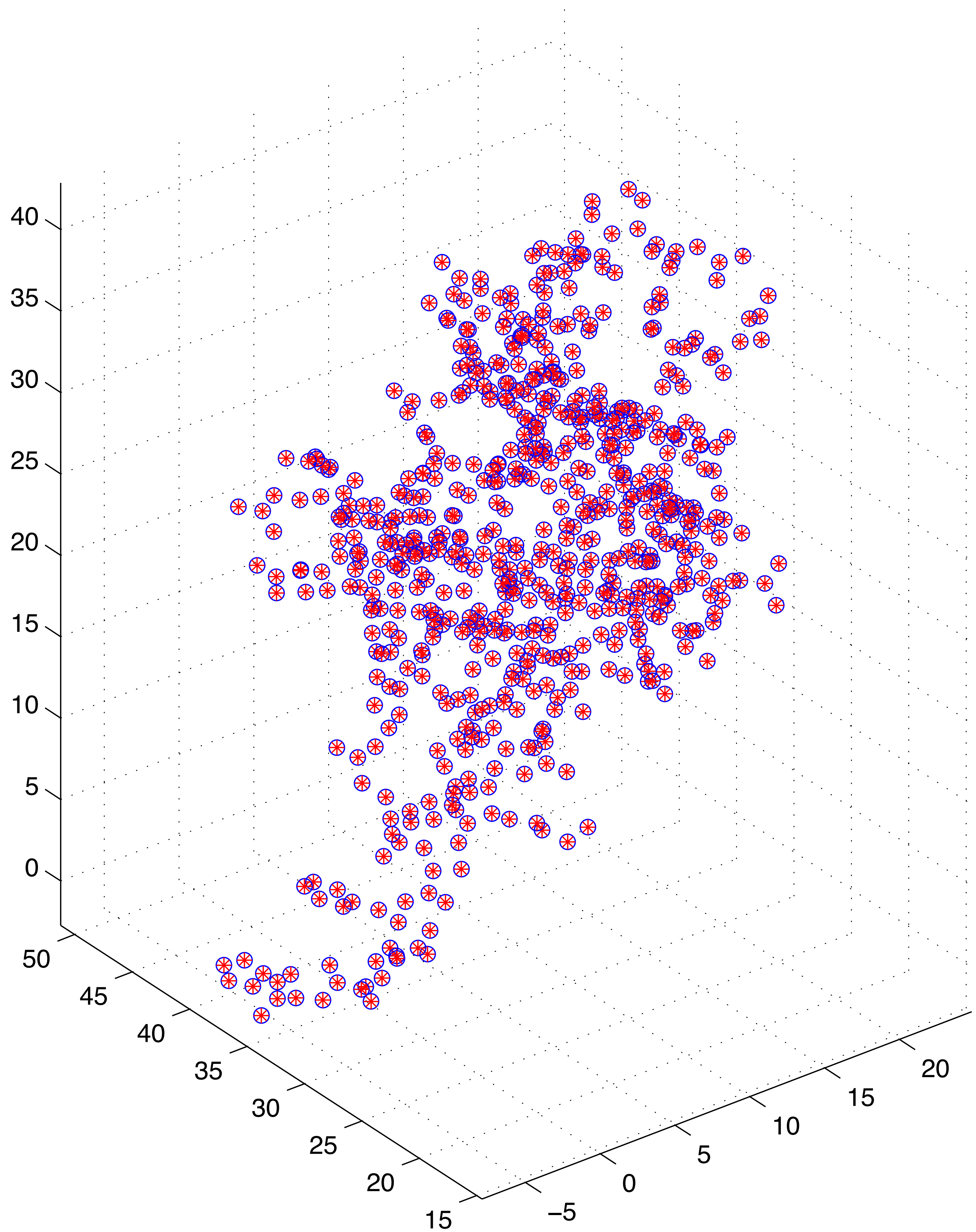
Sensor Network Localization

$$
\begin{array}{lll}
\text{find} & p_1, \ldots, p_n \in \mathbb{R}^3 & \\
\text{such that} & \|p_i - p_j\|_2 = d_{ij}, & (i,j) \in \mathcal{E} \\
& \|p_i - p_j\|_2 \leq u_{ij}, & (i,j) \in \mathcal{U} \\
& \|p_i - p_j\|_2 \geq \ell_{ij}, & (i,j) \in \mathcal{L}
\end{array}
$$

Applications

- when GPS is not possible / practical
- lightweight, low-cost, low-power sensors to monitor environment
- ocean-bottom sensors used in geophysical applications

Protein Structure Determination



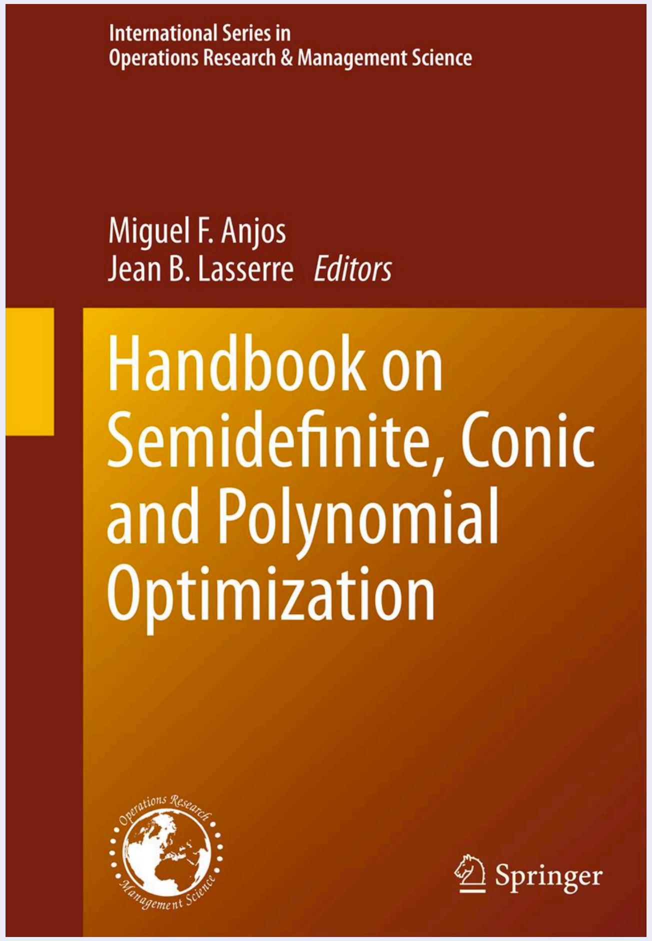
Semidefinite Optimization

A Semidefinite Programming (SDP) Problem

$$(\text{SDP}) \quad \boxed{\begin{array}{ll} \text{minimize} & \langle C, X\rangle \\ \text{subject to} & \langle A_i, X\rangle = b_i, \;\; i = 1, \ldots, m \\ & X \in \mathcal{S}^n_+ \end{array}}$$

Applications

- graph theory
- distance geometry
- combinatorial optimization
- low-rank matrix completion
- polynomial optimization
- quantum computing
- machine learning
- control theory



Euclidean Distance Matrices

Euclidean Distance Matrices (EDMs)

- An $n \times n$ matrix D is an EDM if

$$\exists p_1, \ldots, p_n \in \mathbb{R}^k : D_{ij} = \|p_i - p_j\|_2^2 \qquad (1)$$

- The embedding dimension of D :

$$\dim(D) = \min\{k : (1) \text{ holds}\}$$

- $\mathcal{E}^n$ = the set of all $n \times n$ EDMs

Euclidean Distance Matrices

EDMs and Semidefinite Matrices

- Let $p_1, \ldots, p_n \in \mathbb{R}^k$ and D be their EDM
- Let $Y \in \mathcal{S}^n$ be the Gram matrix:

$$Y_{ij} = p_i^T p_j, \quad \forall ij$$

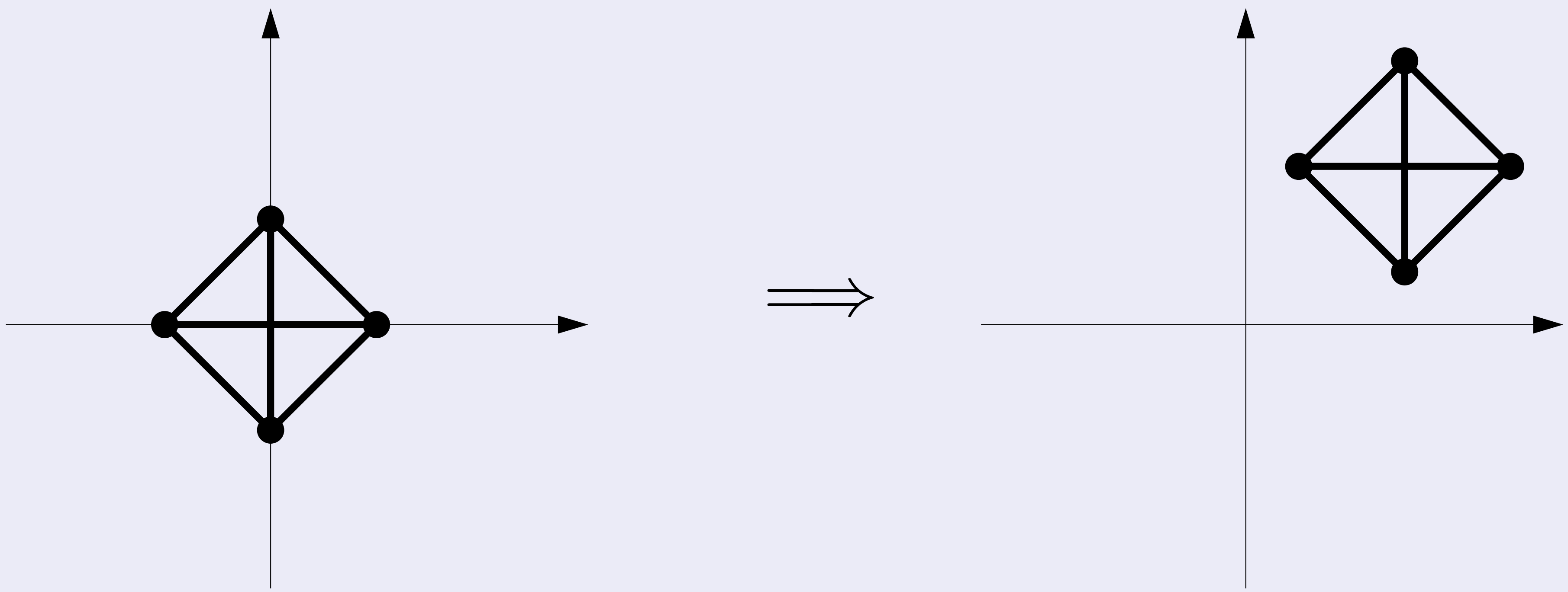
- Then Y is a semidefinite matrix: $Y \in \mathcal{S}_+^n$

$$\begin{aligned} D_{ij} &= \|p_i - p_j\|_2^2 \\ &= p_i^T p_i - 2p_i^T p_j + p_j^T p_j \\ &= Y_{ii} - 2Y_{ij} + Y_{jj} \\ &= \mathcal{K}(Y)_{ij} \end{aligned}$$

- Therefore: $\mathcal{K}(\mathcal{S}_+^n) = \mathcal{E}^n$

Euclidean Distance Matrices

Translation: $D = D'$ but $Y \neq Y'$



$D = \mathcal{K}(Y)$ $\qquad$ $D' = \mathcal{K}(Y')$

Centered matrices: $\mathcal{S}^n_C = \{ Y \in \mathcal{S}^n : \sum_{ij} Y_{ij} = 0 \}$

- $\mathcal{K} \colon \mathcal{S}^n_+ \cap \mathcal{S}^n_C \to \mathcal{E}^n$ is a linear bijection
- If $D = \mathcal{K}(Y)$ and $Y \in \mathcal{S}^n_+ \cap \mathcal{S}^n_C$, then $\dim(D) = \operatorname{rank}(Y)$

Euclidean Distance Matrices

Sensor network localization (vector form)

$$
\begin{array}{ll}
\text{find} & p_1, \ldots, p_n \in \mathbb{R}^3 \\
\text{such that} & \|p_i - p_j\|_2^2 = d_{ij}^2, \quad (i,j) \in \mathcal{E}
\end{array}
$$

Sensor network localization (matrix form)

$$
\begin{array}{ll}
\text{minimize} & \operatorname{rank}(Y) \\
\text{such that} & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \quad (i,j) \in \mathcal{E} \\
 & \sum_{ij} Y_{ij} = 0 \\
 & Y \in \mathcal{S}_+^n
\end{array}
$$

Sensor Network Localization

Sensor network localization (non-convex and NP-hard)

$$
\begin{array}{ll}
\text{minimize} & \operatorname{rank}(Y) \\
\text{such that} & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \quad (i,j) \in \mathcal{E} \\
& \sum_{ij} Y_{ij} = 0 \\
& Y \in \mathcal{S}_+^n
\end{array}
$$

Semidefinite relaxation (convex and poly-time solvable)

$$
\begin{array}{ll}
\text{maximize} & \operatorname{trace}(Y) \\
\text{such that} & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \quad (i,j) \in \mathcal{E} \\
& \sum_{ij} Y_{ij} = 0 \\
& Y \in \mathcal{S}_+^n
\end{array}
$$

Handling Noisy Measurements

$$
\begin{array}{lll}
\text{minimize} & \|\varepsilon\|_2 + \|\eta\|_1 + \|\zeta\|_1 & \\
\text{subject to} & \|p_i - p_j\|_2^2 = d_{ij}^2 + \varepsilon_{ij}, & (i,j) \in \mathcal{E} \\
& \|p_i - p_j\|_2^2 \le u_{ij}^2 + \eta_{ij}, \;\; \eta_{ij} \ge 0, & (i,j) \in \mathcal{U} \\
& \|p_i - p_j\|_2^2 \ge \ell_{ij}^2 - \zeta_{ij}, \;\; \zeta_{ij} \ge 0, & (i,j) \in \mathcal{L} \\
& p_1, \ldots, p_n \in \mathbb{R}^3 &
\end{array}
$$

Handling Noisy Measurements

$$
\begin{array}{lll}
\text{minimize} & \|\varepsilon\|_2 + \|\eta\|_1 + \|\zeta\|_1 - \operatorname{trace}(Y) & \\
\text{subject to} & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij} + \varepsilon_{ij}, & (i,j) \in \mathcal{E} \\
& Y_{ii} - 2Y_{ij} + Y_{jj} \leq u_{ij} + \eta_{ij}, \;\; \eta_{ij} \geq 0, & (i,j) \in \mathcal{U} \\
& Y_{ii} - 2Y_{ij} + Y_{jj} \geq \ell_{ij} - \zeta_{ij}, \;\; \zeta_{ij} \geq 0, & (i,j) \in \mathcal{L} \\
& \sum_{ij} Y_{ij} = 0 & \\
& Y \in \mathcal{S}^n_+ &
\end{array}
$$

Incorporating Angle Information

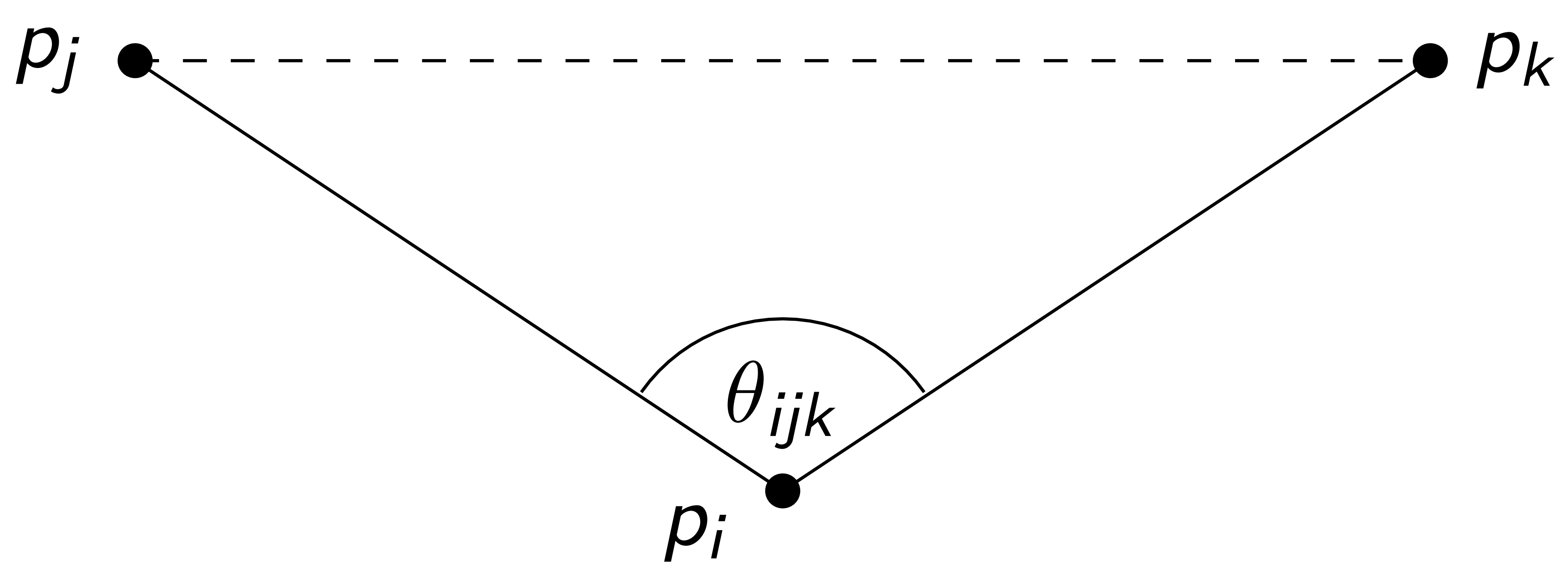
Two methods

- Law of Cosines:

$$\|p_j - p_k\|_2^2 = \|p_i - p_j\|_2^2 + \|p_i - p_k\|_2^2 - 2\,\|p_i - p_j\|_2\,\|p_i - p_k\|_2 \cos(\theta_{ijk})$$

- Inner-Product Formula:

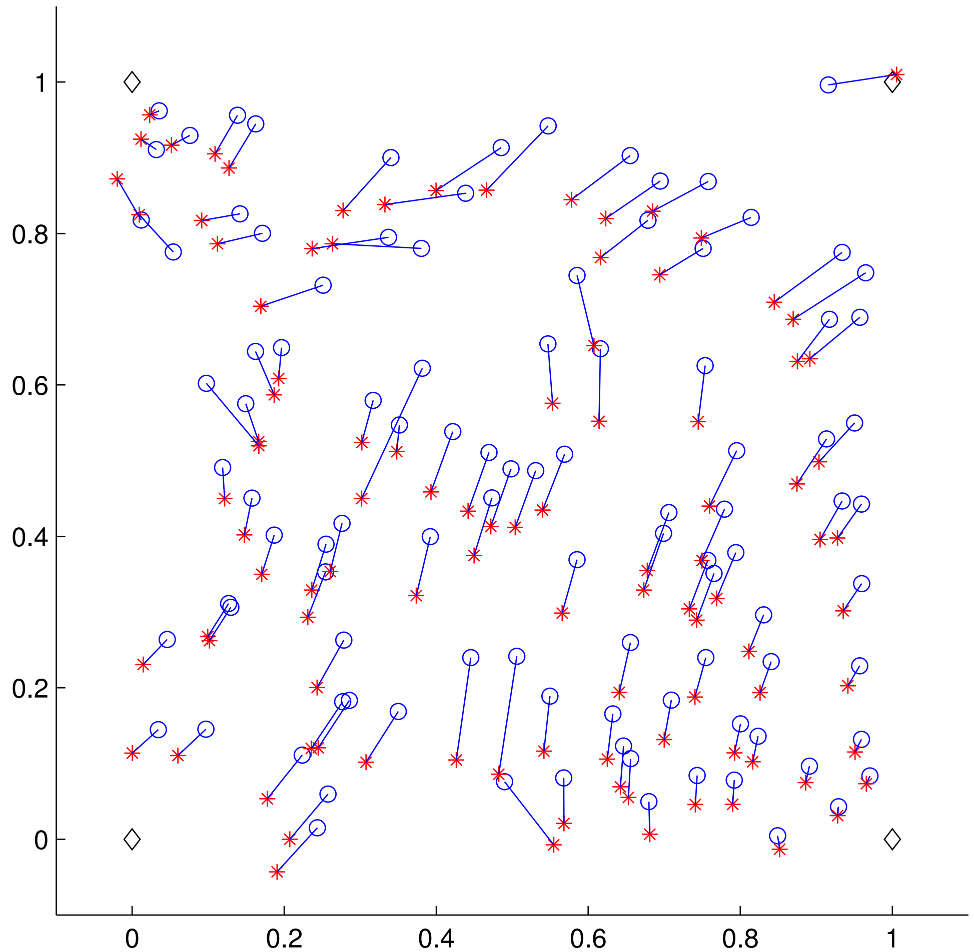
$$(p_i - p_j)^T(p_i - p_k) = \|p_i - p_j\|_2\,\|p_i - p_k\|_2 \cos(\theta_{ijk})$$



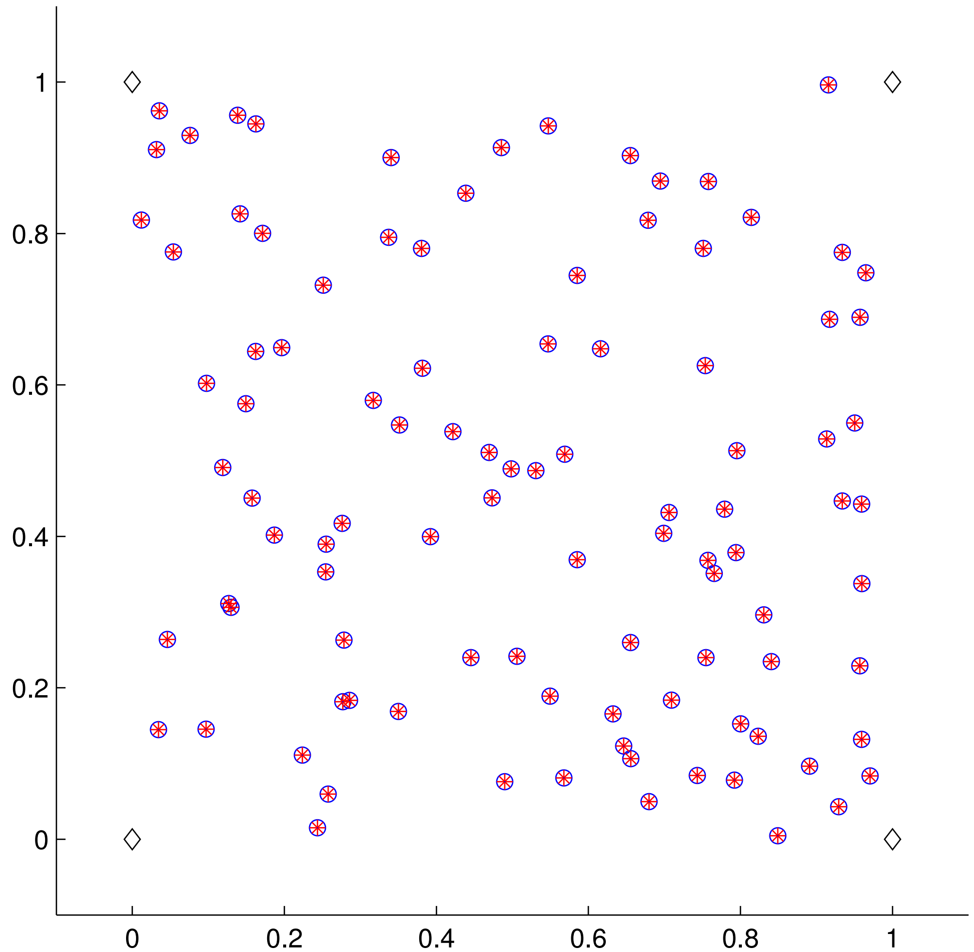
MATLAB/CVX code

```
cvx_begin
variable Y(n,n) symmetric
variable N(n,n) symmetric
maximize( trace(Y) - norm(N,'fro') )
subject to
for ii=1:n
  for jj=ii+1:n
    if D(ii,jj) > 0
      Y(ii,ii) - 2*Y(ii,jj) + Y(jj,jj) == D(ii,jj) + N(ii,jj)
    end
  end
end
sum(Y(:)) == 0
Y == semidefinite(n)
cvx_end
```

Different Radio Range

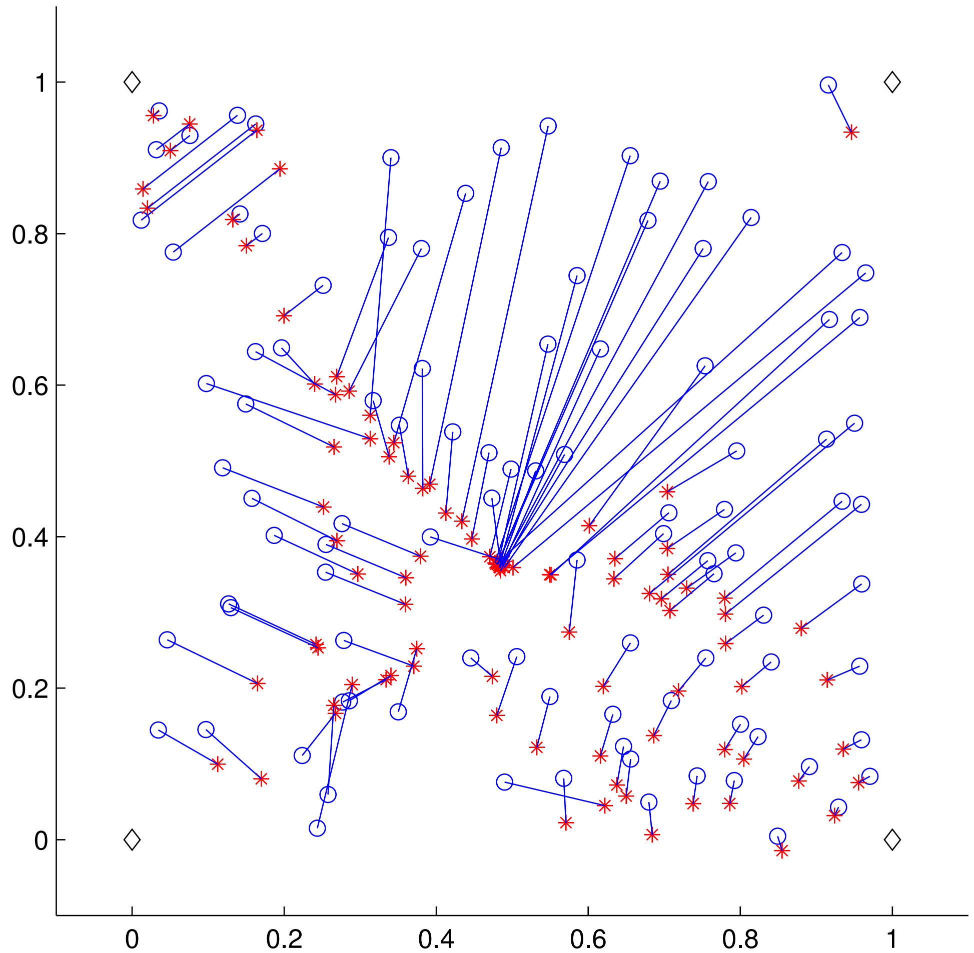


$\rho = .15$

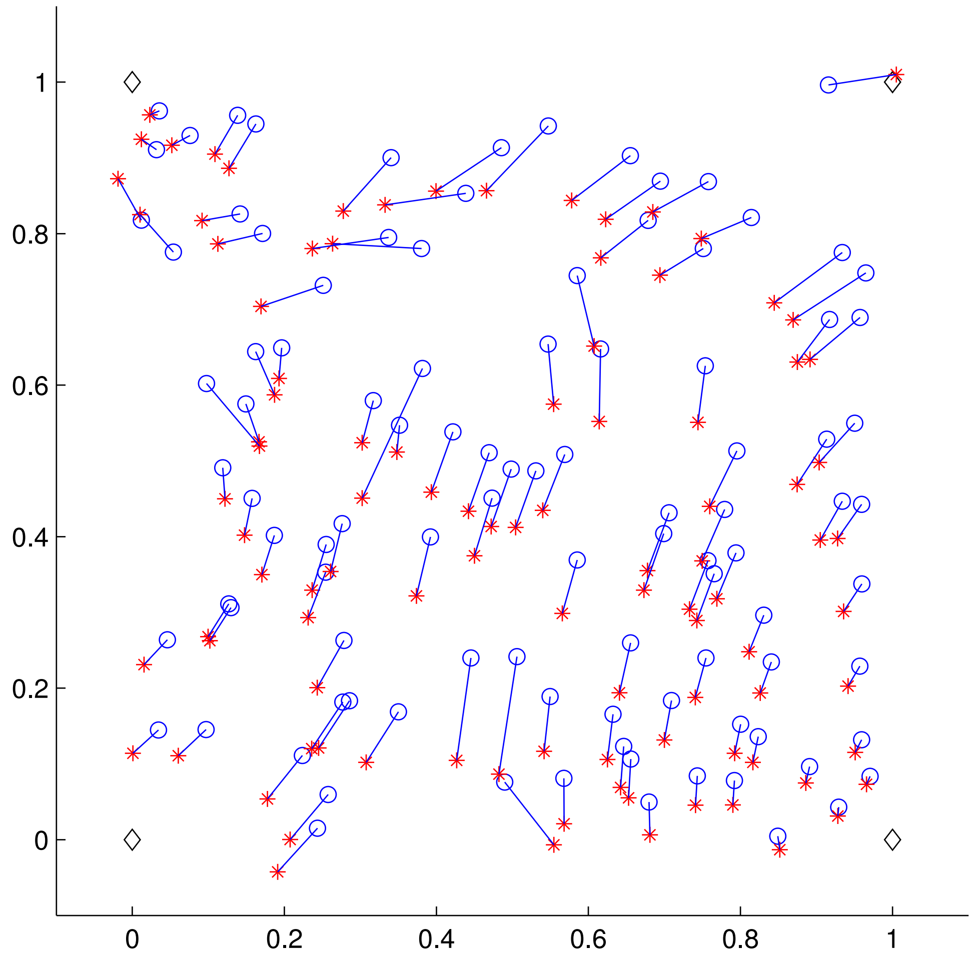


$\rho = .25$

Maximizing trace(Y)

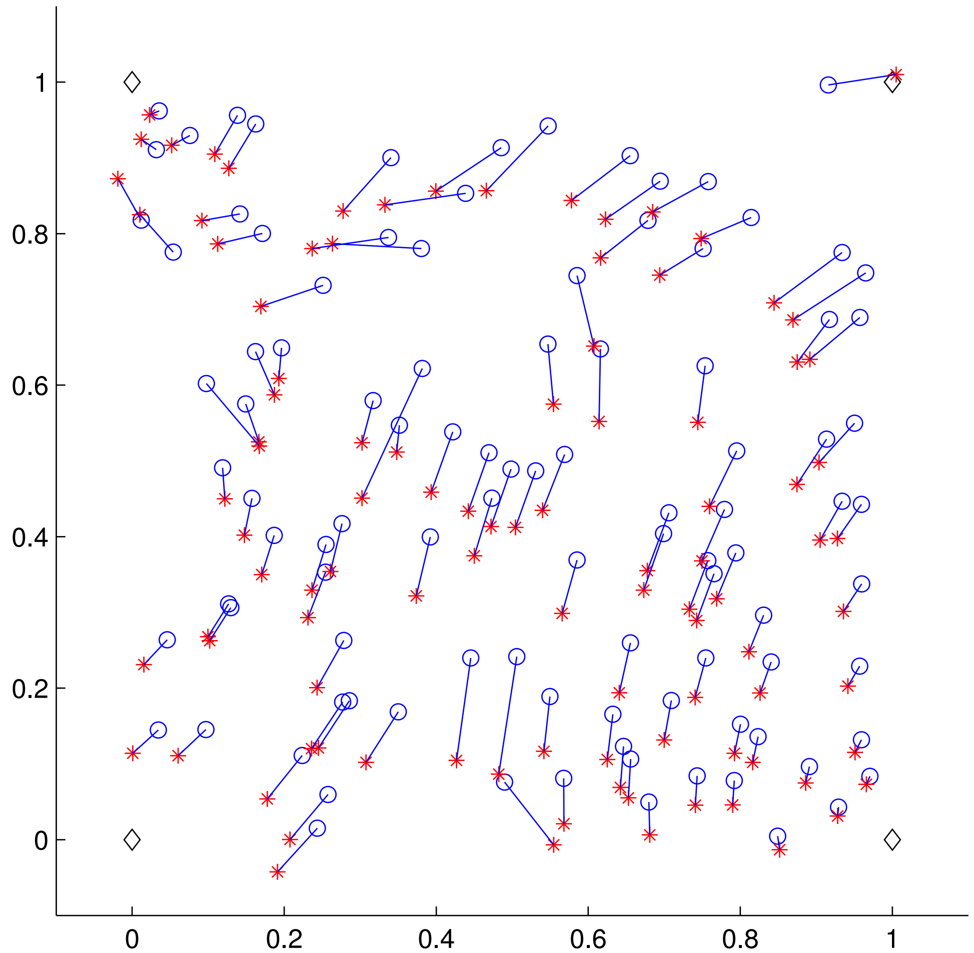


$\rho = .15$, minimum trace

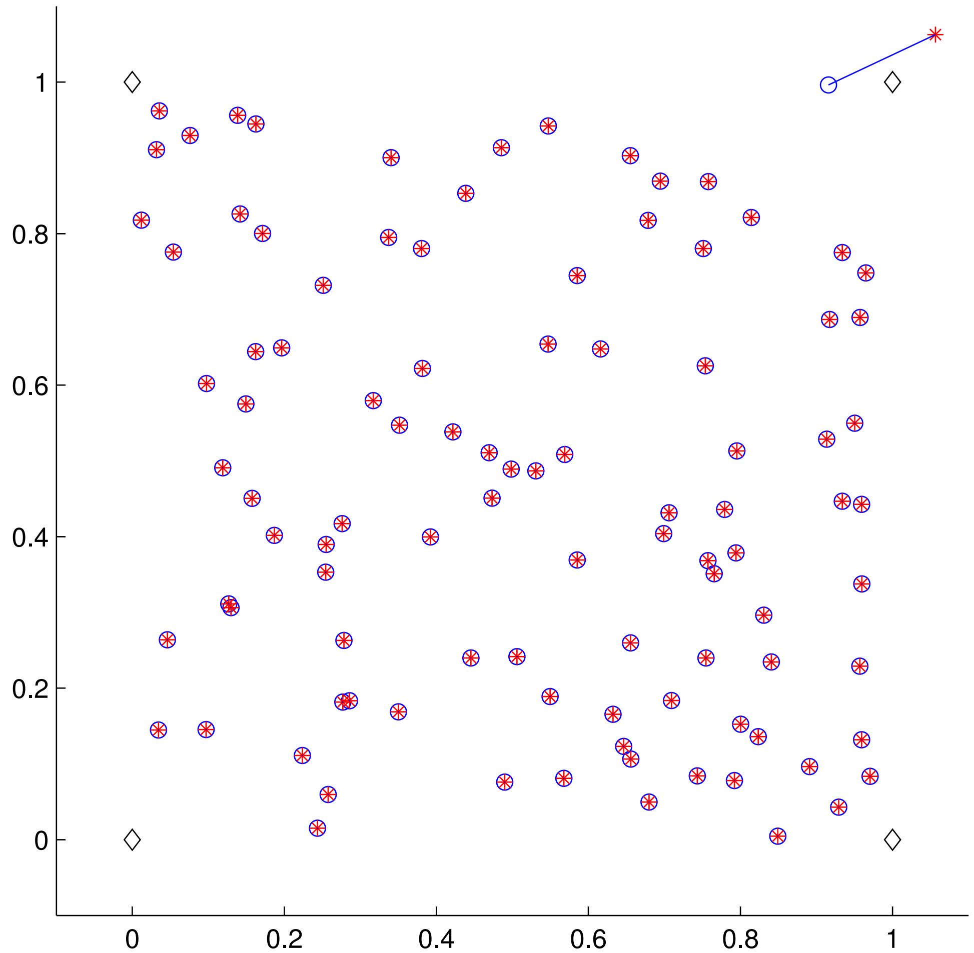


$\rho = .15$, maximum trace

Angle Constraints

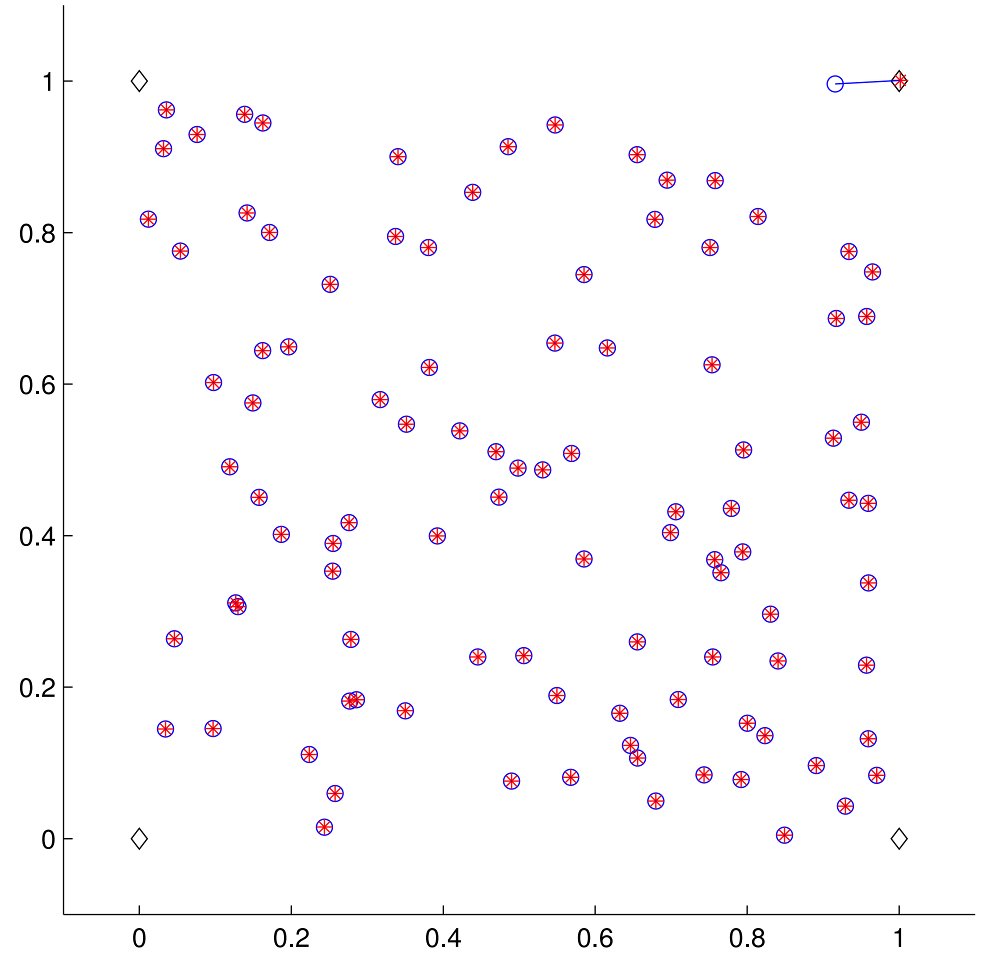


$\rho = .15$, without angles

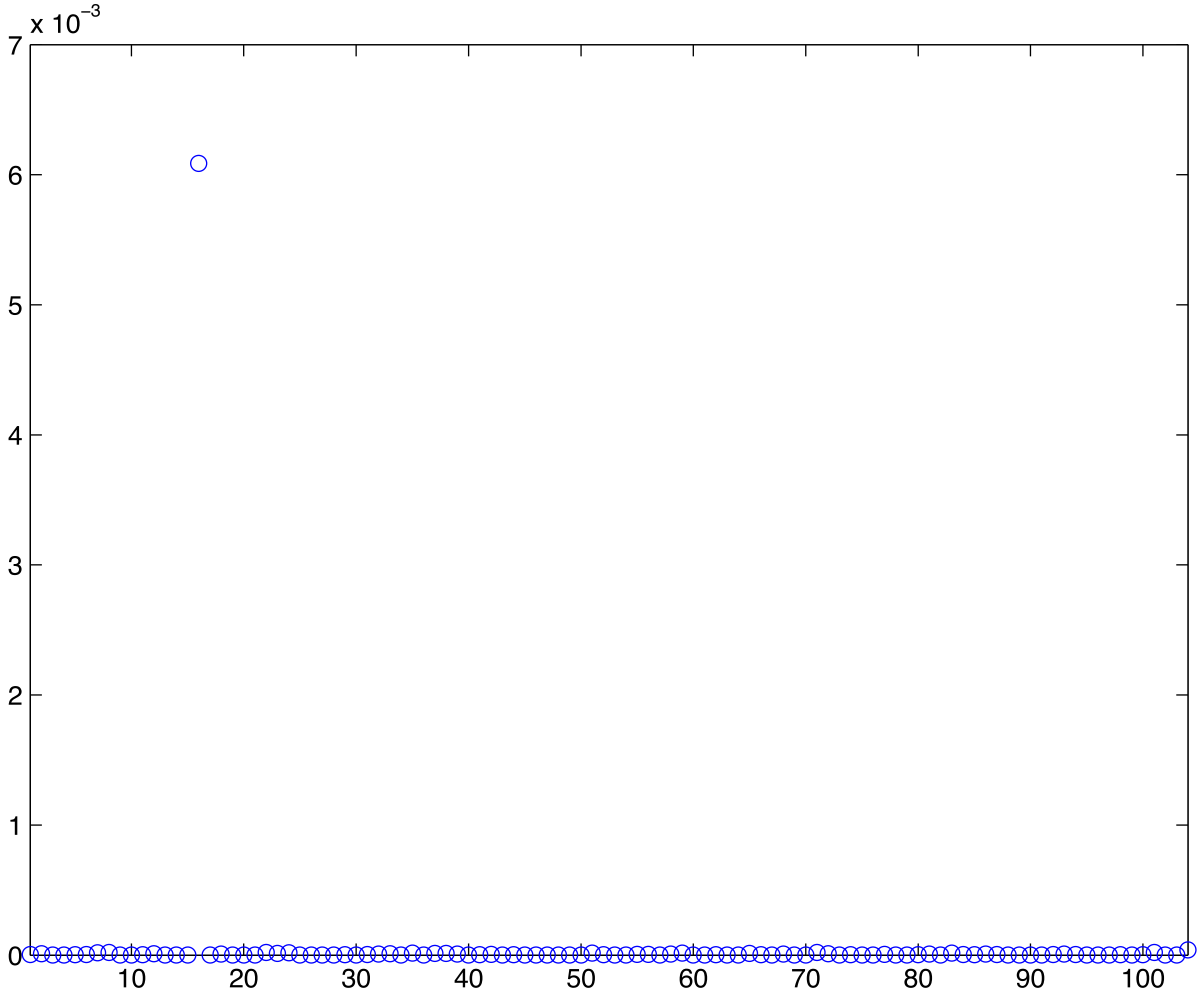


$\rho = .15$, with angles

Determining Incorrect Sensors

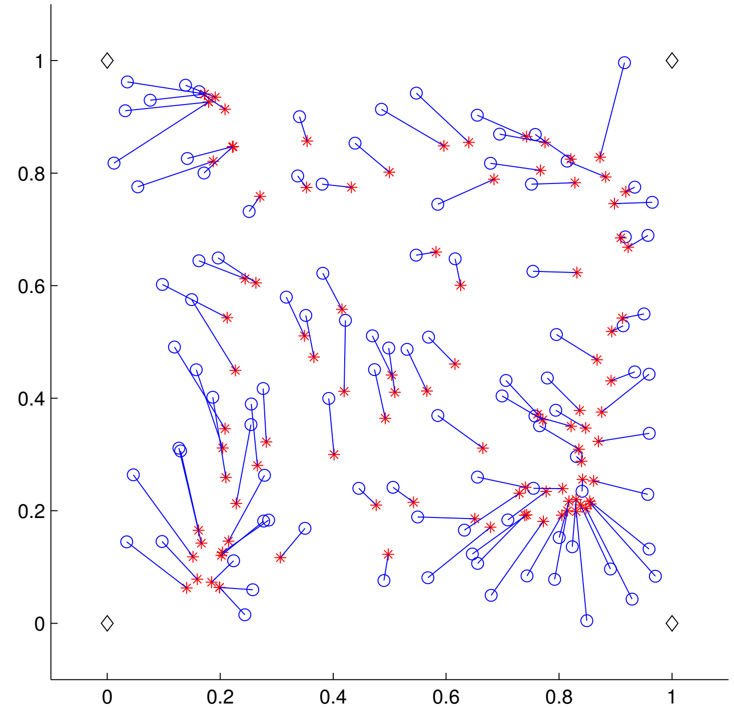
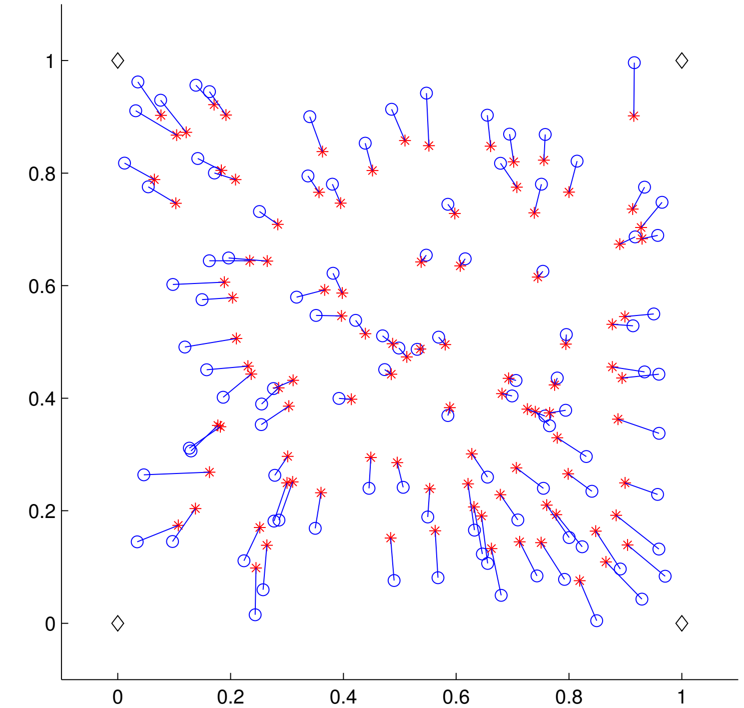
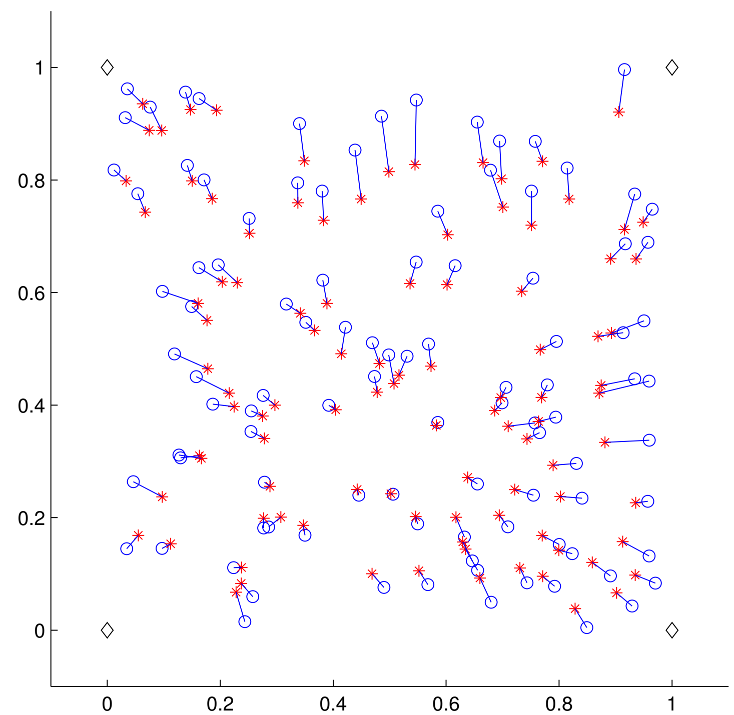
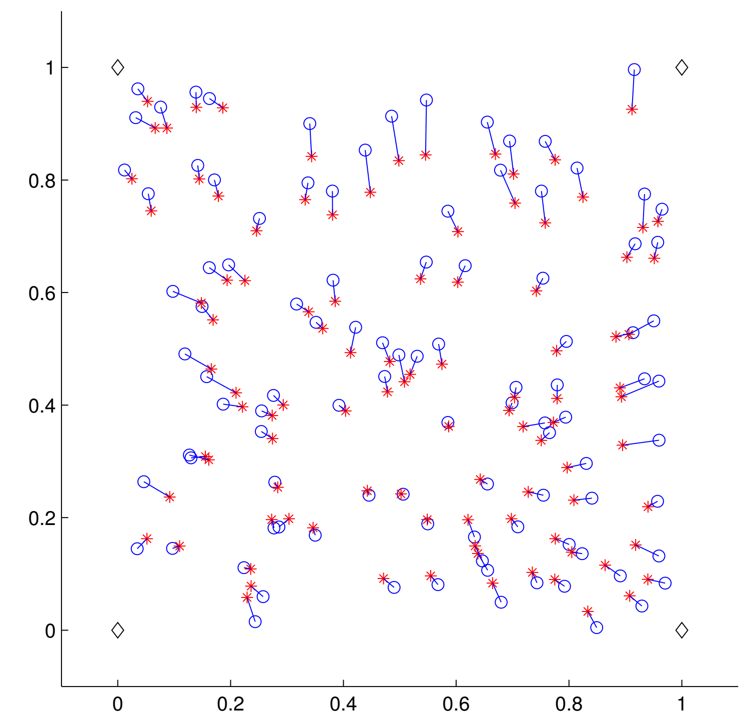


sensor incorrectly placed



$$|\mathrm{diag}(Y - PP^T)|$$

Noisy Measurements (10% multiplicative noise)

	no max trace	max trace
no min error		
min error		

Post-Processing

- Semidefinite optimization provides a high-quality starting solution for local gradient-based optimization:

$$\min_{P \in \mathbb{R}^{n \times 3}} \left\{ \phi(P) = \sum_{(i,j) \in \mathcal{E}} \left(\|p_i - p_j\|_2 - d_{ij} \right)^2 \right\}$$

Large-Scale Problems

- Random problems *without noise*
- Dimension: $d = 2$
- Square region: $[0,1] \times [0,1]$
- 4 anchor nodes
- Root-mean-square error:

$$\text{Error} = \left(\frac{1}{\#\ \text{positioned}} \sum_{i \text{ positioned}} \|p_i - p_i^{\text{true}}\|^2 \right)^{\frac{1}{2}}$$

- The open source software SNLSDPclique is available from my website under the GNU Public License
- MATLAB (MacBook with 2.16 GHz Intel Core 2 Duo and 2 GB RAM)

Large-Scale Problems

# sensors	Radio range of sensors	# sensors positioned	CPU time	Error
2000	.06	1999.9	1 s	3e-13
2000	.05	1996.7	1 s	2e-13
2000	.04	1273.8	3 s	4e-12
6000	.06	6000.0	4 s	7e-14
6000	.05	6000.0	3 s	1e-13
6000	.04	5999.4	3 s	3e-13
10000	.06	10000.0	8 s	1e-13
10000	.05	10000.0	7 s	2e-13
10000	.04	10000.0	6 s	1e-13
20000	.030	20000.0	17 s	2e-13
60000	.015	60000.0	1 m 53 s	7e-13
100000	.011	100000.0	5 m 46 s	9e-11

Figure: Facial representation approach

Nathan Krislock and Henry Wolkowicz. (2010)
Explicit sensor network localization using semidefinite representations and facial reductions.
SIAM Journal on Optimization.

Large-Scale Problems

# sensors	Radio range of sensors	# sensors positioned	CPU time	Error
2000	.06	1999.9	1 s	6e-16
2000	.05	1996.7	1 s	7e-16
2000	.04	1274.4	2 s	7e-16
6000	.06	6000.0	3 s	5e-16
6000	.05	6000.0	3 s	8e-16
6000	.04	5999.4	3 s	6e-16
10000	.06	10000.0	6 s	7e-16
10000	.05	10000.0	6 s	6e-16
10000	.04	10000.0	5 s	1e-15
20000	.030	20000.0	14 s	8e-16
60000	.015	60000.0	1 m 27 s	9e-16
100000	.011	100000.0	3 m 55 s	1e-15

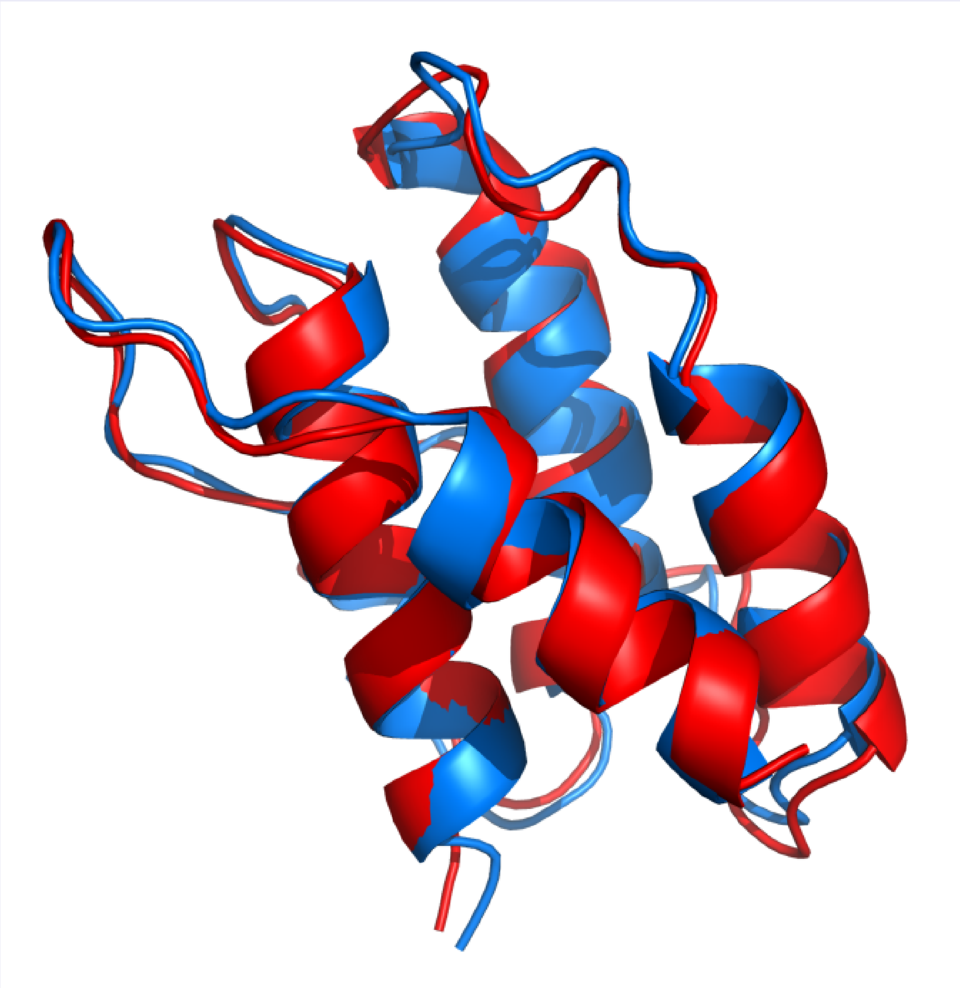
Figure: Point representation approach

Nathan Krislock. (2010)
Semidefinite Facial Reduction for Low-Rank Euclidean Distance Matrix Completion.
PhD thesis, University of Waterloo.

Protein Structure Determination

Protein 2L30 from the Protein Data Bank

- 127 amino acids
- 393 cliques: 269 2D + 124 3D
- SDP size = 1867
- reduced size = 512
- equality constraints = 7143
- reduced number of equality constraints = 1492
- SDP solution time = 61.7s



Babak Alipanahi, Nathan Krislock, Ali Ghodsi, Henry Wolkowicz, Logan Donaldson, and Ming Li. (2012)
Determining protein structures from NOESY distance constraints by semidefinite programming. *Journal of Computational Biology*.

Summary

- we described how to solve localization problems using semidefinite optimization
- semidefinite optimization provides a a high-quality solution to use as a starting point to gradient-based methods
- the high complexity of semidefinite solvers limits their direct use on large-scale problems

References

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