Compressed sensing, random Fourier matrix and jittered sampling

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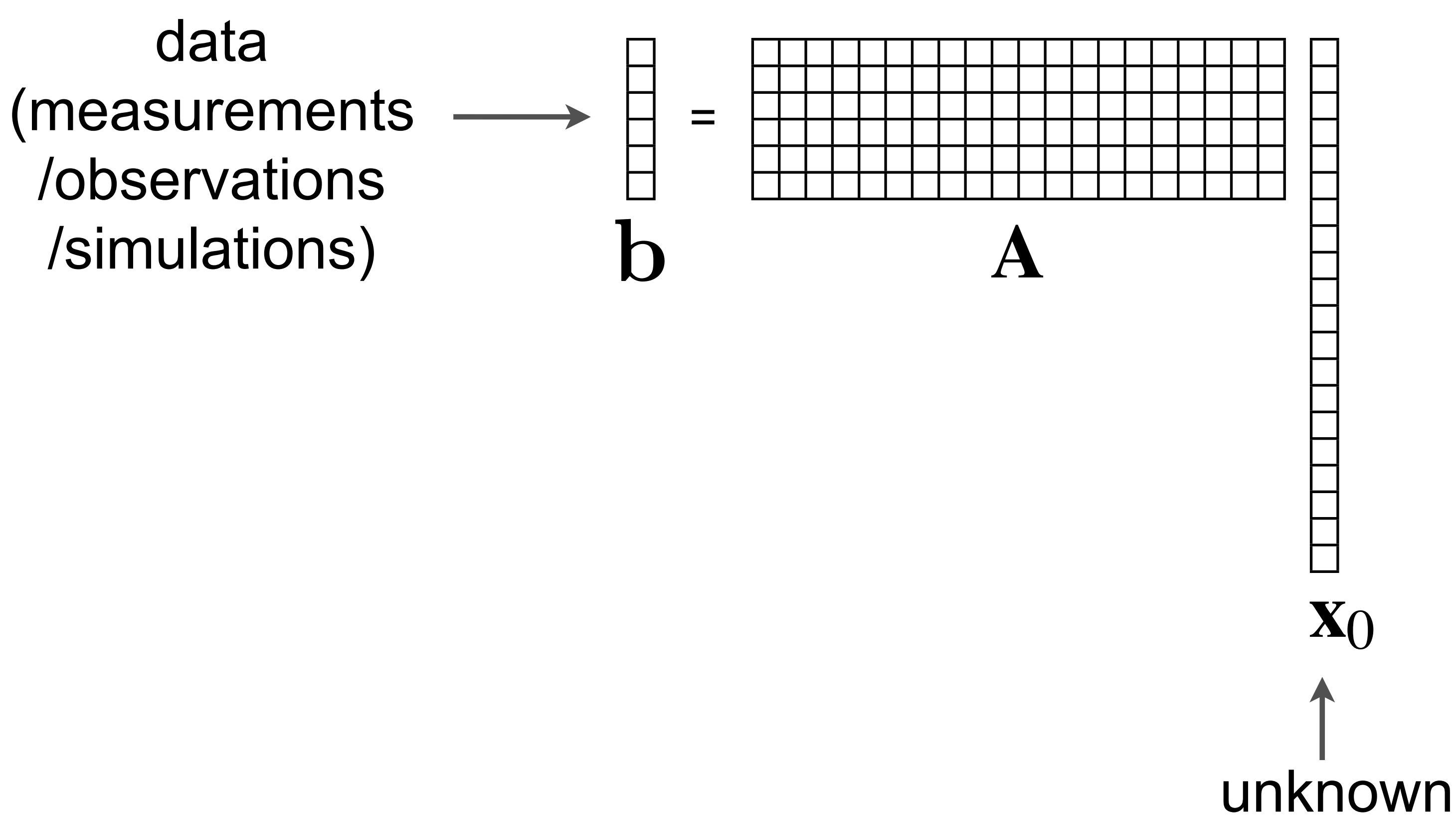
Introduction

Compressed sensing is a powerful technique to reconstruct sparse data.

- What is the main advantage of jittered sampling?
- What do we mean by better results?

Compressed sensing

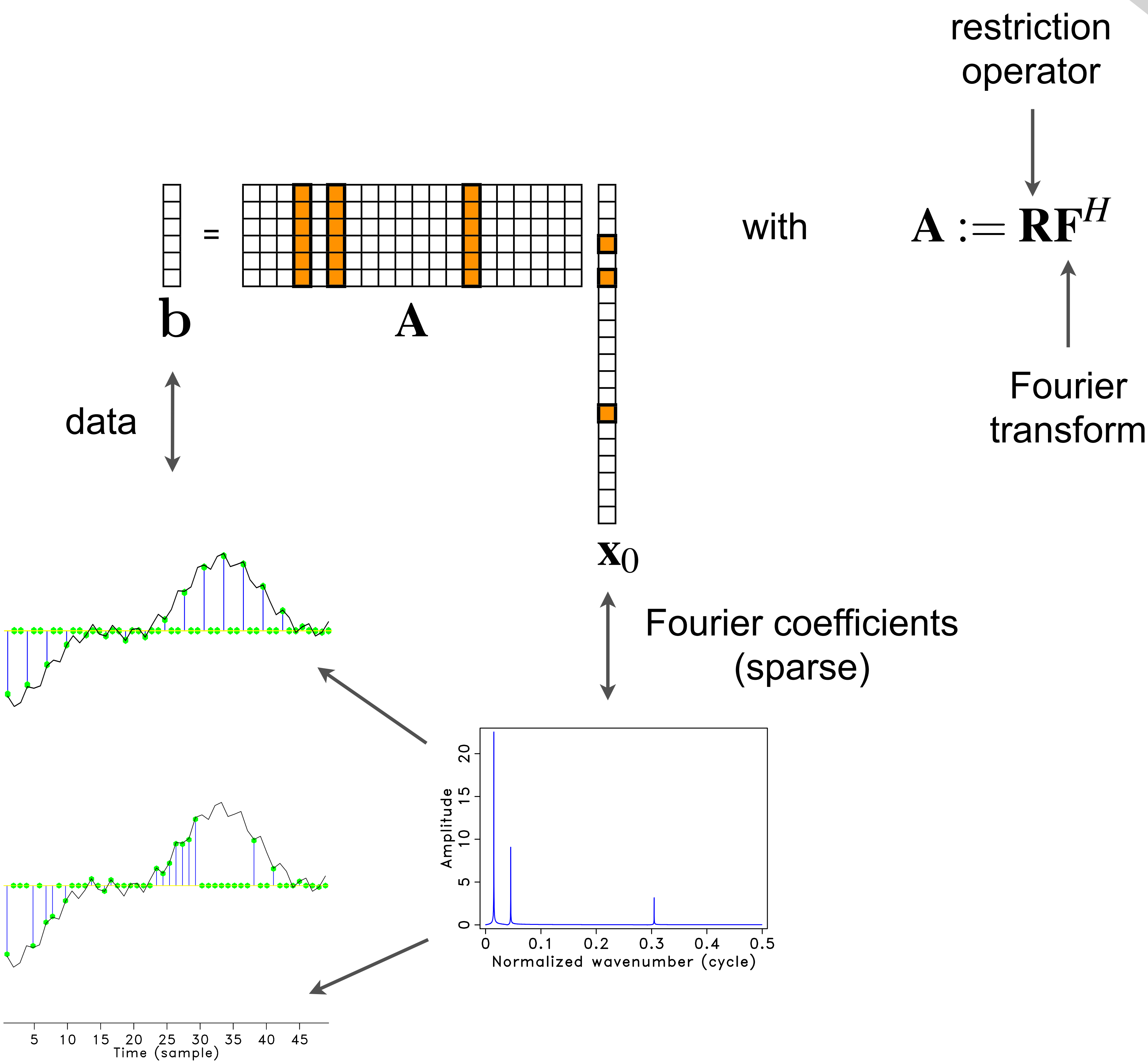
Consider the following (severely) *underdetermined* system of *linear* equations:



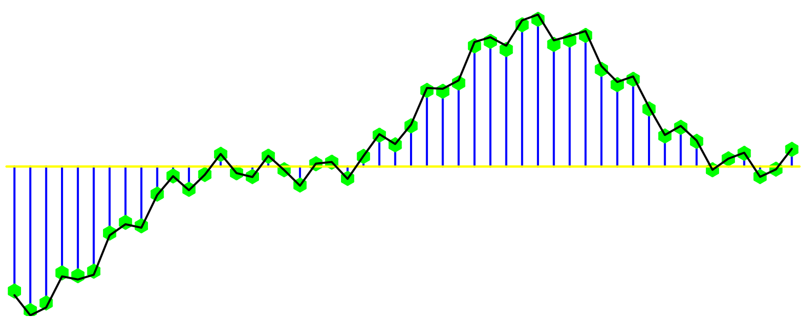
Is it possible to recover $\mathbf{x}_0$ accurately from $\mathbf{b}$?

Compressed Sensing attempts to answer this questions rigorously.

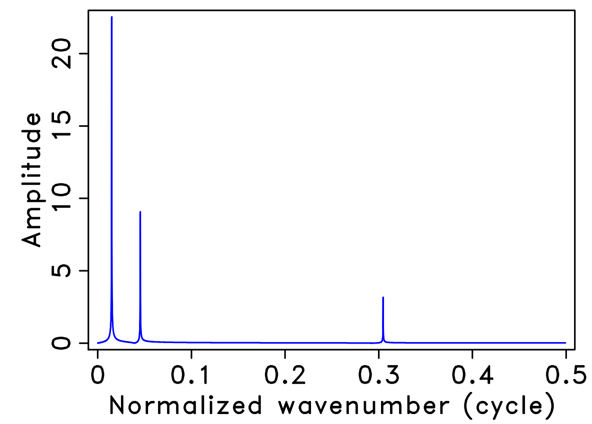
Sparse recovery



Coarse sampling schemes

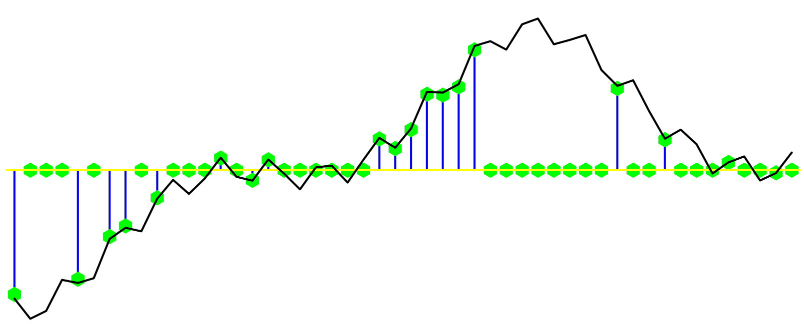


Fourier transform

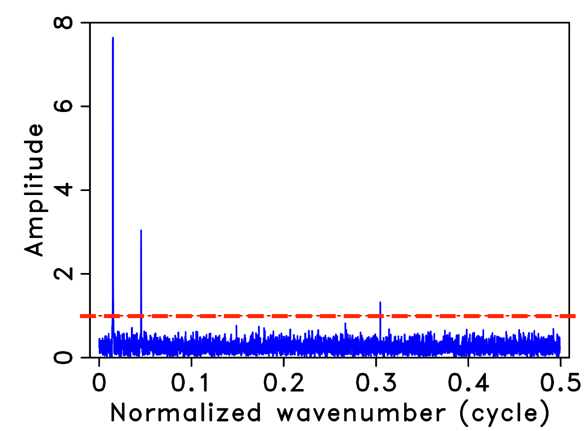


few significant coefficients

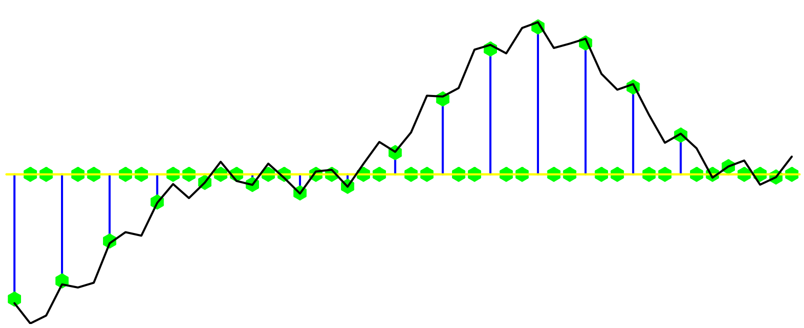
3-fold under-sampling



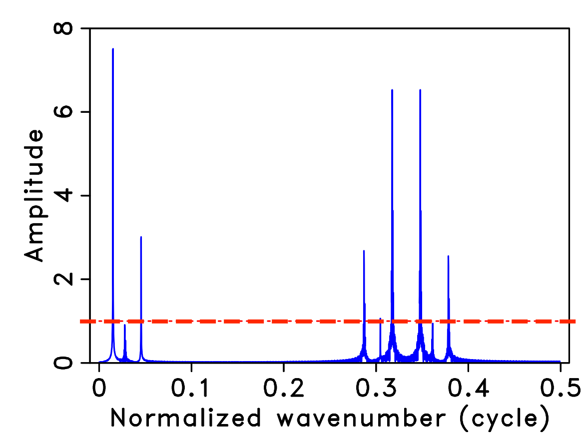
Fourier transform



✓ significant coefficients detected



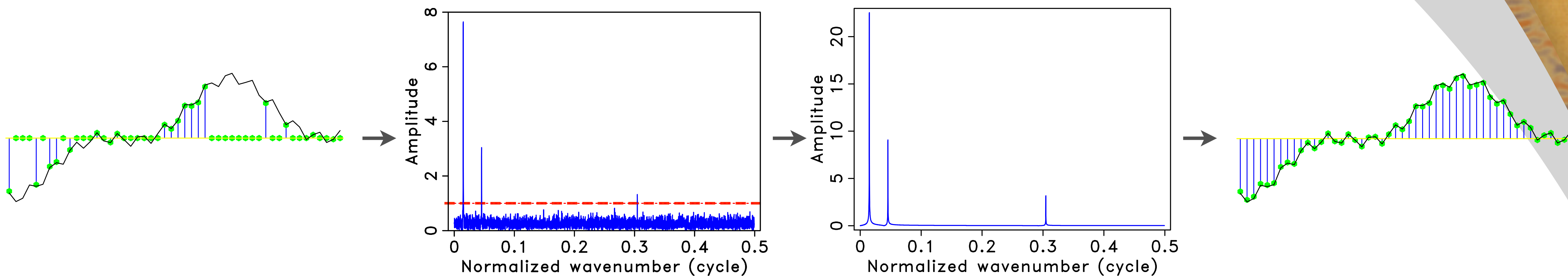
Fourier transform



✗ ambiguity

[Hennenfent & Herrmann, '08]

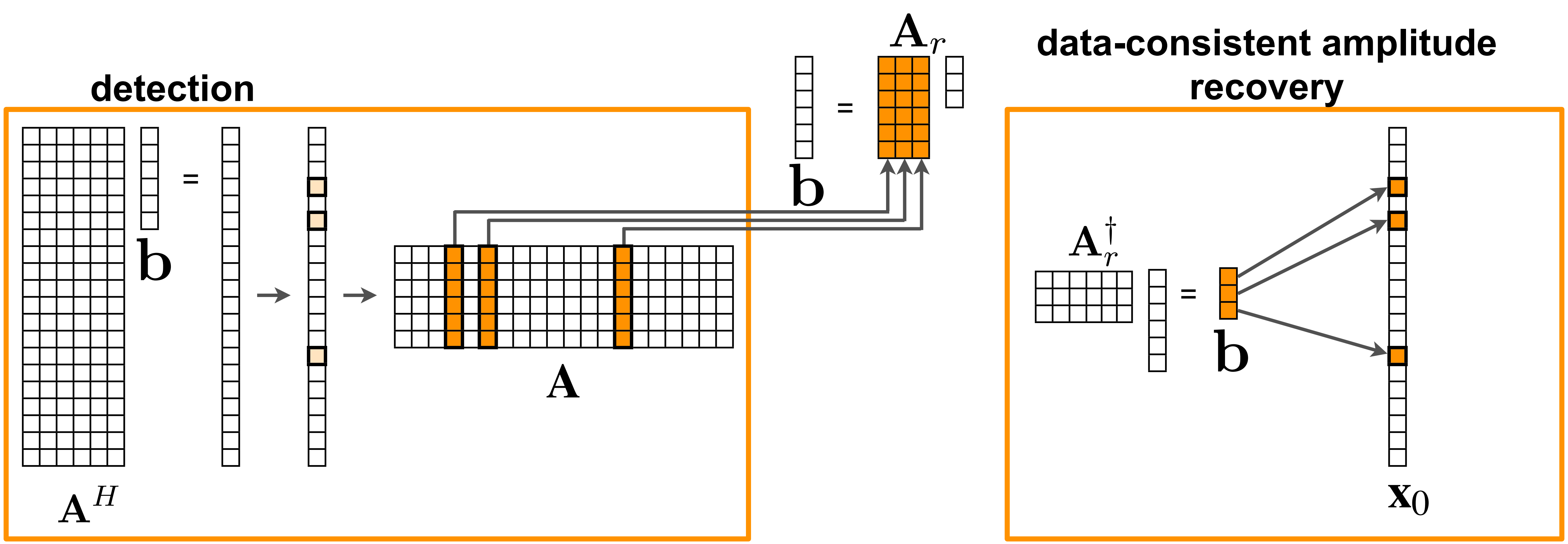
NAIVE sparsity-promoting recovery



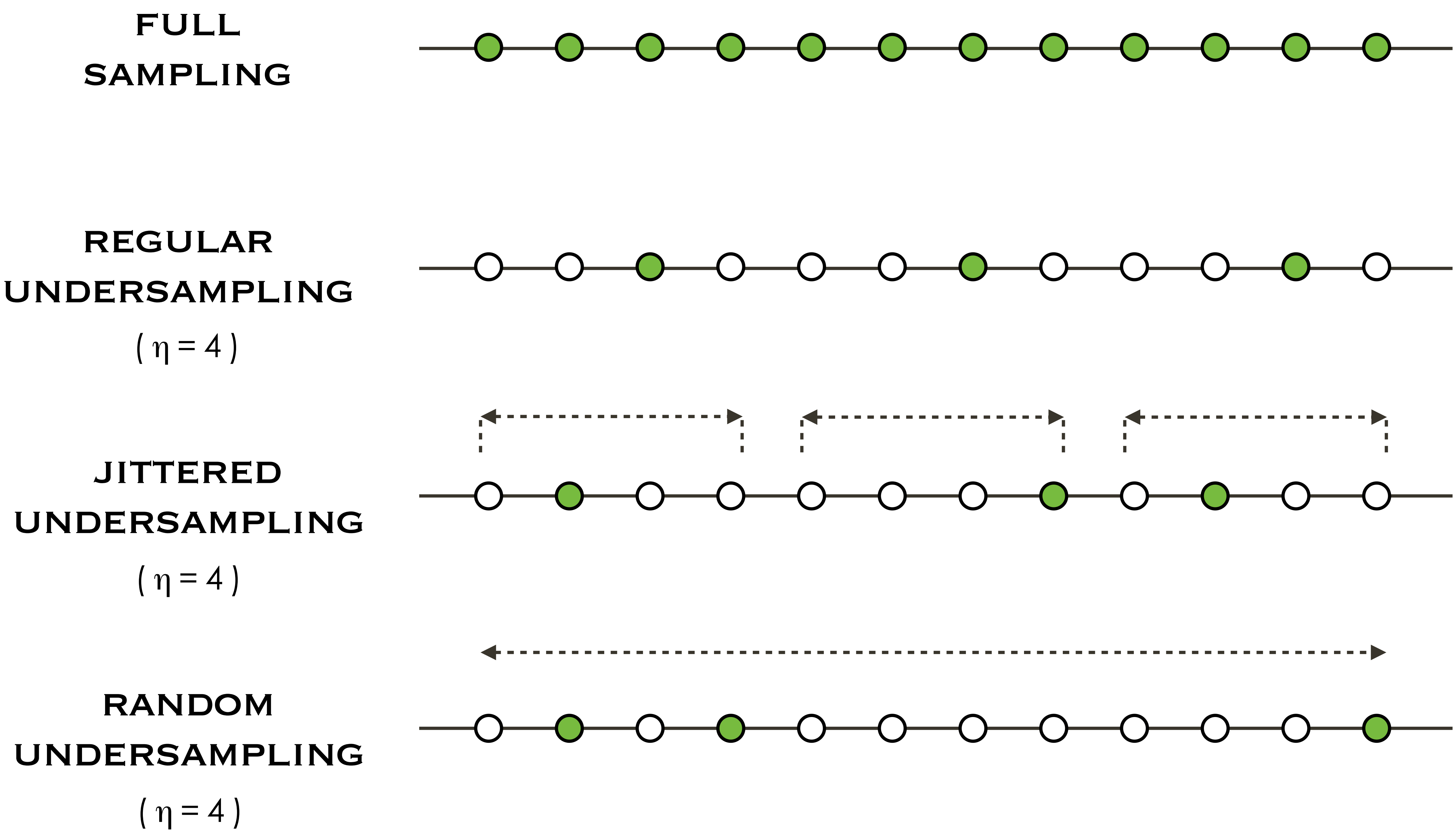
inverse Fourier transform

detection + data-consistent amplitude recovery

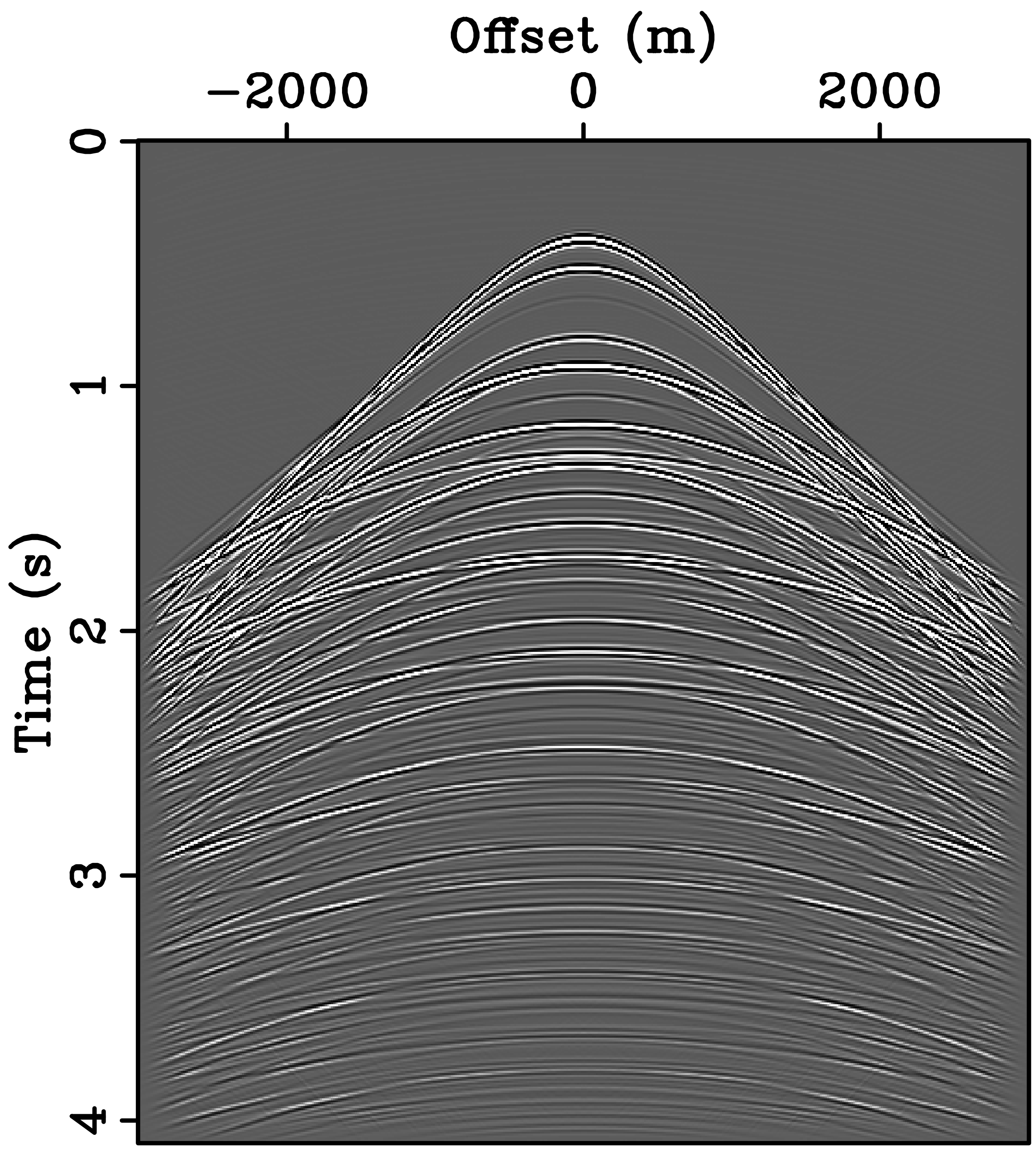
Fourier transform

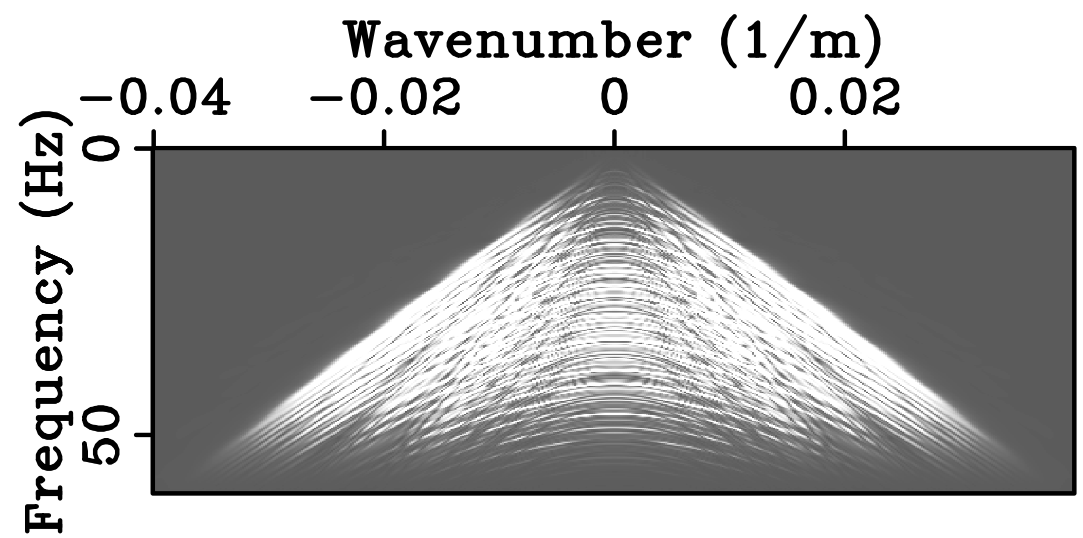


Sampling schemes

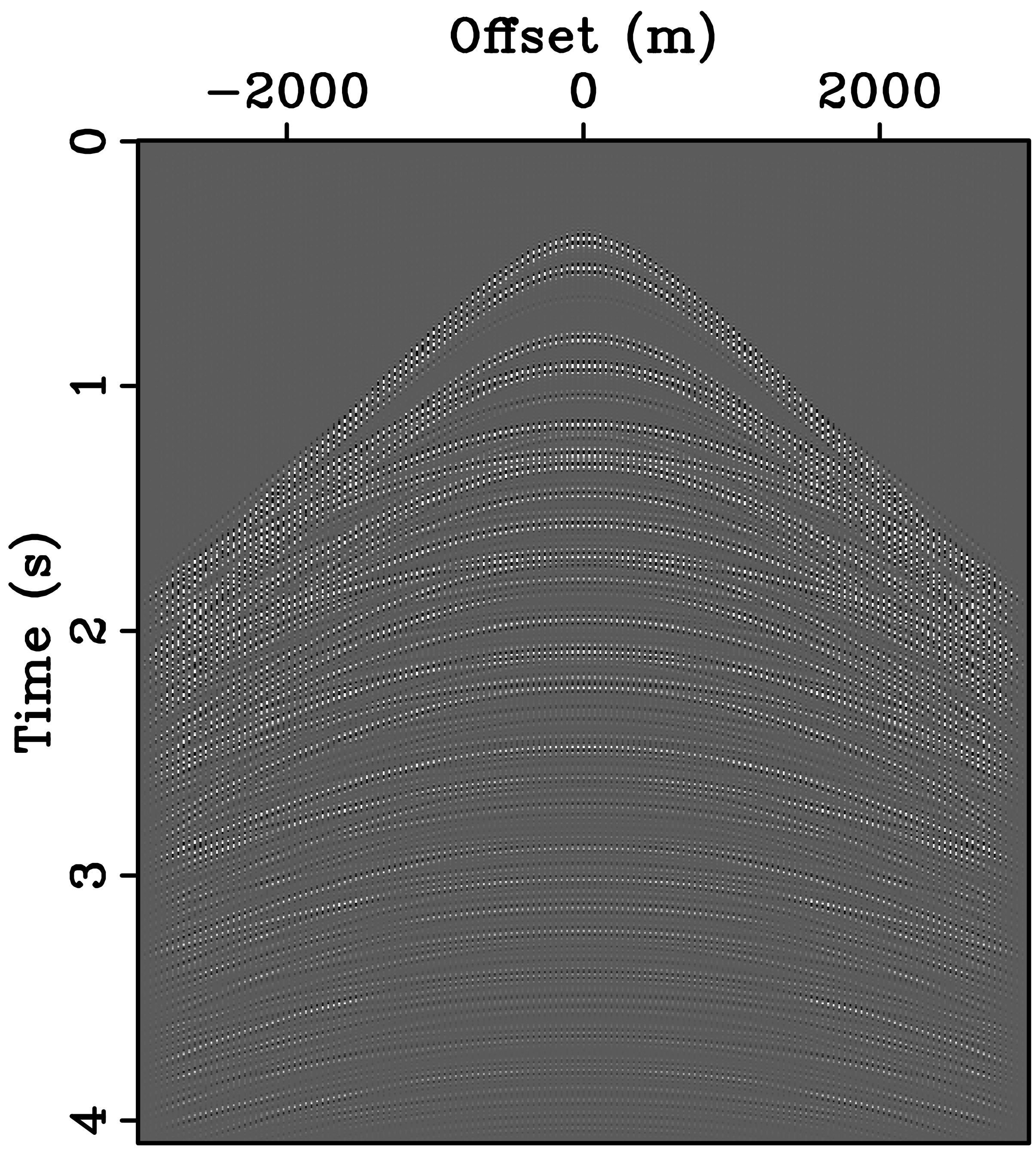


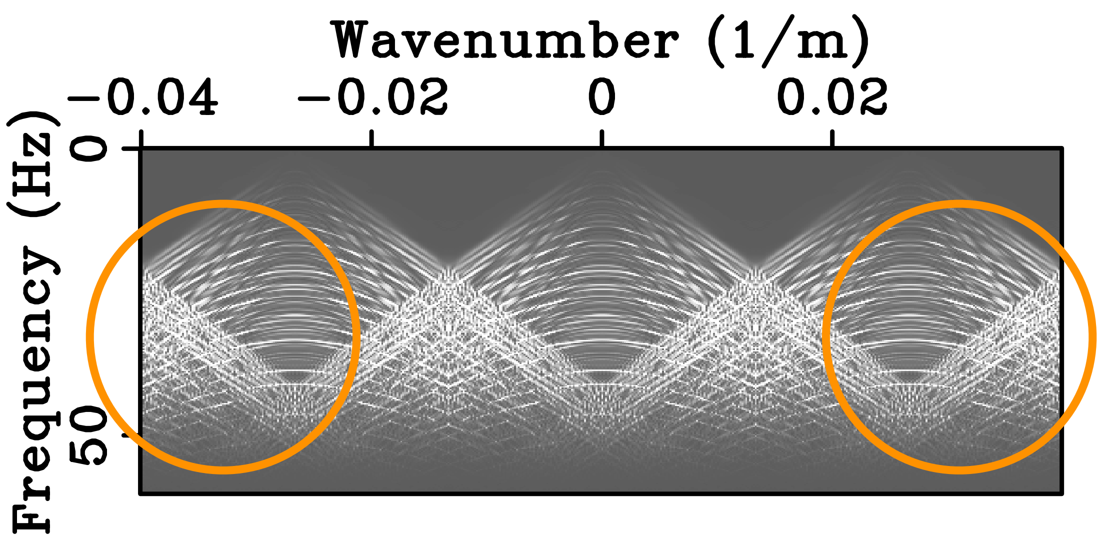
Model


Offset (m)
−2000
0
2000
Time (s)
0
1
2
3
4

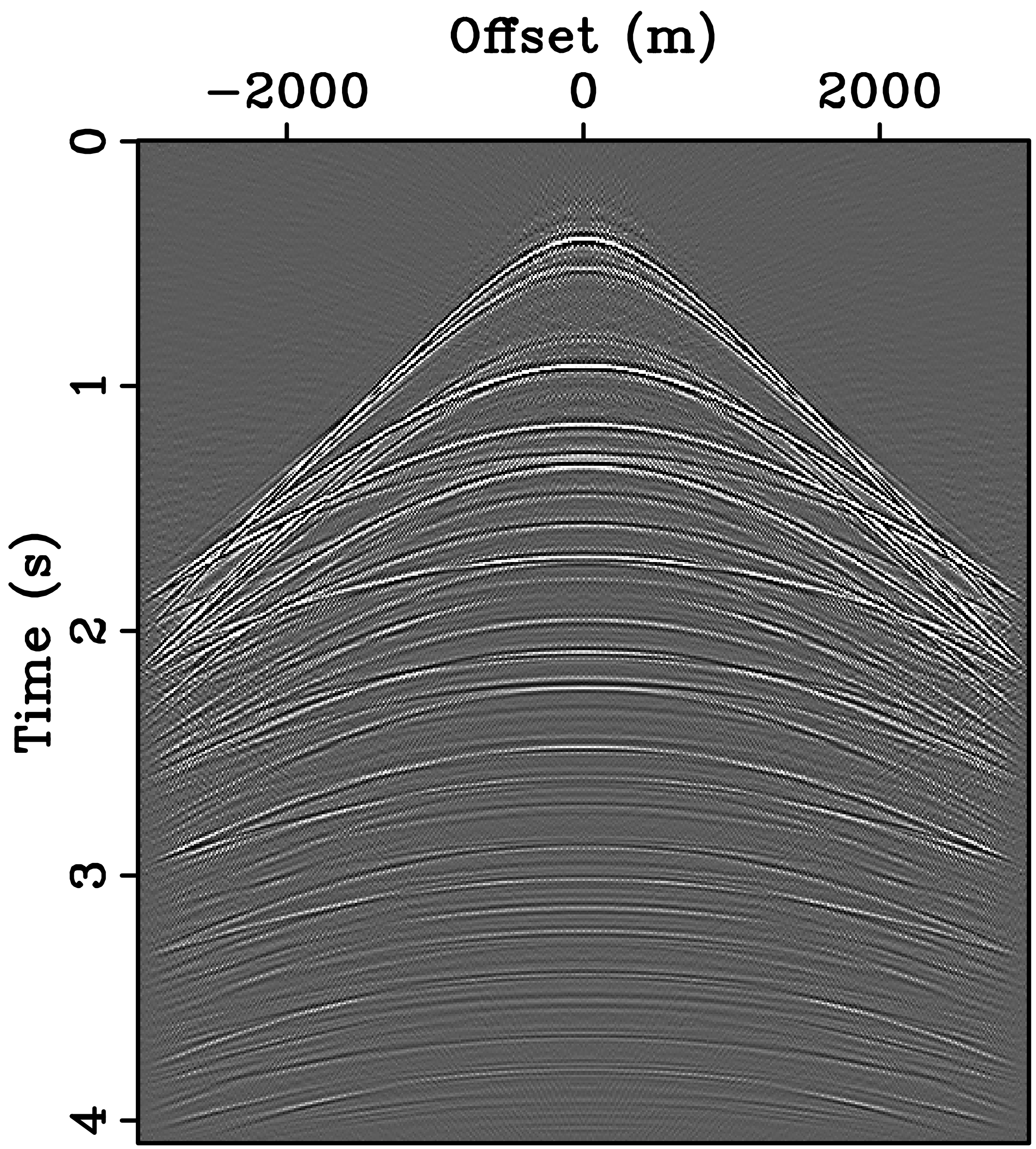

Wavenumber (1/m)
−0.04
−0.02
0
0.02
Frequency (Hz)
0
50

Regular 3-fold undersampling

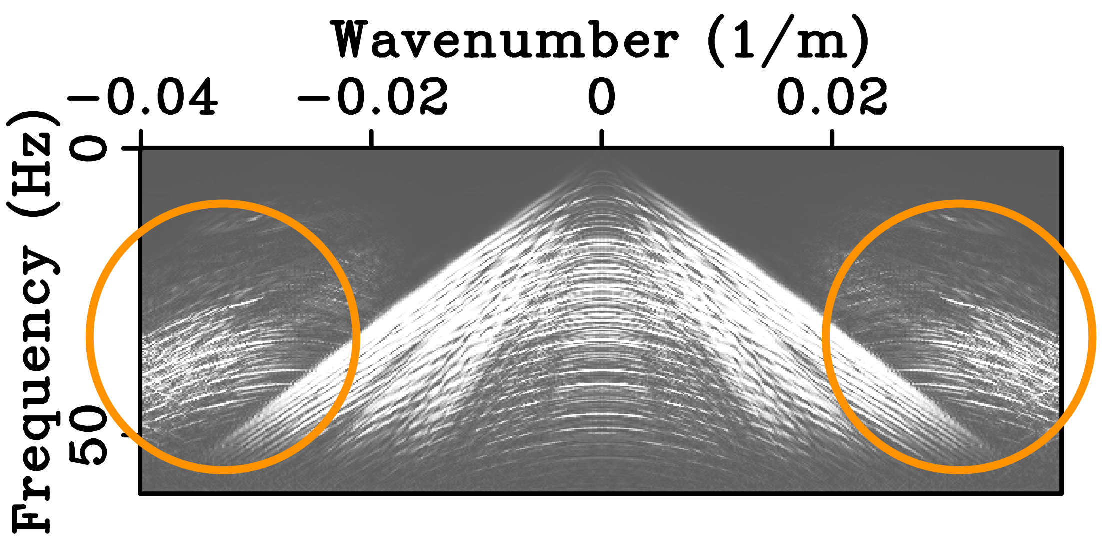




CRSI from regular 3-fold undersampling

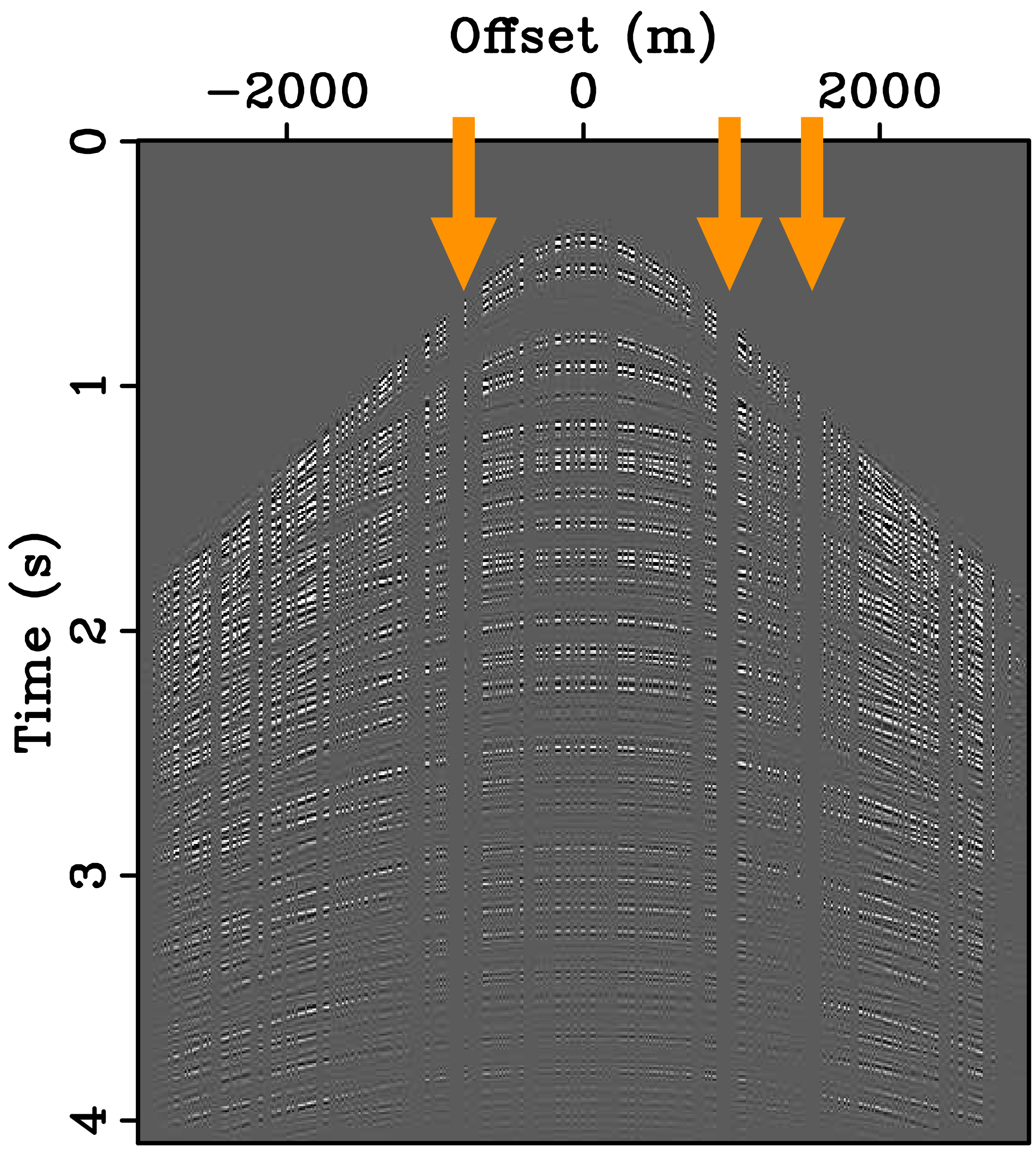


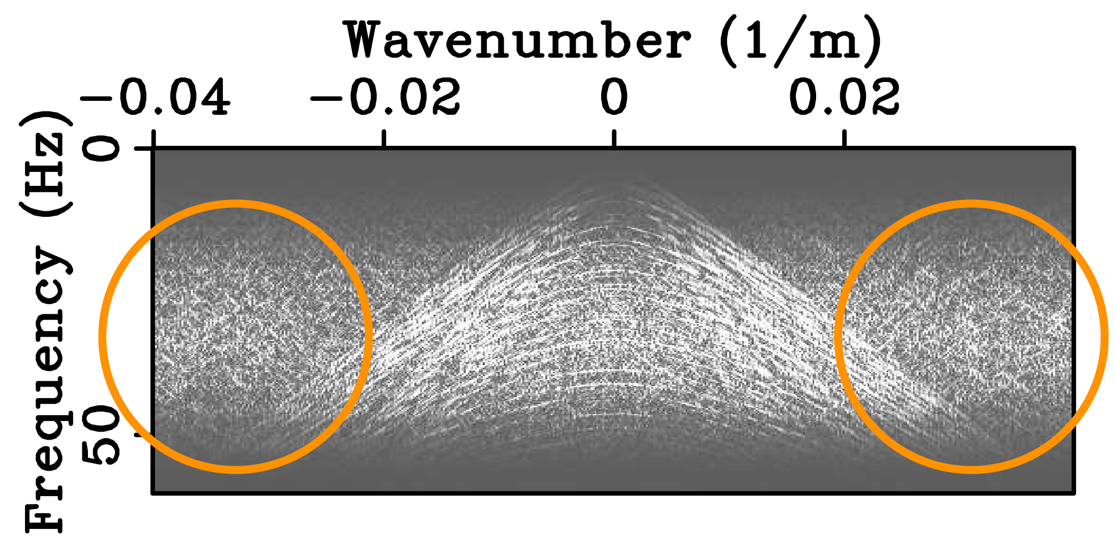
SNR = 6.92 dB



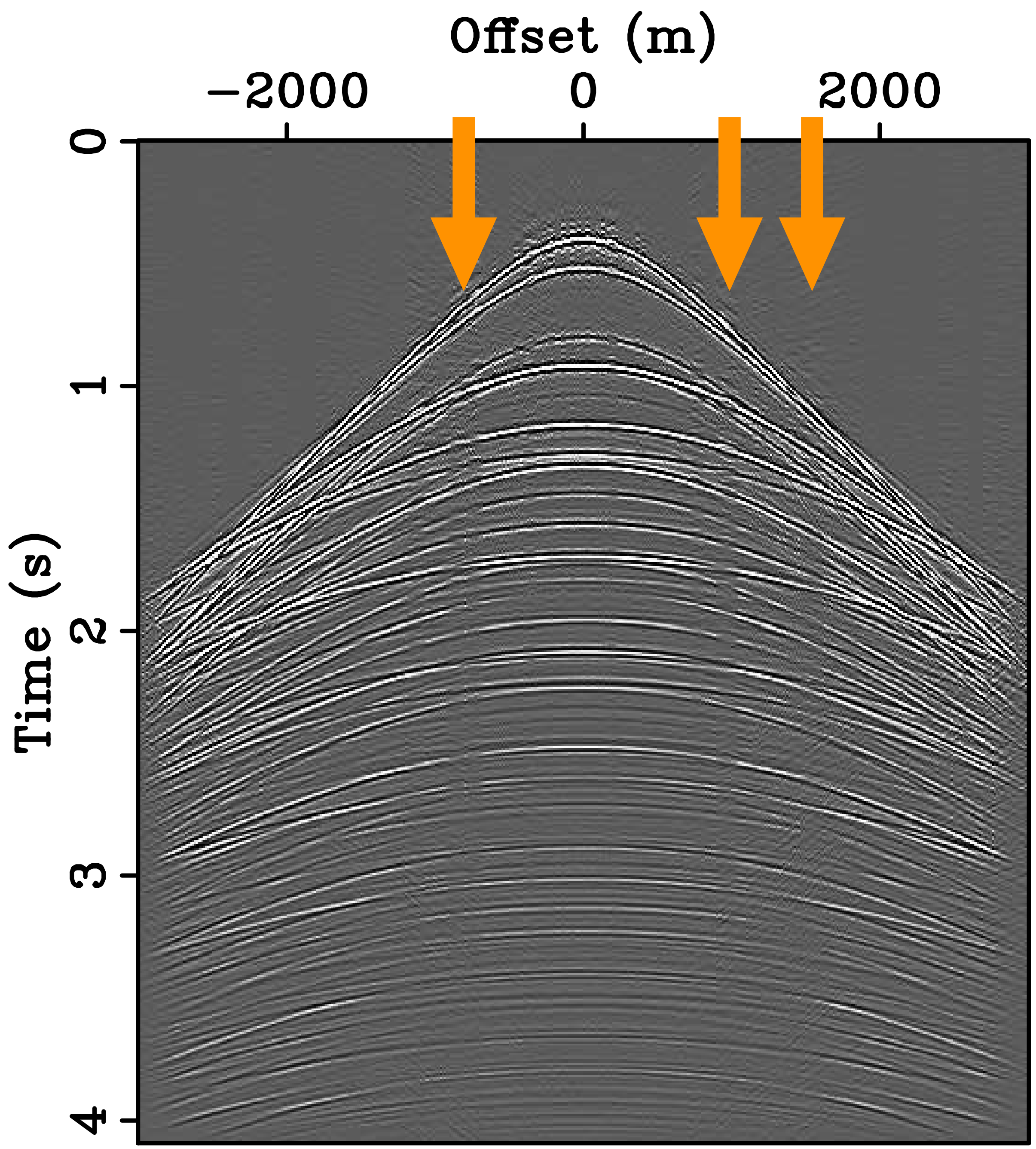
$$\mathrm{SNR} = 20 \times \log_{10}\left(\frac{\|\text{model}\|_2}{\|\text{reconstruction error}\|_2}\right)$$

Random 3-fold undersampling

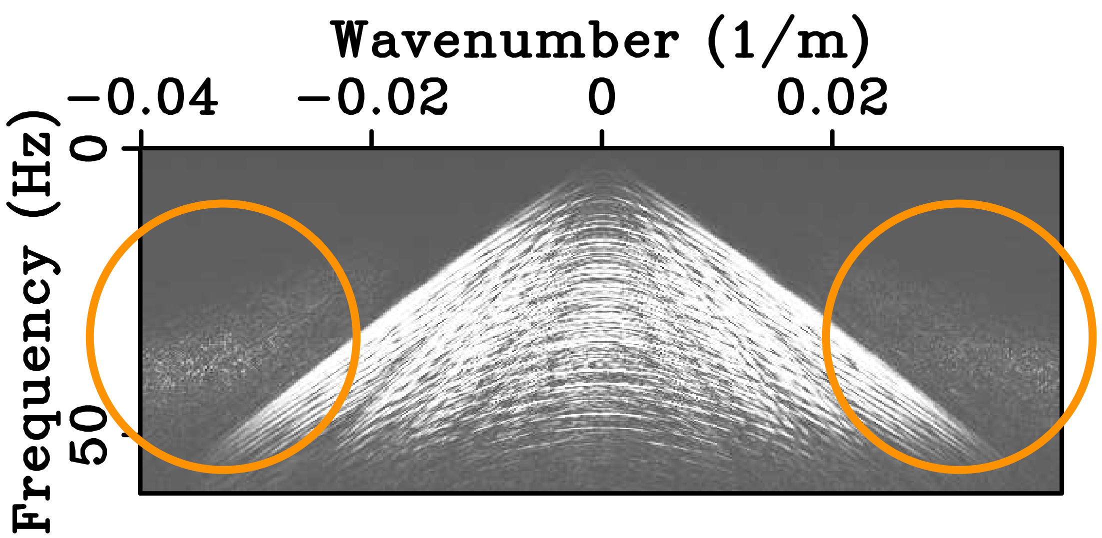




CRSI from random 3-fold undersampling

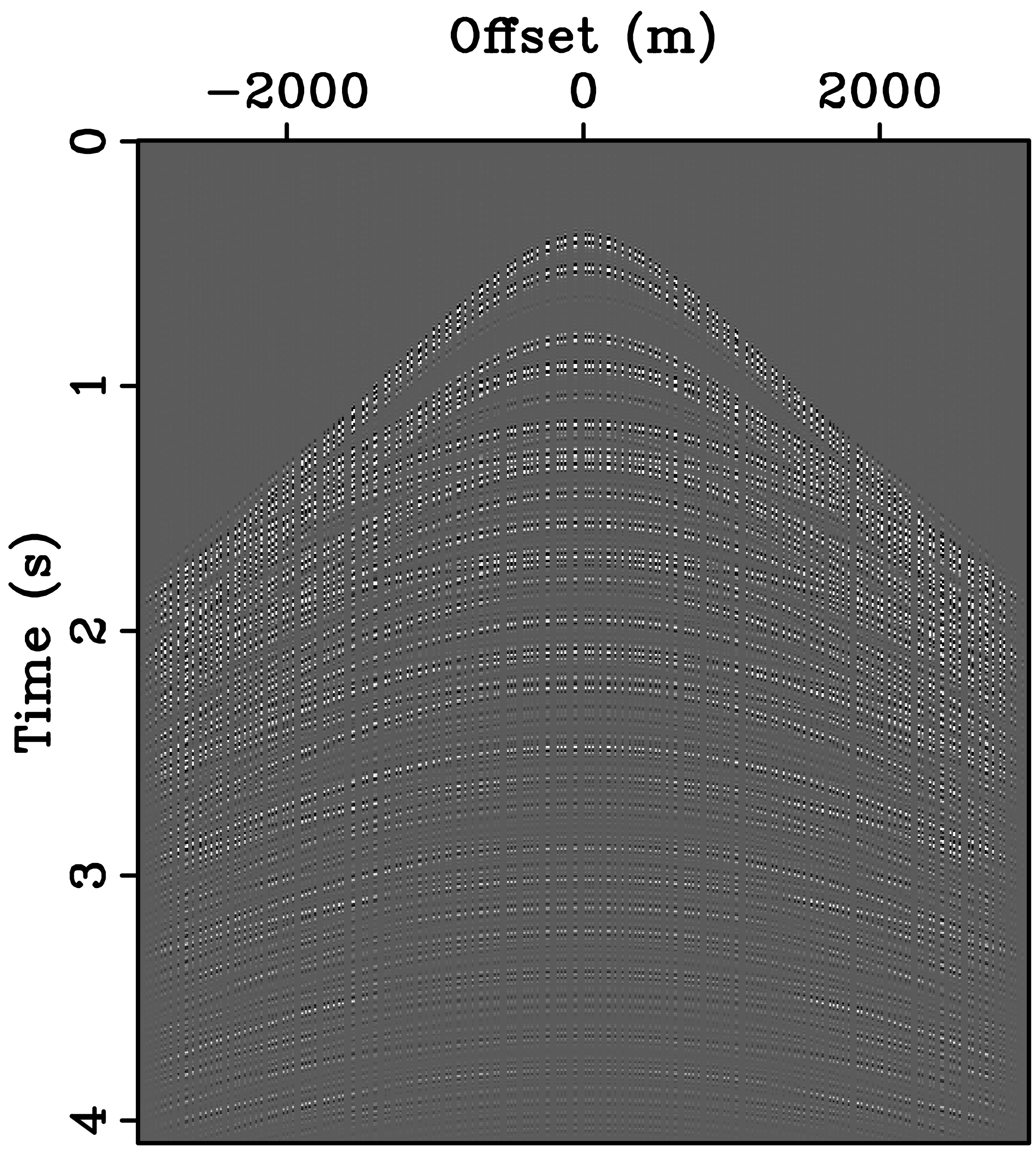


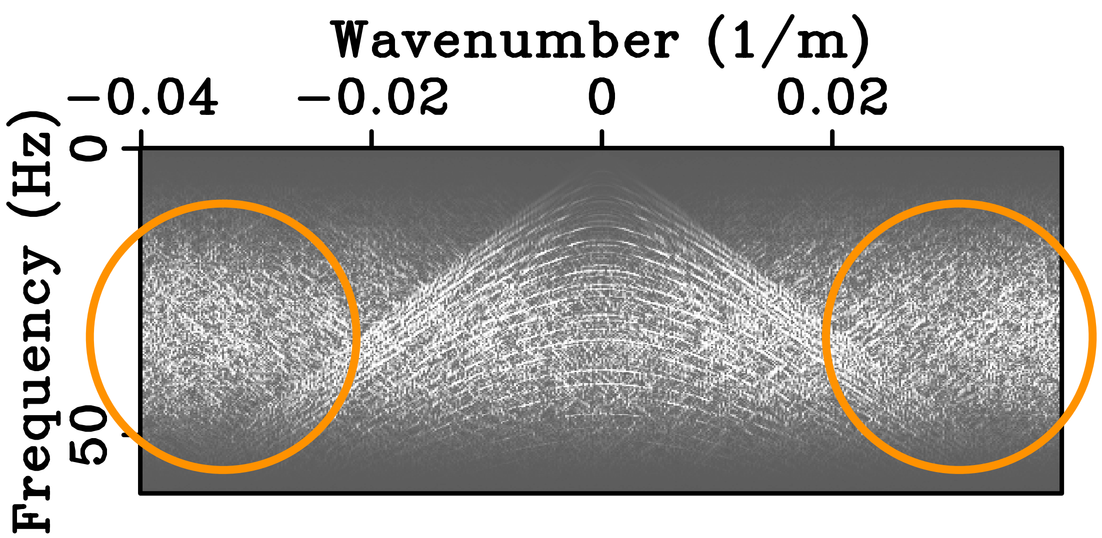
SNR = 9.72 dB



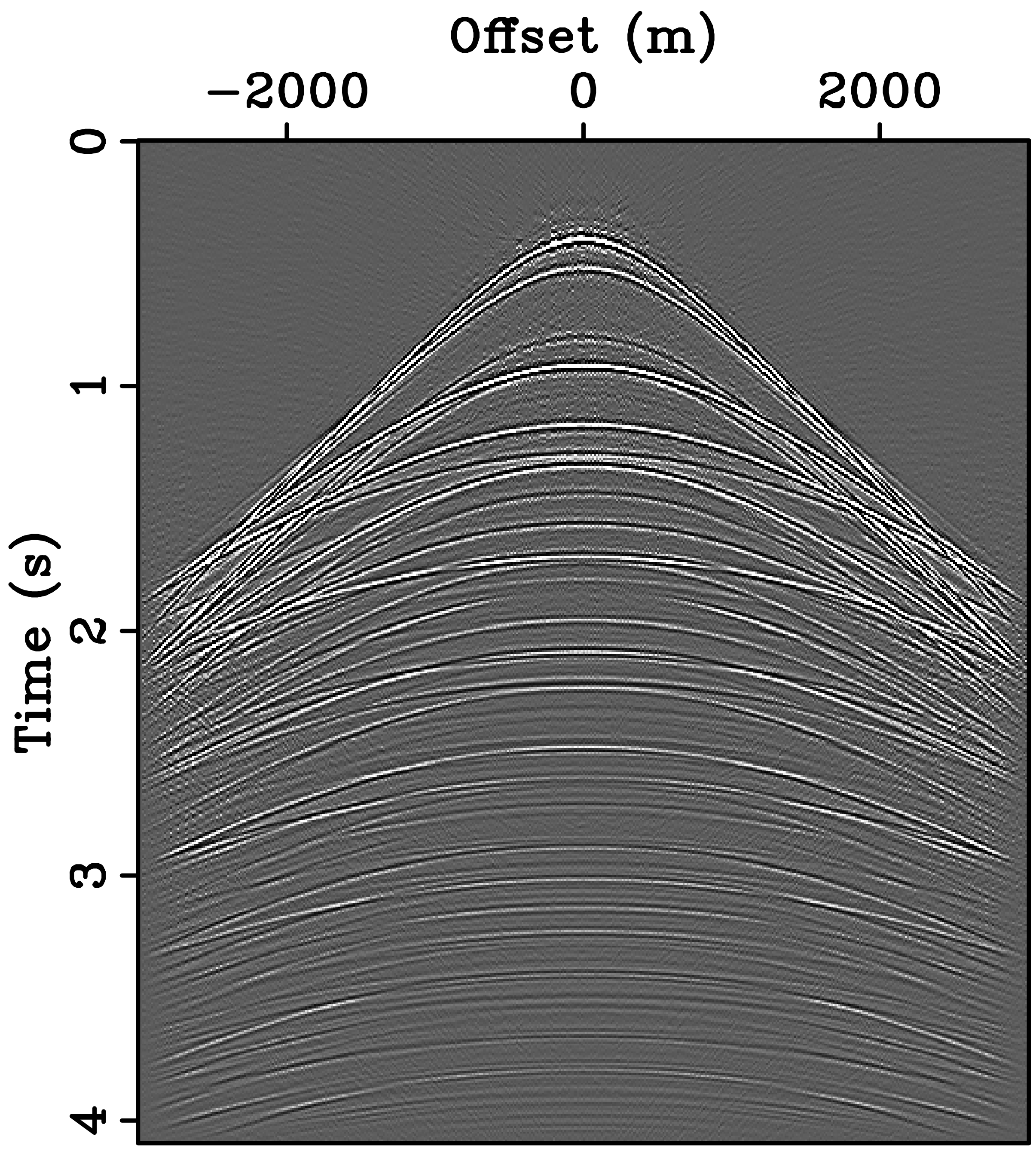
$$\mathrm{SNR} = 20 \times \log_{10}\left(\frac{\|\mathrm{model}\|_2}{\|\mathrm{reconstruction\ error}\|_2}\right)$$

Jittered 3-fold undersampling

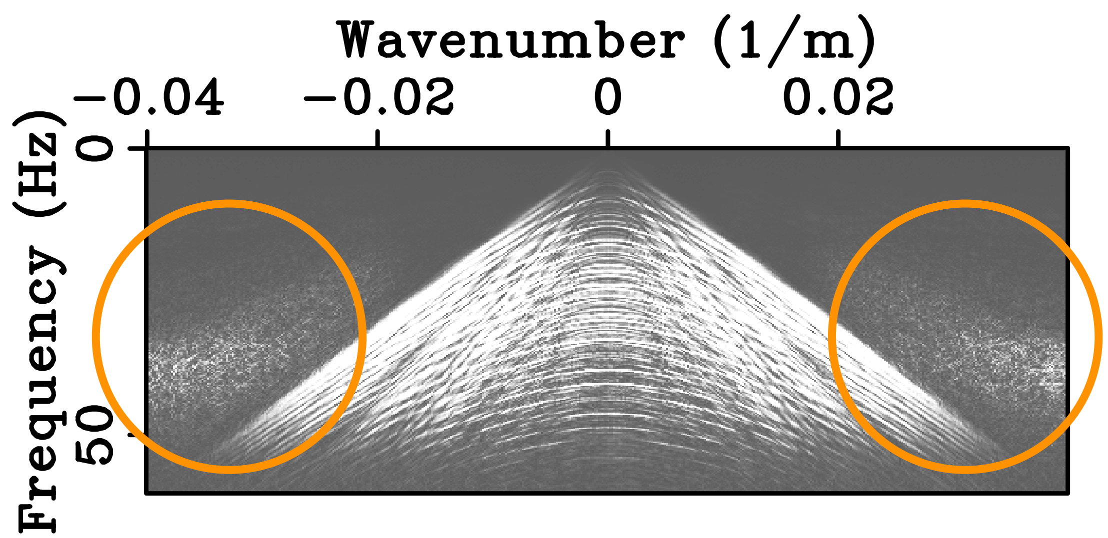




CRSI from jittered 3-fold undersampling



SNR = 10.42 dB



Compressed sensing

How many measurements do we need to make?
Far less than what Shannon tells us.

$$y = Ax$$

x is a vector in $\mathbb{R}^N$
y is a vector in $\mathbb{R}^n$
y is observation
A is a measurement matrix
A has n rows and N columns, where $n \ll M$.

Measurement matrix

$$y = Ax \qquad A \text{ is a matrix}$$

The matrix satisfies the RIP property:
nearly preserve length of sparse vectors.

Fourier matrix is more practical.

Gaussian matrix is easier to analyze .

Random Fourier matrix

- Start with a DFT matrix (N by N)
- Pick n rows at random
- Column entries are not independent
- How to pick the n rows?

Regular sampling:

row 10, 20, 30, 40, 50, 60, ... , 990, 1000

Jittered sampling:

7, 16, 22, 36, 44, 53, 67, ... , 994

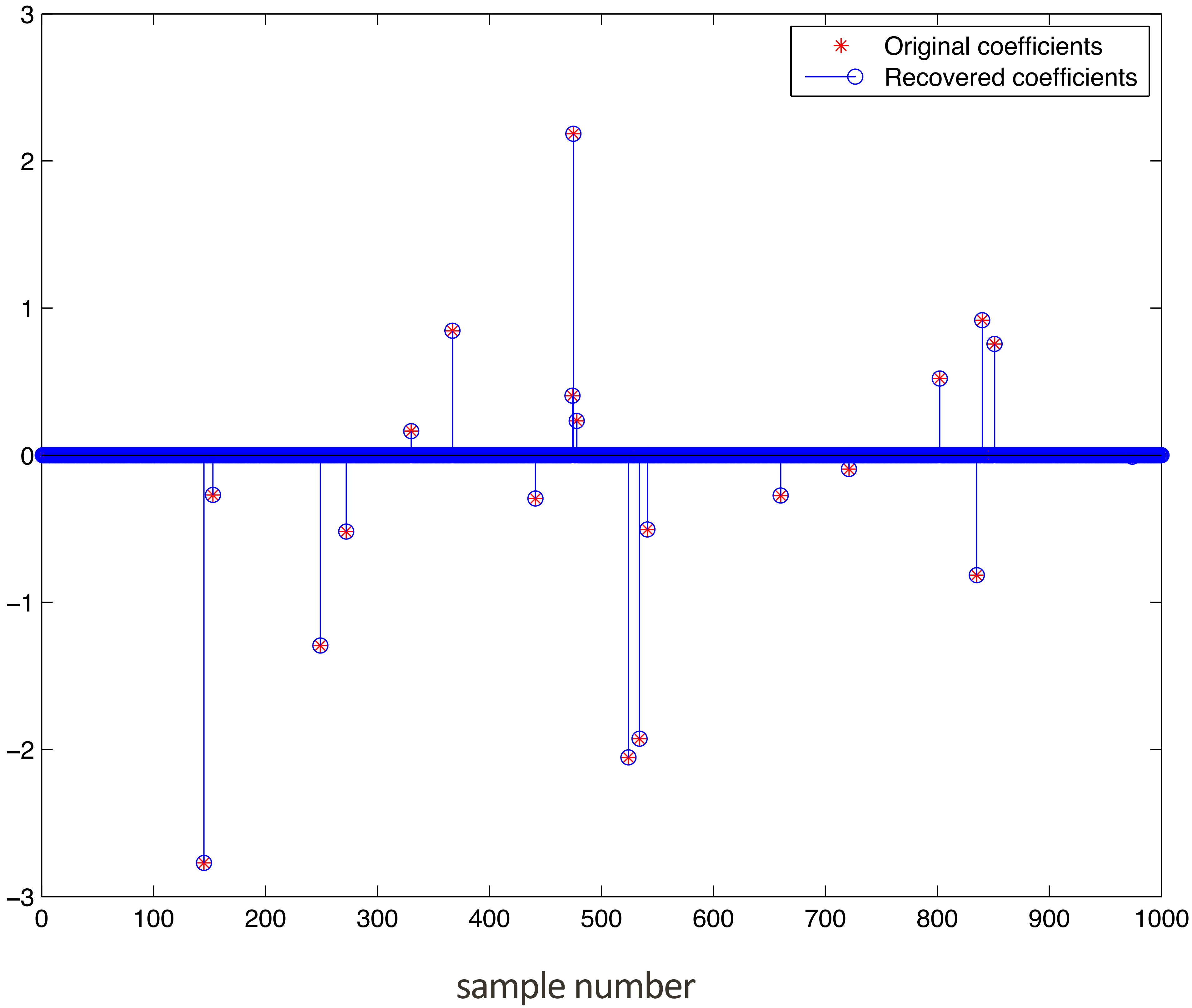
from row 1 to row 10, pick one row

from row 11 to row 20, pick the second row

from row 21 to row 30, pick the third row

Recovery of sparse signal

N = 1000
n = 200
k-sparse
k = 20



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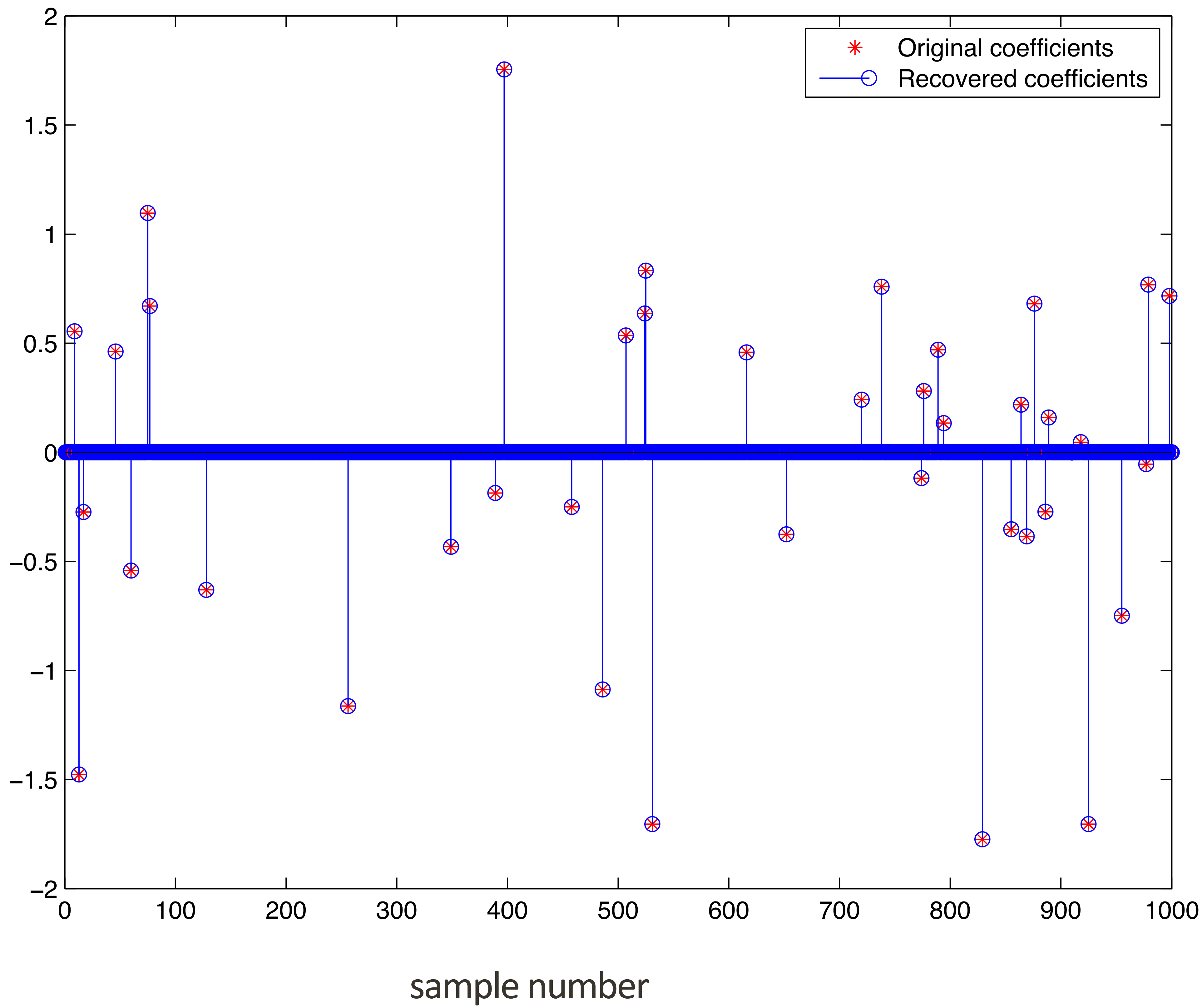
Sparsity

- Compressed sensing can be applied when a signal is sparse.

- What if your signal is not sparse enough?

Recovery of sparse signal

N = 1000
n = 200
k-sparse
k = 40



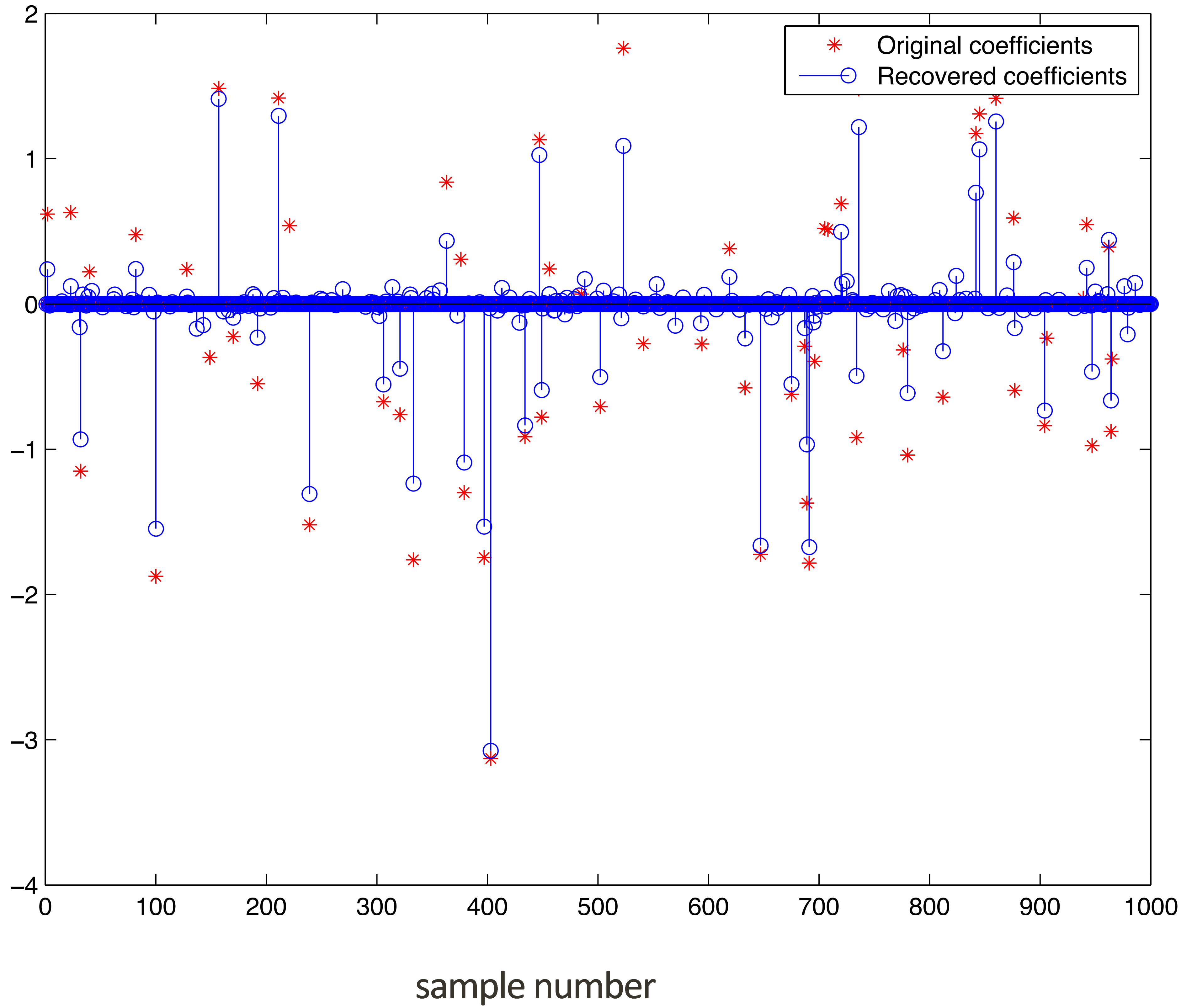
Partial recovery

N = 1000

n = 200

k-sparse

k = 60



When signal is not sparse enough, there can be reconstruction error.

But if most non-zero entries are in the low pass region of the signal, can we recover the signal?

What if most non-zero entries are in low-pass?

Signal in 1000 dimensions is 60-sparse, with 50 entries in the low pass and 10 entries in the high pass.

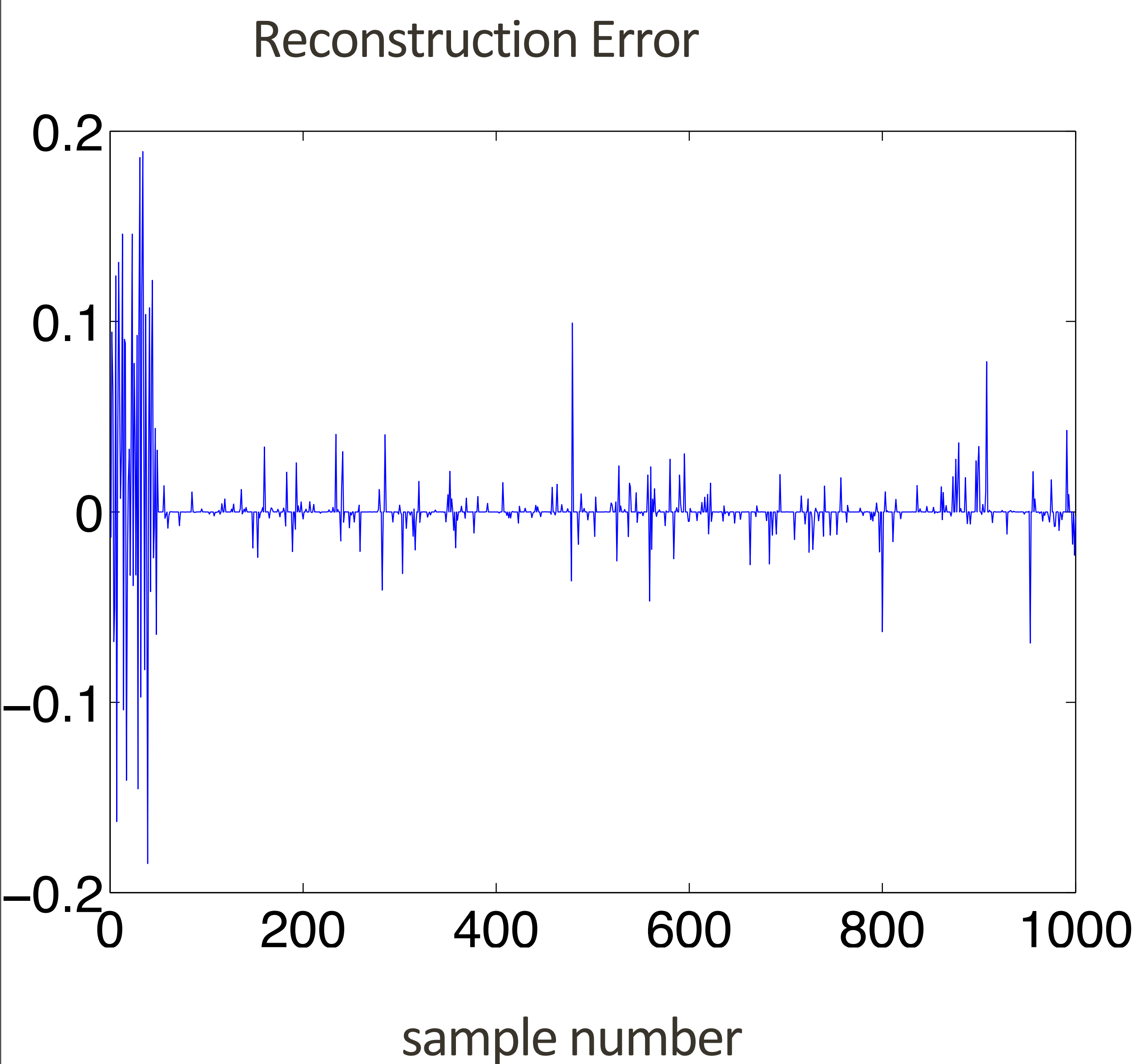
Suppose we make 200 measurements.

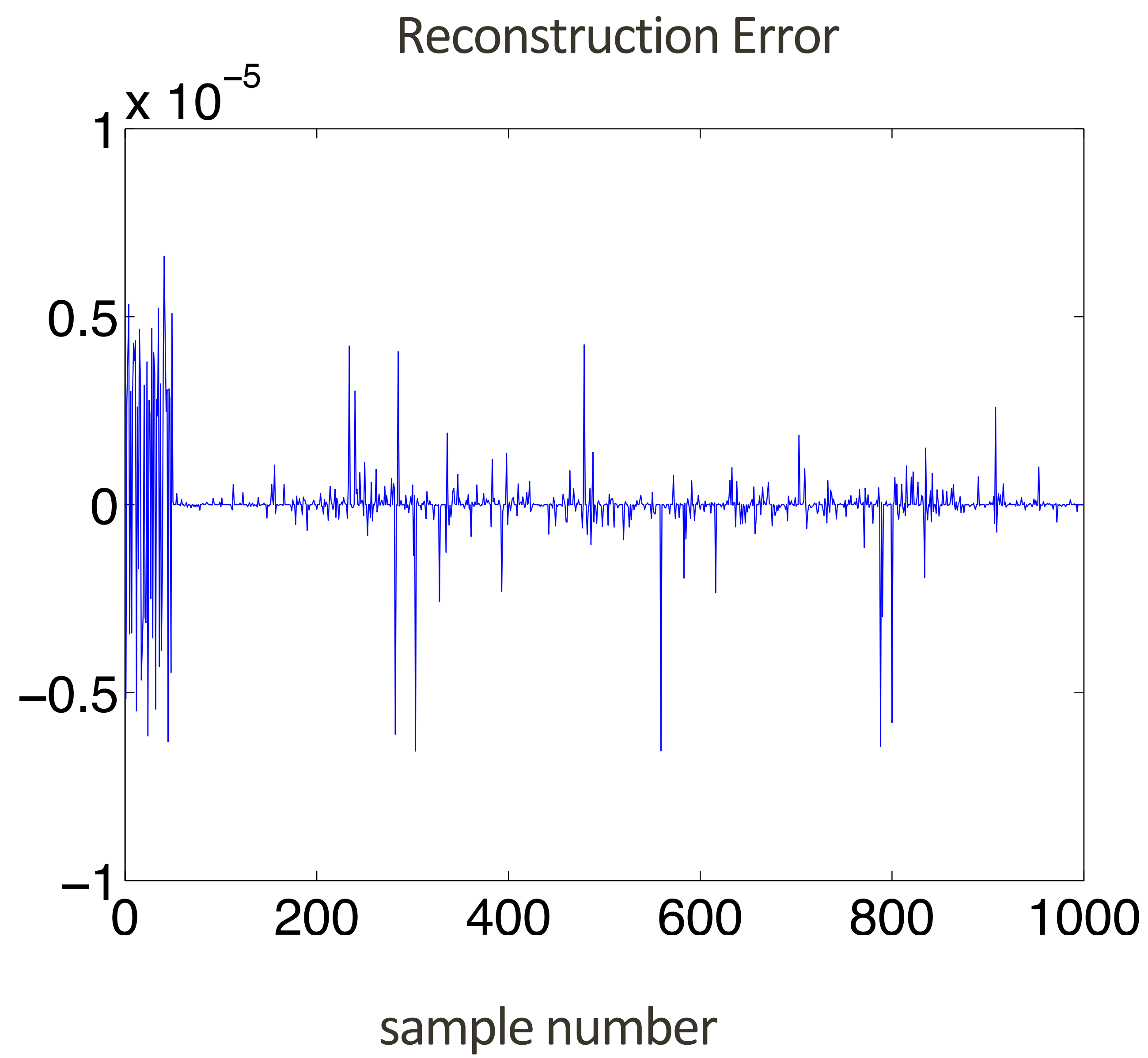
Goal: Compare the reconstruction error in Uniform sampling versus Jittered sampling.

Reconstruction error
= Difference between actual and reconstructed signal

Uniform

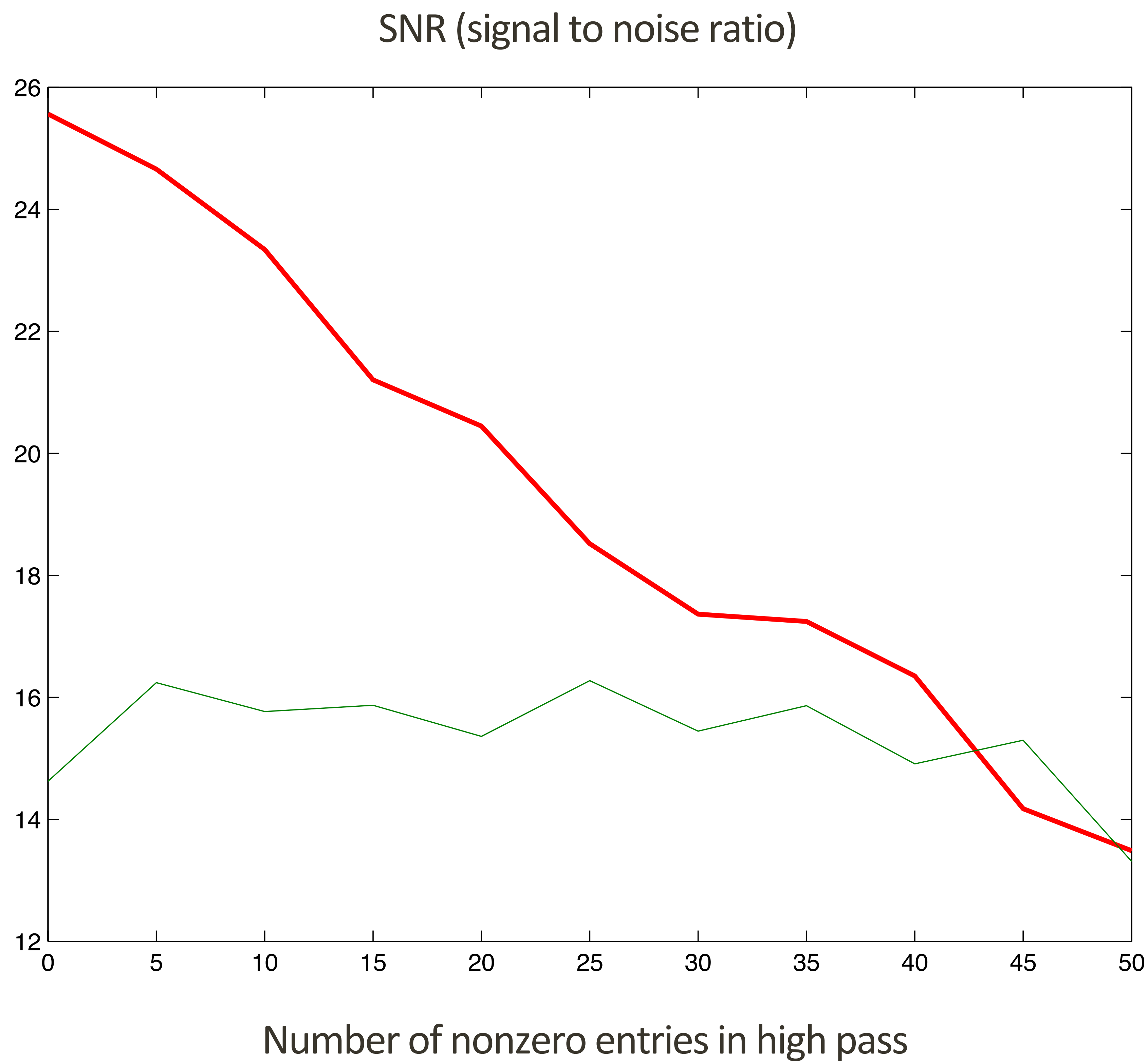
Jittered





Better recovery on lowpass

Jittered and Uniform sampling



N = 1000
n = 200
k-sparse
k = 60

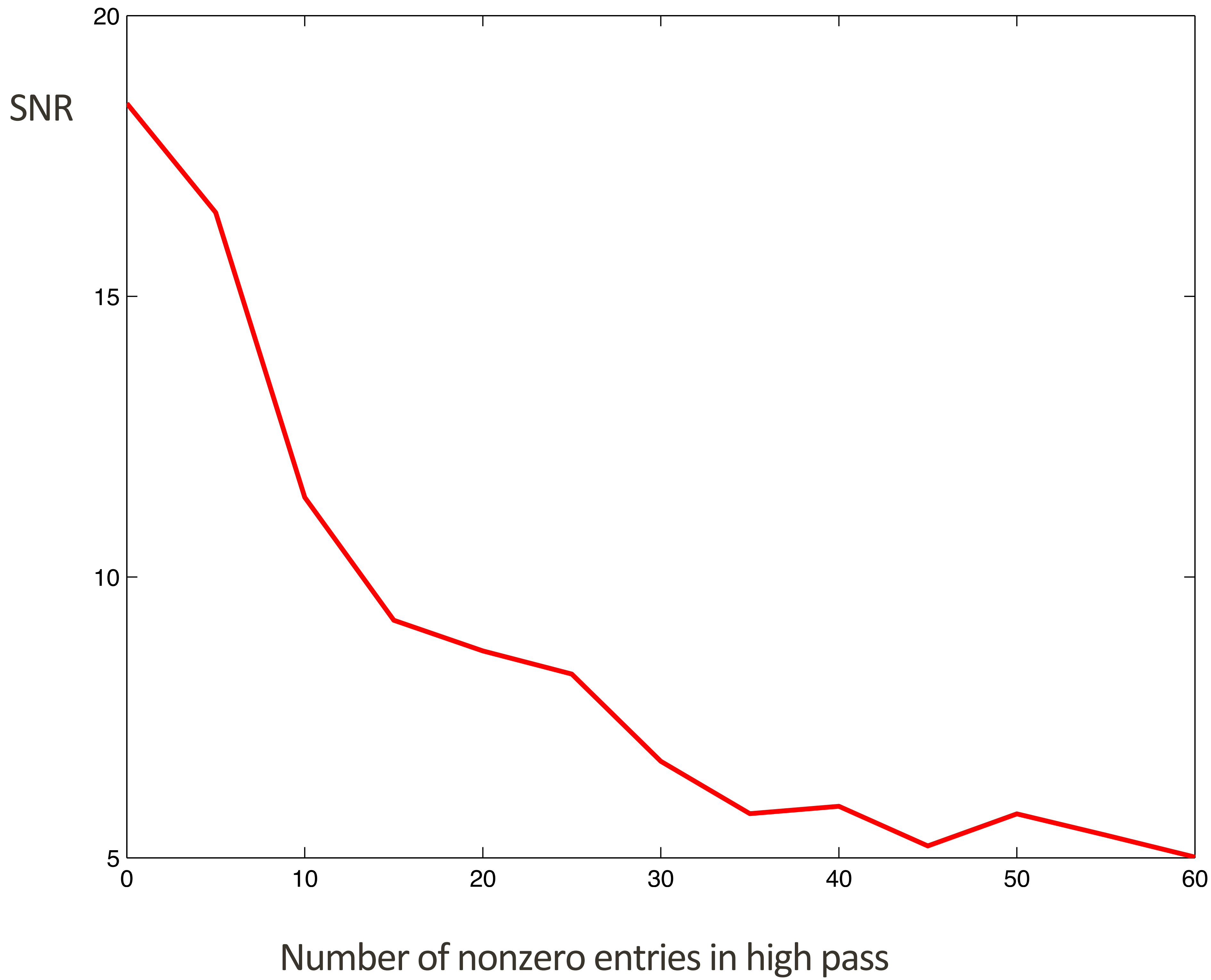
SNR decreases as frequency increases

N = 1000

n = 200

k-sparse,

k = 70



What do you gain?

Uniform: flat error as function of frequency

Jittered: SNR increases as frequency decreases

Moral of the story

What is one main advantage of jittered sampling?

Better reconstruction on low pass and without doing worse on high pass.

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Potential improvement

- Jittered sampling is better than uniform sampling
- Understand why
- Do educated jittered sampling instead of random jitter

Exploit sparsity structure

Data = sum of wave atoms

Each wave atom has the same shape.

$$X = c_1\psi_1 + c_2\psi_2 + \ldots + c_N\psi_N$$

Suppose coefficients are organized in a tree structure.

Better reconstruction

- What does it mean to reconstruct a signal with high probability?
- Uniform recovery implies non-uniform recovery, but converse is not true

Uniform recovery

Uniform recovery means that once the random matrix is chosen, then with high probability, all k-sparse signals can be recovered.

Non-uniform recovery states only that each k-sparse signal can be recovered with high probability using a random draw of the DFT matrix.

Theory behind the scene

$$B = \{x \in \mathbb{R}^N : \|x\|_2 = 1\}$$

Theorem: Let x, y be any two vectors in B.
Let A be a random projection into k-dimensional subspace.
Then the following holds.

$$Pr\left(| \; \|Ax\|_2 - \|Ay\|_2 \; | > u\sqrt{\frac{k}{n}} \; \|x - y\|_2 \right) \leq \exp(-cu^2 k)$$

Dimensional reduction

What does the theorem say in plain English?

$$N = 1000 \qquad k = 60 \qquad \mathbb{R}^N \to \mathbb{R}^k$$

Points in very high dimension that were near to each other, when you project them to lower dimension, (most likely) there will not be too much distortion.

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Theorem (on Dimensional reduction) applies to every random projection. It is stronger than the following theorem:

Given any two nearby points x and y, there exists a random projection matrix A such that Ax and Ay will not be distorted too much.

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Theorem (on Dimensional reduction) applies to every random projection. It is stronger than the following theorem:

Given any two nearby points x and y, there exists a random projection matrix A such that Ax and Ay will not be distorted too much.

This A can be taken to be a Gaussian matrix.
[Concentration of Measure argument does not apply to random Fourier matrix.]

Conclusion

Jittered sampling leads to better reconstruction on low pass, without doing worse on high pass.

Thank You!

Acknowledgement

- Ozgur Yilmaz
- Felix Herrmann
- Haneet Wason, Hassan Mansour
- All siblings from the SLIM family

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