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# Robust FWI Using Student's t-distribution

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# Full Waveform Inversion

- **The Full Waveform Inversion (FWI) problem is to find solutions to the Helmholtz PDE that match data from source experiments on the surface**
- **Problems are typically very large: billions of variables and terabytes of data.**
- **Typically formulated as a Nonlinear Least Squares (NLLS) problem:**

$$\min_{\mathbf{m}} \quad \left\{ f(\mathbf{m}) := \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m};\mathbf{Q}]\|_F^2 \right\}$$

$$\begin{aligned}
\mathbf{D} &:= \text{data} \\
\mathbf{m} &:= \text{model parameters (speed or slowness squared)} \\
\mathbf{Q} &:= \text{multiple source experiments} \\
\boldsymbol{\mathcal{F}} &:= \text{solution operator of Helmholtz eqn. with absorbing boundary}
\end{aligned}$$

# Statistical Implications

- The NLLS formulation is equivalent to the following statistical model:

$$\begin{aligned} \mathbf{D} &= \mathcal{F}[\mathbf{m};\mathbf{Q}] + \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon} &\sim \mathbf{N}(0, I) \end{aligned}$$

- Equivalence follows from maximum likelihood estimate for model parameters:

$$\mathcal{L}(\mathbf{m}) \quad \propto \quad \exp\left(-\frac{1}{2}\left\|\mathbf{D} - \mathcal{F}[\mathbf{m};\mathbf{Q}]\right\|_F^2\right)$$

- Minimizing the negative log likelihood is exactly the FWI problem.

- Q: So what?

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# Practical Consequences

- **Large deviations from the mean are VERY unlikely in the Gaussian model:**

| | Gaussian |
|---|---|
| $p(\|x-\mu\| > 4\sigma)$ | $6.3 \times 10^{-5}$ |
| $p(\|x-\mu\| > 8\sigma)$ | $1.3 \times 10^{-15}$ |

- **Observations more than 4 standard deviations away from the mean occur less than .006 percent of the time.**
- **As we get further away, the likelihood shrinks astronomically.**
- **Low likelihood values correspond to HIGH penalties for outliers.**

# 'Outliers' in FWI??

- Mathematical model cannot distinguish 'artifacts' from 'outliers'. Any unexplained events in the residual will have a strong effect on the final image.

- Examples:
  1) Modeling Inelastic/Anisotropic data with Acoustic PDE
  2) Ignoring Acquisition Models

  [Brossier '10]

- Key point: models are improving all the time, but are never perfect. It is worthwhile to have methods that still perform well when models are wrong.

- Q: How do we design such methods?

# Statistical Modeling

- **We can tweak the assumptions on the NOISE in the model:**

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\mathcal{F}}[\mathbf{m};\mathbf{Q}] + \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon} &\sim \text{Heavy Tailed Distribution} \end{aligned}$$

- **The parametric form of the distribution then determines the optimization formulation:**

$$\min_{m} \quad -\log(\boldsymbol{\mathcal{L}}(\mathbf{m})) := -\log\Big(f\left(\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m};\mathbf{Q}]\right)\Big)$$

- **Q: Which distribution do we choose, and how do we solve the problem?**

# A Simple Comparison

- **We present a comparison with two other distributions:**

| | Gaussian | $L(\lambda = 1)$ | $T(df = 3)$ |
|---|---|---|---|
| $p(\|x - \mu\| > 4\sigma)$ | $6.3 \times 10^{-5}$ | $1.8 \times 10^{-2}$ | $0.6 \times 10^{-2}$ |
| $p(\|x - \mu\| > 8\sigma)$ | $1.3 \times 10^{-15}$ | $3.3 \times 10^{-4}$ | $8.1 \times 10^{-4}$ |

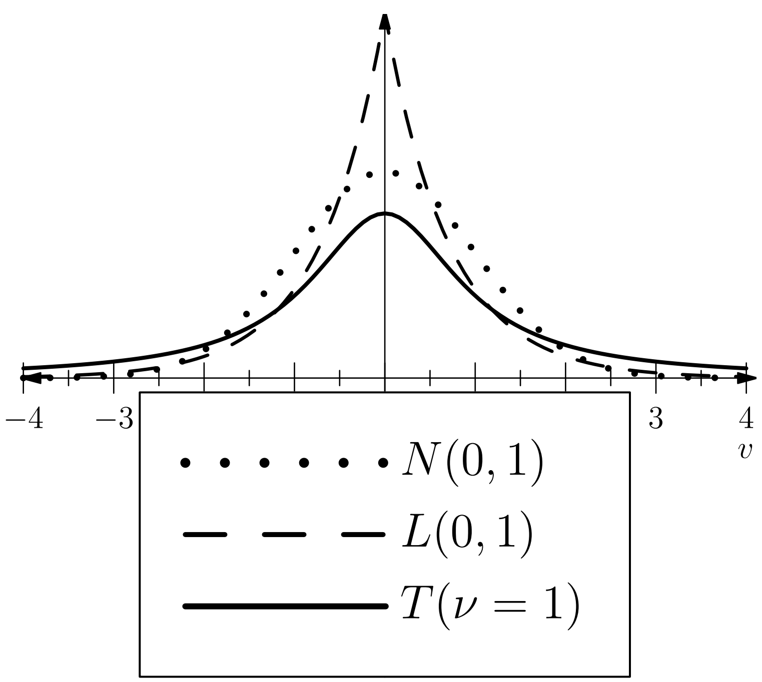
- **The Laplace distribution corresponds to the L1 penalty on the misfit:**

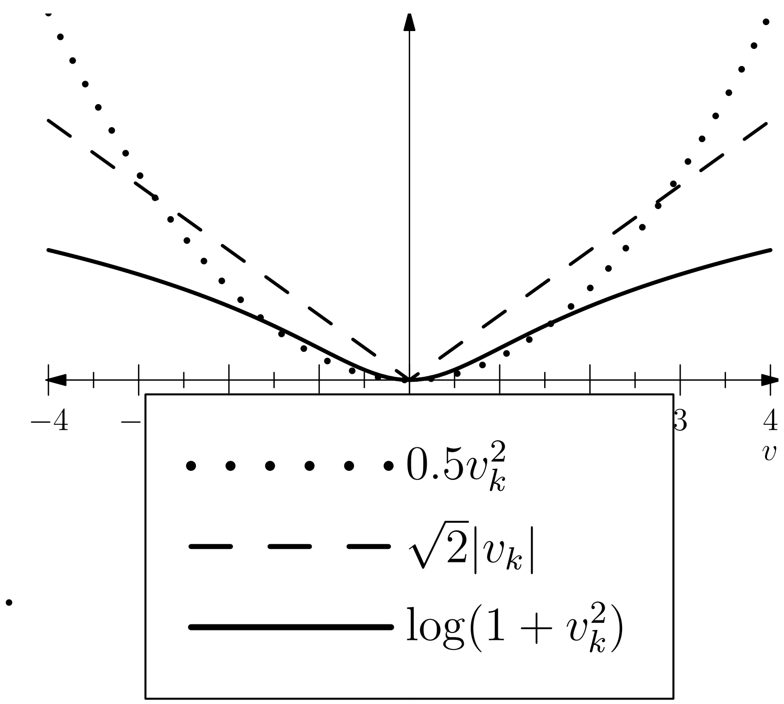
$$\|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m}; \mathbf{Q}]\|_1$$

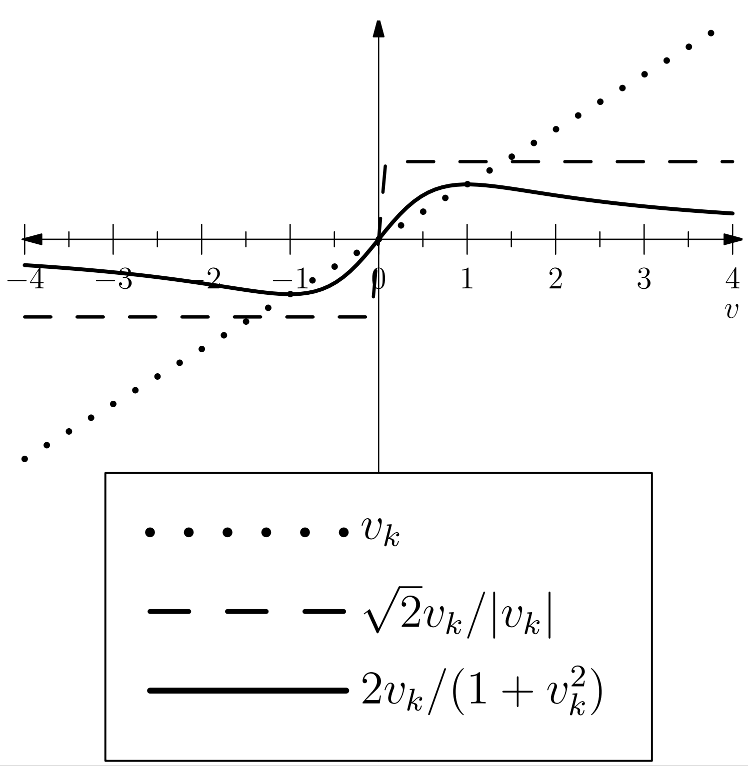
- **In the class of CONVEX negative log likelihoods, it has the heaviest tail (as does the distribution corresponding to the Huber misfit).**

- **But the full problem is non-convex anyway, so let's consider Student's t!**

# Densities, Penalties, and Gradients



# Student's t-Distribution

- **Student's t-density**

$$\mathbf{p}(\epsilon|\mu, M) \quad = \quad \frac{\Gamma(\frac{s+l}{2})}{\Gamma(\frac{s}{2})\det[\pi s M]^{1/2}}\left(1 + \frac{\|\epsilon - \mu\|^2_{M^{-1}}}{s}\right)^{\frac{-(s+l)}{2}}$$

- **Student's t-objective for Seismic case (negative log likelihood)**

$$\min_m \quad \phi_T(m) := \sum_i \frac{s+l}{2} \log\left(s + (D - PH[m]^{-1}Q)^2_i\right)$$

- **Student's t-gradient**

$$\nabla_m \phi_T(m) = \sum_i \frac{s+l}{2} \frac{\nabla_m F[m;Q]^T_i (F[m;Q] - D)_i}{s + (D - F[m;Q])^2_i} \, .$$

# Gradient Comparison:

NLLS

$$\sum_i \frac{1}{2} \nabla_m F[m;Q]_i^T (F[m;Q]-D)_i$$

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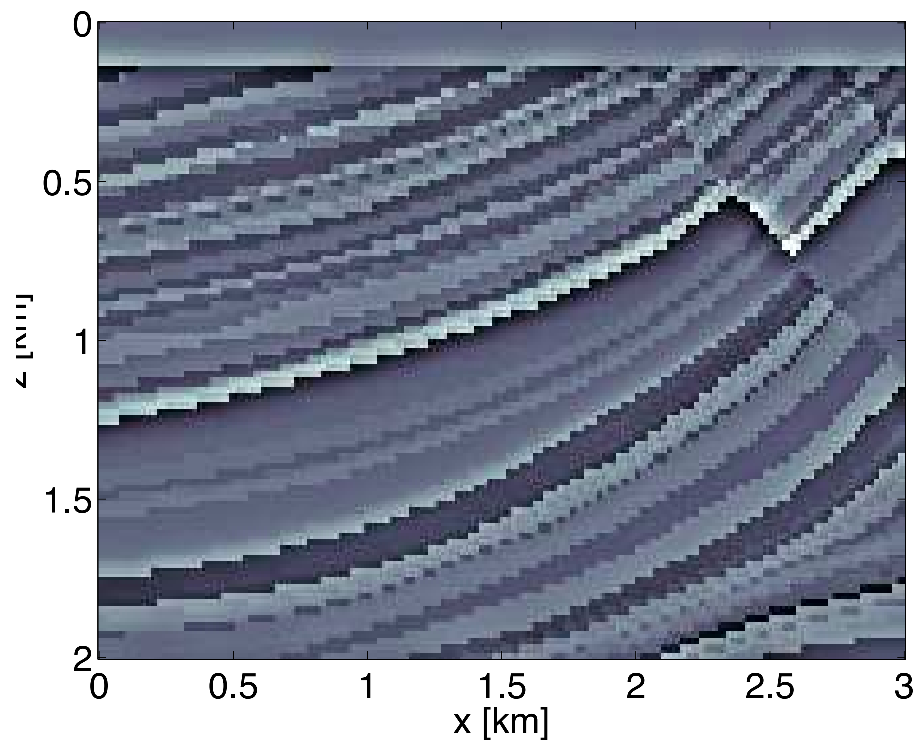
$$\sum_i \frac{s+l}{2} \frac{\nabla_m F[m;Q]_i^T (F[m;Q]-D)_i}{s+(D-F[m;Q])_i^2} \, .$$

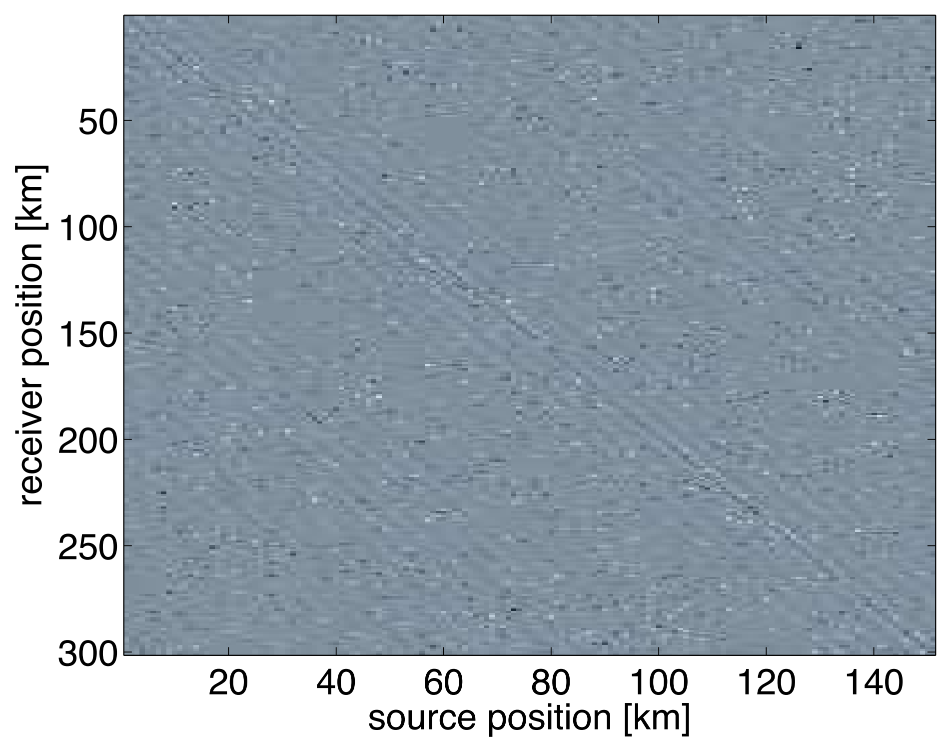
# Marmoussi Example

- We consider a subset of the Marmoussi model
- 151 shots, 301 receivers
- 9 pt. discretization of Helmholtz operator with absorbing boundary; 10 m. spacing on grid
- Sample of Frequencies [5.0, 6.0, 11.5, 14.0, 15.5, 17.5, 23.5] Hz

# Experiment I

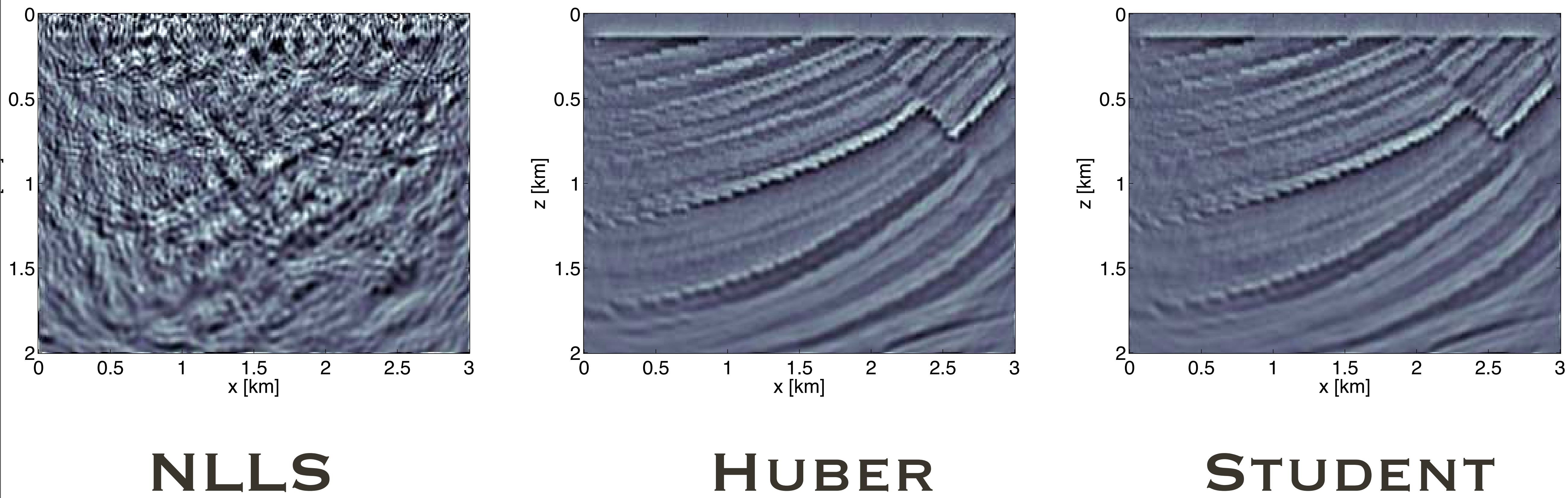



True Reflectivity

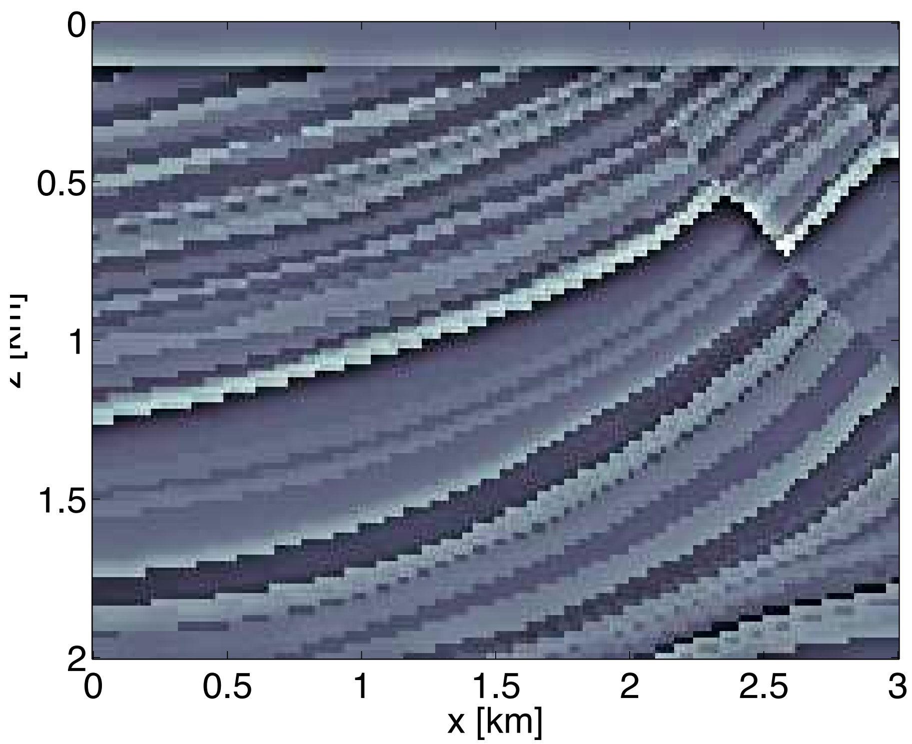



15 hz data slice
with spiky noise

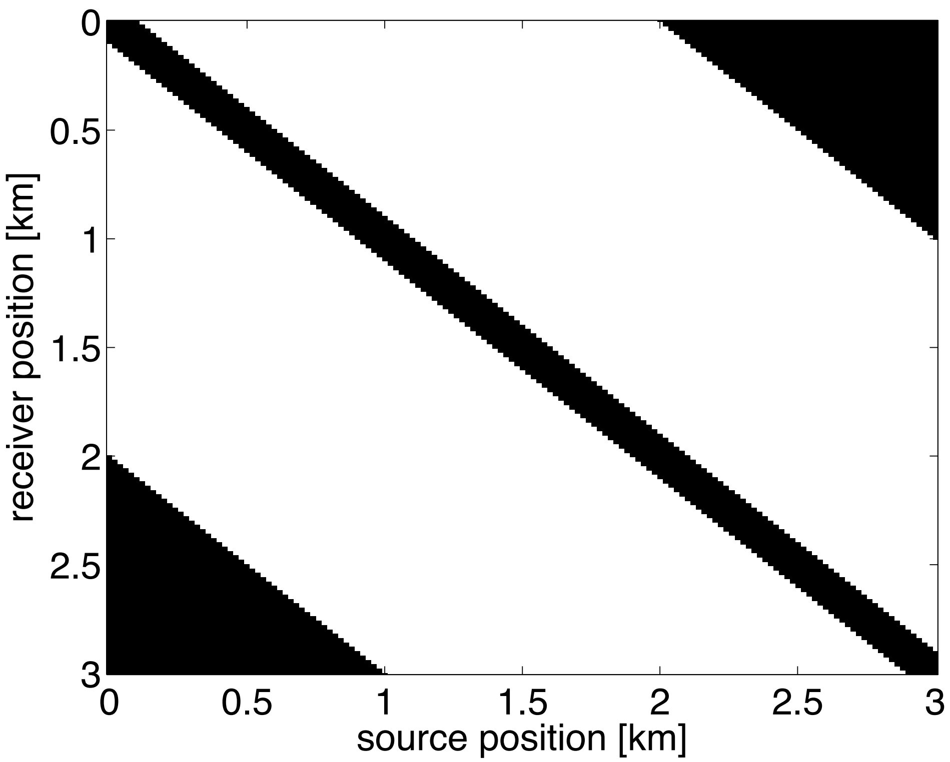
# Results I



# Experiment II

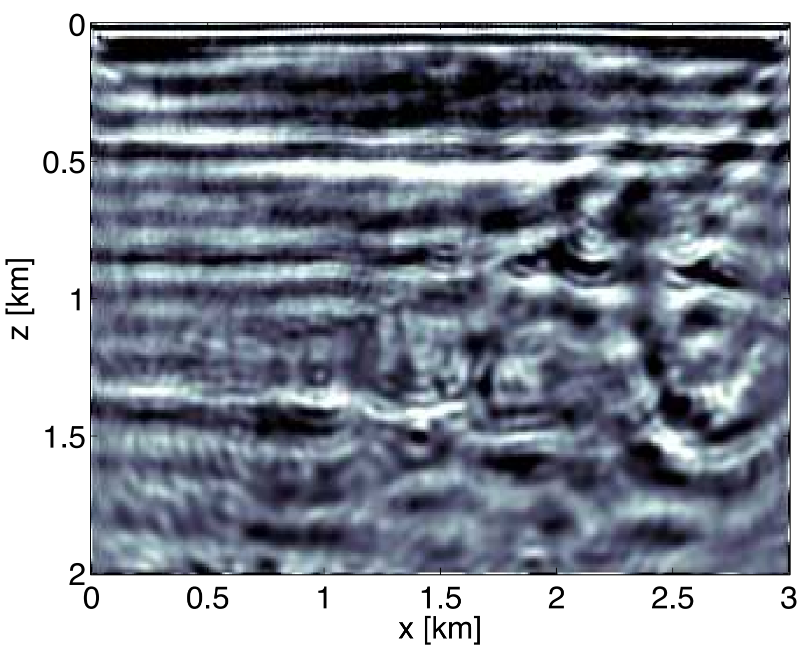



True Reflectivity

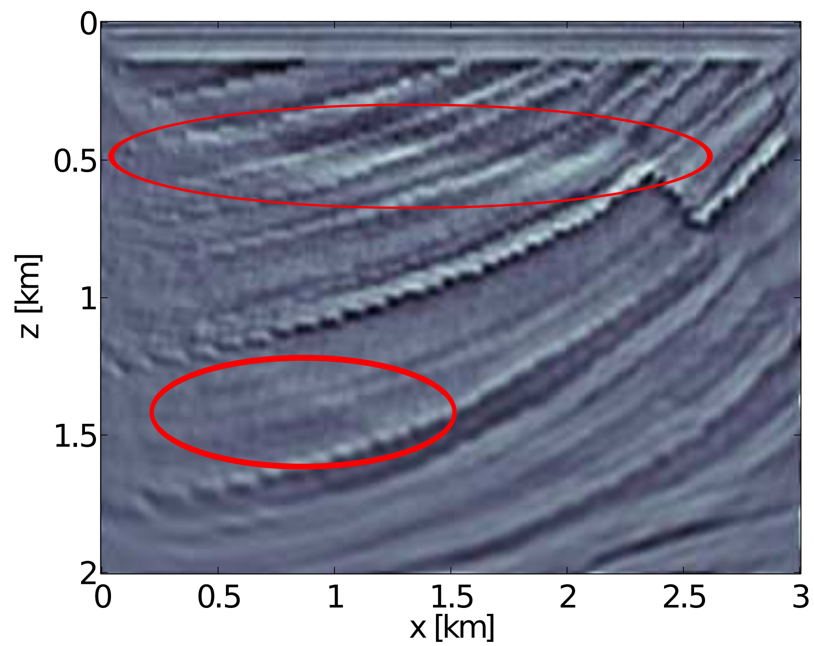



Marine Acquisition Mask

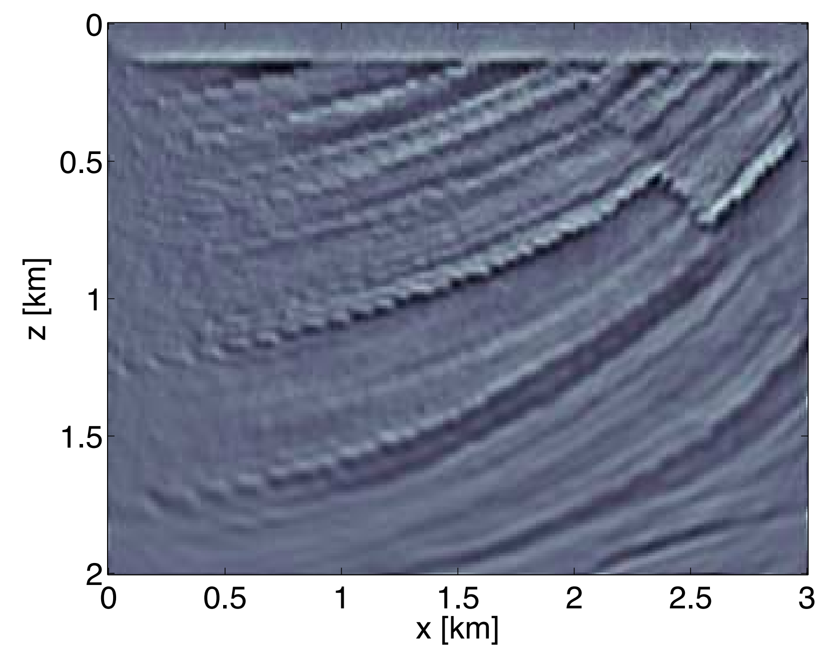
# Results II




NLLS




Huber

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# Conclusions

- Robust formulations allow good recovery even with poor modeling
- 'Mistakes' are typically thought of as 'outliers' in the data, but can also be events unexplained (or ignored) by the modeling
- Since the FWI problem is non-convex, we can feel free to exploit distributions with non-convex negative log likelihoods
- Future direction: combining robust and sparse recovery.

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