Recent Developments in Preconditioning the FWI Hessian

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Preconditioning the FWI Hessian

- Work done by L. Demanet et. al - Matrix probing: a randomized preconditioner for the wave-equation Hessian

Motivation

- Gradient-based methods converge *slowly*
- Gauss-Newton type methods require inverting the Hessian
- In seismic applications, the Hessian is large, dense, and poorly conditioned

Motivation

- Solely optimization-based approaches to dealing with the Hessian (or its inverse) are too expensive
- The Hessian is not strictly low rank in the SVD sense

Motivation

- The Hessian has a (somewhat) known mathematical structure that we can exploit

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Previous Work

- Previous work by Symes, Hermann, others on *scaling-based* methods (diagonal approximations to the Hessian)
- Exploiting the near-diagonal nature of the Hessian acting on curvelets

Overview

- Optimization context
- Pseudo-differential operators
- Matrix probing

Optimization Context

Problem:

$$\min_m \phi(m) = \frac{1}{2}\|F[m] - d\|_2^2$$

where

$$F[m]\big|_{\text{receivers}} = u$$

$$\nabla^2 u + \omega^2 m^2(x) u = f$$

Optimization Context

$$J = \frac{\delta F}{\delta m}$$

$$\nabla_m \phi(m) = J^*(F[m] - d)$$

$$H_{GN} = J^* J$$

Optimization Context

Looking to solve:

$$Hp = -\nabla_m \phi(m)$$

Left Preconditioning

$$PHp = -P\nabla_m \phi(m)$$

Preconditioning

- Ideally:

$$P = H^{-1}$$

- Realistically:
  PH should have *clustered* eigenvalues and/or a lower condition number

Hessian

- Under certain conditions, the Hessian is known to behave like a *pseudo-differential operator*
- As does its inverse, assuming it exists

Pseudo-differential Operators

- A generalization of differential operators
- Differential operators are polynomials in the Fourier domain (filters with no spatial dependence)

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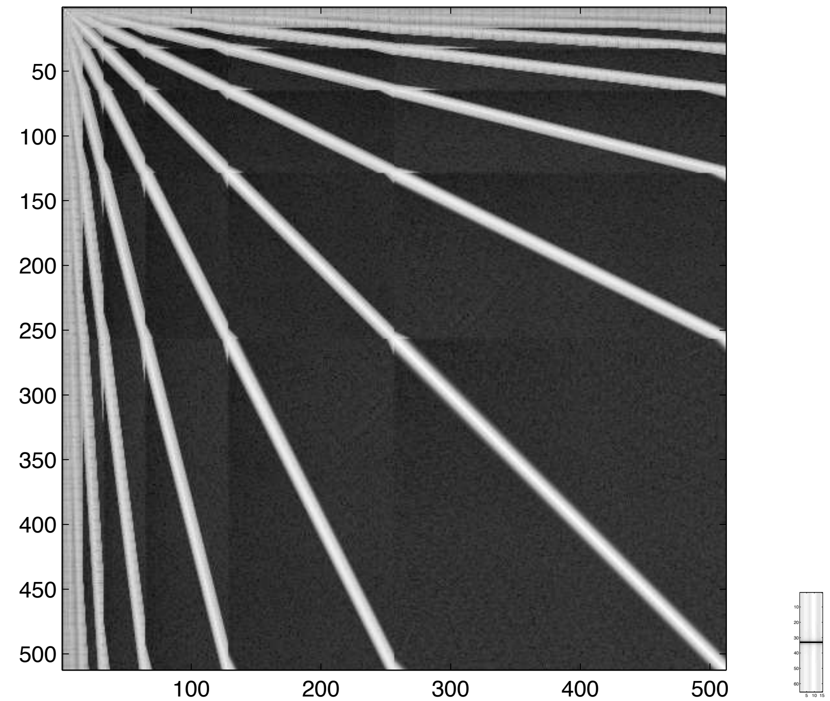
Pseudo-differential Operators

- A generalization of differential operators
- Pseudo-differential operators are polynomial-like in the Fourier domain (act as spatially dependent filters)

Pseudo-differential Operators

- Their particular Fourier-domain behaviour makes them amenable to computation
- Can be stored in an extremely compressed form, efficiently applied to functions

Pseudo-differential Operators



(a) (b)

adapted from Demanet '08

a) $O(N)$ storage in the wavelet basis,

b) $O(\log N)$ storage in the basis described in *Discrete Symbol Calculus*

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Hessian

$$Hf(x) \approx \int a(x,k)\hat{f}(k)e^{2\pi i k\cdot x}dk$$

- $a(x,k)$ is the *symbol* of the pseudo-differential operator

Inverse Hessian

$$H^{-1}f(x) \approx \int b(x,k)\hat{f}(k)e^{2\pi i k\cdot x}dk$$

- Knowledge of $b(x,k)$ gives us knowledge of H^{-1} applied to any vector (which is what we aim for in a preconditioner)

Matrix Probing

- A significant amount of information about the effective range of a matrix can be gleaned simply from observing its action on a set of random vectors

Matrix Probing

- If we can write

$$H^{-1} \approx \sum_i c_i B_i$$

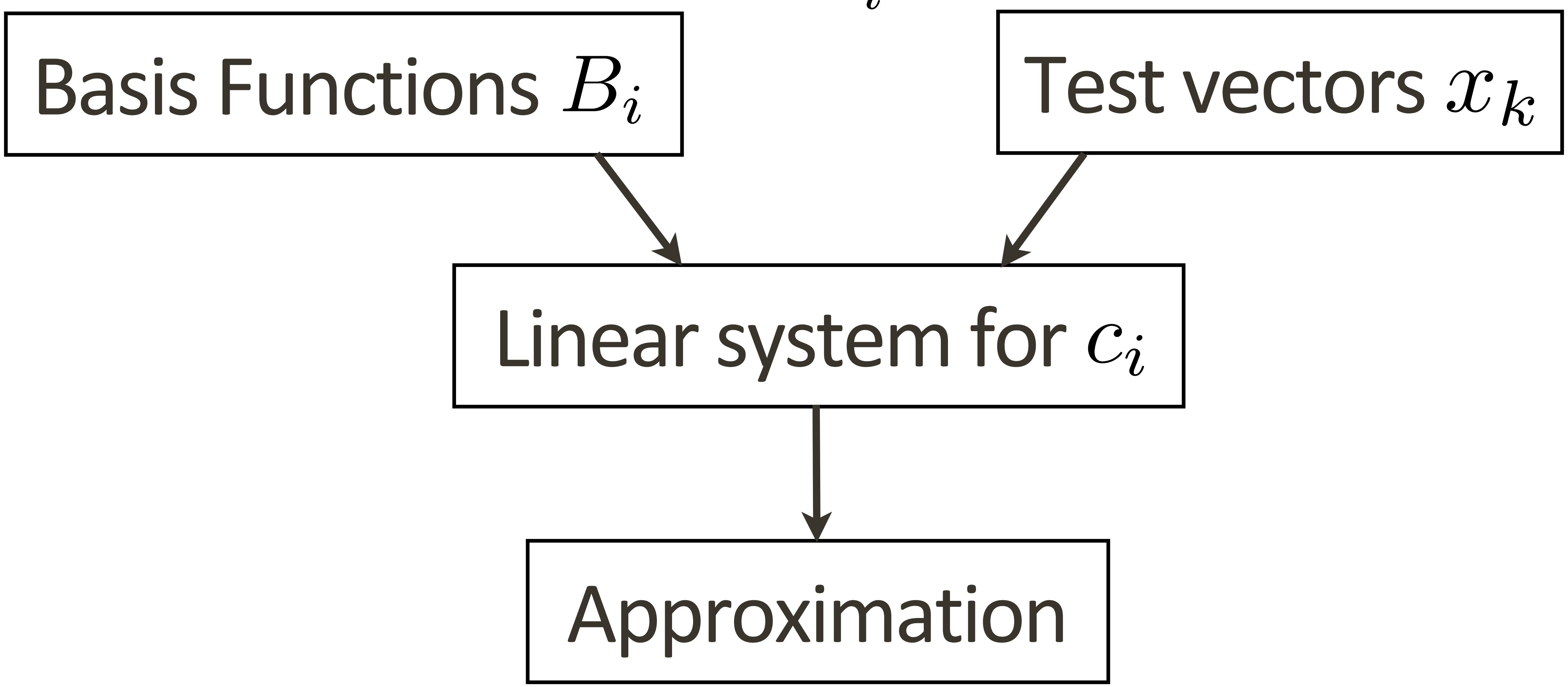
for known operators B_i,
unknown coefficients c_i

Matrix Probing

- We can recover the (approximate) action of H^{-1} by forming the linear system
$$H^{-1}x_k = \sum_i c_i B_i x_k$$
(x_k are random test vectors) and solving for c_i

Matrix Probing

$$H^{-1}x_k = \sum_i c_i B_i x_k$$



Matrix Probing

$$Hx_k = y_k$$

- x_k are test vectors

Matrix Probing

$$H_{approx}^{-1} y_k = x_k$$

- Knowledge of the action of the inverse Hessian can be obtained from observing the behaviour of the forward Hessian

SVD Approach

$$H^{-1} \approx \sum_{i=1}^{p} \lambda_i u_i v_i^T$$

- u_i, v_i are vectors, so $u_i v_i^T$ are rank-1 matrices
- unfortunately both H and H^{-1} have high-rank in this basis

Basis for PDOs

- Using the basis

$$B_{\lambda,q_1,q_2} = \int e^{2\pi i x\cdot\lambda} e^{2\pi i q_1 \theta} T L_{q_2}(|k|)|k| dk$$

results in a fast converging expansion for a large class of PDOs

Coefficient Recovery

$$H^{-1} \approx \sum_{(\lambda, q_1, q_2)} c_{\lambda, q_1, q_2} B_{\lambda, q_1, q_2}$$

- B_{λ, q_1, q_2} are known, c_{λ, q_1, q_2} are unknown but theoretically quickly decreasing

Random Test Vectors

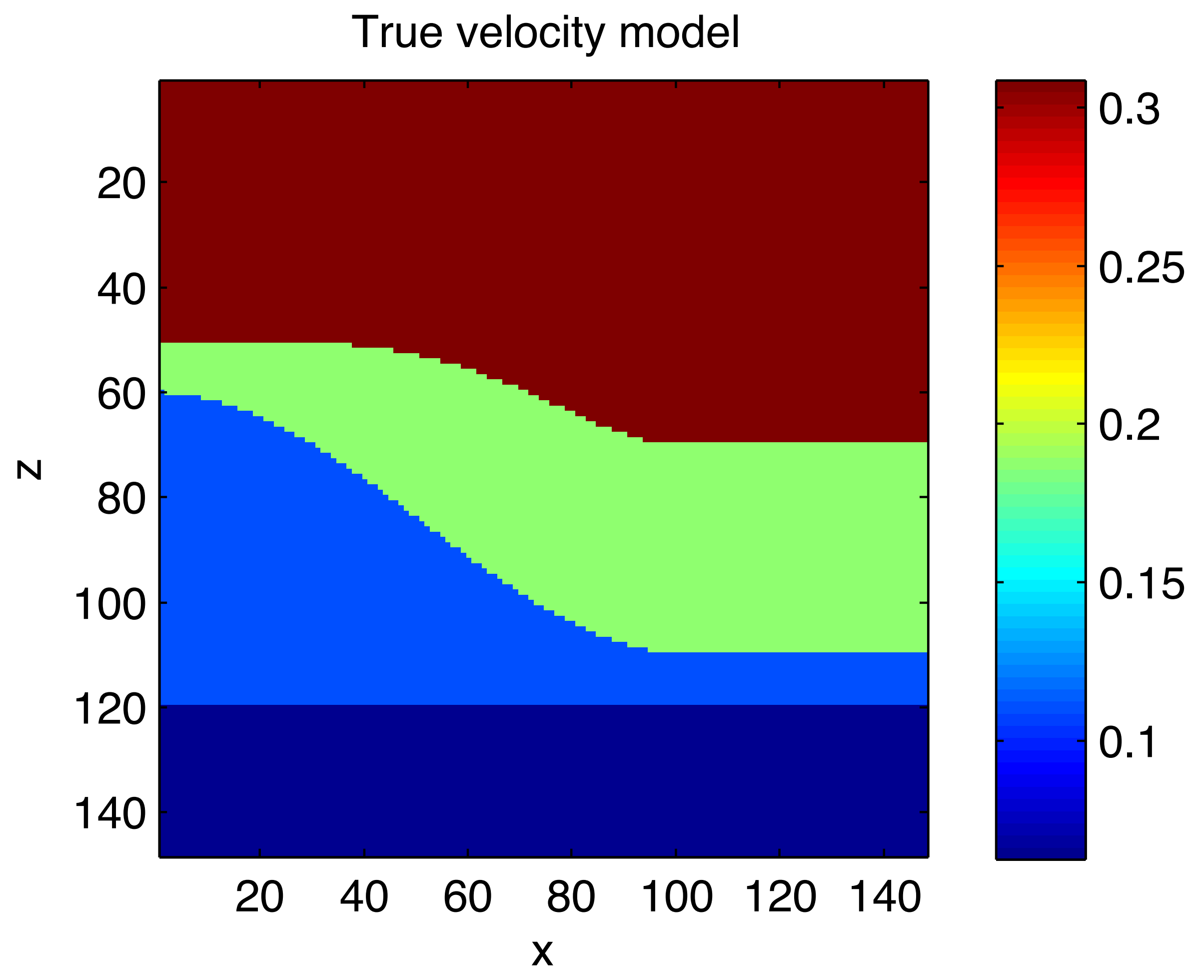
- Various choices of test vectors
- Gaussian noise - too much energy in the nullspace of H
- Gaussian noise to which the Hessian is applied - avoids the nullspace, but expensive to compute

Random Test Vectors

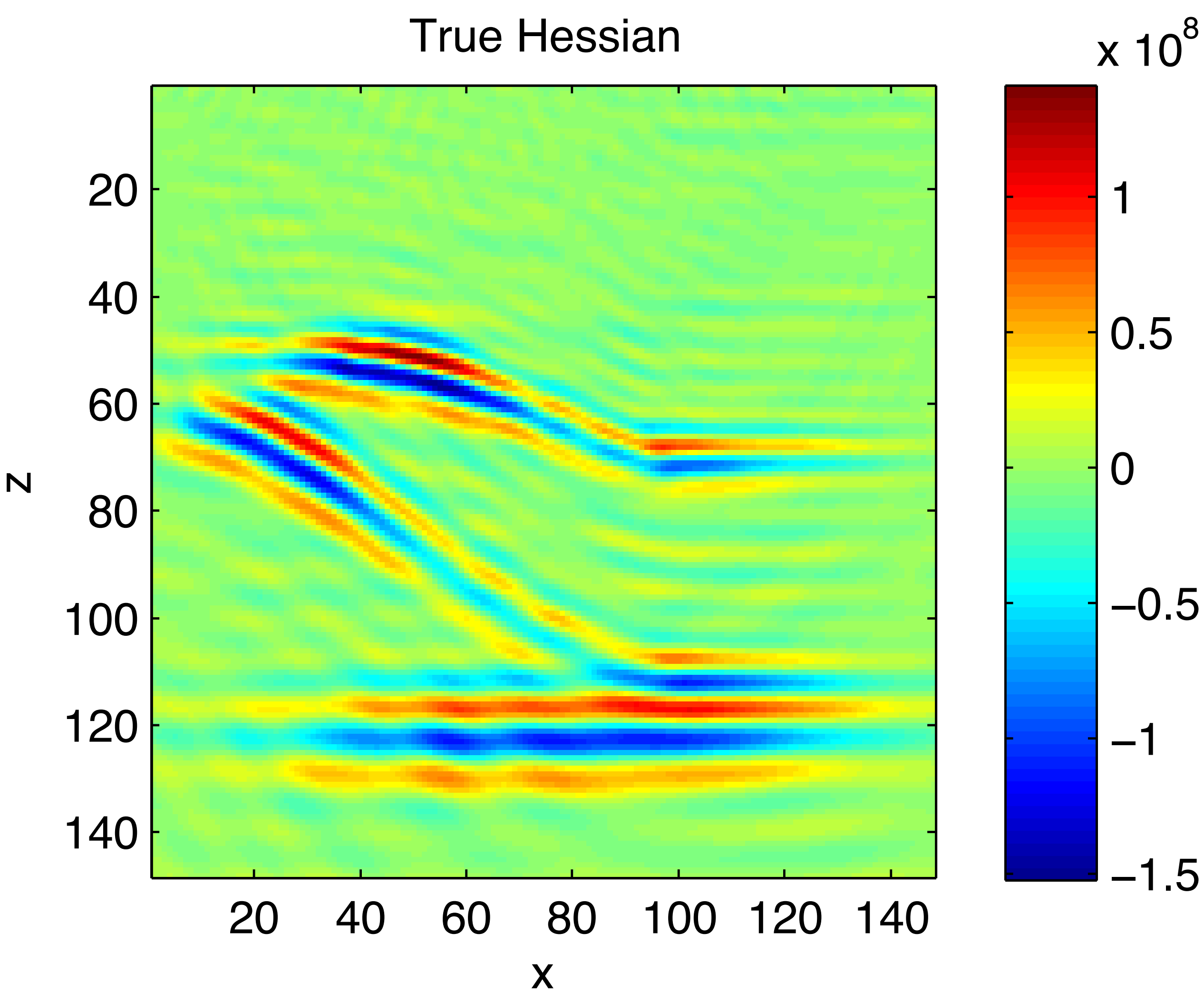
- Various choices of test vectors
- Noise in the Curvelet domain taken with a 1-0 mask
- Curvelets discriminate space-frequency regions where H is effectively zero
- More details in L. *Demanet Et Al. - 2011*

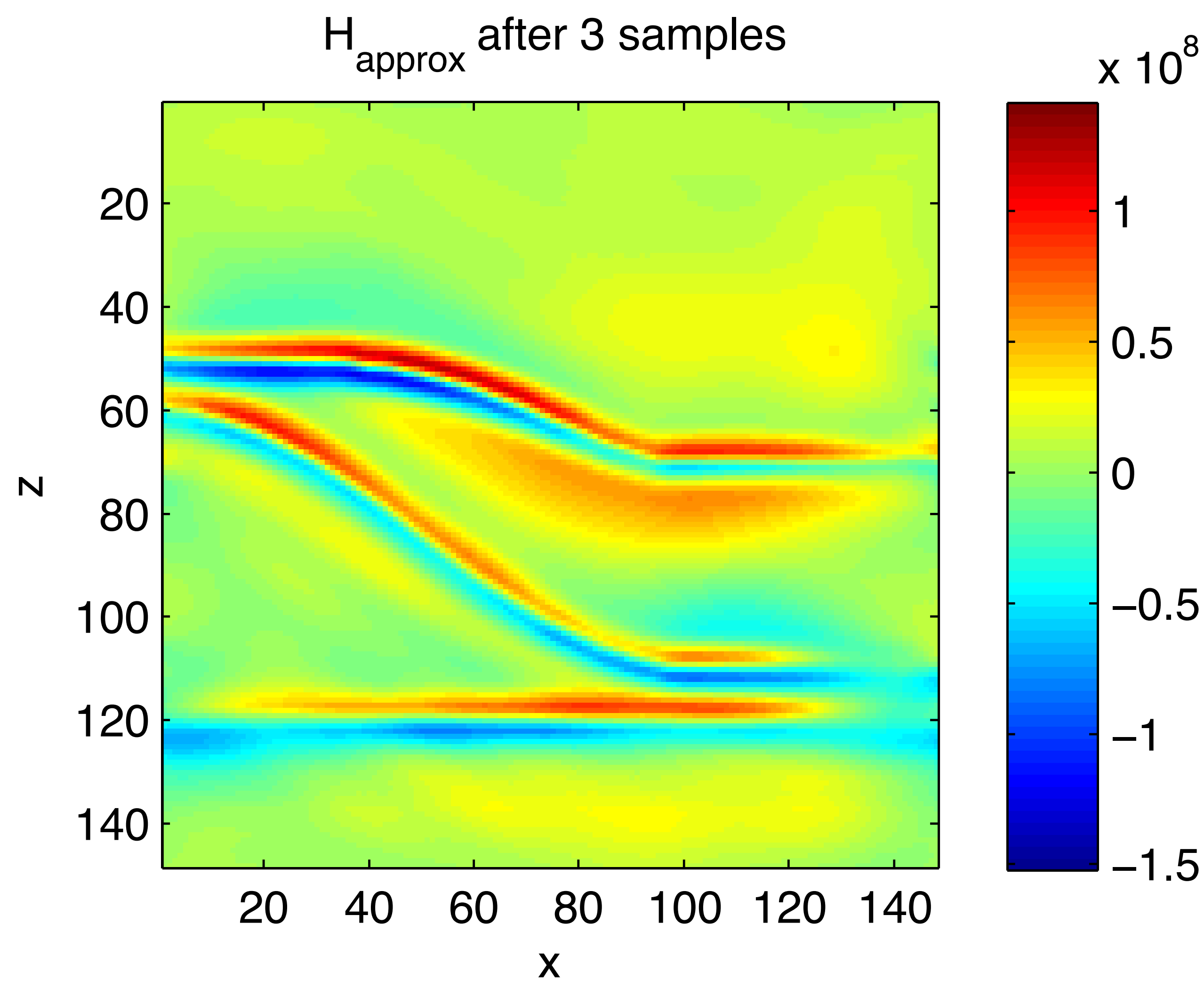
Results

- Synthetic Clay Model
- Forward Hessian probing

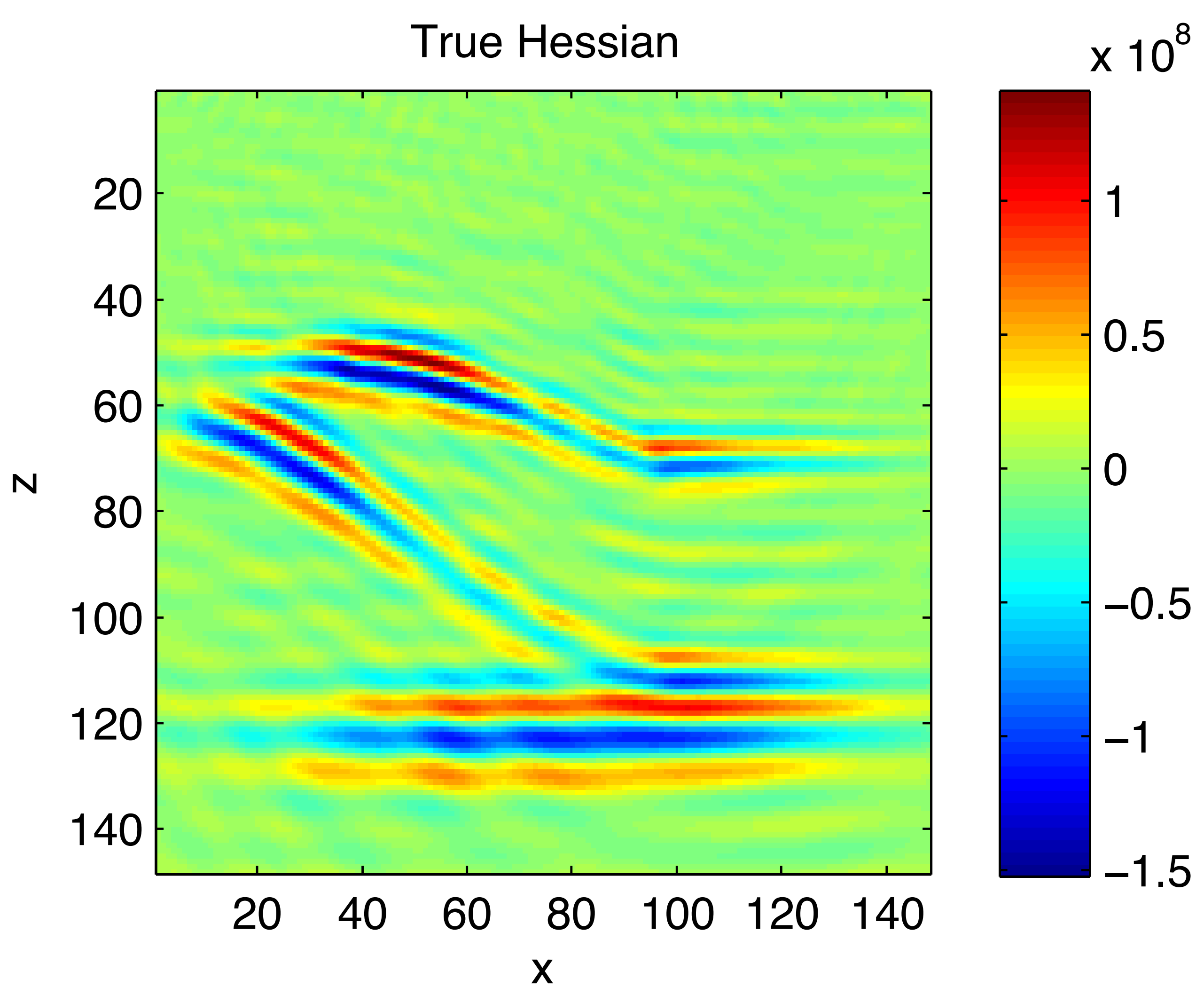


Results


True Hessian
x 10^8
1
0.5
0
−0.5
−1
−1.5
20
40
60
80
100
120
140
z
x


H approx after 3 samples
x 10^8
1
0.5
0
−0.5
−1
−1.5
20
40
60
80
100
120
140
z
x

Results



True Hessian
x 10^8
1
0.5
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z
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40
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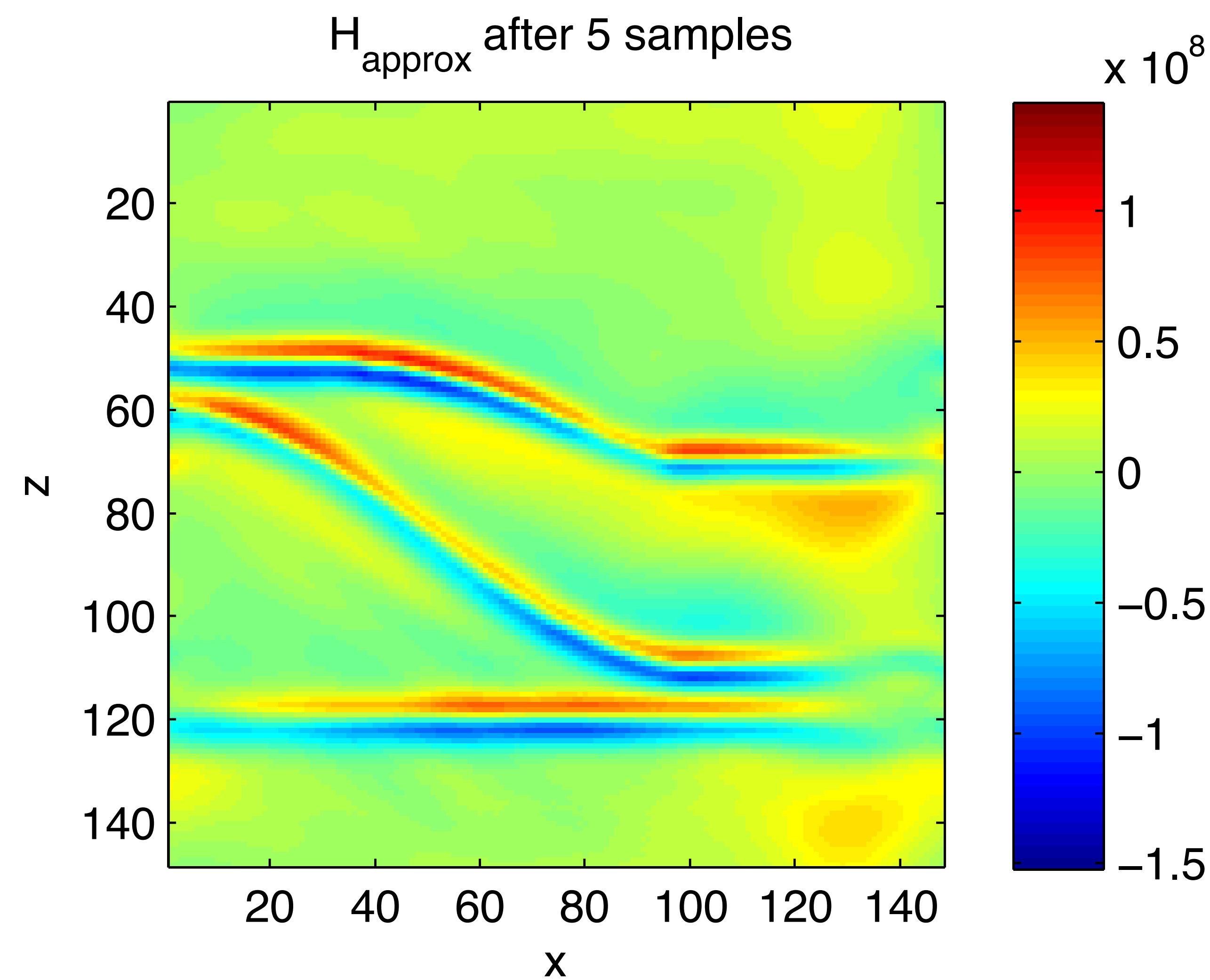
120
140
x



H approx after 5 samples
x 10^8
1
0.5
0
−0.5
−1
−1.5
z
20
40
60
80
100
120
140
x

Conclusion

- Use existing mathematical knowledge of the structure of the Hessian to obtain information about the inverse Hessian
- Fewer overall optimization iterations as a result

Future Directions

- Dimensionality reduction
- Probing the full Hessian (of which the Gauss-Newton Hessian is a part)
- Adapting this technique to large scale problems

Future Work

- Dealing with low-frequencies
- Incorporating work into a hierarchical low-rank matrix framework

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