Robust FWI using Student's T & Robust Source Estimation

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Joint work with

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Implementations and tests at Total with

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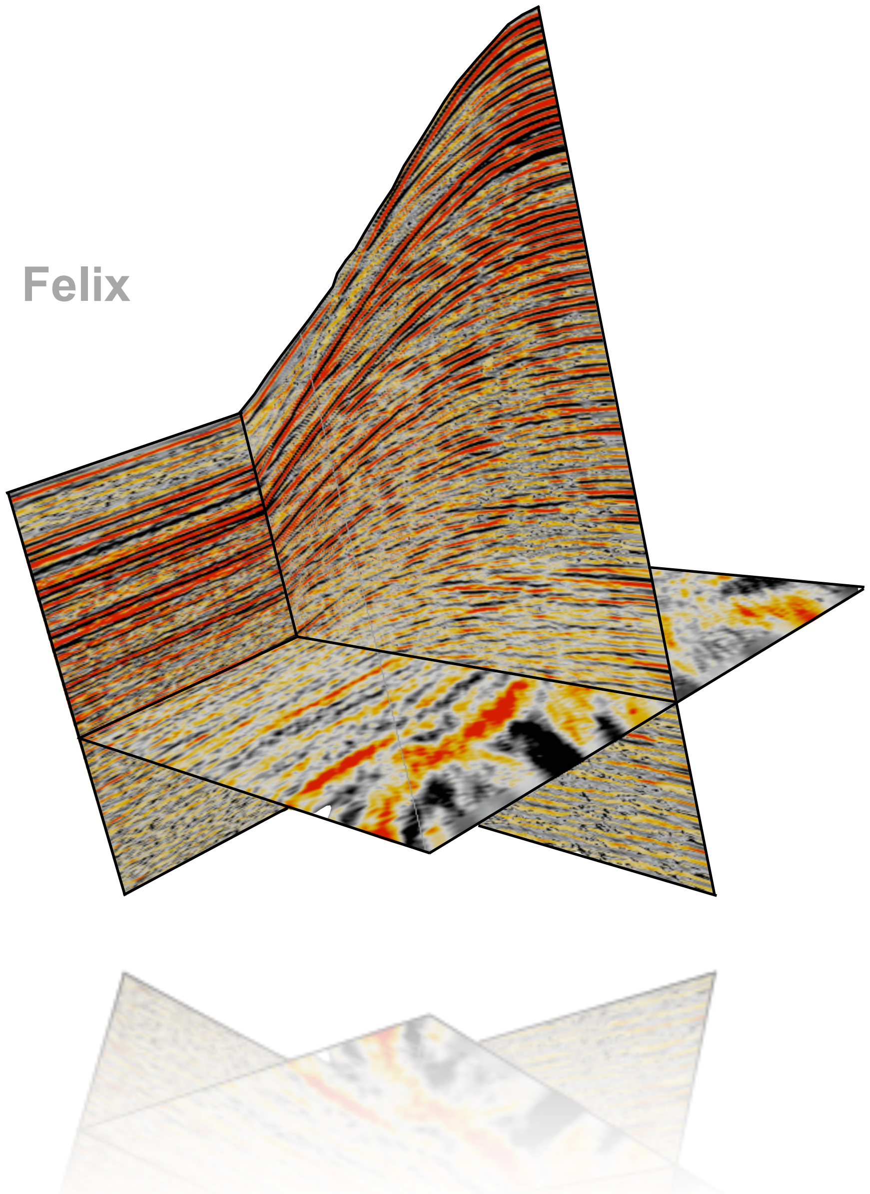


Sinbad Consortium
Whistler, 2011

Outline

- Robust FWI
  - Motivation and statistical insight
  - Robust FWI formulations, with focus on **Student's t**
  - Results on synthetic data (including implementation at Total)

- Robust Source Estimation
  - General formulation (includes Student's t, Huber, hybrid, etc formulations).
  - Generalized Variable Projection Approach
  - Specific implementations and examples
  - Proof of concept numerics

Nonlinear Least Squares Formulation

- We consider inverse problems of the form

$$\mathbf{D} = \mathcal{F}(\mathbf{m};\mathbf{Q}) + \boldsymbol{\epsilon}$$

$\mathbf{D}$	$n \times m$ matrix of observations
$\mathbf{Q}$	$l \times m$ array of source parameters
$\mathbf{m}$	parameters to be recovered
$\mathcal{F}(\mathbf{m};\mathbf{Q})$	Forward model (calculated data)
$\boldsymbol{\epsilon}$	Model for error, typically Gaussian i.i.d.

- Choice of Gaussian error leads to least squares formulation:

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) = \| \underbrace{\mathbf{D} - \mathcal{F}(\mathbf{m};\mathbf{Q})}_{\mathbf{R}(\mathbf{m})} \|_F^2 = \sum_{i=1}^{m} \| \underbrace{\mathbf{d}_i - \mathcal{F}(\mathbf{m})\mathbf{q}_i}_{\mathbf{r_i}(\mathbf{m})} \|_2^2$$

Statistical Perspective for Least Squares

- The NLLS formulation is equivalent to the following statistical model:

$$\begin{aligned} \mathbf{D} &= \mathcal{F}[\mathbf{m};\mathbf{Q}] + \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon} &\sim \mathbf{N}(0, I) \end{aligned}$$

- Equivalence follows from maximum likelihood estimate for model parameters:

$$\mathcal{L}(\mathbf{m}) \quad \propto \quad \exp\left(-\frac{1}{2}\left\|\mathbf{D} - \mathcal{F}[\mathbf{m};\mathbf{Q}]\right\|_F^2\right)$$

- Minimizing the negative log likelihood is exactly the FWI problem.

- Statistical perspective explains why least squares are sensitive to outliers and artifacts in the data!

Sensitivity to Outliers in Gaussian Regime

- Large deviations from the mean are VERY unlikely in the Gaussian model:

	Gaussian
$p(\|x\| > 4\sigma)$	6.3×10^{-5}
$p(\|x\| > 8\sigma)$	1.3×10^{-15}
$p(\|x\| > 8\sigma \mid \|x\| > 4\sigma)$	2.1×10^{-11}

- Observations more than 4 standard deviations away from the mean occur less than .006 percent of the time.

- Even when we KNOW we have an outlier 4 standard deviations away, we still believe it is impossible for the outlier to be more than 8 standard deviations away!

- Low likelihood values correspond to HIGH penalties for outliers.

Motivation for Robust Formulations

- Errors in measurement, e.g. equipment malfunction
- Missing data: measurement instruments may fail to record
- Even more important: unexplained “artifacts” in the data! A lot of effort is routinely devoted to
  - Data cleaning to remove unexplained artifacts
  - Complex forward model design to explain such artifacts e.g. acoustic vs. elastic vs. anisotropic
- Why not use robust fitting methods with cheaper modeling?

Modeling from the Statistical Perspective

- We can alter the assumptions on the model error:

$$\mathbf{D} \quad = \quad \mathcal{F}[\mathbf{m};\mathbf{Q}] + \boldsymbol{\epsilon}\,, \quad \boldsymbol{\epsilon} \text{ has density } \mathbf{p}$$

	Gaussian	$L(\lambda = 1)$	$T(k = 3)$
$p(\lvert x\rvert > 4\sigma)$	6.3×10^{-5}	$\mathbf{0.02}$	$\mathbf{0.6 \times 10^{-2}}$
$p(\lvert x\rvert > 8\sigma)$	1.3×10^{-15}	3.3×10^{-4}	8.1×10^{-4}
$p(\lvert x\rvert > 8\sigma \mid \lvert x\rvert > 4\sigma)$	2.1×10^{-11}	$\mathbf{0.02}$	$\mathbf{0.14}$

- Laplace/Huber have heavier tails than the Gaussian...

- But Student's t-density (3rd column) is **heavy tailed**.

Some Previous Work

- Robust statistical work has a long history (I've seen references to 1930's). A few useful 'Robust statistics' books:
  - Huber 1981
  - Hampel et al (2003)
  - Marona et al, (2006)
- For robust penalties in Seismic, see
  - Huber: Guitton & Symes, 2003
  - Huber and L1: Brossier, Operto, Virieux 2009, 2010
  - Hybrid: Bube, 2007.
- We are particularly interested in Student's t distribution. See
  - Lange 1989, general paper applying student's t formulations to regression
  - Fahrmeir 1998, Robust kalman smoothing using Student's t
- In our experience, Student's t works well for structured inverse problems in nonlinear Kalman smoothing, computer vision applications, and FWI.

From the Statistics to the Formulation

- Formulate *maximum a posteriori* (MAP) problem:

$$\begin{aligned} \mathbf{D} &= \mathcal{F}[\mathbf{m};\mathbf{Q}] + \boldsymbol{\epsilon}\,, \quad \boldsymbol{\epsilon} \text{ has density } \mathbf{p} \\ f(\mathbf{R}) &= -\log(\mathbf{p}) \end{aligned}$$

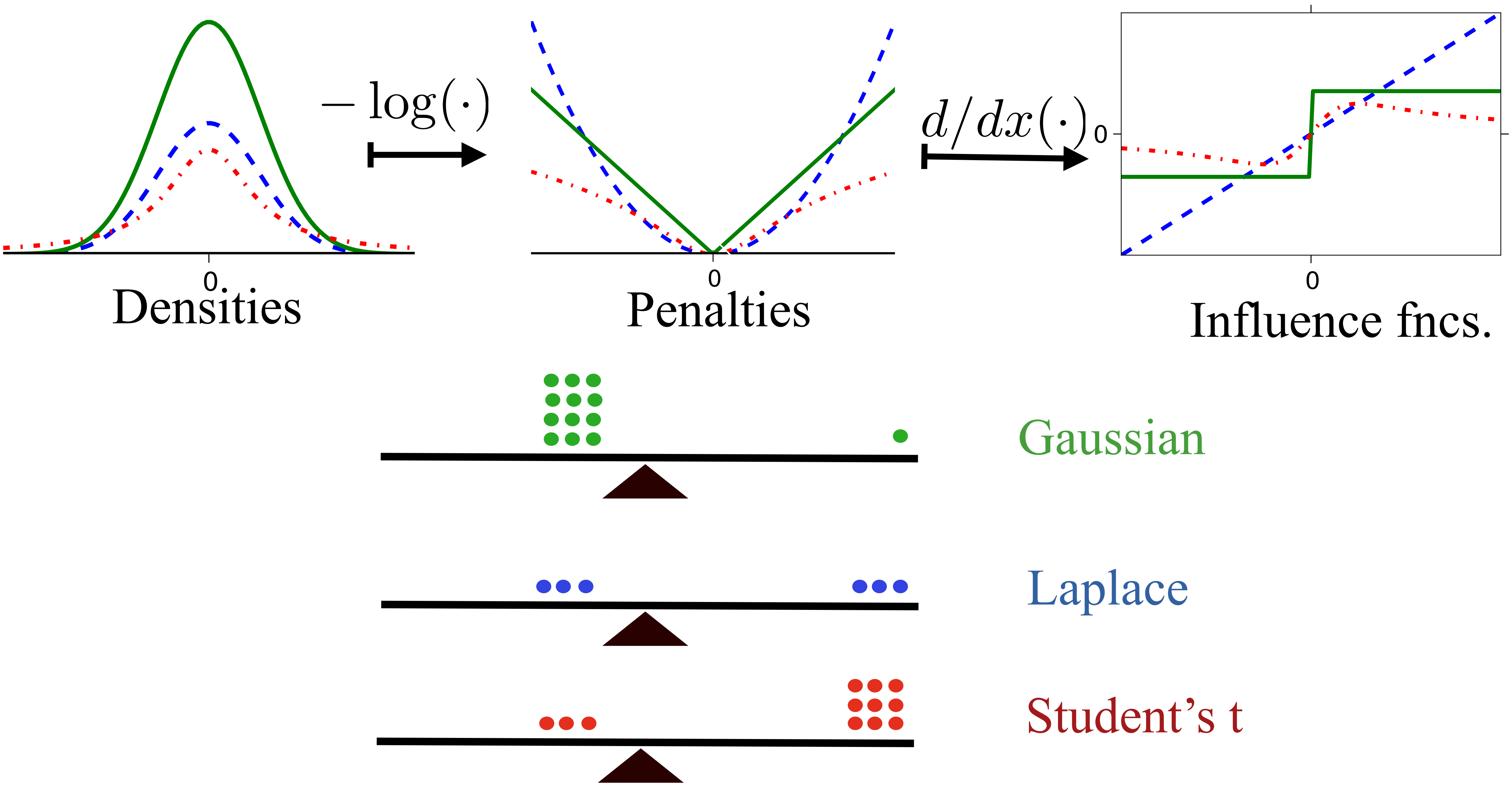
- MAP solution can be found by solving

$$\min_{\mathbf{m}} \Phi(\mathbf{m}) := f\left(\mathbf{D} - \mathcal{F}[\mathbf{m};\mathbf{Q}]\right)$$

  – NLLS: $\boldsymbol{\epsilon} \sim \mathbf{N(0,I)} \iff \Phi(\mathbf{m}) = \|\mathbf{D} - \mathcal{F}[\mathbf{m};\mathbf{Q}]\|_F^2$

- Theorem: **heavy tailed** densities correspond to **nonconvex f!**
  – Aravkin, Friedlander, Herrmann, and van Leeuwen, 2011.

Densities, Penalties, and Influence Functions



FWI Using Student's t-distribution

DENSITY:

$$\mathbf{p}(\epsilon|\mu,\sigma,k) = \frac{\Gamma(\frac{k+1}{2})}{\sigma\Gamma(\frac{k}{2})\sqrt{\pi k}}\left(1+\frac{(\epsilon-\mu)^2}{k\sigma^2}\right)^{\frac{-(k+1)}{2}}$$

FOR FWI:

$$\mathbf{p}(\epsilon|\mu=0,\sigma=1,k) \propto \left(k+\epsilon^2\right)^{\frac{-(k+1)}{2}}$$

ROBUST OBJECTIVE:

$$\min_{\mathbf{m}} \quad \mathbf{\Phi}_{St}(\mathbf{m}) := \frac{k+1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\log\left(k+(\mathbf{r_{ij}})^2\right)$$

Gradient Comparison

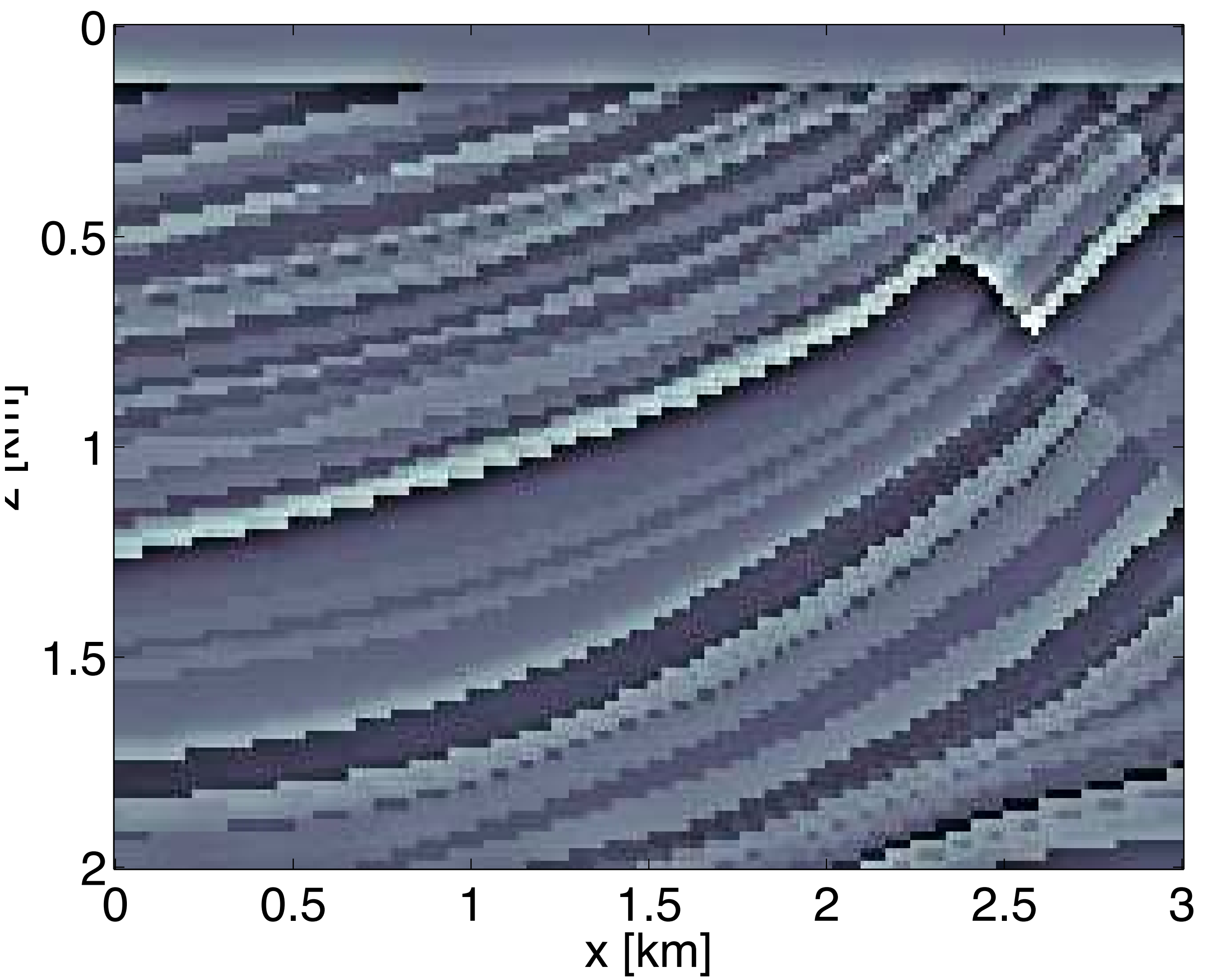
Least Squares:

$$\nabla \boldsymbol{\Phi}(\mathbf{m}) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \nabla \boldsymbol{\mathcal{F}}[\mathbf{m}, \mathbf{q}_{ij}]^T \left(\mathbf{D}_{ij} - \boldsymbol{\mathcal{F}}[\mathbf{m}, \mathbf{q}_{ij}] \right)$$

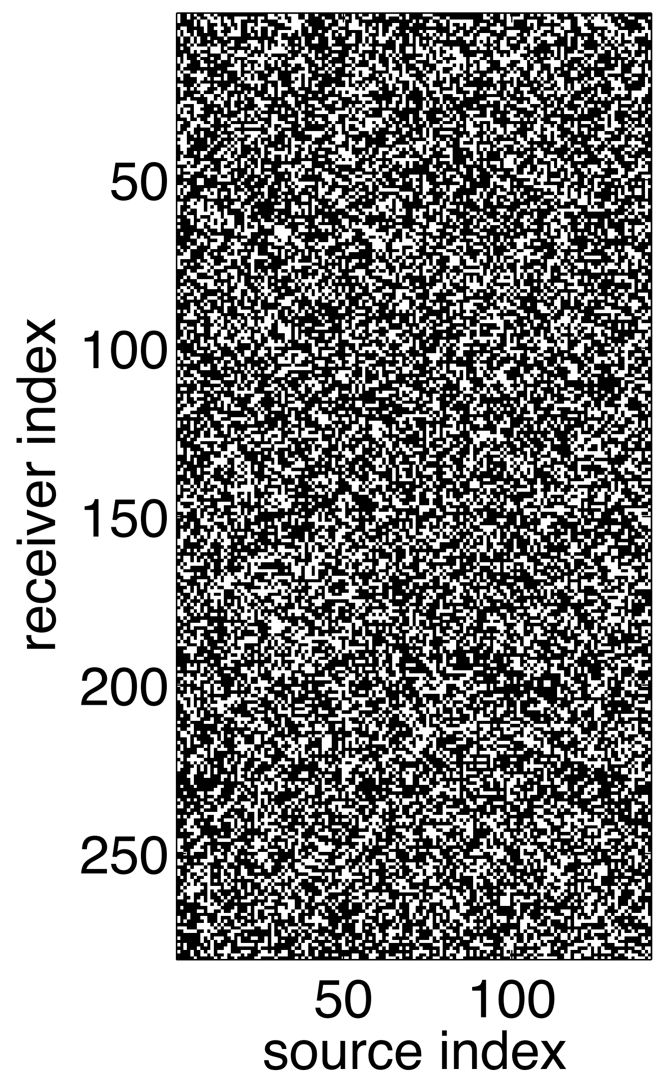
Student's t:

$$\nabla \boldsymbol{\Phi}_{St}(\mathbf{m}) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\nabla \boldsymbol{\mathcal{F}}[\mathbf{m}, \mathbf{q}_{ij}]^T \left(\mathbf{D}_{ij} - \boldsymbol{\mathcal{F}}[\mathbf{m}, \mathbf{q}_{ij}] \right)}{k + (\mathbf{D}_{ij} - \boldsymbol{\mathcal{F}}[\mathbf{m}, \mathbf{q}_{ij}])^2}$$

Marmoussi with 50% data corrupted at random

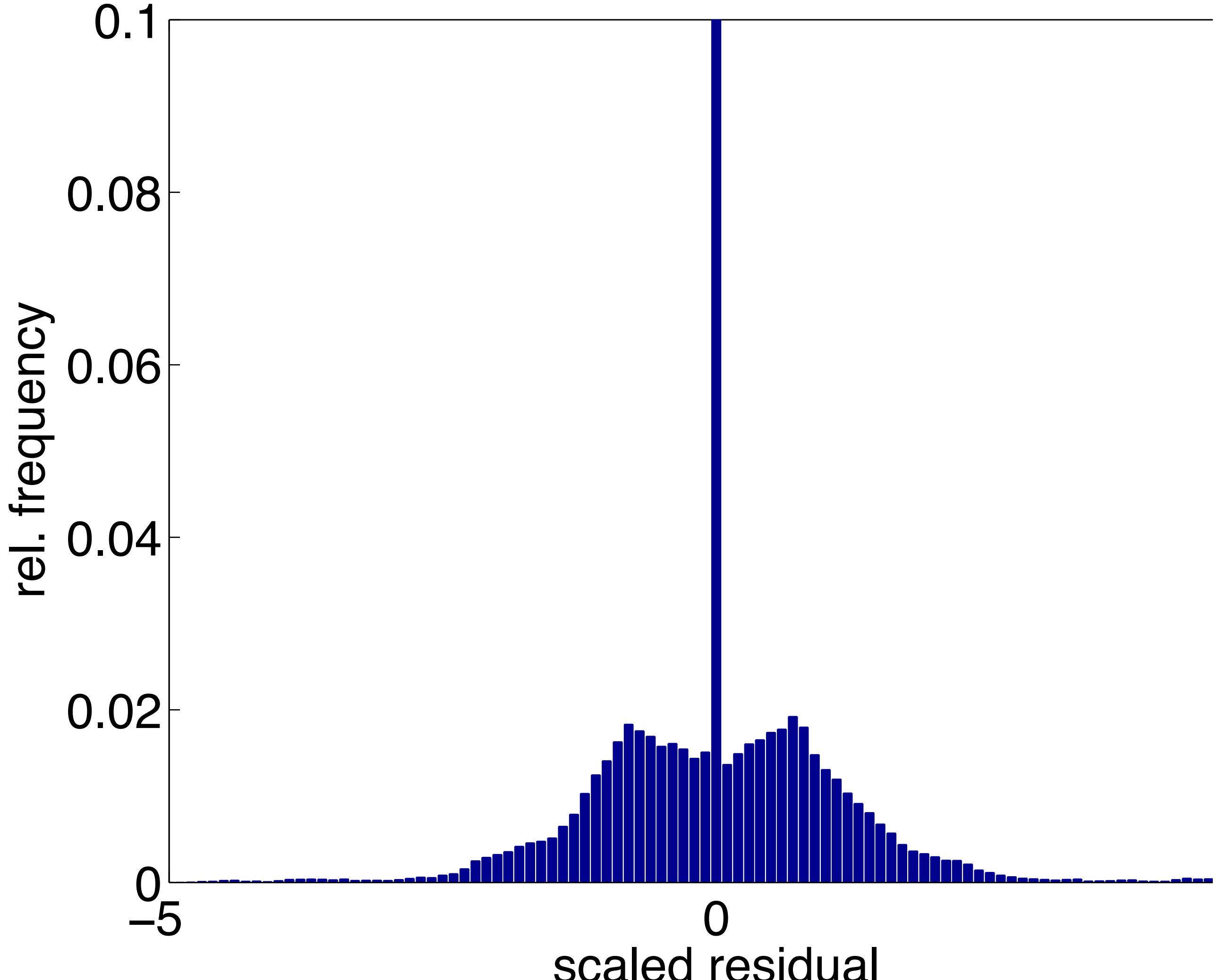


True Reflectivity

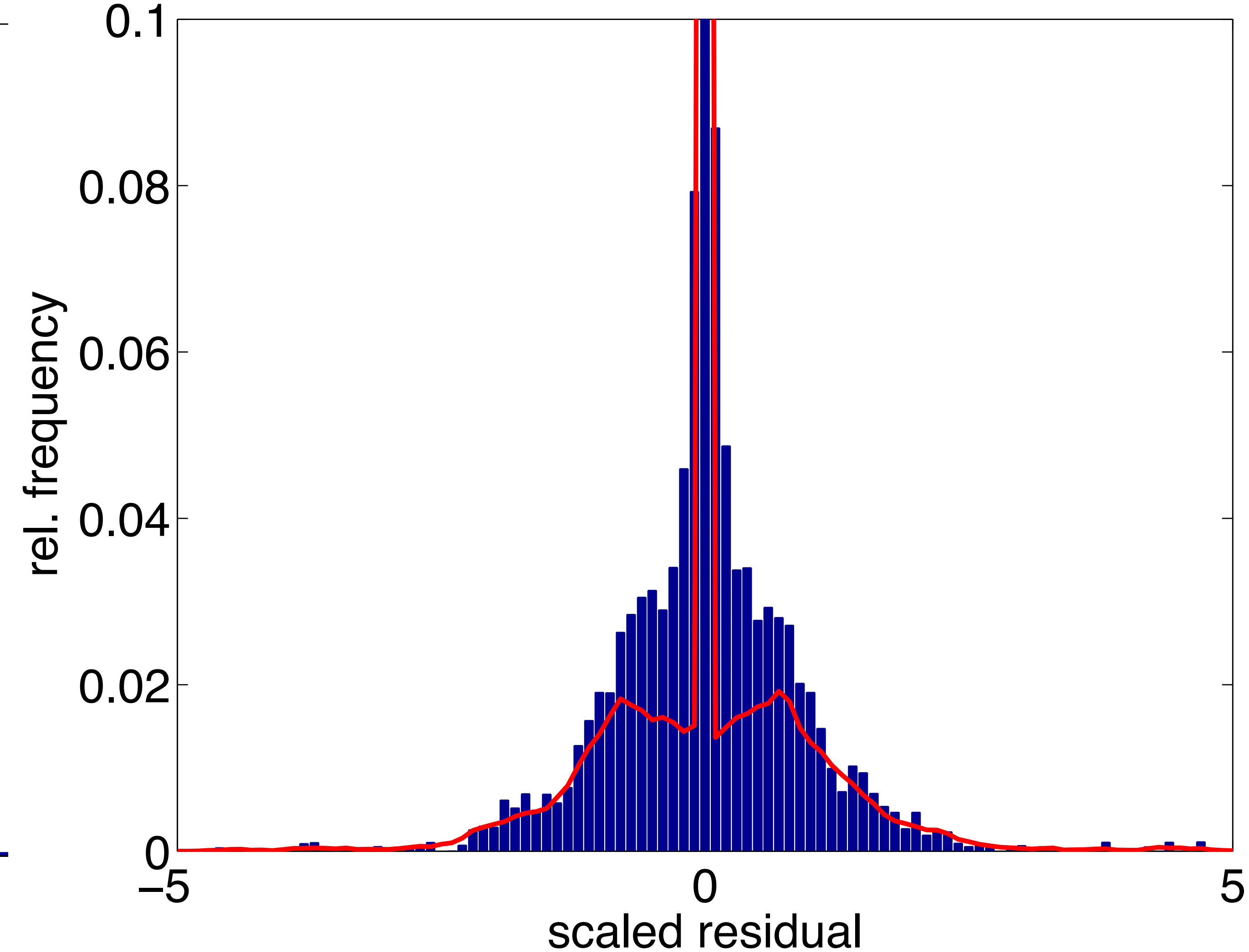


50% missing data

Histograms of residual magnitudes:

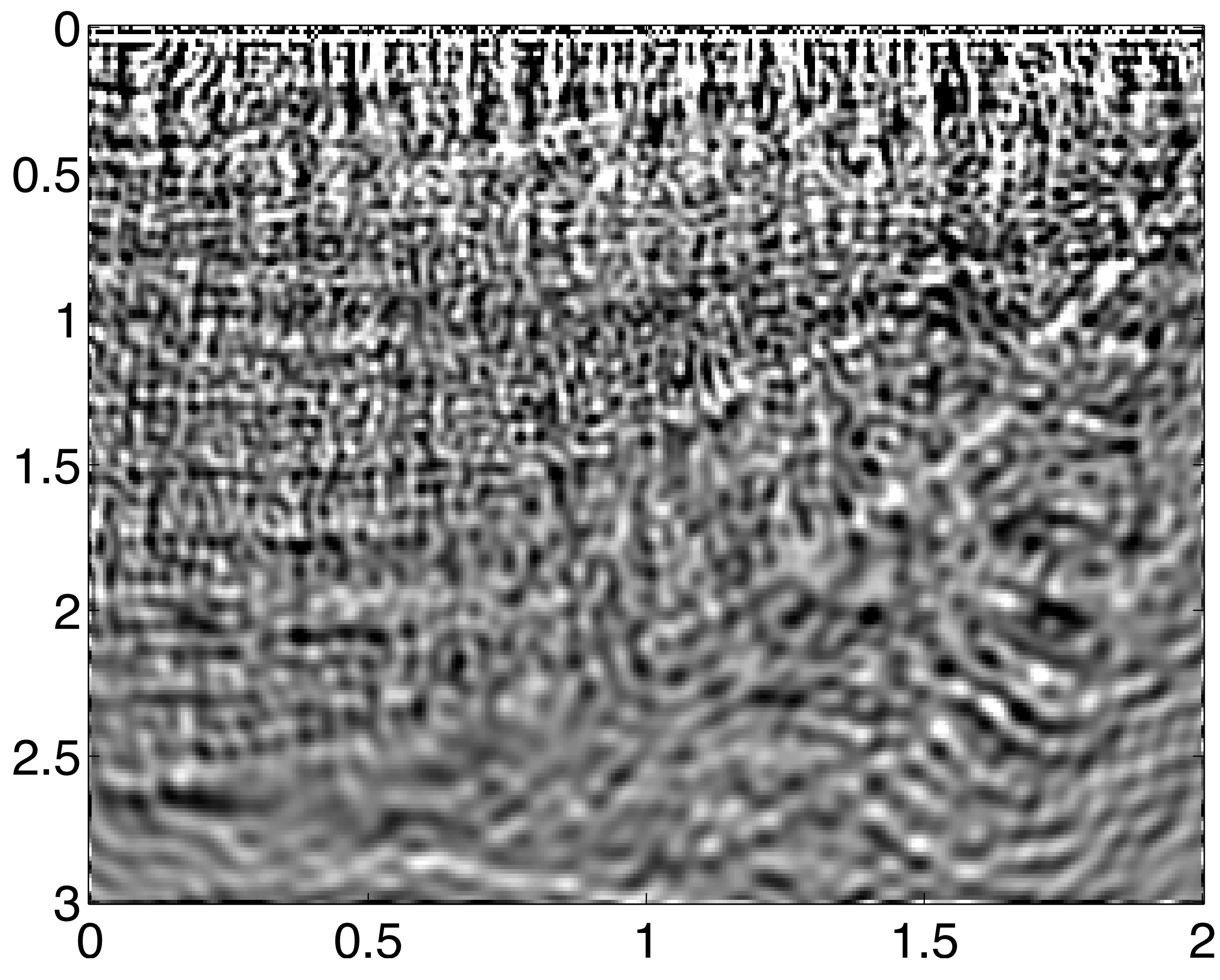


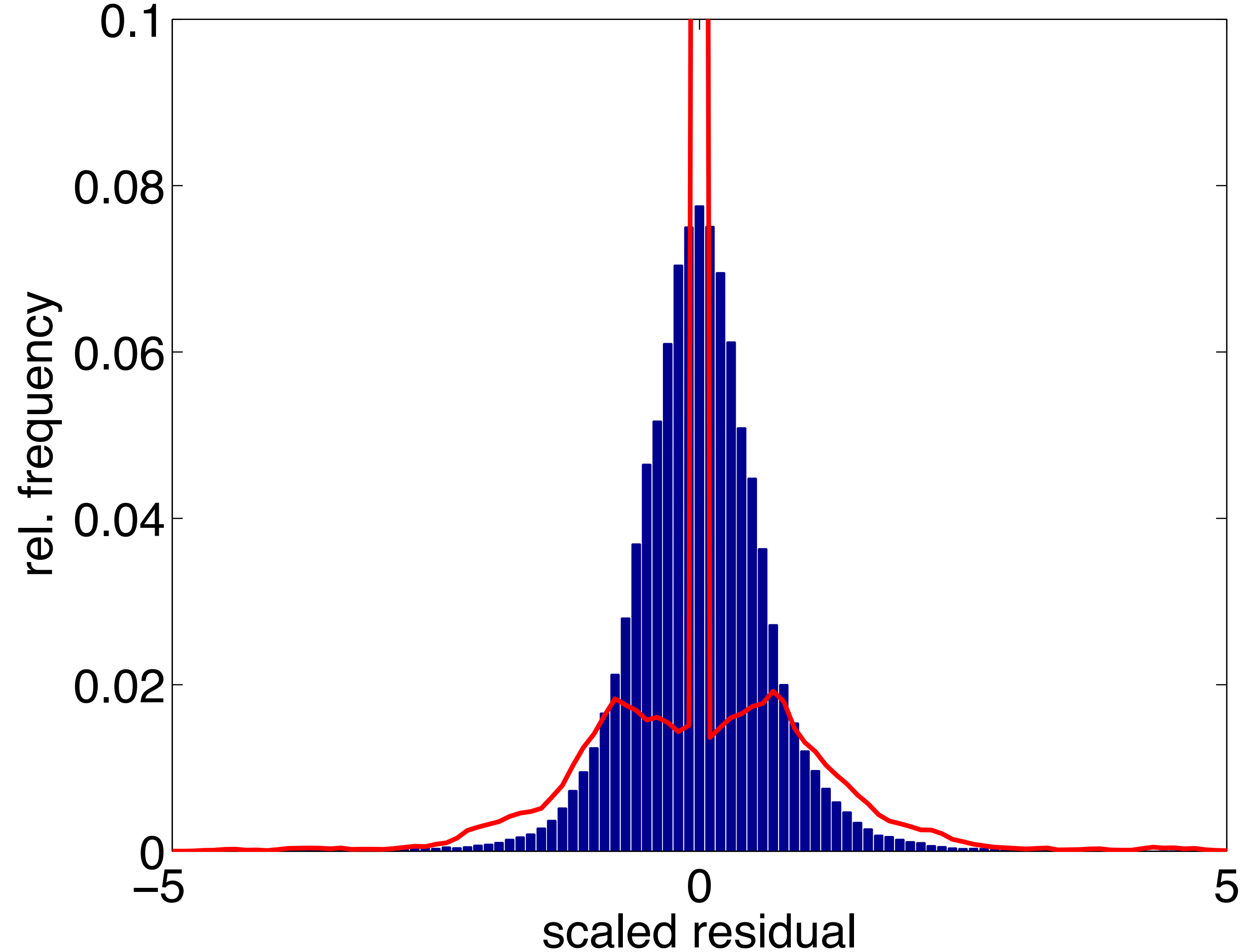
TRUE RESIDUALS



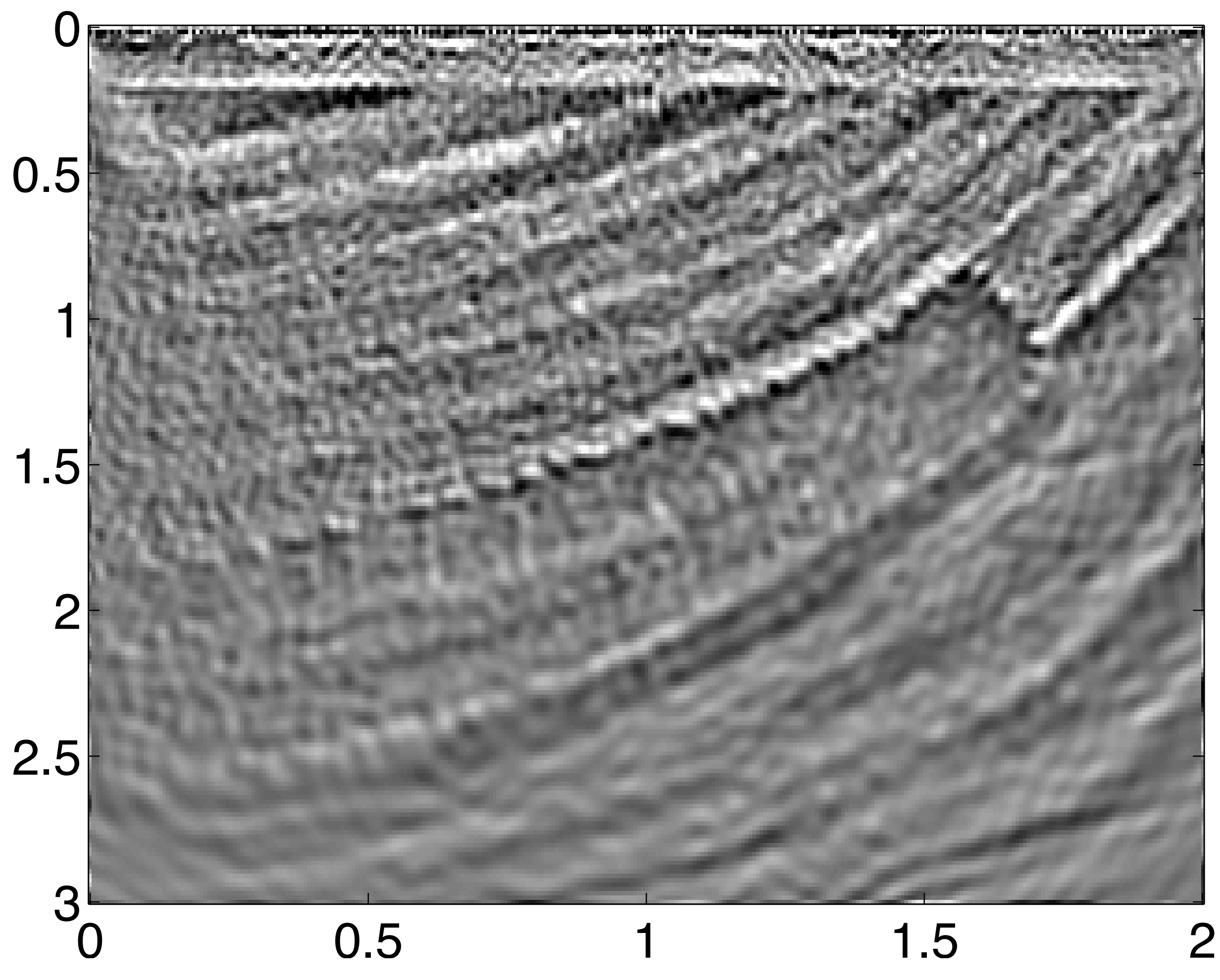
INITIAL RESIDUALS

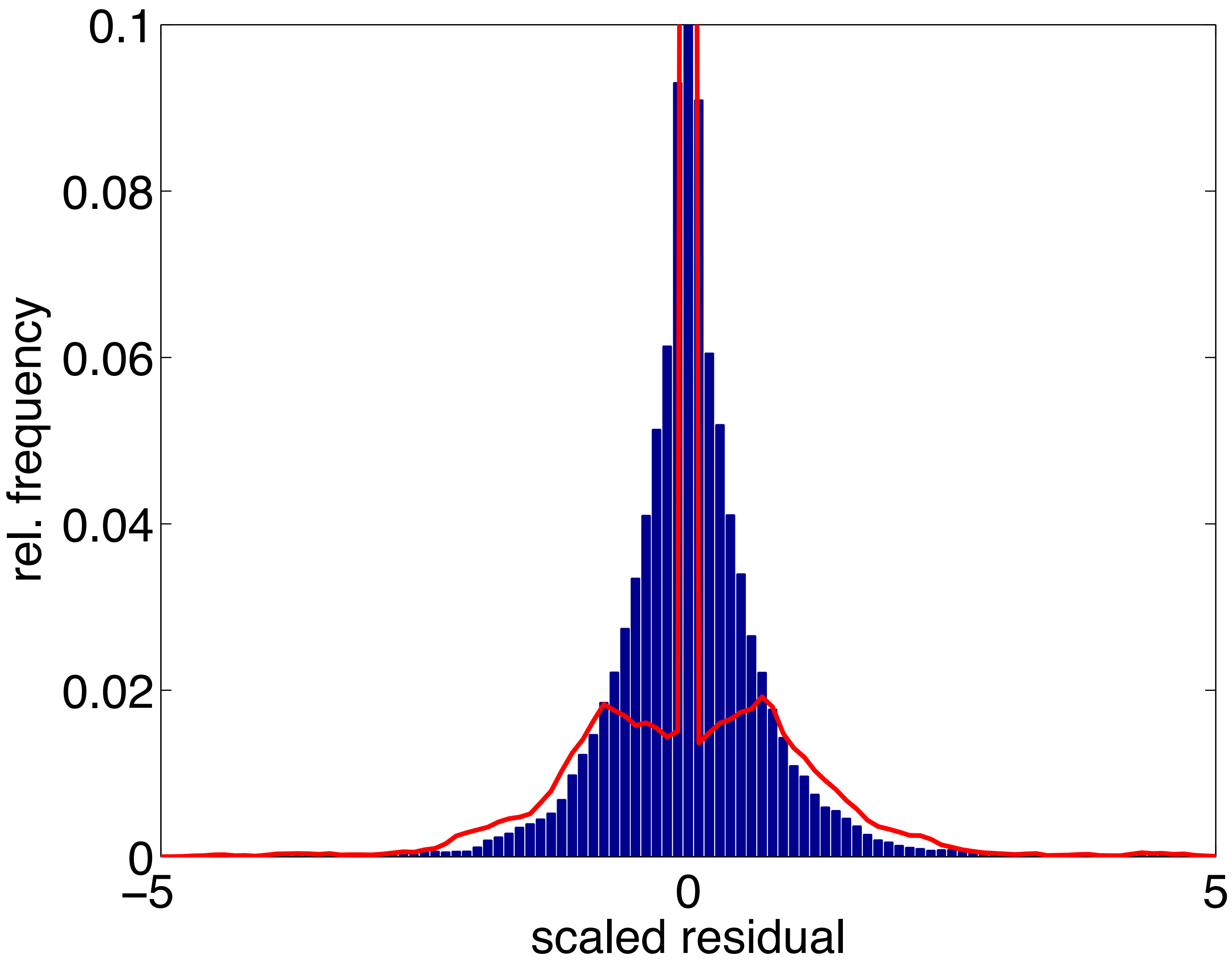
Marmoussi, LS fit, 50% corrupted data



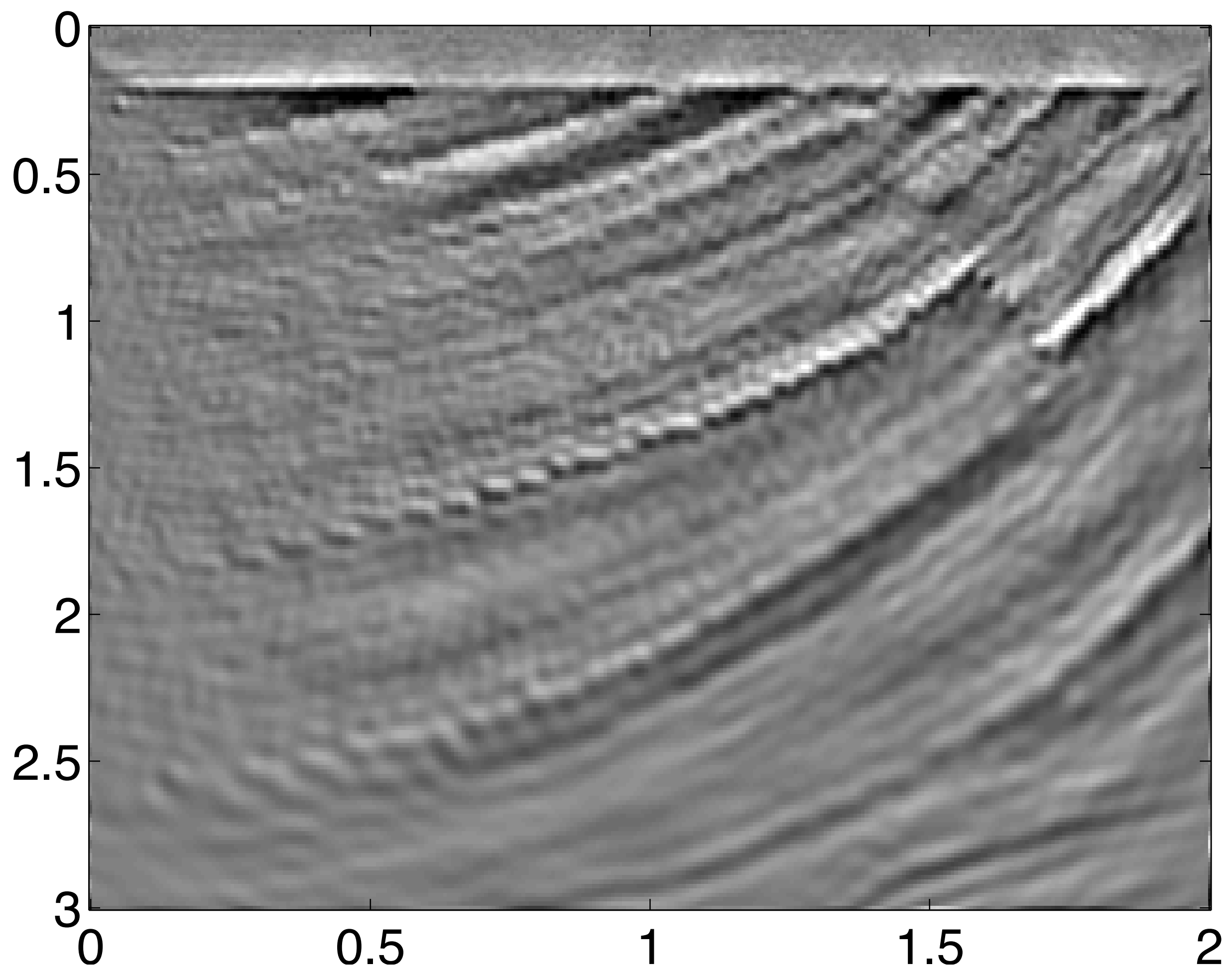


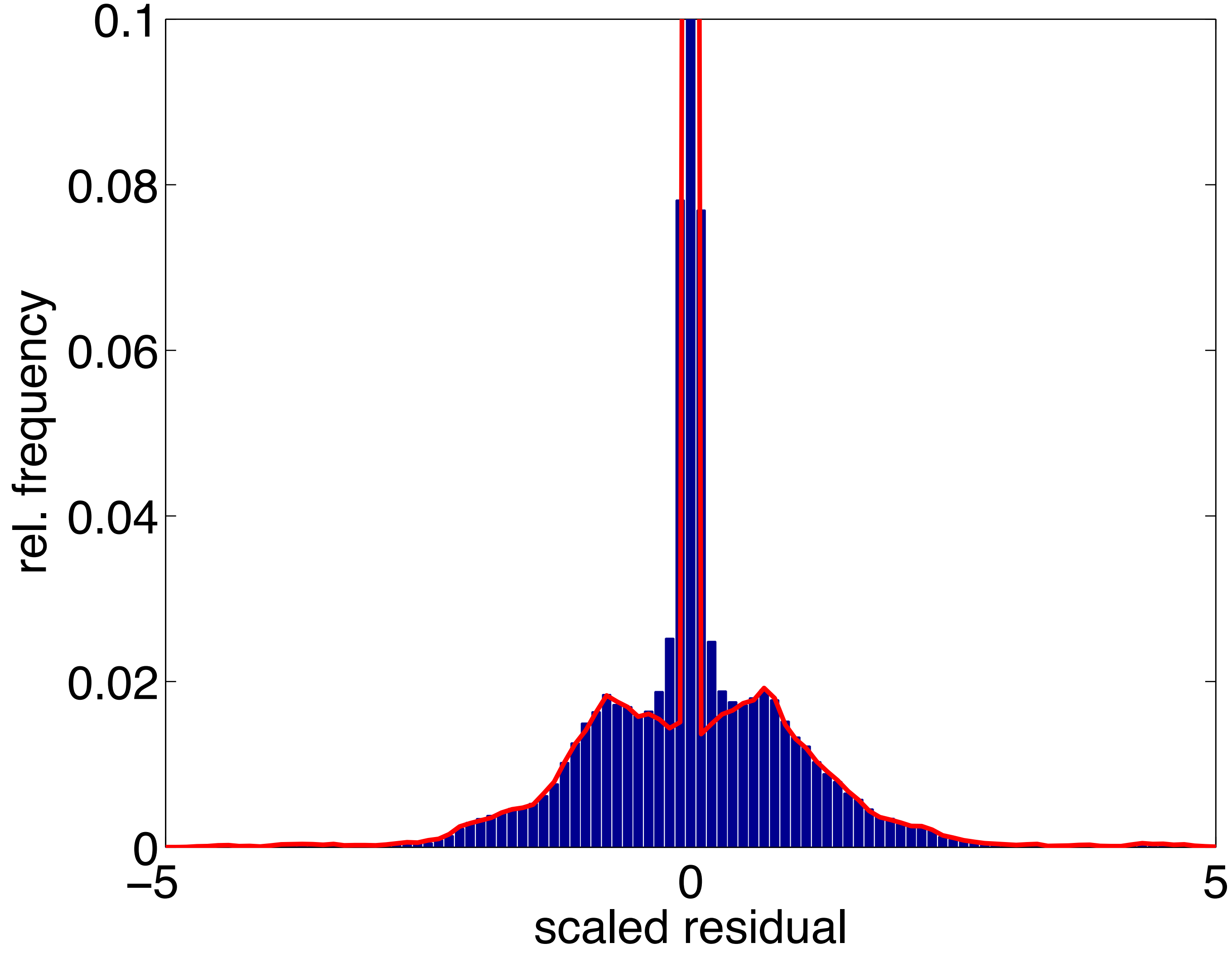
Marmoussi, Huber fit, 50% corrupted data



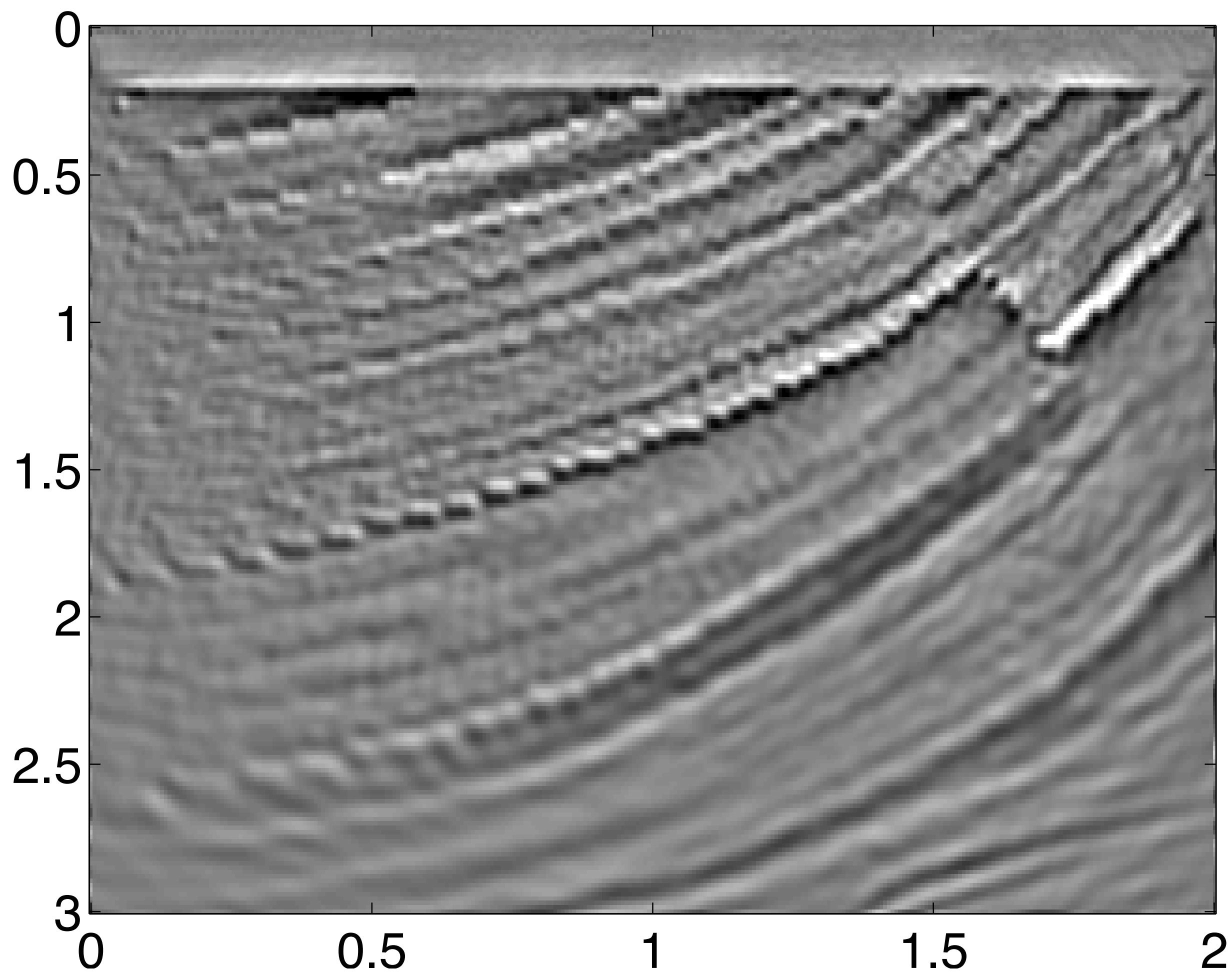


Marmoussi: T fit, 50% corrupted data

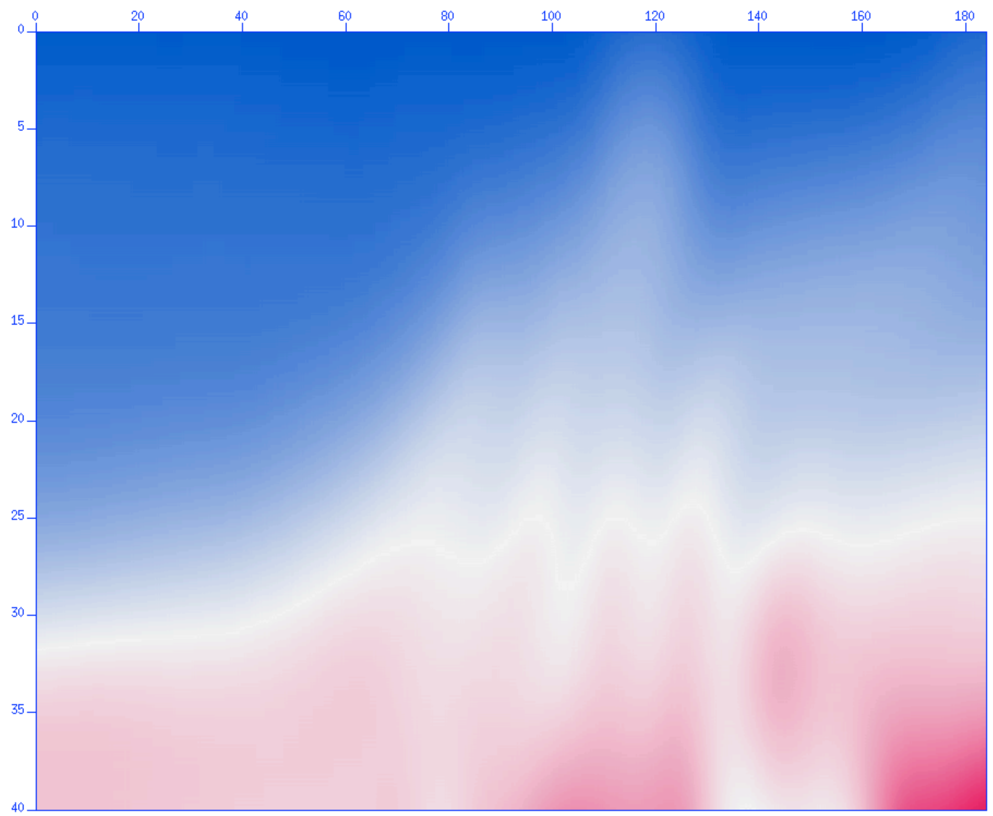




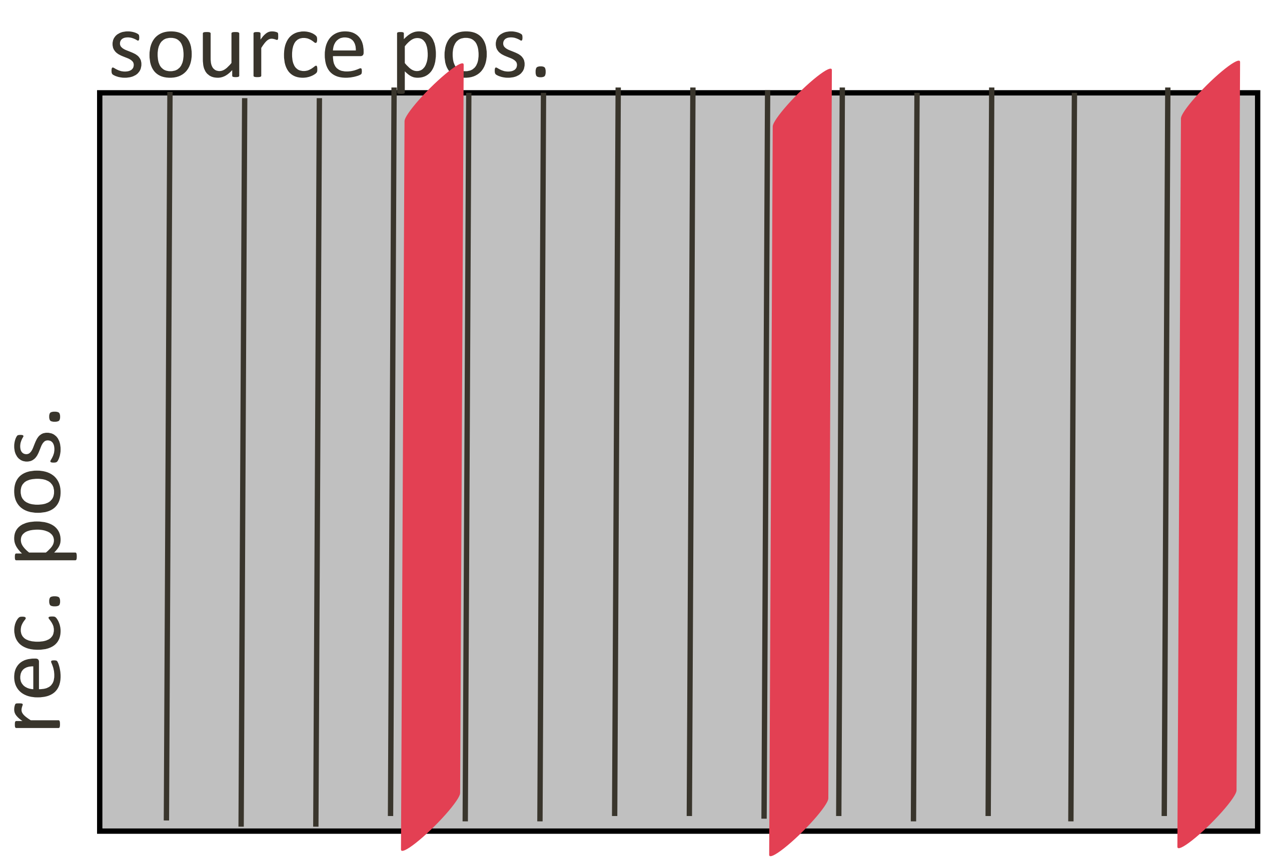
Marmoussi: LS fit, corrupted data ignored



Marmoussi II: Total Implementation

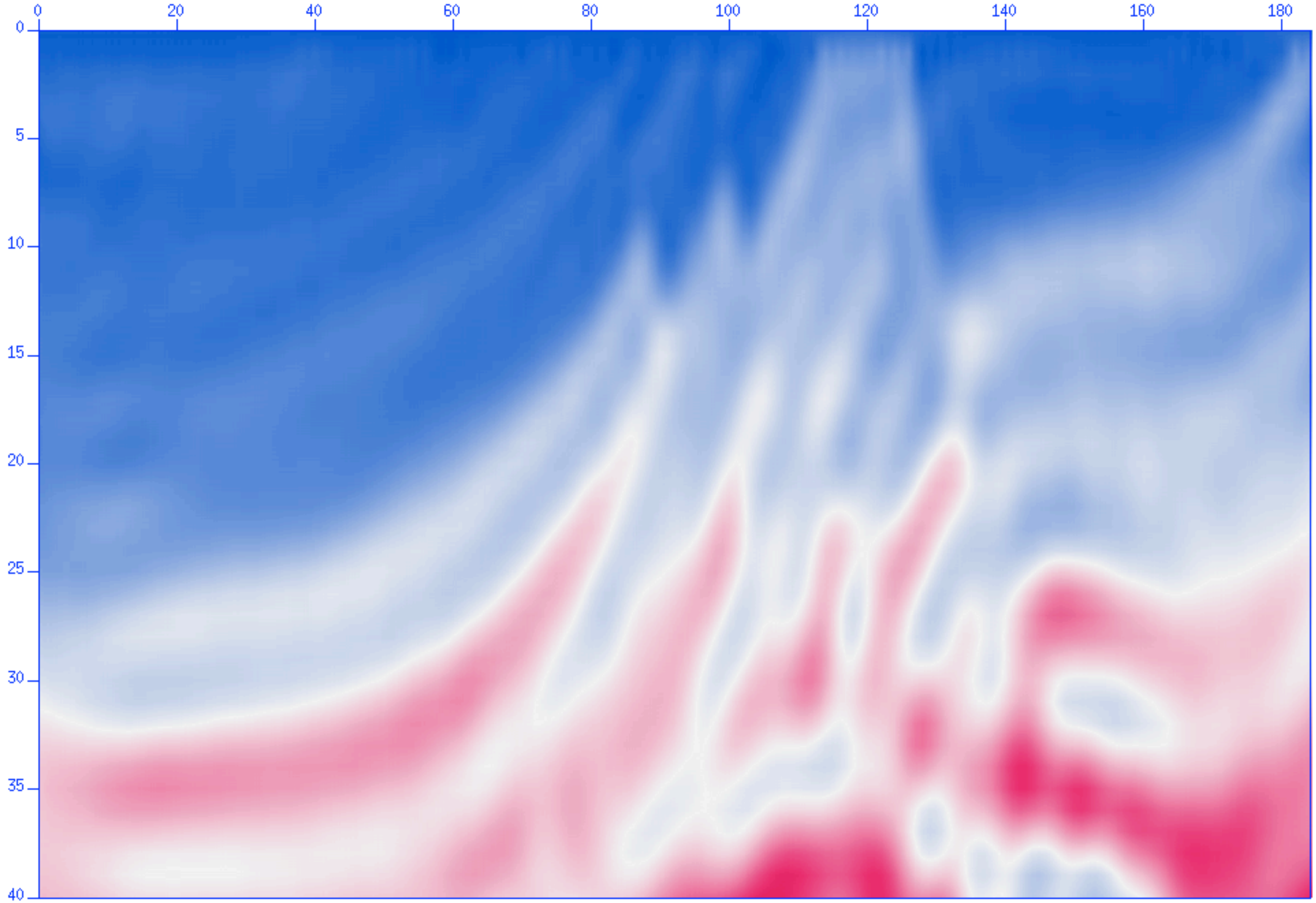


Initial Model, 4 Hz

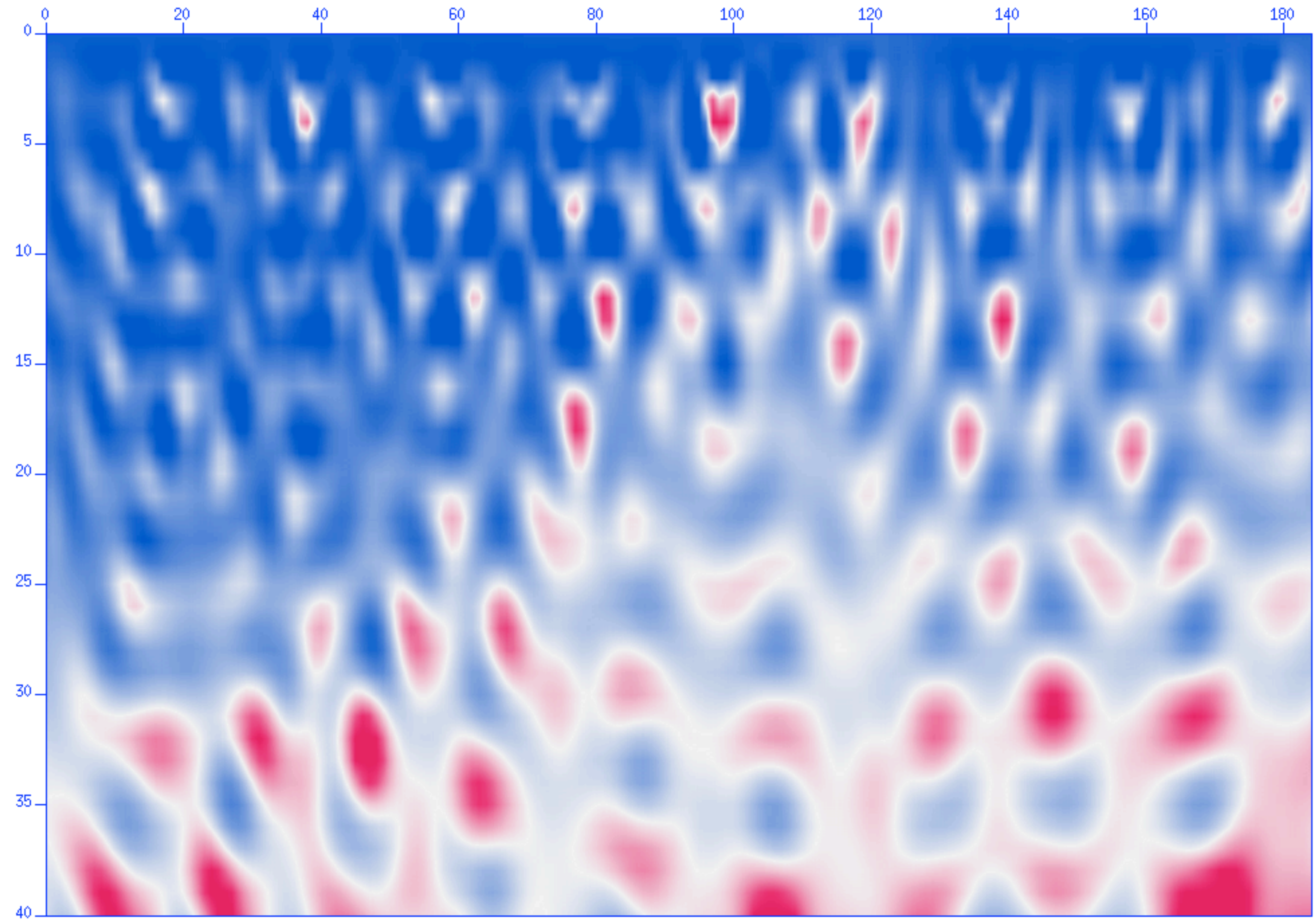


20% Corrupted Data

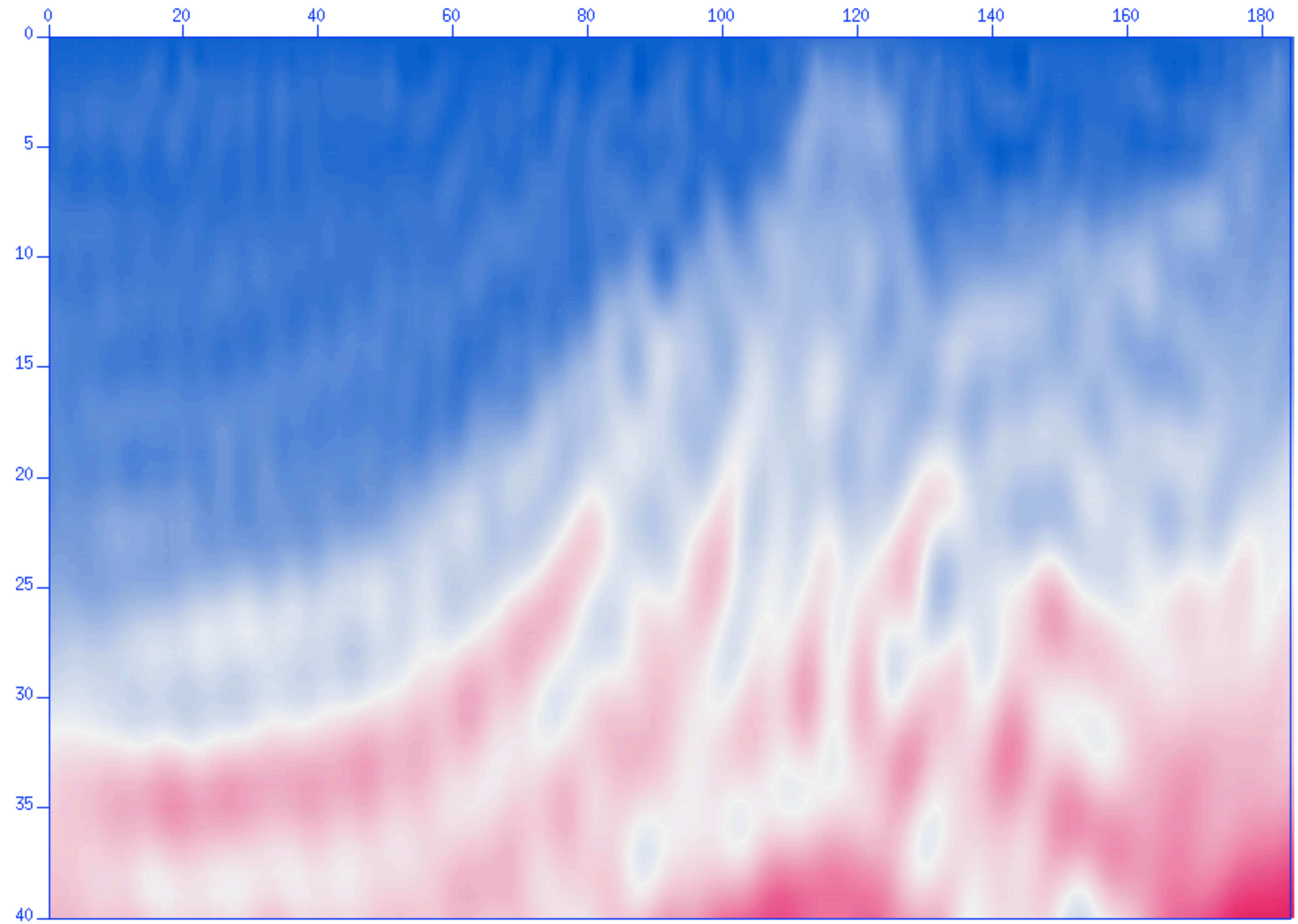
Results: Least Squares with GOOD data, 4Hz



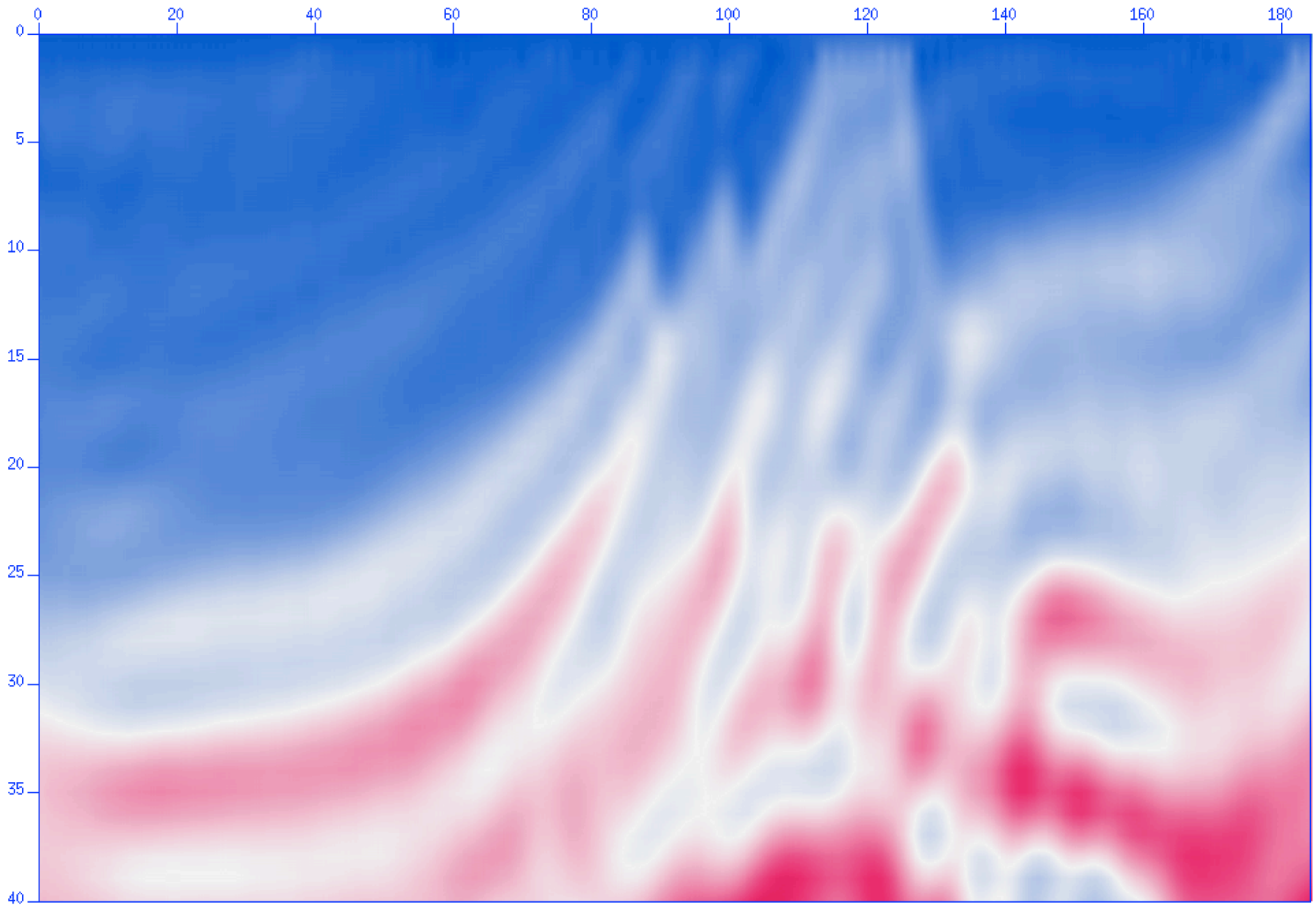
Results: Least Squares with BAD data, 4 Hz



Results: Student's t with BAD data, 10 DF, 4Hz



Results: Least Squares with GOOD data



Source Estimation for Robust Formulations

- We consider general inverse problems of the form

$$\min_{\mathbf{m},\boldsymbol{\alpha}} \Phi(\mathbf{m}, \boldsymbol{\alpha}) = \sum_{i=1}^{m} \phi_i \left(\mathbf{r_i}(\mathbf{m},\, \alpha_i)\right) \;,$$

$$\mathbf{r_i}(\mathbf{m},\, \alpha_i) := \mathbf{d_i} - \alpha_i \boldsymbol{\mathcal{F}}_\mathbf{i}(\mathbf{m})\mathbf{q_i}$$

$\mathbf{d_i}$	$n \times 1$ shot record
$\mathbf{q_i}$	$l \times 1$ source
$\mathbf{m}$	parameters to be recovered
$\boldsymbol{\mathcal{F}}_\mathbf{i}(\mathbf{m})$	Forward model (calculated data)
α_i	Unknown source amplitude
ϕ_i	Smooth misfit function (robust)

- We estimate source amplitudes and model parameters **jointly.**

Generalized Variable Projection Approach

- For fixed model parameters, obtain a function of amplitudes only:

$$\rho(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \phi_i(\mathbf{r_i}(\hat{\mathbf{m}}\ , \alpha_i))$$

- Find the optimal amplitudes by minimizing this function:

$$\hat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{\alpha}} \sum_{i=1}^{m} \phi_i(\mathbf{r_i}(\hat{\mathbf{m}}\ , \alpha_i))$$

- One can easily show that as long as misfit is smooth,

$$\nabla\Phi(\mathbf{m}, \boldsymbol{\alpha}(\mathbf{m})) = \nabla_{\mathbf{m}}\Phi(\mathbf{m}, \hat{\boldsymbol{\alpha}})$$

  - As long as **amplitudes are re-estimated at each step**, we iterate as usual, and still converge to a minimum of the **joint** objective.
  - The key now is to solve the amplitude-only problem **FAST**.

Newton Method for Amplitude-Only Problem

- For each amplitude, implement (scalar) Newton's method:

$$
\begin{aligned}
\alpha_i^{k+1} &= \alpha_i^k + s_i^k d_i^k \\
d_i^k &= -\frac{\nabla \phi_i(\mathbf{r_i}(\hat{\mathbf{m}}, \alpha_i^k))^T \boldsymbol{\mathcal{F}}_i(\hat{\mathbf{m}})\mathbf{q_i}}{\|\boldsymbol{\mathcal{F}}_i(\hat{\mathbf{m}})\mathbf{q_i}\|_H^2}
\end{aligned}
$$

  - When misfit is least squares, this method converges in one iteration and reduces to standard source estimation formula.
  - In general, good solutions are obtained in 5 to 10 iterations, so general source estimation requires only a little more effort than LS source estimation.
  - When the Hessian is not positive definite (Student's t), it is easy to adjust.
  - Updates to amplitudes to not require any forward modeling - note that model is fixed throughout the entire process.

Source Estimation for Student's t-Formulation

- Full algorithm:

$$\begin{aligned}
\alpha_i^{k+1} &= \alpha_i^k - \sum_j \frac{r_{ij}^k f_{ij}}{k + (r_{ij}^k)^2} \Big/ \sum_j \frac{f_{ij}^2}{k + (r_{ij}^k)^2} \\
f_{ij} &= (\boldsymbol{\mathcal{F}}_i(\mathbf{m})\mathbf{q_i})^j \\
r_{ij}^k &= d_{ij} - \alpha_i(\boldsymbol{\mathcal{F}}_i(\mathbf{m})\mathbf{q_i})^j
\end{aligned}$$

- Hessian for Student's t misfit is NOT positive definite, but we use a simple modification to design a nice method.
- In practice, the method converges in just a few iterations, without a line search.
- It took just a few hours to implement Student's t source estimation in a massively parallel FWI code at Total, thanks to Henri Calandra.

Proof of Concept

- We perform a simple experiment:
  - (1) We generate vectors of observed and predicted data randomly.
  - (2) We estimate source amplitude using LS and Student's t methods.
  - (3) We add 'outliers' to a portion of the data
  - (4) We re-estimate the source using LS and Student's t, and compare.

- Results: for vectors of 100,000 measurements, we generated outliers 10000 times larger than actual data points.

	LS Error (%)	T Error (%)
Good data	**0**	$\approx 1\%$
10 outliers	$\approx$ 1000%	$\approx 1\%$
100 outliers	$\approx$ 9000%	$\approx 1\%$

Conclusions

- Robust formulations to FWI that are able to ignore LARGE unexplained artifacts in the data.
- A particularly robust approach is obtained using **heavy tailed** densities, such as the Student's t, and corresponding **non-convex** penalties.
  - It is easy to modify an existing code base to solve the Student's t FWI problem!
- We have also derived a general methodology for robust source estimation using variable projection, and applied it to Robust FWI with Student's t.
- Future work includes strategies for automated estimation of degrees of freedom parameters **k** and uncertainty quantification.

SLIM

Acknowledgements

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Marmoussi Example

- We consider a subset of the Marmoussi model
- 151 shots, 301 receivers
- 9 pt. discretization of Helmholtz operator with absorbing boundary; 10 m. spacing on grid
- Sample of Frequencies [5.0, 6.0, 11.5, 14.0, 15.5, 17.5, 23.5] Hz