# Introduction to compressive (wavefield) computations

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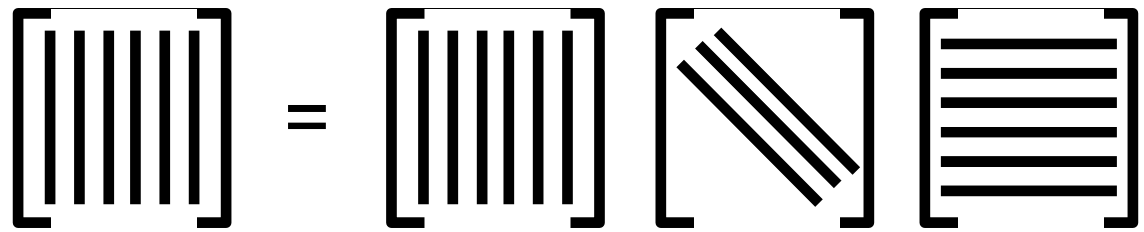
# Motivation

"Holy grail" has been to find transforms that "near diagonalize" wavefield extrapolation operators



$$\mathbf{W}^{\pm} = \mathbf{C}^{H} \hat{\mathbf{W}}^{\pm} \mathbf{C}$$

For smooth media curvelets "remain" near diagonal.

Efforts are made to correct for curvelet dispersion.

Problems:

- wavefield extrapolation operators difficult to compute in transformed domain
- complex media tend to fill the extrapolation matrix
- hampered by large constants

Propose a different approach ....

# Compressive
## wavefield computations

Near diagonality corresponds to preservation of sparsity (Hart Smit's original motivation)

Use this property to "smash" wavefield extrapolation & imaging operators

- compressively sample the solution & recover by
- exploiting sparsity &
- "incoherence" between measurement basis & sparsity representations

Smashed operators correspond to operators that are

- restricted in angular frequencies
- restricted in eigenvalues
- etc. etc.

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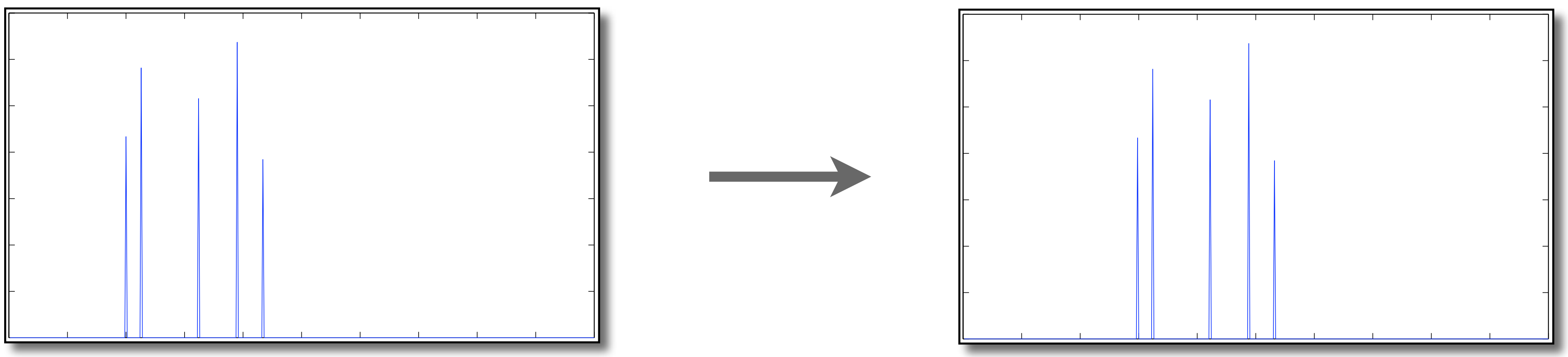
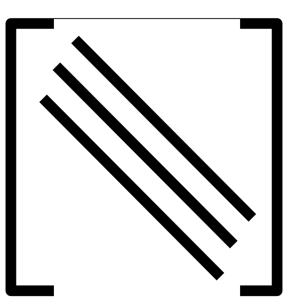
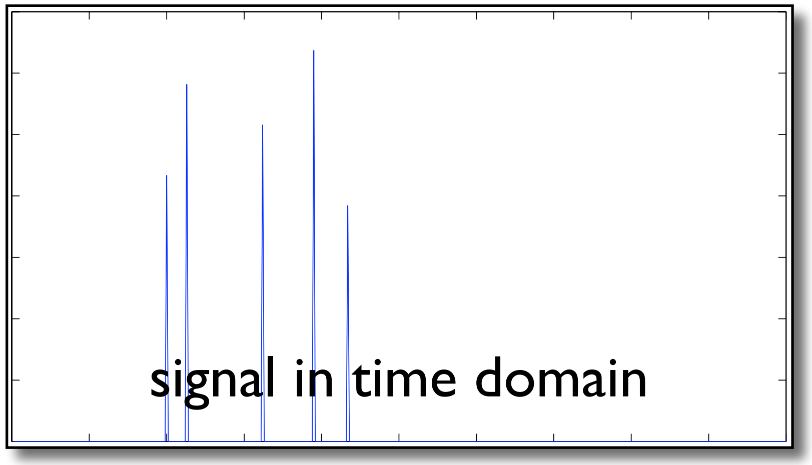
# Our approach

- Consider a related, but simpler problem: shifting (or translating) signal



- operator is $\mathbf{S} = e^{-j\frac{\Delta x}{2\pi}\mathbf{D}}$
- $\mathbf{D}$ is differential operator

$\mathbf{D} =$ 

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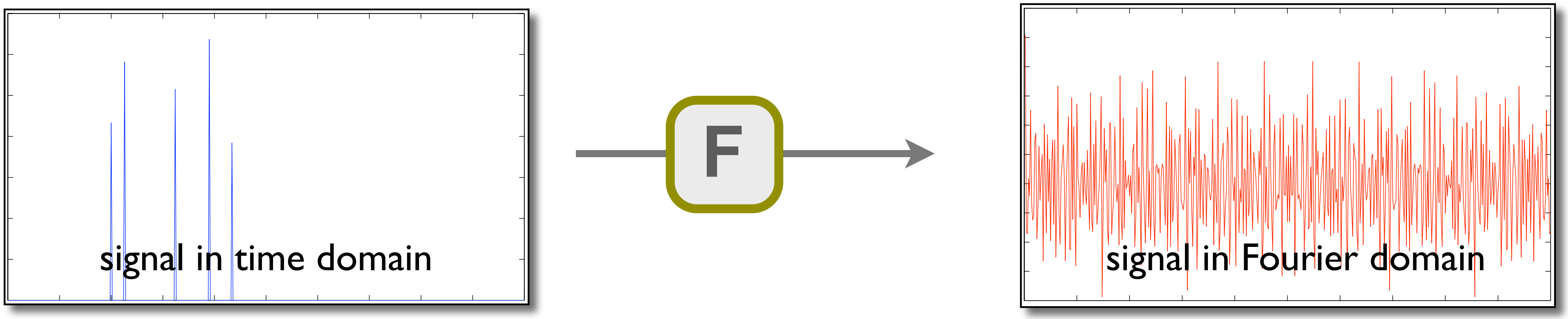
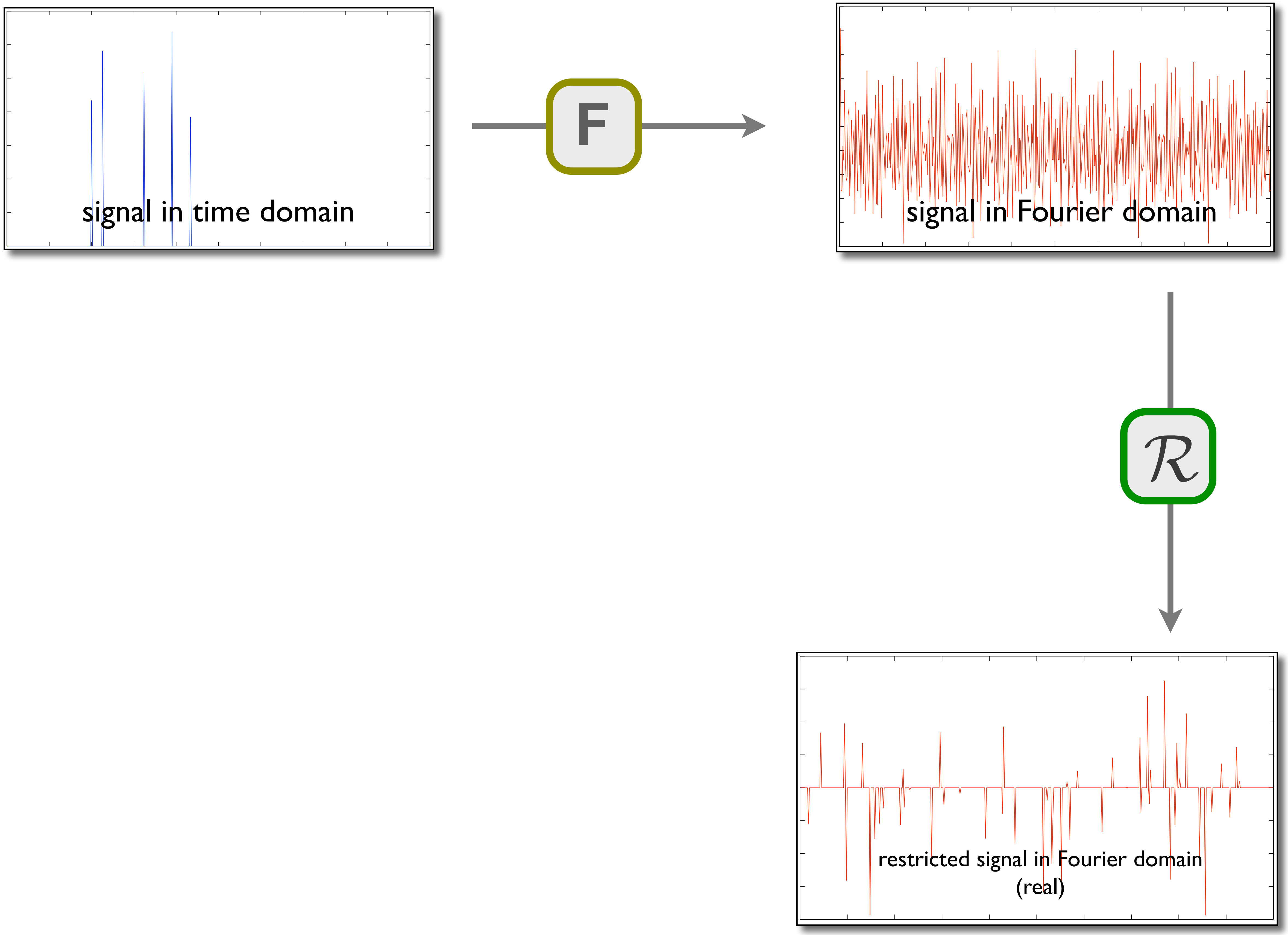

signal in time domain
F
signal in Fourier domain

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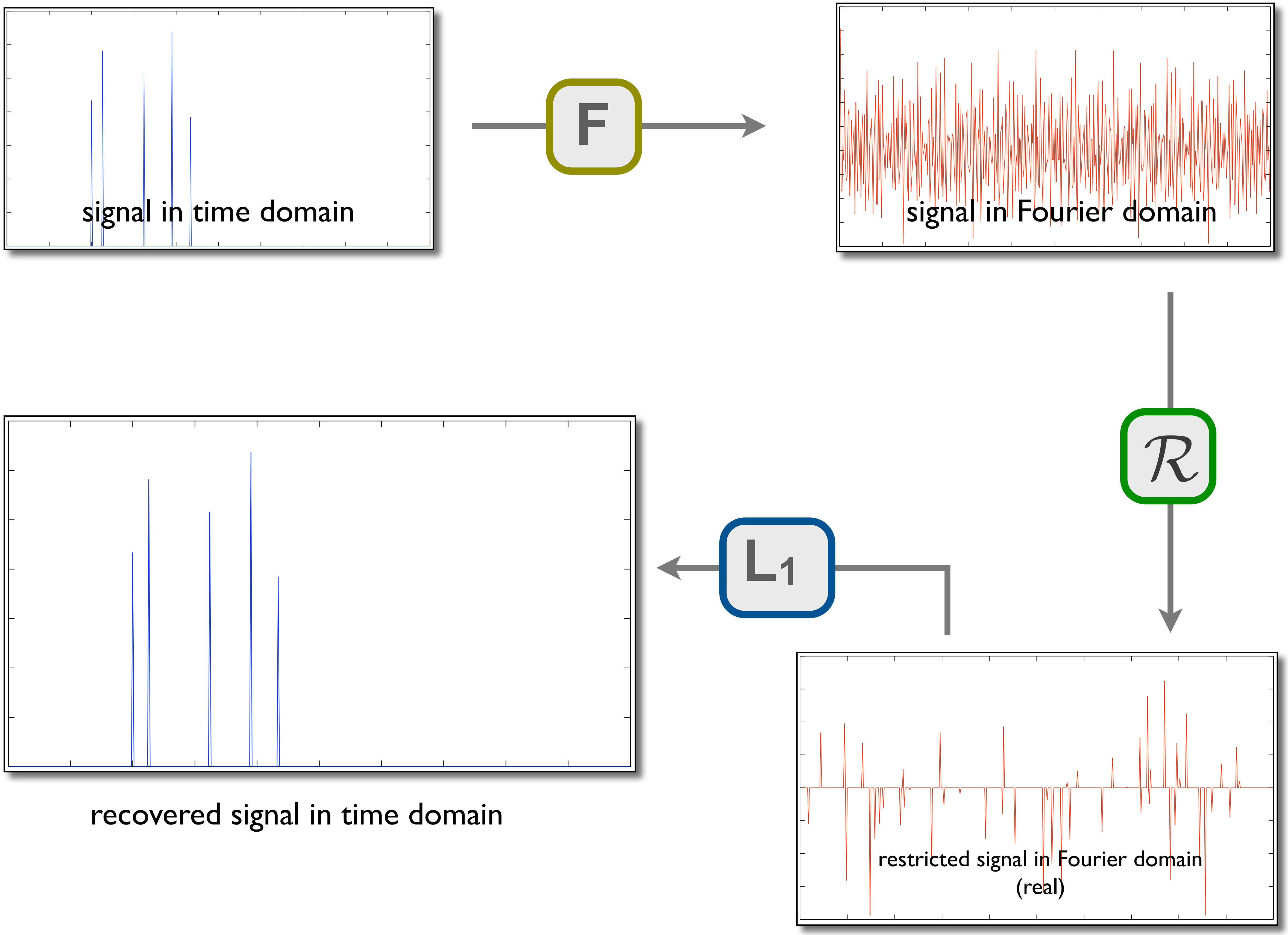
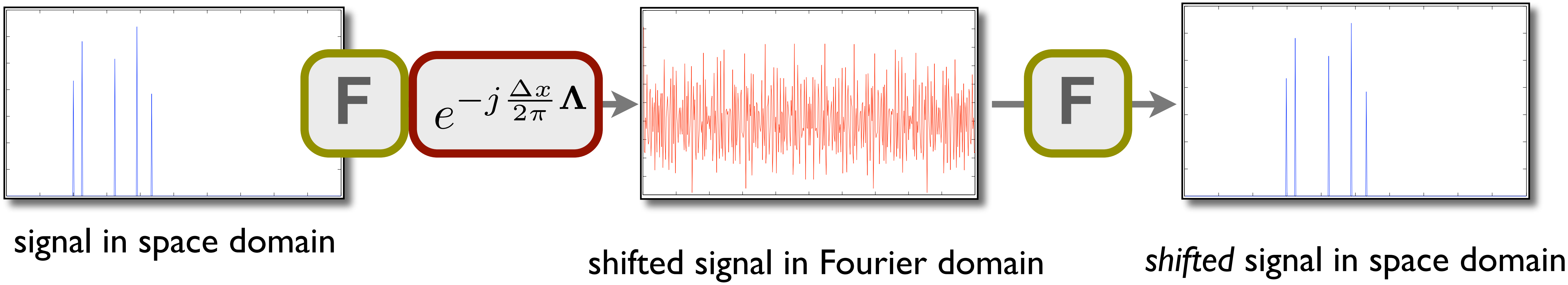

signal in time domain
F
signal in Fourier domain
$\mathcal{R}$
restricted signal in Fourier domain
(real)
L1
recovered signal in time domain

# Straightforward Computation




# Compressed Processing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# Compressive imaging

Seismic data is highly redundant.

Relies on redundancy to reduce SNR.

Based on multidimensional *correlations* of extrapolated wavefields.

*Correlations* are ‘poor man’s inverses.

Opens the possibility to leverage *compressive* sampling

- extend the *focal* transform to include *compressive* sampling
- reduce the computational burden of computing the gradient in adjoint state methods

# Tensor extension of compressive sampling

# Compressive sampling

Based on linear "forward" model

$$\begin{aligned} \mathbf{y} &= \mathbf{RMf}_0 \\ &= \underbrace{\mathbf{RMS}^H}_{\mathbf{A}} \mathbf{x}_0 \end{aligned}$$

with

$$\begin{aligned} \mathbf{R} &= \text{flat restriction matrix} \\ \mathbf{M} &= \text{measurement basis} \\ \mathbf{S}^H &= \text{synthesis matrix} \\ \mathbf{y} &= \mathbf{RMf}_0 \quad \text{(measured data)} \\ \mathbf{f}_0 &= \text{original function} \\ \mathbf{x}_0 &= \text{transform-domain representation.} \end{aligned}$$

# Tensor CS

"Wavefield deconvolution" at heart of

- focal transform
- imaging (*prior* to applying imaging conditions)

Can be formulated as a CS problem

$$
\begin{aligned}
\mathbf{P} &= \mathbf{\Delta P P}_0 \\
\mathbf{Y} &= \underbrace{\mathbf{R \Delta P S}^H}_{\mathbf{A}} \mathbf{x}_0
\end{aligned}
$$

- for the focal transform
- multiple right-hand sites
- **R** is a 2-D picking matrix

Special case of more general **tensor** formulation of CS for matrices ...

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# Tensor CS

Use Kronecker product and the vector identity

$$
\begin{aligned}
\mathbf{UVW} &= \mathbf{Z} \\
(\mathbf{W}^H \otimes \mathbf{U})\text{vec}(\mathbf{V}) &= \text{vec}(\mathbf{Z})
\end{aligned}
$$

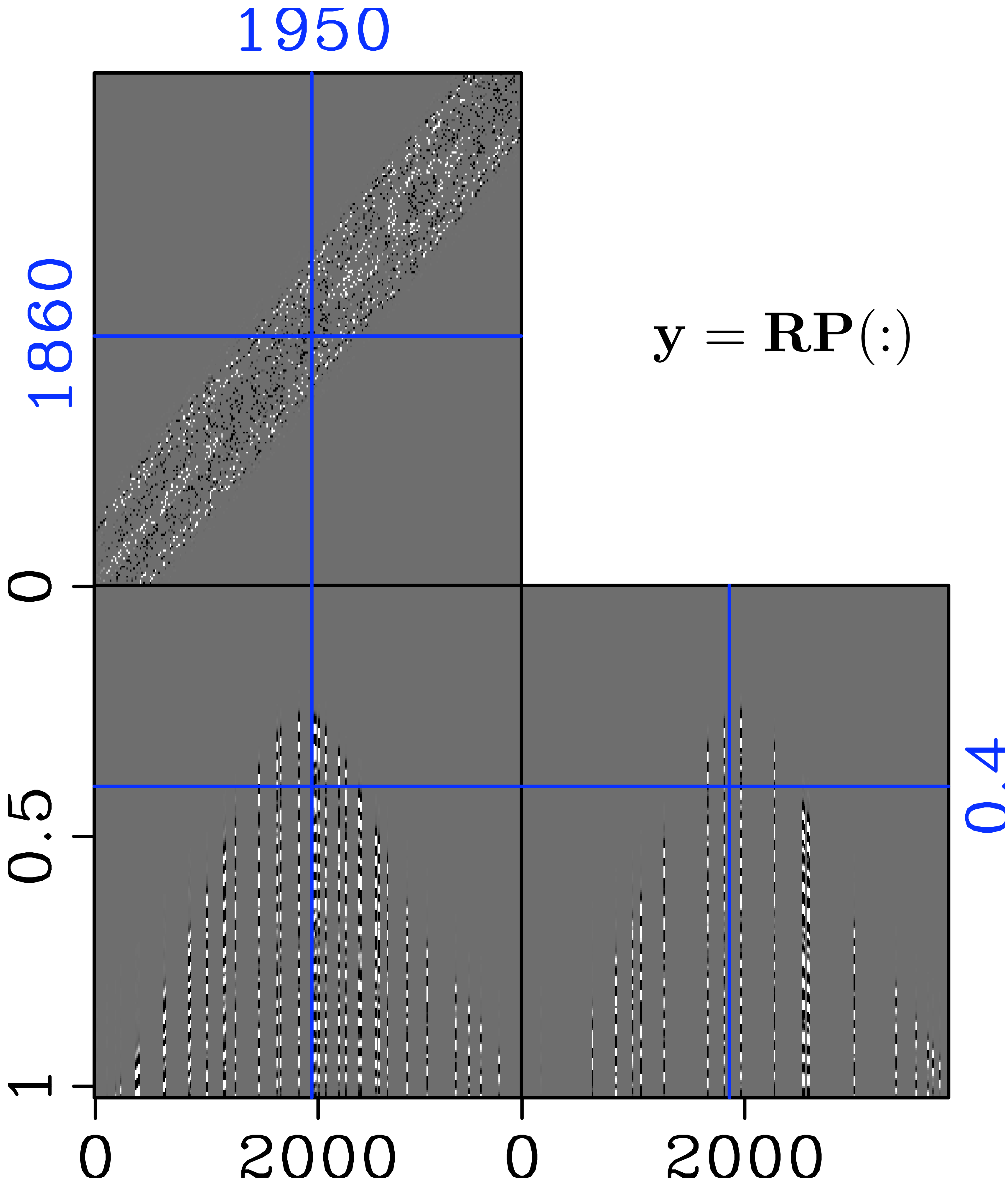
yielding the "forward model" for CS

$$
\mathbf{y} = \overbrace{(\mathbf{R}_1\mathbf{M}_1 \otimes \mathbf{R}_2\mathbf{M}_2)\boldsymbol{\Delta}\mathbf{P}\mathbf{C}^H}^{\mathbf{A}}\text{vec}(\mathbf{x}_0)
$$

$$
\mathbf{y} = (\mathbf{R}_1\mathbf{M}_1 \otimes \mathbf{R}_2\mathbf{M}_2)\text{vec}(\mathbf{P})
$$

- so far we used **M=Id**
- possibility to measure & restrict more generally

Opens possibility to image compressively compute wavefields (CCW) ....




$$\mathbf{y} = \mathbf{RP}(:)$$

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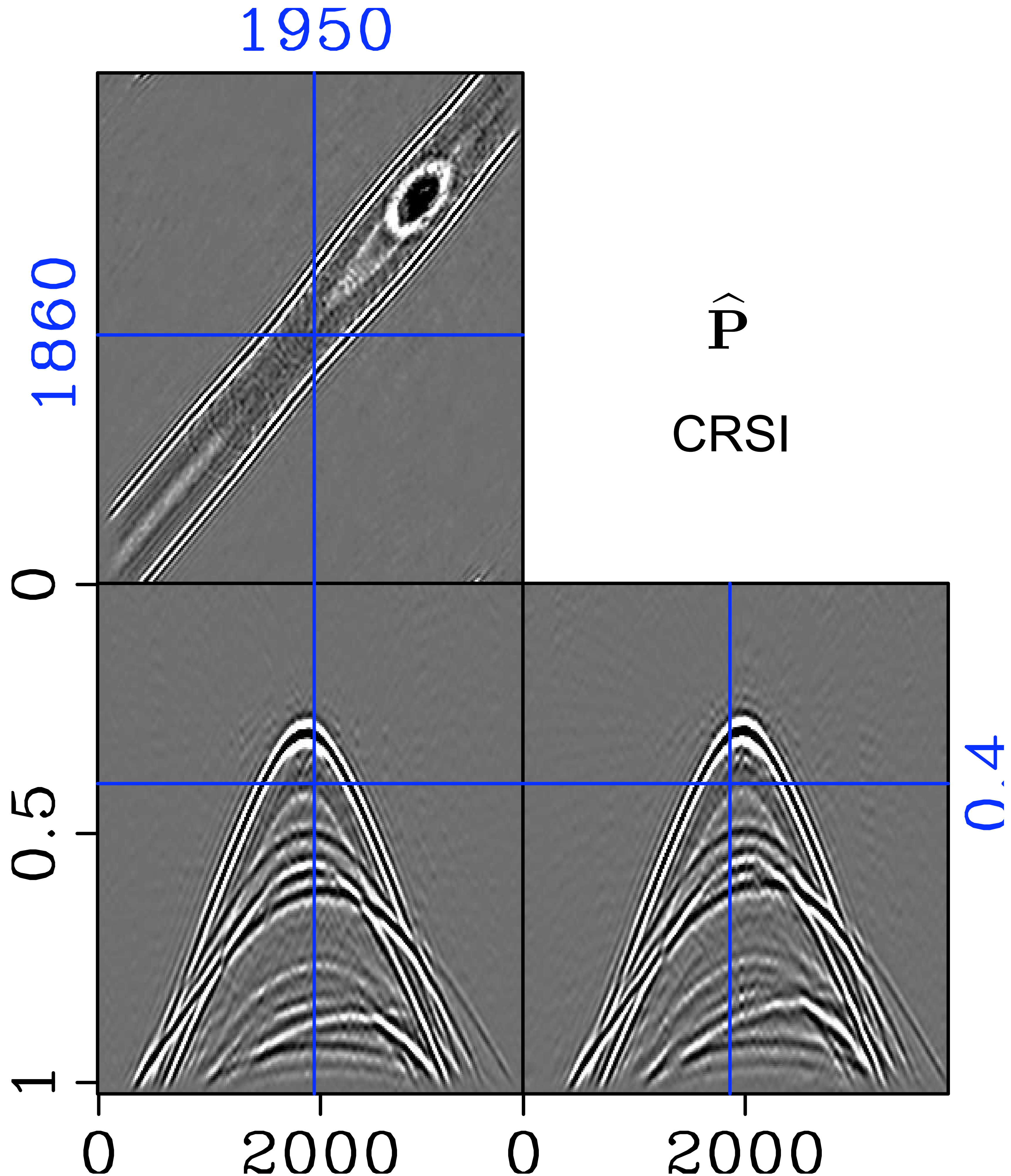

1950
1860
$\widehat{\mathbf{P}}$
CRSI
0
0.5
1
0.4
0
2000
0
2000
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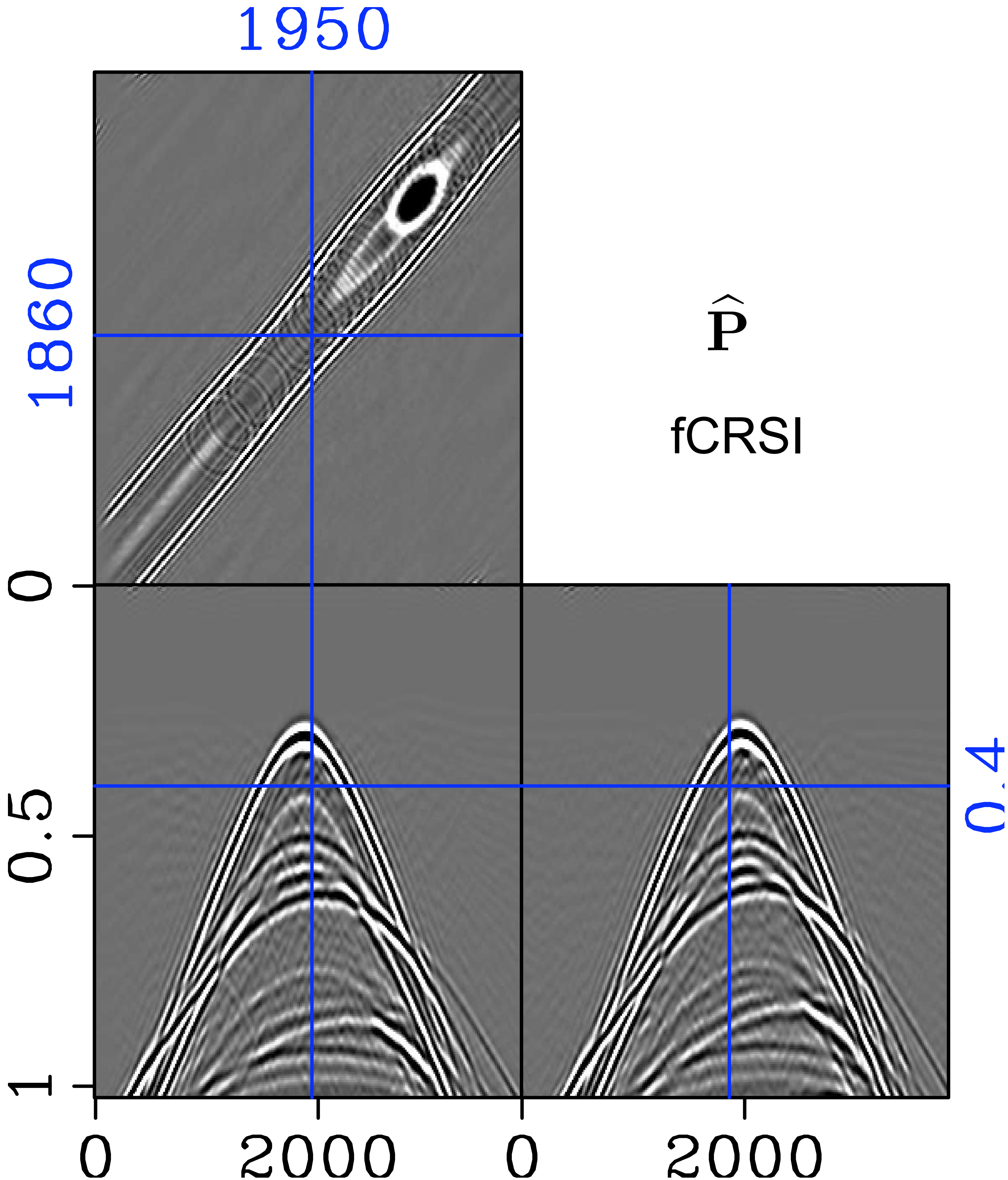
1950
1860
0
0.5
1
0.4
0
2000
0
2000
$\widehat{\mathbf{P}}$
fCRSI

# Compressive imaging

# Seismic imaging

Current paradigm:

- based on adjoint state methods for the wave equation
- predominantly implemented in finite difference => reverse-time wave-equation migration
- most time spend on **marching** wavefields
- imaging done through multidimensional **correlations**

New developments

- implicit spectral methods for Helmholtz
- new preconditioners
- compressive sensing

Propose a new nonlinear paradigm ...

# Compressive seismic imaging

New paradigm: compressively

- forward propagate the source wavefield
- backward propagate the residual wavefield
- multidimensionally invert ("deconvolve") the backward propagated wavefield from the forward propagated wavefield => image

Benefits from

- current preconditioners for Helmholtz
- reduced system size due to CS
- improved image through inversion
- intuitive divide-and-conquer
  - compressive linear wavefield extrapolation
  - nonlinear image recovery

# Adjoint state methods

Migration corresponds to the Jacobian of a PDE constraint optimization problem.

Forward model:

$$\mathbf{A}[\mathbf{m}]\mathbf{u} = \mathbf{f}$$

With observed data

$$\mathbf{d} = \mathbf{D}\mathbf{u} + \mathbf{n}$$

Inverse problem

$$\min_{\mathbf{u}\in\boldsymbol{\mathcal{U}},\, \mathbf{m}\in\boldsymbol{\mathcal{M}}} \frac{1}{2}\|\mathbf{D}\mathbf{u} - \mathbf{d}\|_2^2 \quad \text{subject to} \quad \mathbf{A}[\mathbf{m}]\mathbf{u} = \mathbf{f}$$

# Adjoint state methods

Reformulate in unconstrained nonlinear LS problem

$$\min_{\mathbf{m}\in\boldsymbol{\mathcal{M}}} \frac{1}{2}\|\mathbf{F}[\mathbf{m}]-\mathbf{d}\|_2^2$$

with

$$\mathbf{F}[\mathbf{m}] = \mathbf{D}\mathbf{A}^{-1}[\mathbf{m}]\mathbf{f}$$

and the gradient = migrated image

$$\nabla J(\mathbf{m}) = -\Re\left(\sum_{\omega}\sum_{s}\langle \mathbf{v}_s, \frac{\partial \mathbf{A}}{\partial \mathbf{m}}\mathbf{u}_s\rangle\right)$$

involving for each shot the solution of

$$\mathbf{A}[\mathbf{m}]\mathbf{u} = \mathbf{f} \quad \text{and} \quad \mathbf{A}^H[\mathbf{m}]\mathbf{v} = \mathbf{r}$$

with

$$\mathbf{r} = \mathbf{D}^H(\mathbf{F}[\mathbf{m}]-\mathbf{d})$$

# Compressed adjoint state method

Replace observations by compressed observations:

$$\mathbf{y} = \mathbf{RMDu}$$

Change representation forward model:

$$\mathbf{A}[\mathbf{S}^H\mathbf{x}]\mathbf{u} = \mathbf{f}$$

yielding the inverse problem

$$\min_{\mathbf{u}\in\hat{\mathcal{U}},\,\mathbf{x}\in\mathcal{X}} \frac{1}{2}\|\mathbf{RMDu}-\mathbf{y}\|_2^2 + \alpha\|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{A}[\mathbf{S}^H\mathbf{x}]\mathbf{u} = \mathbf{f}$$

Compressed systems for **R** along source & receiver & omega

$$\mathbf{A}[\mathbf{S}^H\mathbf{x}]\mathbf{u} = \mathbf{f} \quad \text{and} \quad \mathbf{A}^H[\mathbf{S}^H\mathbf{x}]\mathbf{v} = \mathbf{r}$$

with residue

$$\mathbf{r} = -\mathbf{D}^H\mathbf{M}^H\mathbf{R}^H(\underbrace{\mathbf{RMDA}^{-1}[\mathbf{S}^H\mathbf{x}]\mathbf{f}}_{\mathbf{F}[\mathbf{S}^H\mathbf{x}]} - \mathbf{y})$$

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# Compressed adjoint state method

Opens the way to formulate 'post-stack' imaging as

$$\min_{\mathbf{x}\in\boldsymbol{\mathcal{X}}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \underbrace{\mathrm{diag}\big(\mathbf{u}\big)\mathbf{S}^H\mathbf{x} = \mathbf{v}}_{\mathbf{A}\mathbf{x}=\mathbf{y}}$$

with descent updates on the l_1 ball

$$\mathbf{x}^{n+1} = \mathbf{S}_\alpha\big(\mathbf{x}^n + \mathbf{A}^H\big(\mathbf{y} - \mathbf{A}\mathbf{x}^n\big)\big)$$

Special case of prestack imaging

$$\min_{\mathbf{x}\in\boldsymbol{\mathcal{X}}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \underbrace{\big(\mathbf{RMU}\big)\mathbf{S}^H\mathbf{x} = \mathbf{RMV}}_{\mathbf{A}\mathbf{x}=\mathbf{y}}$$

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# Conclusions

CS opens perspectives on compressing operators.

New field in scientific computing.

Next talks will are precursors of what to come ...

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# Extensions imaging

part of SINBAD II