Stable recovery and separation of seismic data

“Non-parametric seismic data recovery with curvelet frames”
with Gilles Hennenfent
to be submitted

Context

- Linear least-squares data continuation and L1 [Claerbout, 73-92]
- Discrete & unequally sampled Fourier Transforms [Sacchi, 96; Schoneville, 01; Zwartjes, 04]
- Inpainting with Morphological Component Analysis (MCA) using Redundant Directional Frames such as Curvelets [Candes; Donoho; Demanet; Ying, 05; Elad, 05]
- Stable signal recovery with uniform uncertainty principles [Candes, Romberg and Tao 2004-2005]
- Compressed sensing [Donoho 04-06]
- Iterative thresholding [Daubechies et al 2004]
- A Hardy space for Fourier integral operators [Smit, 97]

Seismic recovery

Incomplete and noisy measurements:

$$\mathbf{y} = \overbrace{\mathbf{R}}^{\text{Restriction}} \underbrace{\mathbf{M}}_{Measurement} \mathbf{m} + \mathbf{n}$$

y incomplete and noisy data
m the unknown model
M measurement/modeling matrix
R restriction matrix
n noise

Our approach

- Sparsity promotion by norm-one minimization.
- Sparsity promoting transforms.
- Seek a sparse representation for the model space.
- Forward model:

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0$$

with

$$\mathbf{A} = \mathbf{R}\mathbf{M}\mathbf{S}^H$$

restriction matrix (R), **measurement matrix** (M), **sparsity matrix** (S^H)

Our approach

Sparsity representation:

- prototype waveforms that locally match
- redundant

Invert an underdetermined system.

Regularized by sparsity.

Machinery works when

- subset of columns act as an orthobasis
- there is sufficient mixing

Difference with existing methods

- seek sparsity transform
- provide conditions for recovery

Recovery

Include a sparsity matrix S:

$$\mathbf{y} = \mathbf{R} \overbrace{\mathbf{M}\mathbf{S}^H}^{\text{multiscale modeling}} \mathbf{x}_0 + \mathbf{n}$$

or

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{n}$$

Main challenge recover m-sparsity vector $\mathbf{x}_0$ from noisy *n*-vector y for *n*<<*m*

- **underdetermined problem**
- **depends on sparsity x, size y and properties A**

Use recent results from information theory.

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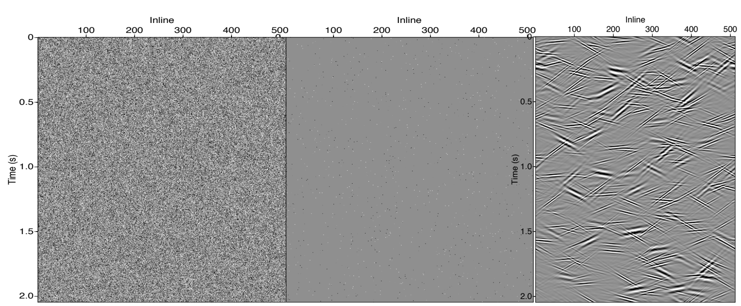
Prior information

on the model



l_2 in Dirac basis l_1 in Dirac basis l_1 in curvelet frame

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Our program

Theoretical:

- selection of the appropriate sparsity & measurement matrices.
- conditions for recovery.
- measures for performance.

Applied:

- translate seismic problems into recovery framework
- tailor to seismic situation
- built computational framework
- test on real size problems
- apply to real data

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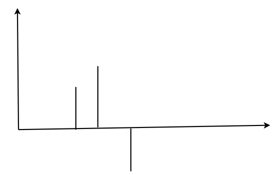
Canonical problem

Suppose one would like to recover a ***sparse* spike** train



What would be the best strategy when we can measure by taking inner products?

Canonical problem

Could measure with a spike/Dirac basis

- would have to look everywhere

$$\mathbf{f} = \mathbf{I}^T\mathbf{I}\mathbf{f}$$

Could measure with complex exponentials (Fourier)

$$\mathbf{f} = \mathbf{F}^T\mathbf{F}\mathbf{f}$$

- would also have to measure everywhere

True when signal is recovered linearly ...

Nonlinear recovery

Candes, Tao, Romberg and Donoho show

$$(P_1): \begin{cases} \min_{\mathbf{x}\in\mathbb{R}^m} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2 \leq \epsilon \\ \hat{\mathbf{m}} = \mathbf{S}^H\hat{\mathbf{x}} \end{cases}$$

reconstructs for specific A within the noise level

$$\|\hat{\mathbf{x}} - \mathbf{x}_0\|_2 \leq C_3 \cdot \epsilon + C_4 \cdot S^{-r+1/2}.$$

when $n \propto \mu^2 \cdot S$

with S the # largest S entries in x and μ the mutual coherence between **M** and **S.**

Recovery conditions

Mutual coherence:

$$\mu(\mathbf{M},\mathbf{S}) = \sqrt{m} \max_{(k,l)\in[1\cdots m]\times[1\cdots m]} |\langle m_k,\, s_l\rangle|$$

Success recovery depends on interplay

- Sparsity <-> compression rate

$$|\mathbf{x}_{i\in I}| \leq Ci^{-r},\, r \geq 1 \quad \text{and} \quad x_{I(1)} \geq x_{I(2)} \geq \cdots \geq x_{I(m)}$$

- Mutual coherence

Constants are known for certain combinations **M** & **S**.

Recovery example

Start with a spike train <-> Dirac basis = sparsity matrix

$$\mathbf{S} = \mathbf{S}^H = \mathbf{I}$$

What would be a good measurement basis M?

$$\mathbf{M} = \mathbf{I}?$$

NO! This implies $\mu \to \sqrt{n}$ look everywhere ....

Let's take something completely different Fourier perhaps?

Framework

YES YES! Because incoherent Spike & Fourier give

$$\mu = 1$$

Can **EXACTLY** recover for truly sparse x & no noise

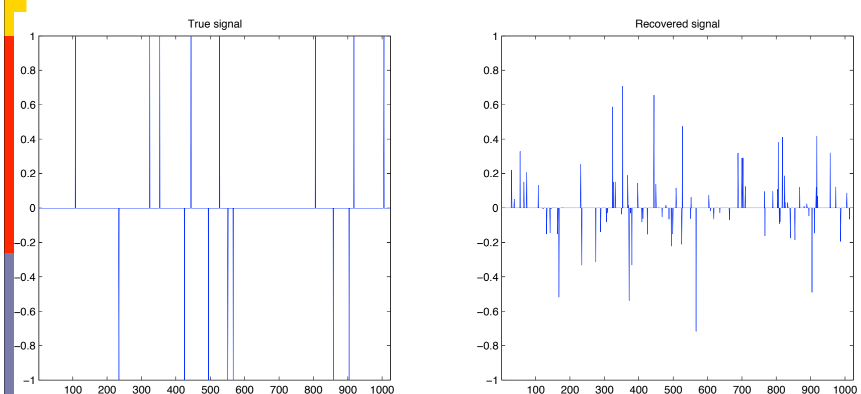
Can approximately recover when noise and x compressible, i.e.

$$|\mathbf{x}_{i \in I}| \leq C i^{-r}, \, r \geq 1 \quad \text{and} \quad x_{I(1)} \geq x_{I(2)} \geq \cdots \geq x_{I(m)}$$

for large as possible r**.**

Experiment I:

Sharpness Of The Recovery Condition



Thanks to Romberg

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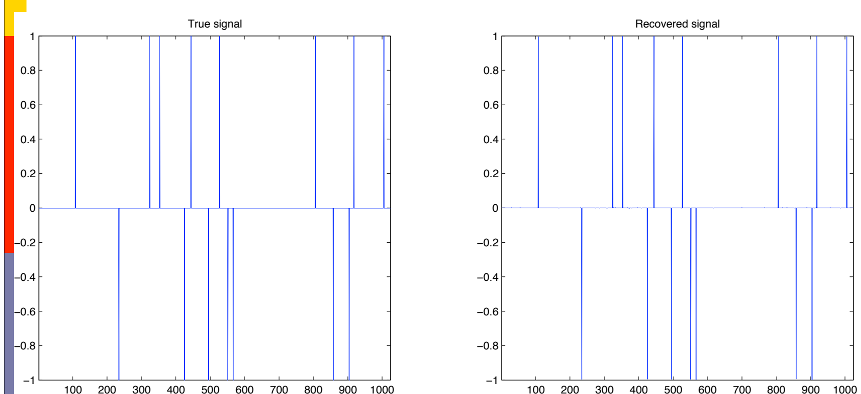
Sharpness Of The Recovery Condition



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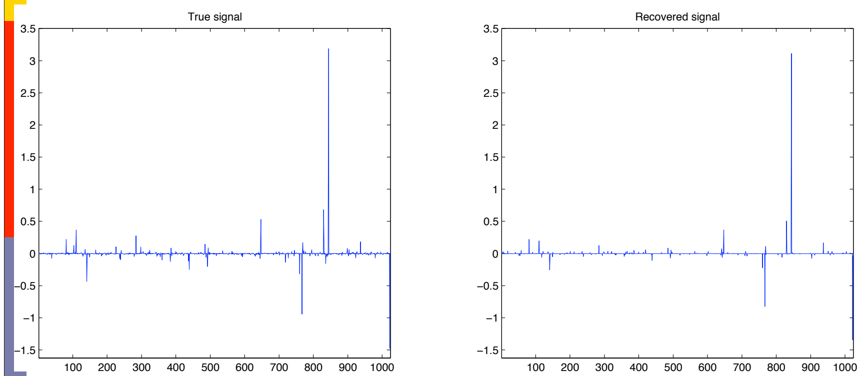
Experiment II:

Compressibility vs Incoherency



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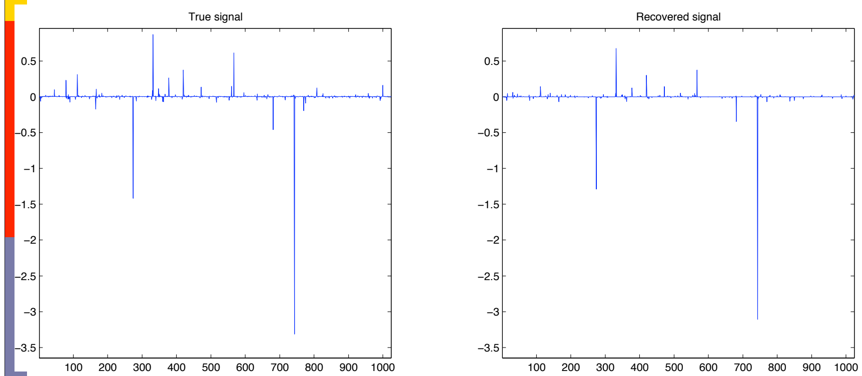
Compressibility vs Incoherency



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Questions

Given an incomplete seismic acquisition geometry how much can we recover and with what accuracy?

Given a certain accuracy, how much do we need to measure?

Can we convince management ....

Does the sharpness of the recovery tell us something?

Can the results from stable recovery with uniform uncertainty principles be extended to

- sparsity and measurement frames?
- migration-like sparsity frames?