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# Sparsity- and continuity-promoting seismic image recovery with curvelet frames

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submitted

# UPDATE WITH PEYMAN'S LATEST

# Context

- An optimal true-amplitude least-squares prestack depth-migration operator [Chavent & Plessix, 99]
- Frequency-domain finite difference amplitude preserving migration [Plessix & Mulder, 99]
- A microlocal analysis of migration [ten Kroode, Verdel & Smit, 98]
- Iterative thresholding [Daubechies et al 2004]
- A Hardy space for Fourier integral operators [Smit, 97]
- Stable signal recovery with uniform uncertainty principles [Candes, Romberg and Tao 2004-2005]

# Problem

Seismic data volumes are very large

- computation of the Hessian is prohibitive
- image amplitudes are not preserved

Utilize

- invariance curvelets under the Hessian
- smoothness of the symbol
- nonlinear signal recovery techniques

# Seismic imaging

Forward model:

$$\mathbf{d} = \mathbf{K}\mathbf{m} + \mathbf{n}$$

Imaging:

$$\mathbf{y} = \mathbf{K}^T\mathbf{d}$$

Pseudo inverse:

$$\mathbf{m} = \overbrace{\left(\mathbf{K}^T\mathbf{K}\right)^{-1}}^{\Psi\mathrm{DO}} \underbrace{\mathbf{K}^T}_{\mathrm{FIO}} \mathbf{d} = \mathbf{K}^{\dagger}\mathbf{d}.$$

# Seismic imaging

Misfit functional:

$$J(s) = \frac{1}{2}\|\mathbf{d} - \mathbf{F}[\mathbf{s}]\|_2^2.$$

Linearized Jacobian:

$$\mathbf{y} = -\nabla_{\mathbf{m}} J(\mathbf{m}) = \mathbf{K}[\bar{\mathbf{s}}]^T \mathbf{d}$$

Amplitude correction with inverse Hessian($\mathbf{\Psi} := \mathbf{K}^T\mathbf{K}$)

$$\mathbf{m} = -\mathbf{\Psi}^{\dagger} \nabla_{\mathbf{m}} J(\mathbf{m})$$

# Seismic imaging

Inversion Hessian infeasible

Two approaches to invert (precondition)

- high-frequency asymptotics <=> Microlocal analysis <=> diagonal approximation in Fourier domain
- diagonal approximation in space domain

Make strong assumptions on acquisition and freq. content

Our approach holds the middle <=> diagonal approximation in the curvelet domain.

# Seismic imaging

Imaging operators are Fourier Integral Operators

Make zero order via

$\mathbf{K} \mapsto \mathbf{K}\,(-\Delta)^{-1/2}$ and $\mathbf{m} \mapsto (-\Delta)^{1/2}\,\mathbf{m}$ with $((-\Delta)^{\alpha} f)^{\wedge}(\mathrm{k}) = |\mathrm{k}|^{2\alpha} \cdot \hat{f}(\mathrm{k})$.

Hessian zero-order pseudo differential operator

$$\mathbf{T} f(\mathrm{x}) = \int e^{i\mathrm{k}\cdot\mathrm{x}} a(\mathrm{x},\mathrm{k}) \hat{f}(\mathrm{k}) d\mathrm{k}$$

Curvelets are invariant under FIO's and pseudos.

**Candes & Demanet; Smit**

# Approximation of the Hessian

**Approximate**

$$\begin{aligned} \mathbf{y} &= \mathbf{K}^H\mathbf{K}\mathbf{m} + \mathbf{e} \\ &\simeq \mathbf{A}\mathbf{A}^H\mathbf{m} + \mathbf{e} \\ &= \mathbf{A}\mathbf{x}_0 + \mathbf{e} \end{aligned}$$

**using the following diagonal approximation**

$$\mathbf{K}^H\mathbf{K}\mathbf{r} \simeq \mathbf{C}^H\mathbf{\Gamma}[\mathbf{r}_0]\mathbf{\Gamma}[\mathbf{r}_0]\mathbf{C}\mathbf{r} := \mathbf{A}\mathbf{A}^H\mathbf{r}$$

**Approximation error decays with $2^{j/2}$ with the scale j**

# Denoising

Denoising problem:

$$\mathbf{y} = \mathbf{m} + \mathbf{n}$$

Sparsity representation

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{n}$$

Stable recovery

$$\mathbf{P}_1 : \quad \begin{cases} \min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \epsilon \\ \hat{\mathbf{m}} = \mathbf{A}\mathbf{x}, \end{cases}$$

# Denoising

A=S$^T$ orthonormal solve

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \|\mathbf{y} - \mathbf{m}\|_2^2 + \lambda \|\mathbf{S}\mathbf{m}\|_1$$

with soft thresholding

$$\hat{\mathbf{m}} = \mathbf{S}^T S_\lambda^s\left(\mathbf{S}\mathbf{y}\right)$$

with the soft thresholding operator given by

$$S_\lambda^s(x) := \begin{cases} x - \text{sign}(x)\lambda & |x| \geq \lambda \\ 0 & |x| < \lambda. \end{cases}$$

# Denoising

Extends to A redundant:

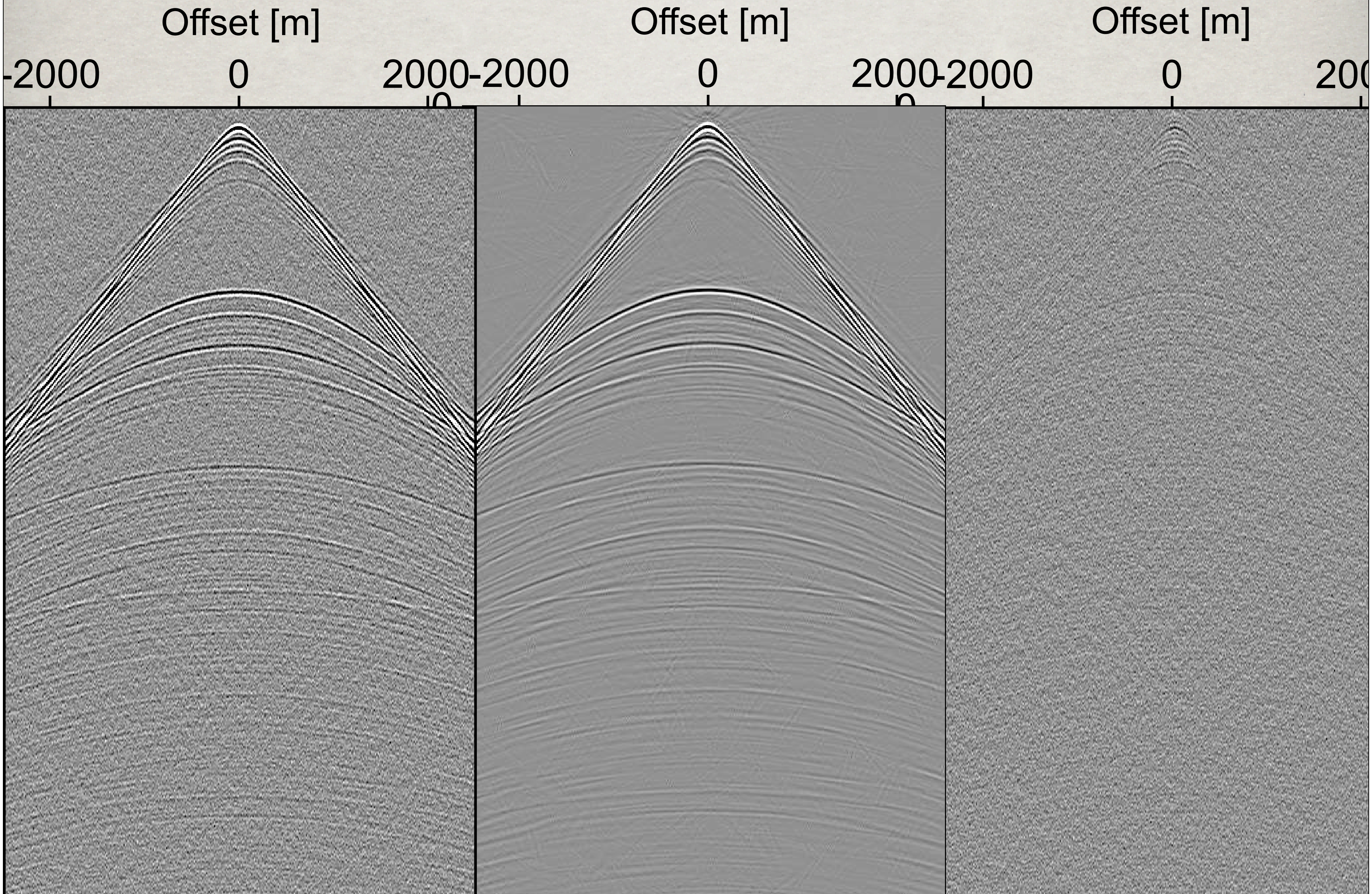
$$\mathbf{P}_\lambda : \quad \begin{cases} \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_1 \\ \hat{\mathbf{m}} = \mathbf{A}\hat{\mathbf{x}}. \end{cases}$$

solve with iterative soft thresholding

$$\mathbf{x}^{m+1} = S_\lambda^s \left( \mathbf{x}^m + \mathbf{A}^T (\mathbf{y} - \mathbf{A}\mathbf{x}^m \right)$$

**Candes et al, Donoho et al; Daubechies**

# Denoising



# Denoising

Initialize:

$m = 0$; $\mathbf{x}^0 = \mathbf{A}^T\mathbf{y}$;

Choose: $L$, $\lambda_1 > \lambda_2 > \cdots$

**while** $\|\mathbf{y} - \mathbf{A}\mathbf{x}^m\|_2 > \epsilon$ **do**

$m = m + 1$;

$\mathbf{x}^m = \mathbf{x}^{m-1}$;

**for** $l = 1$ to $L$ **do**

$\mathbf{x}^m = S^s_{\lambda_m}\left(\mathbf{x}^m + \mathbf{A}^T(\mathbf{y} - \mathbf{A}\mathbf{x}^m)\right)$ {Iterative thresholding}

**end for**

**end while**

$\hat{\mathbf{m}} = \mathbf{A}\mathbf{x}^m$.

# Seismic imaging

Return to the normal equation

$$\mathbf{y} = \mathbf{\Psi m} + \mathbf{e} \qquad \text{with} \qquad \mathbf{e} = \mathbf{K}^T\mathbf{n}$$

Suppose $\mathbf{A} := \mathbf{W}^T\mathbf{\Gamma}$ with W orthonormal and $\mathbf{\Psi r} = \mathbf{A}\mathbf{A}^T\mathbf{r}$ then

$$\mathbf{\Psi}^{-1}\mathbf{r} = \mathbf{S}^T\mathbf{S}\mathbf{r}$$

from which it follows that

$$\hat{\mathbf{m}} = \mathbf{S}^H S_\lambda^s(\mathbf{S}\mathbf{y})$$

which corresponds to a "weighted" denoising, i.e. Donoho's Wavelet-Vaguelette Estimators (WVD).

# Seismic imaging

By **analogy** generalize to redundant curvelet frame.

Set $\mathbf{A} := \mathbf{C}^T\mathbf{\Gamma}$

and solve for the reflectivity with

$$\mathbf{P}'_\lambda : \quad \begin{cases} \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1 \\ \hat{\mathbf{m}} = \left(\mathbf{A}^T\right)^\dagger \hat{\mathbf{x}}. \end{cases}$$

Remains to estimate the diagonal.

# Diagonal estimation

Exploit smoothness of the symbol of the pseudo.

Solve a regularized Least-squares problem w.r.t. $\mathbf{d}$

$$\begin{bmatrix} \mathbf{C}^H \mathbf{diag}\, \mathbf{r}_0 \\ \mathbf{Q} \end{bmatrix} \mathbf{d} = \begin{bmatrix} \mathbf{K}^H \mathbf{K} \mathbf{r}_0 \\ \mathbf{0} \end{bmatrix}$$

with

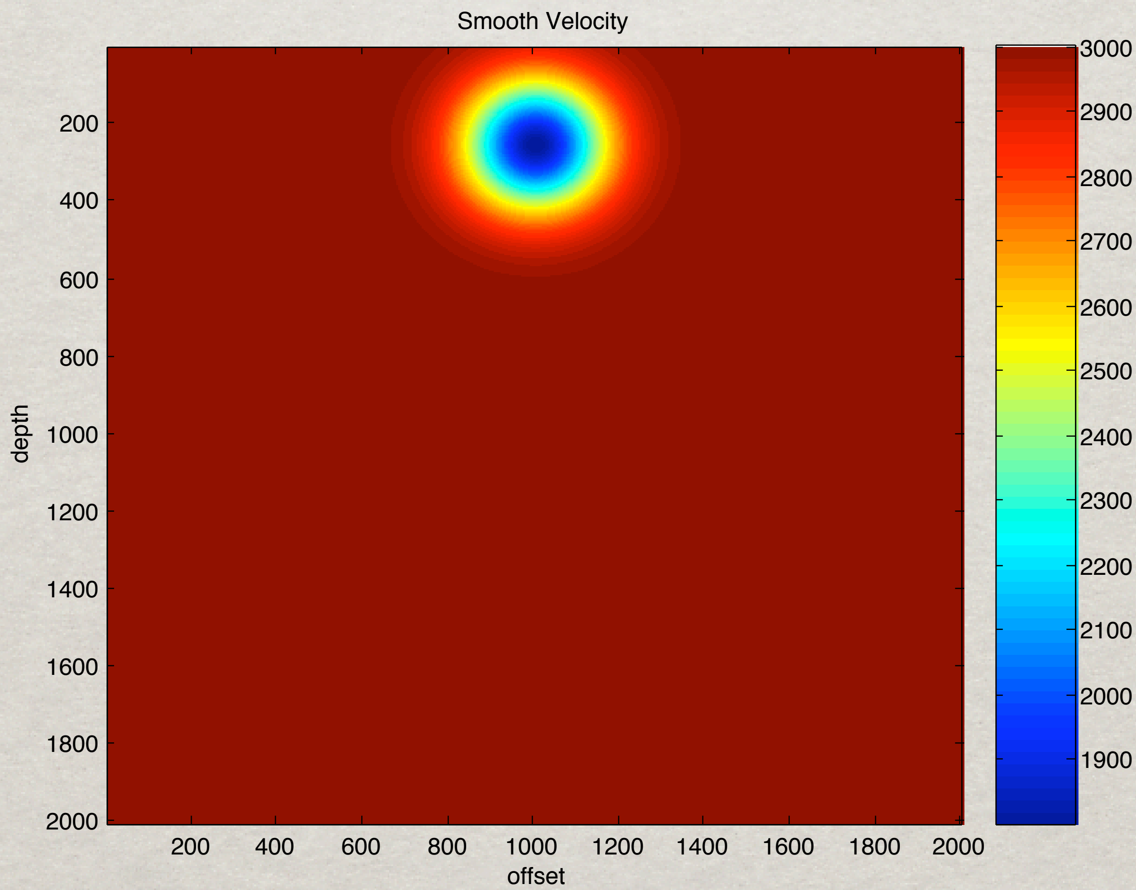
$$\mathbf{Q} = \begin{bmatrix} \lambda_1 \mathbf{D}_x \\ \lambda_2 \mathbf{D}_y \\ \lambda_3 \mathbf{D}_\theta \end{bmatrix}$$

and $\mathbf{r}_0$ an appropriate reference vector. The weighting matrix is given

$$\mathbf{\Gamma}^2 := \mathbf{diag}\, \hat{\mathbf{d}}.$$

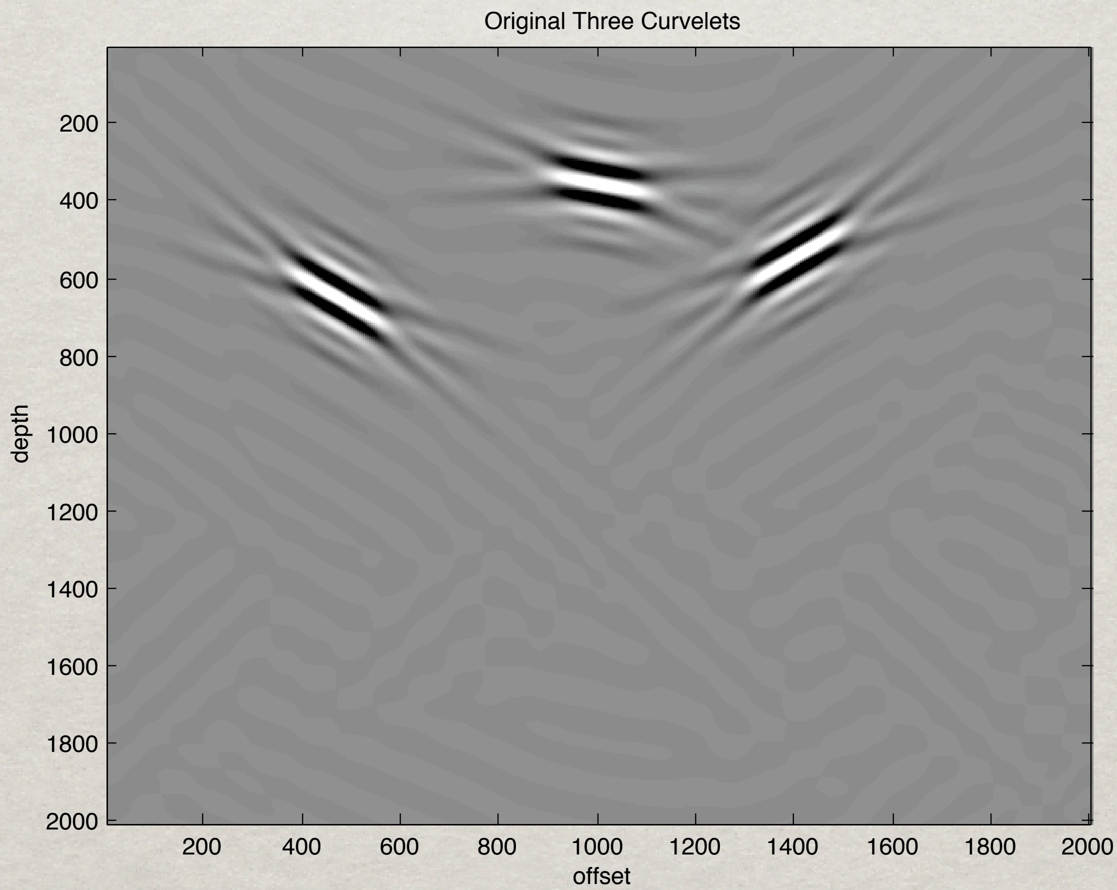
# ‘Wave-equation’ migration

## Lens velocity model



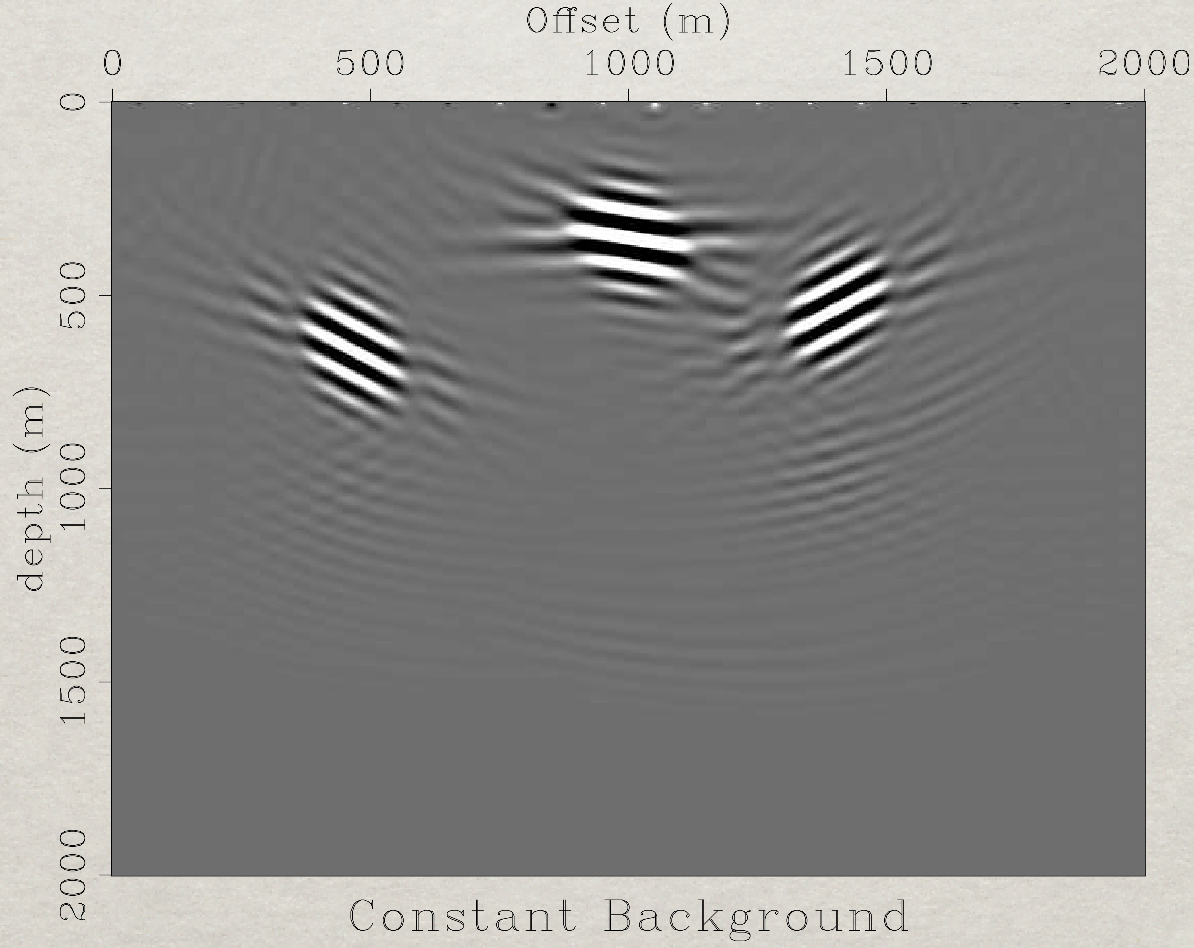
# 'Wave-equation' migration

## Three curvelets



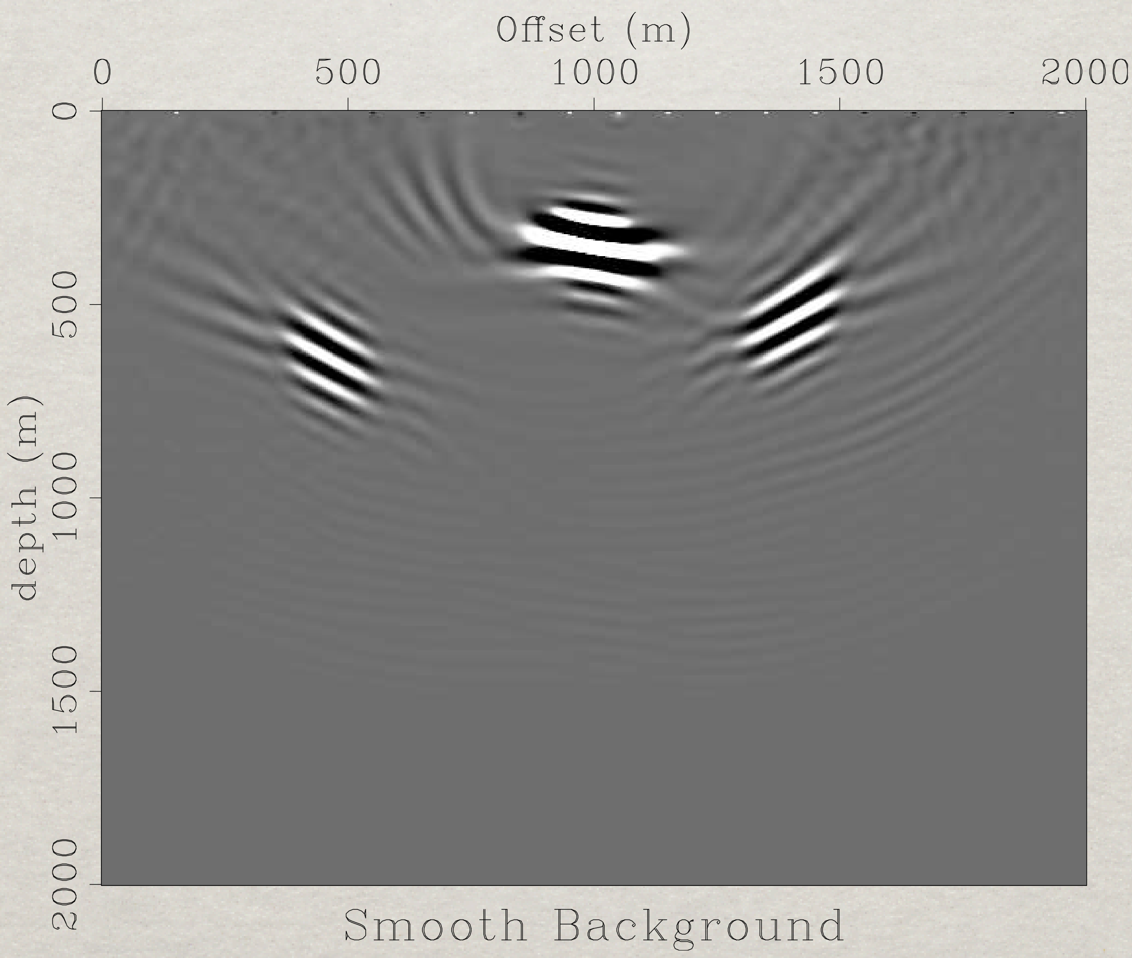
# ‘Wave-equation’ migration

## Constant medium



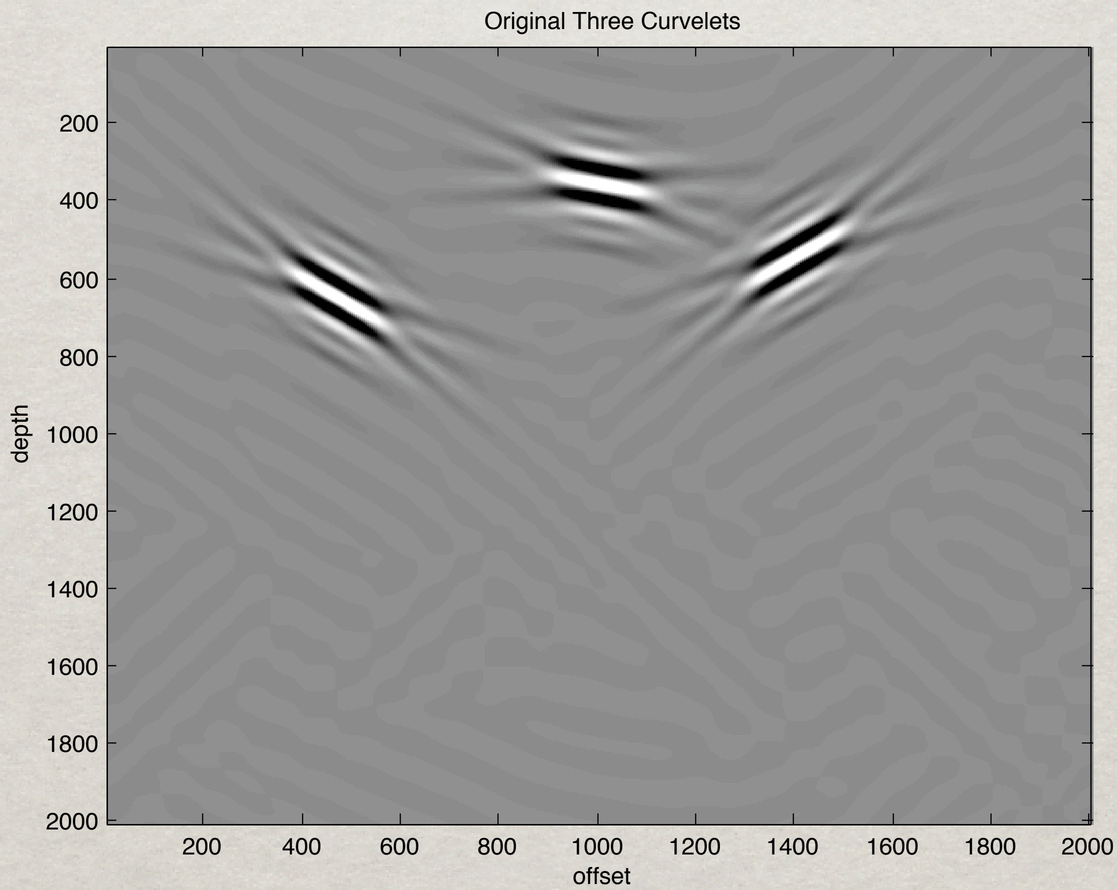
# ‘Wave-equation’ migration

**lens model**



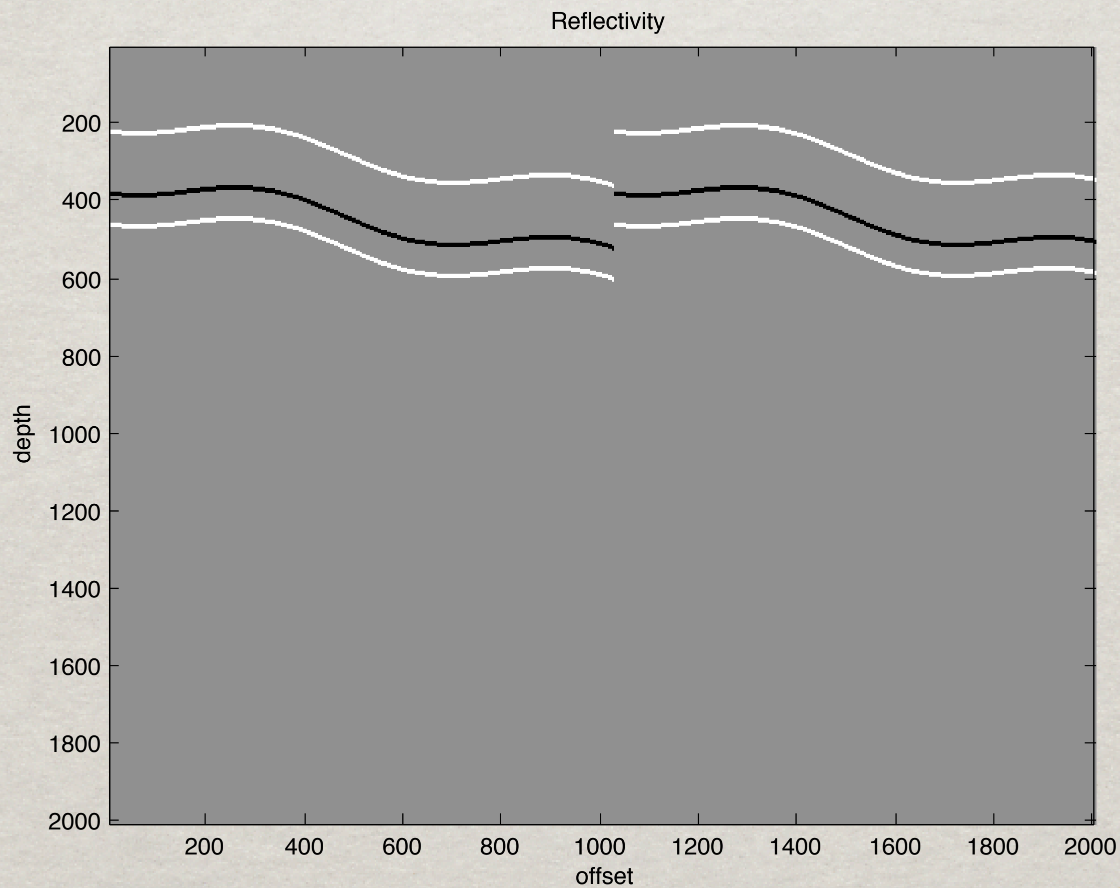
# 'Wave-equation' migration

## Three curvelets

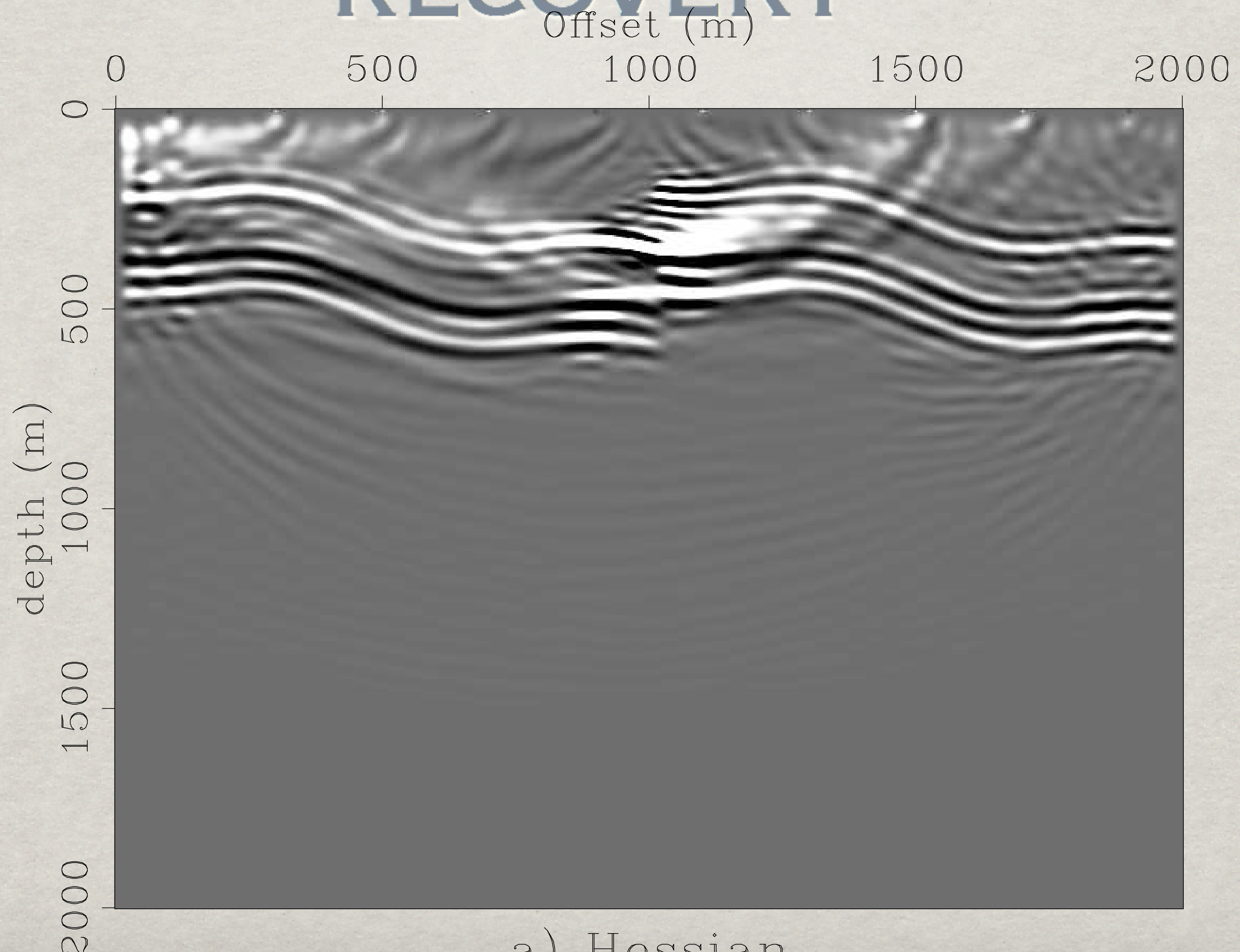


# Preliminary results

## original reflectivity

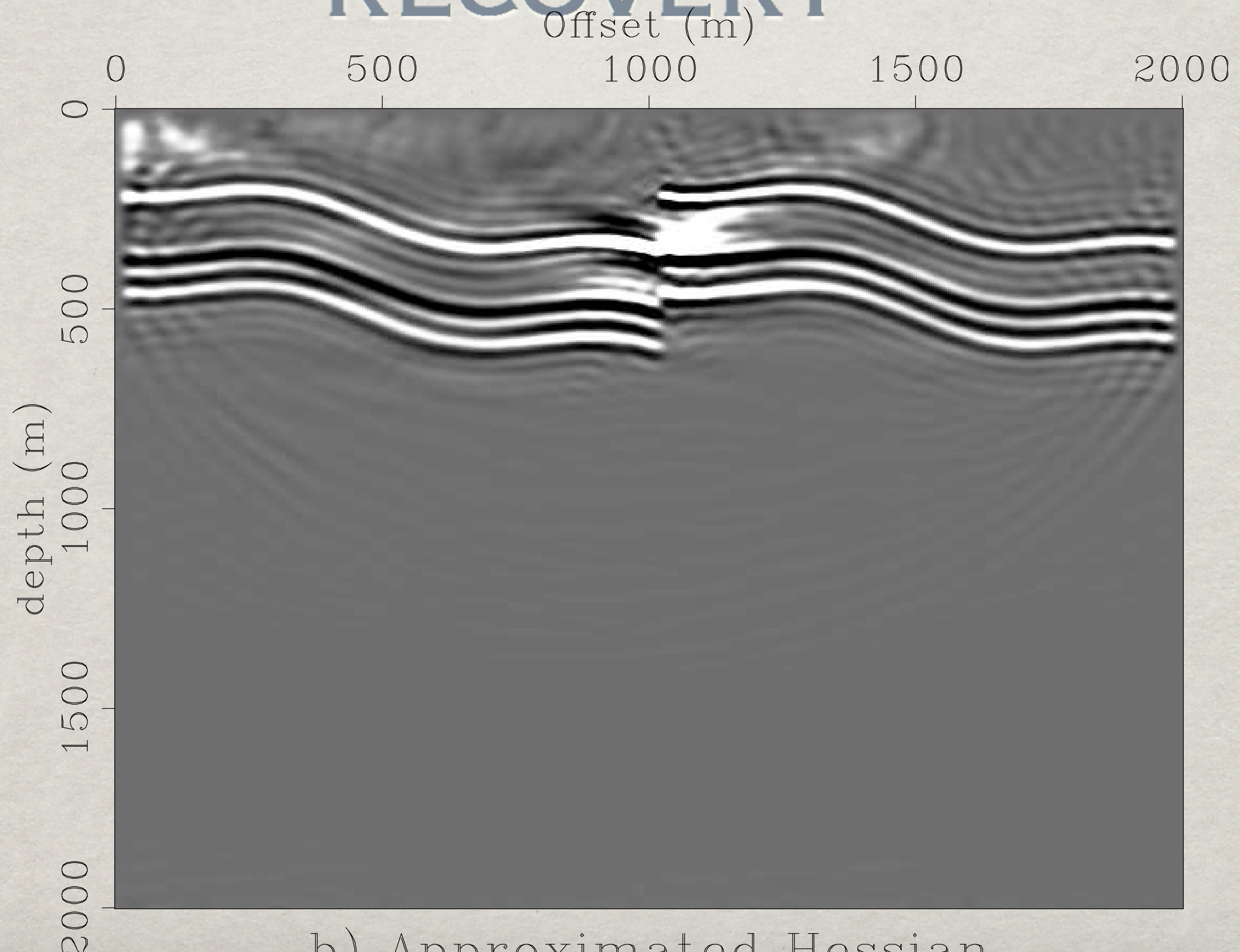


# Image amplitude recovery




a) Hessian

# Image amplitude recovery




b) Approximated Hessian

# Image recovery

**Solve for x**

$$\mathbf{P}: \quad \begin{cases} \min_{\mathbf{x}} J(\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \epsilon \\ \\ \tilde{\mathbf{m}} = (\mathbf{A}^{\mathbf{H}})^{\dagger} \tilde{\mathbf{x}} \end{cases}$$

with

$$J(\mathbf{x}) = \overbrace{\alpha \|\mathbf{x}\|_1}^{\text{sparsity}} + \beta \underbrace{\|\mathbf{\Lambda}^{1/2} \left(\mathbf{A}^{H}\right)^{\dagger} \mathbf{x}\|_p}_{\text{continuity}} .$$

**Exploit sparsity and continuity.**

# Recovery problem

Initialize:

$m = 0;$

$\mathbf{x}^0 = \mathbf{A}^H \mathbf{y};$

$\mathbf{y} = \mathbf{K}^H \mathbf{d};$

Choose:

$M$ and $L$

$\lambda_1 > \lambda_2 > \cdots > \lambda_M$

**while** $\|\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}\|_2 > \epsilon$ and $m < M$ **do**

$\mathbf{x}^m = \mathbf{x}^{m-1};$

$m = m + 1;$

**for** $l = 1$ to $L$ **do**

$\mathbf{x}^m = S_{\lambda_m}\left(\mathbf{x}^m + \mathbf{A}^H\left(\mathbf{y} - \mathbf{x}^m\right)\right)$ {Iterative thresholding}

**end for**

$\mu_m = \arg\min_\mu \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\alpha\|\mathbf{x}\|_1 + \mu\beta J_c(\mathbf{x})$ {Line search};

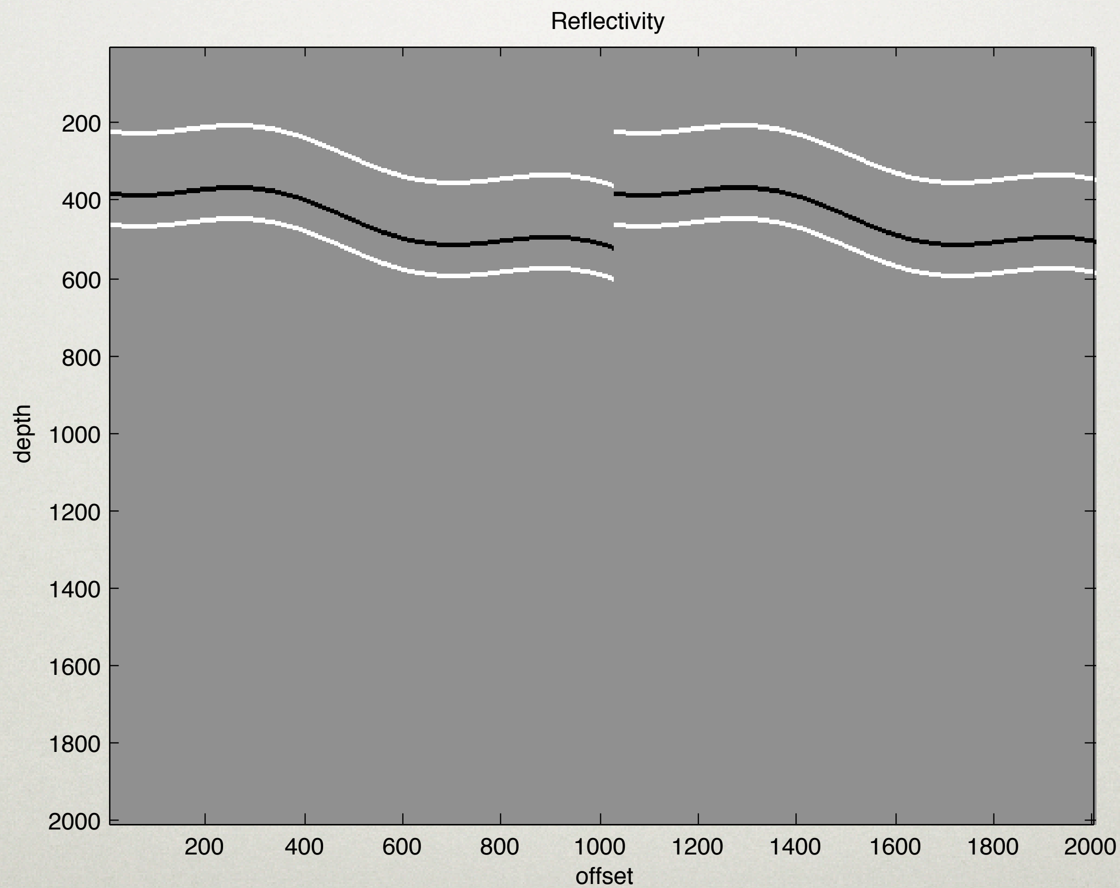
$\mathbf{x}^m = \mathbf{x}^m - \mu J_c(\mathbf{x}^m);$

**end while**

$\hat{\mathbf{m}} = \left(\mathbf{A}^H\right)^\dagger \hat{\mathbf{x}}.$

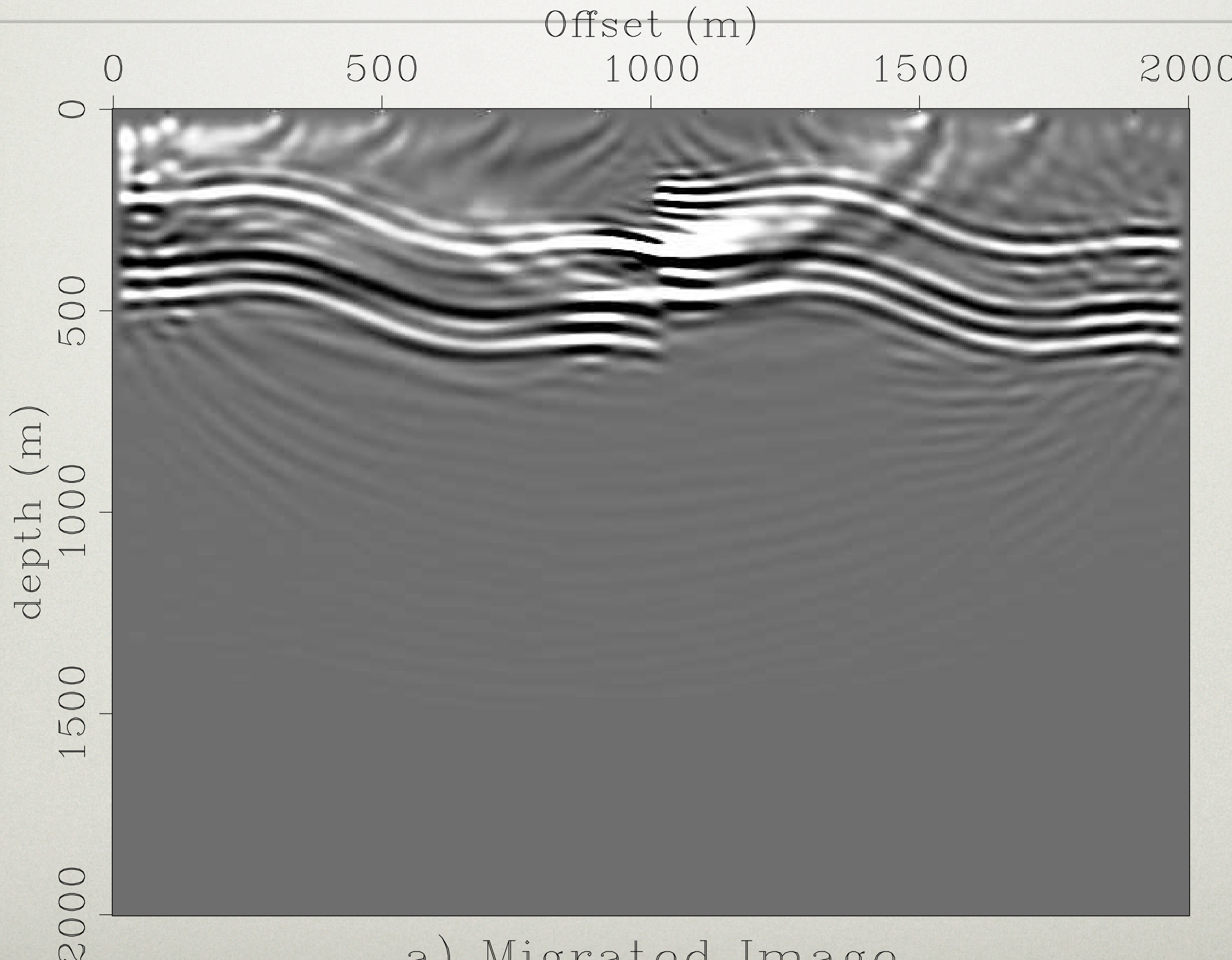
# Image amplitude recovery

## original reflectivity



# Preliminary results

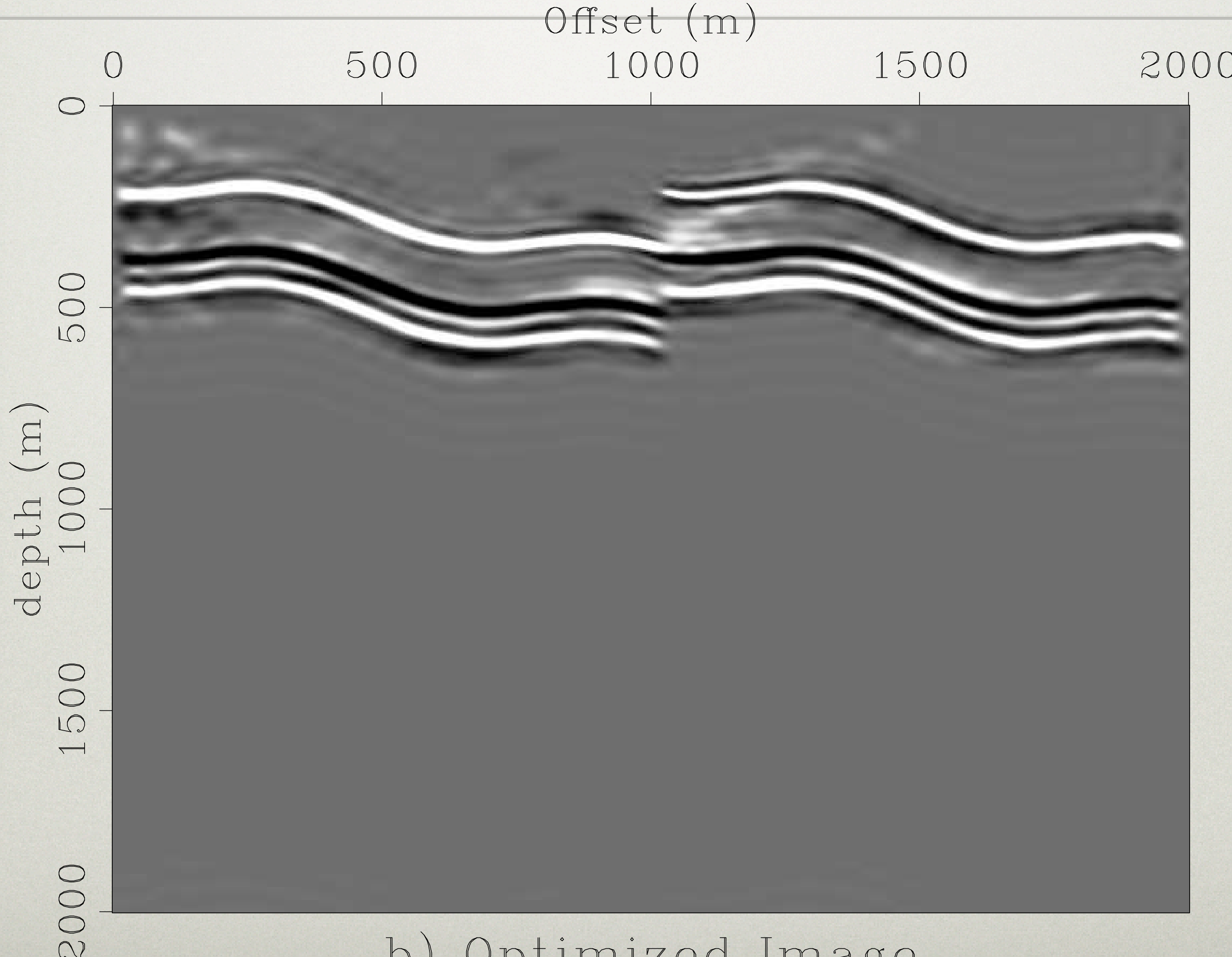
## imaged reflectivity




a) Migrated Image

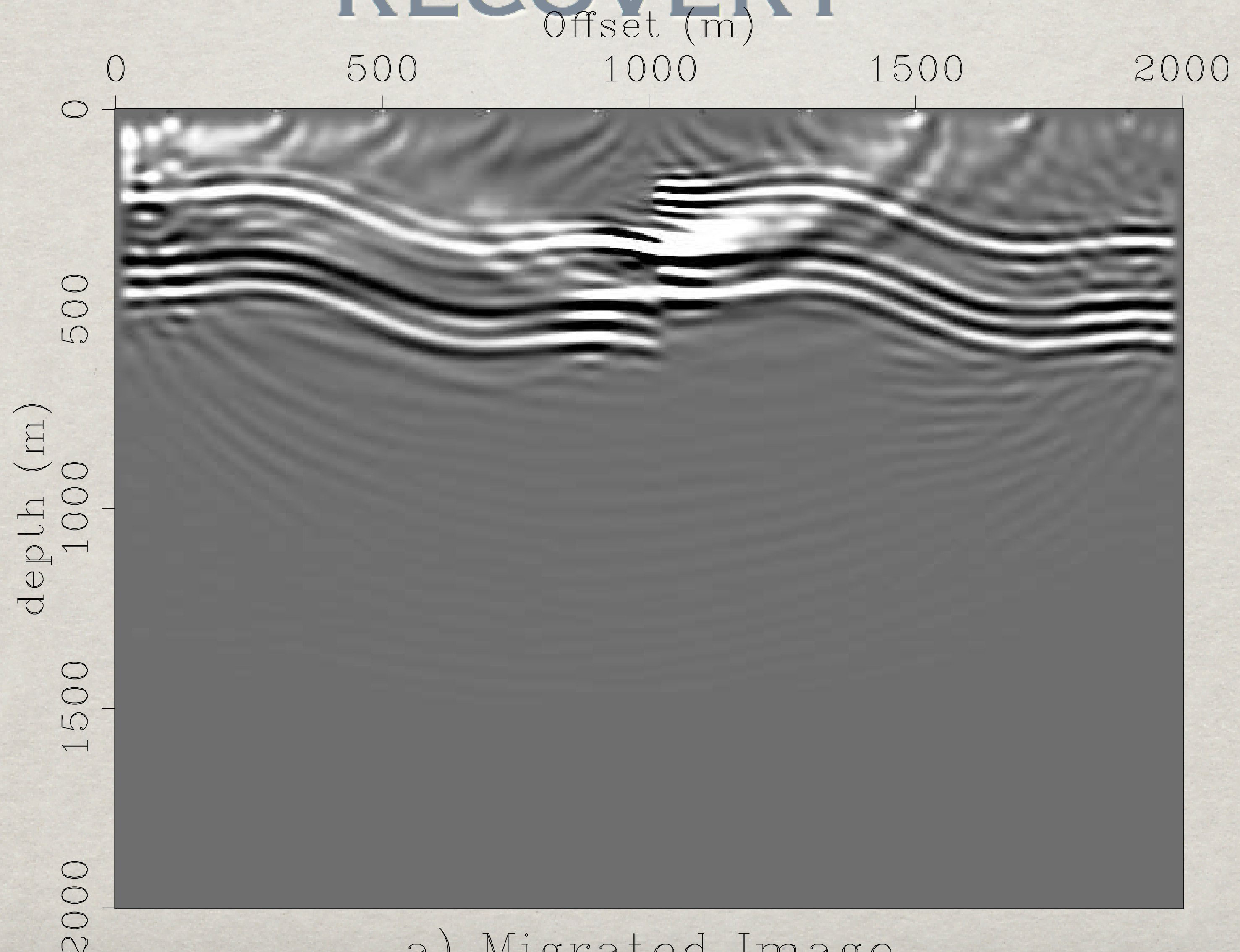
# Preliminary results

## improved reflectivity



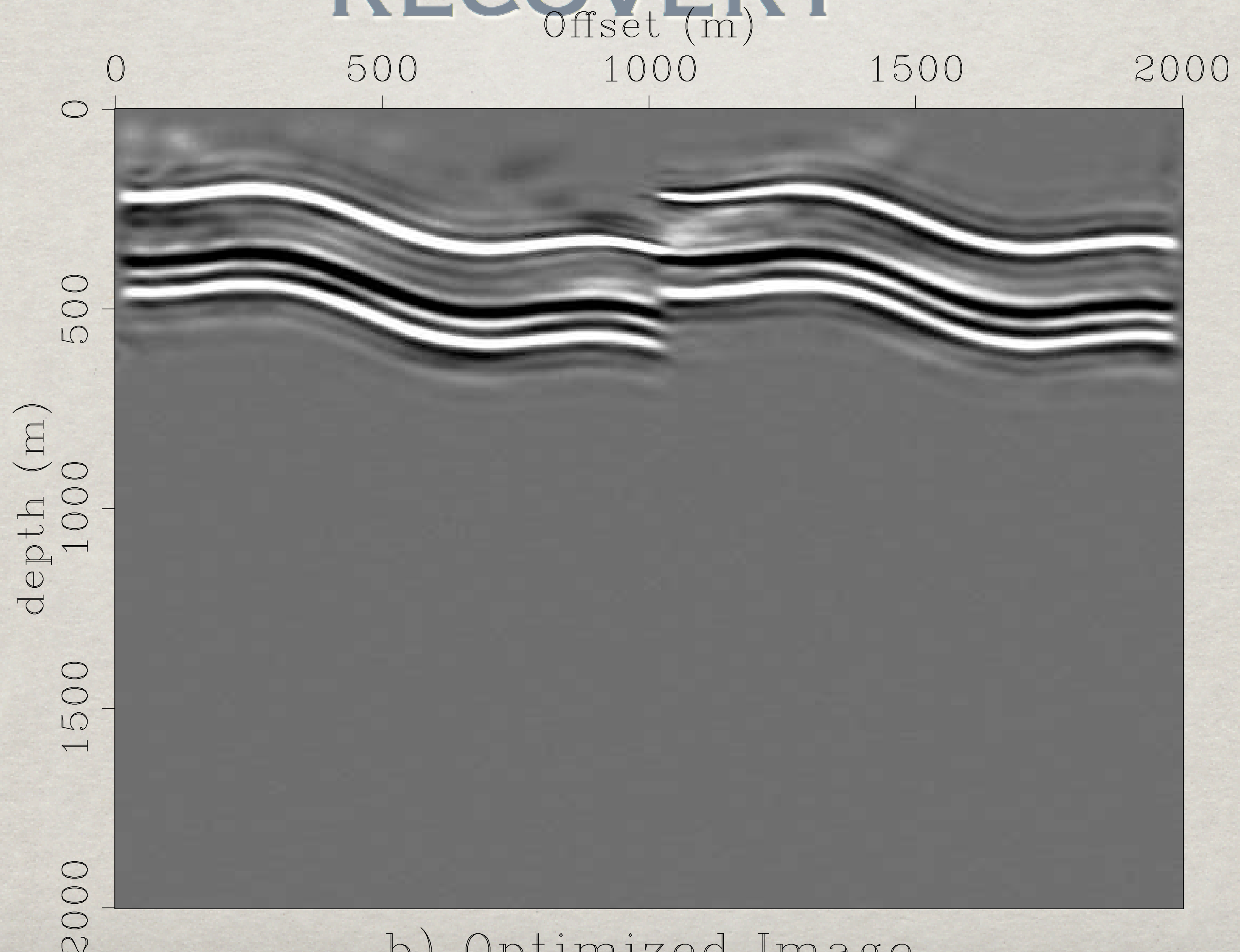
b) Optimized Image

# Image amplitude recovery




a) Migrated Image

# Image amplitude recovery




b) Optimized Image

# Image recovery

Amplitudes are partially recovered

Spurious artifacts remain due to

- side-band effects (Candes)
- instabilities due to bad illumination

Exploit smoothness along wavefronts via anisotropic norm.

Use smoothed migrated image (reference vector).

# Image recovery

**Solve for x**

$$\mathbf{P}: \quad \begin{cases} \min_{\mathbf{x}} J(\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \epsilon \\ \\ \tilde{\mathbf{m}} = (\mathbf{A}^{\mathbf{H}})^{\dagger} \tilde{\mathbf{x}} \end{cases}$$

with

$$J(\mathbf{x}) = \overbrace{\alpha \|\mathbf{x}\|_1}^{\text{sparsity}} + \beta \underbrace{\|\mathbf{\Lambda}^{1/2} \left(\mathbf{A}^{H}\right)^{\dagger} \mathbf{x}\|_p}_{\text{continuity}}.$$

**Exploit sparsity and continuity.**

# Image recovery

**Sparsity norm**

$$J_s(\mathbf{x}) = \|\mathbf{x}\|_{1,\mathbf{w}} := \sum_{j=1}^{N} |w_j x_j|$$

**with**

$$\begin{cases} w_j = \infty \text{ for } j \in \mathcal{M}_0 \\ w_j = 1 \text{ otherwise} \end{cases}$$

**where**

$$\mathcal{M}_0 = \{j : \Gamma_j \leq \delta\}$$

# Image recovery

**Anisotropic continuity-promoting norm**

$$J_c(\mathbf{m}) = \|\boldsymbol{\Lambda}^{1/2}\nabla_d \mathbf{m}\|_p$$

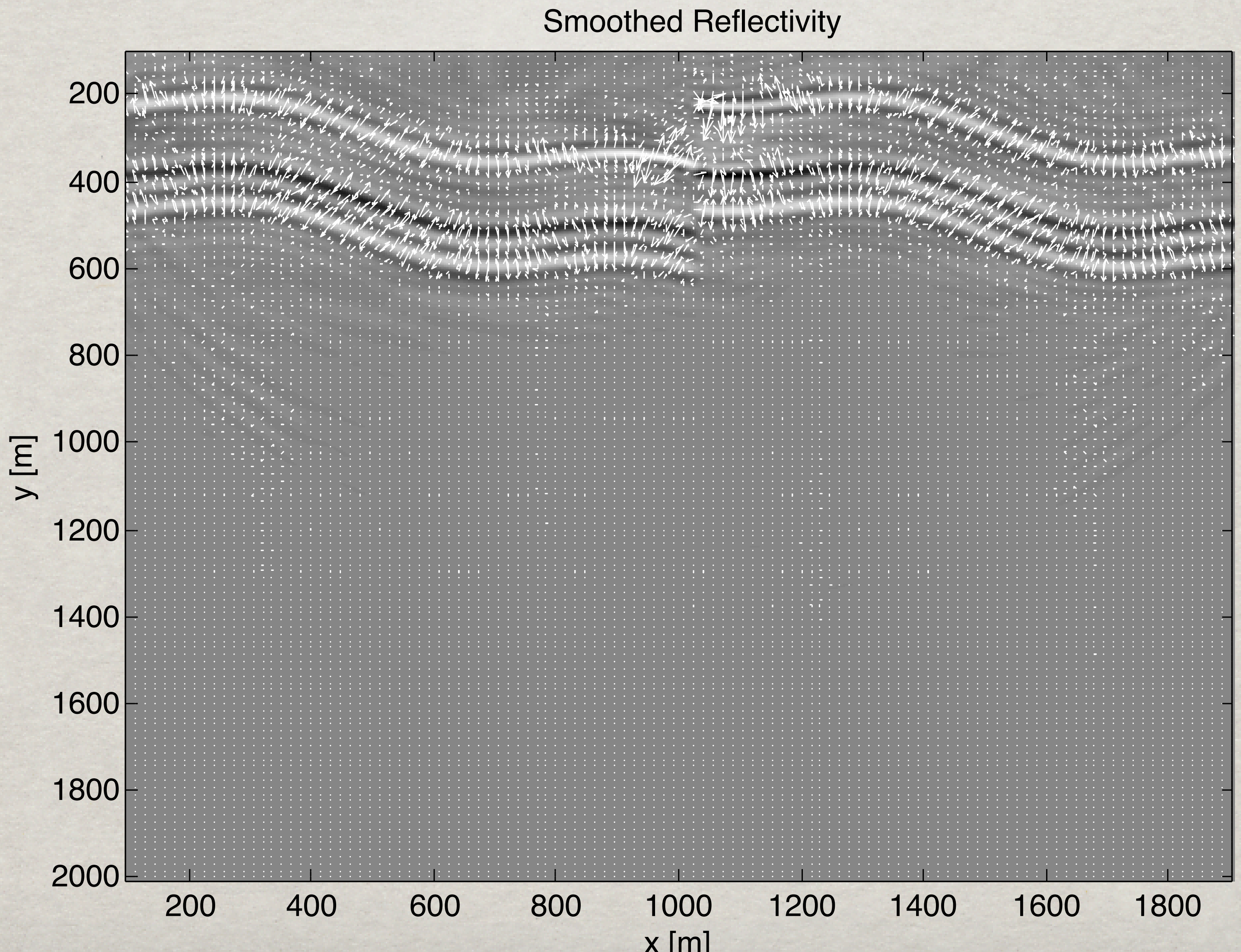
**with**

$$\boldsymbol{\Lambda}[\bar{\mathbf{b}}] = \frac{1}{\|\nabla_d \bar{\mathbf{b}}\|_2^2}\left\{\begin{pmatrix}\mathbf{D}_2\bar{\mathbf{b}} \\ -\mathbf{D}_1\bar{\mathbf{b}}\end{pmatrix}\begin{pmatrix}\mathbf{D}_2\bar{\mathbf{b}} & -\mathbf{D}_1\bar{\mathbf{b}}\end{pmatrix}\right\}$$

**p=2 <=> anisotropic diffusion**

**p=1 <=> anisotropic TV**

# GRADIENT DIRECTIONS



# Image recovery

**Step** 1: Update of the Jacobian of $\frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$:

$$\mathbf{x} \leftarrow \mathbf{x} + \mathbf{A}^T\left(\mathbf{y}-\mathbf{A}\mathbf{x}\right); \tag{33}$$

**Step** 2: projection onto the $\ell_1$ ball $S = \{\|\mathbf{x}\|_1 \leq \|\mathbf{x}_0\|_1\}$ by soft thresholding

$$\mathbf{x} \leftarrow S^s_{\lambda\mathbf{w}}(\mathbf{x}); \tag{34}$$

**Step** 3: projection onto the anisotropic diffusion ball $C = \{\mathbf{x}: J(\mathbf{x}) \leq J(\mathbf{x}_0)\}$ by

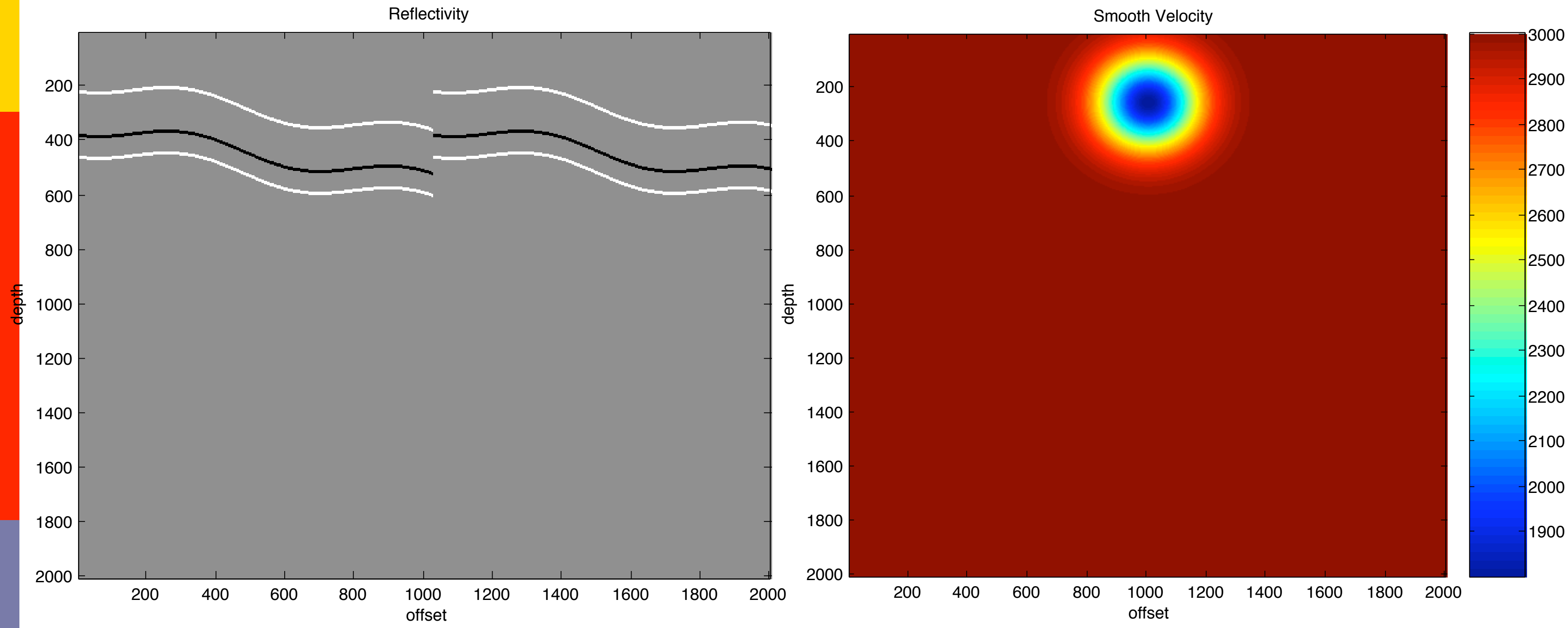
$$\mathbf{x} \leftarrow \mathbf{x} - \mu\nabla_{\mathbf{x}} J_c(\mathbf{x}) \tag{35}$$

with

$$\nabla_{\mathbf{x}} J_c(\mathbf{x}) = 2\mathbf{A}^{\dagger}\nabla\cdot\left(\mathbf{\Lambda}\nabla\left(\left(\mathbf{A}^T\right)^{\dagger}\mathbf{x}\right)\right). \tag{36}$$

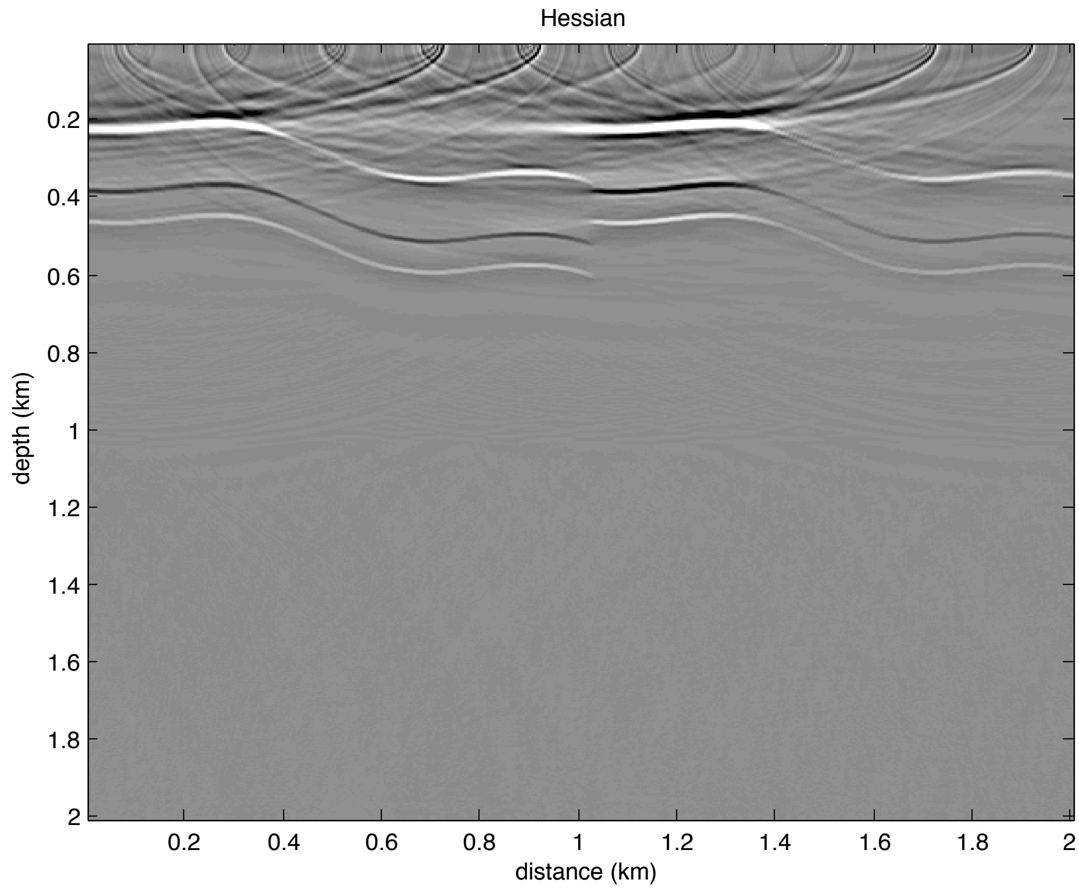
The $\mu$ is found by conducting a line search

$$\min_{\mu} \frac{1}{2}\|\mathbf{y}-\mathbf{A}(\mathbf{x}-\mu\nabla_{\mathbf{x}}J_c(\mathbf{x}))\|_2^2 + \lambda\alpha\|\mathbf{x}-\mu\nabla_{\mathbf{x}}J_c(\mathbf{x})\|_1 + \lambda\beta J_c(\mathbf{x}-\mu\nabla_{\mathbf{x}}J_c(\mathbf{x})). \tag{37}$$



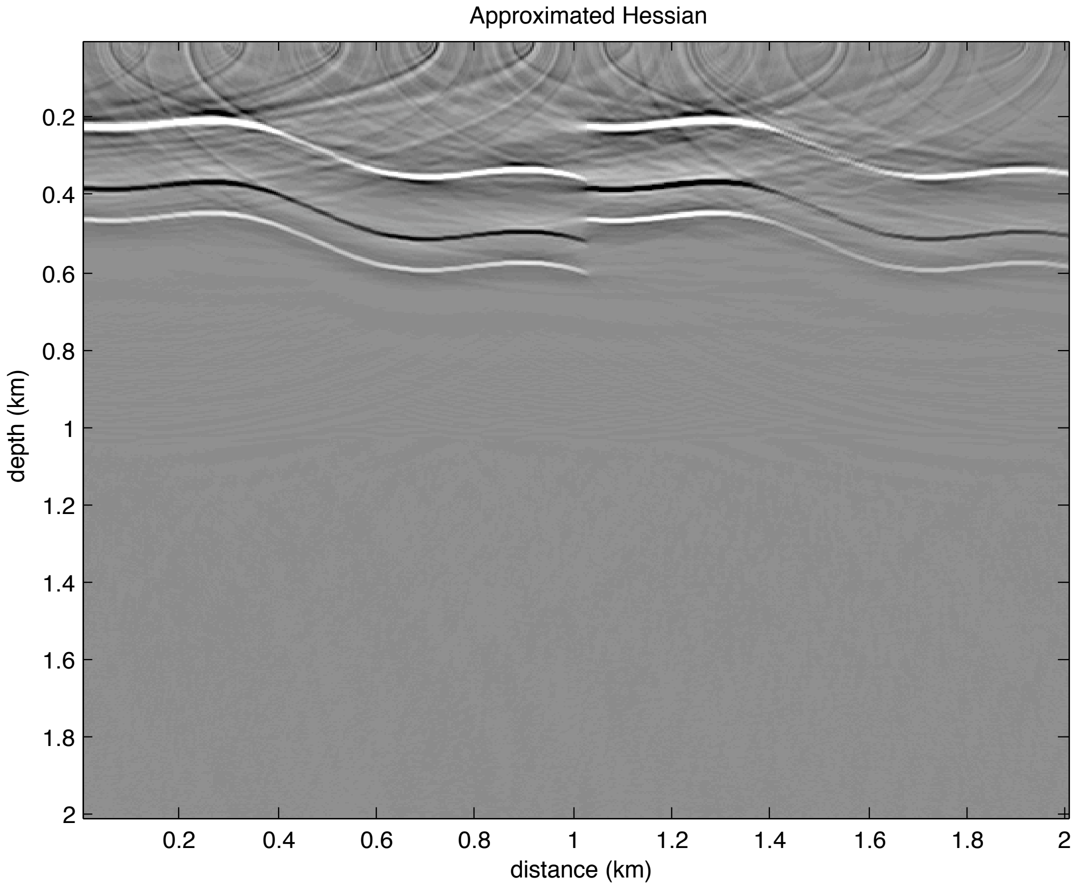

# Velocity Model and Reflectivity

SLIM
Seismic Laboratory for Imaging and Modeling



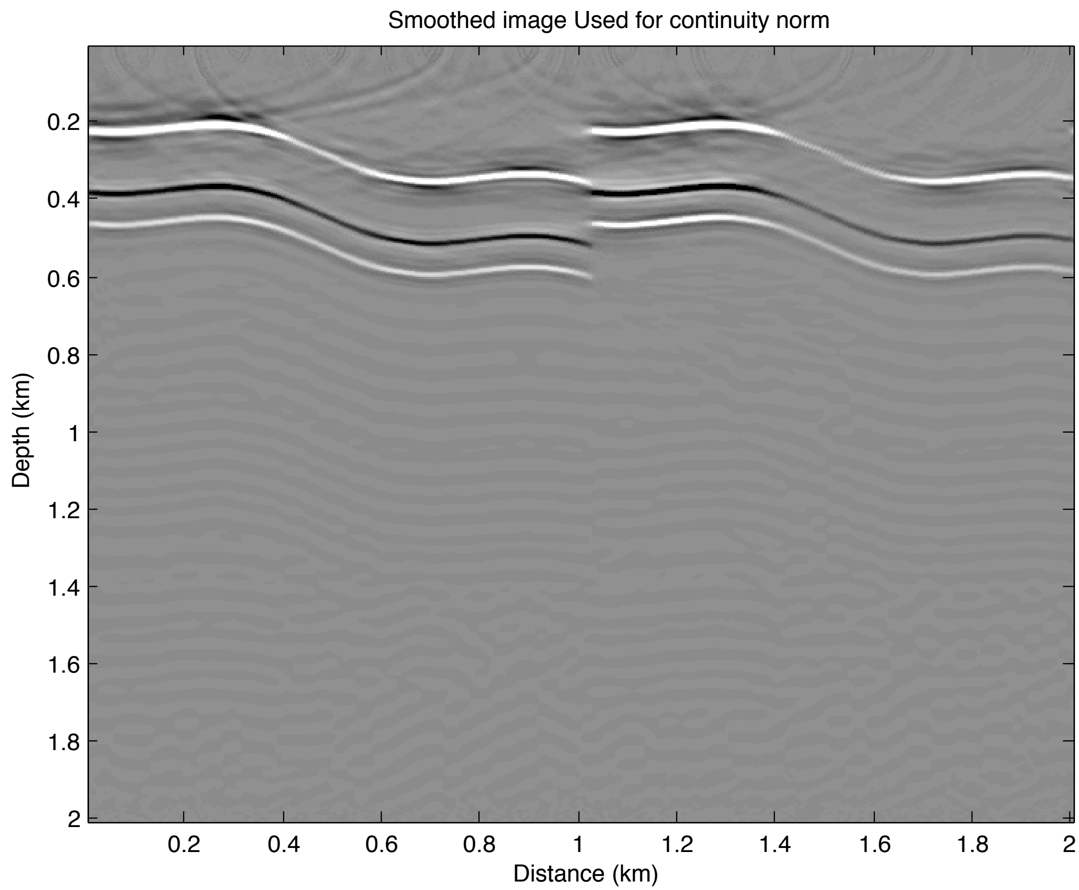

# Hessian

SLIM
Seismic Laboratory for Imaging and Modeling



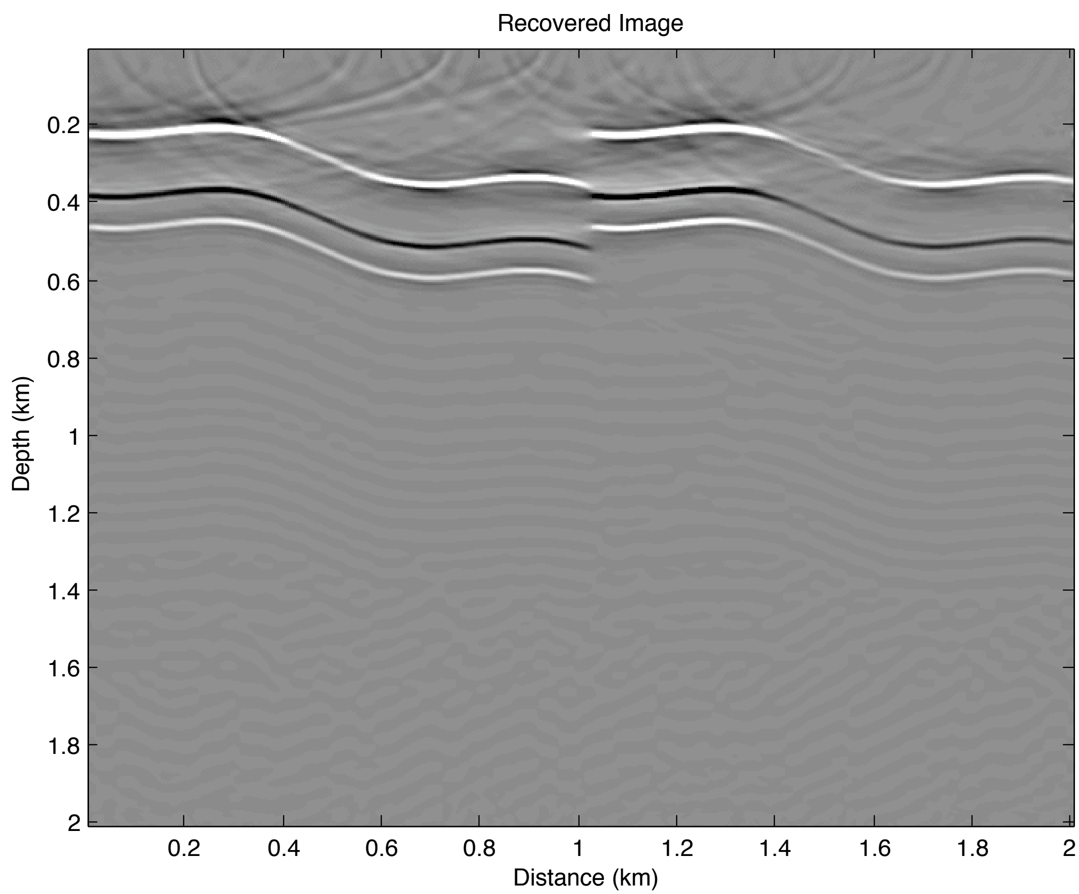

# Approximated Hessian

SLIM Seismic Laboratory for Imaging and Modeling



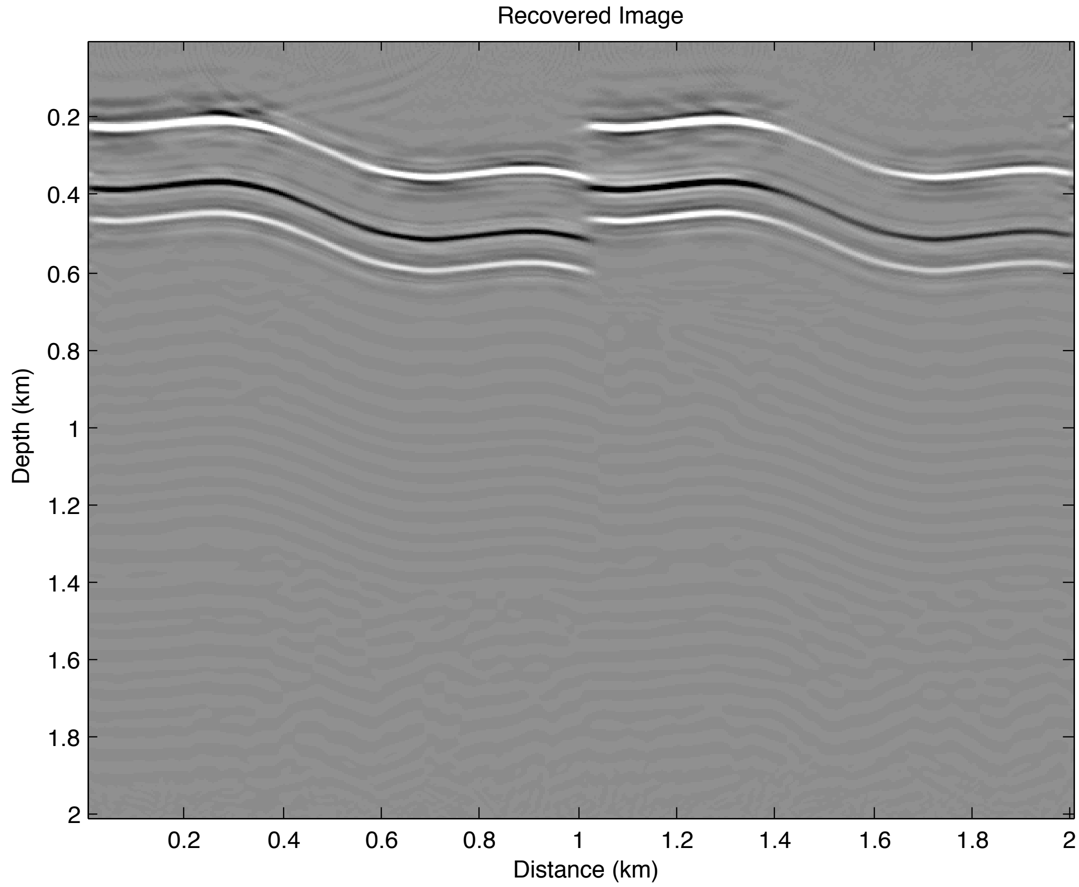

# Smoothed image used for anisotropic diffusion norm

Seismic Laboratory for Imaging and Modeling




# Recovered Image (Sparsity Only)

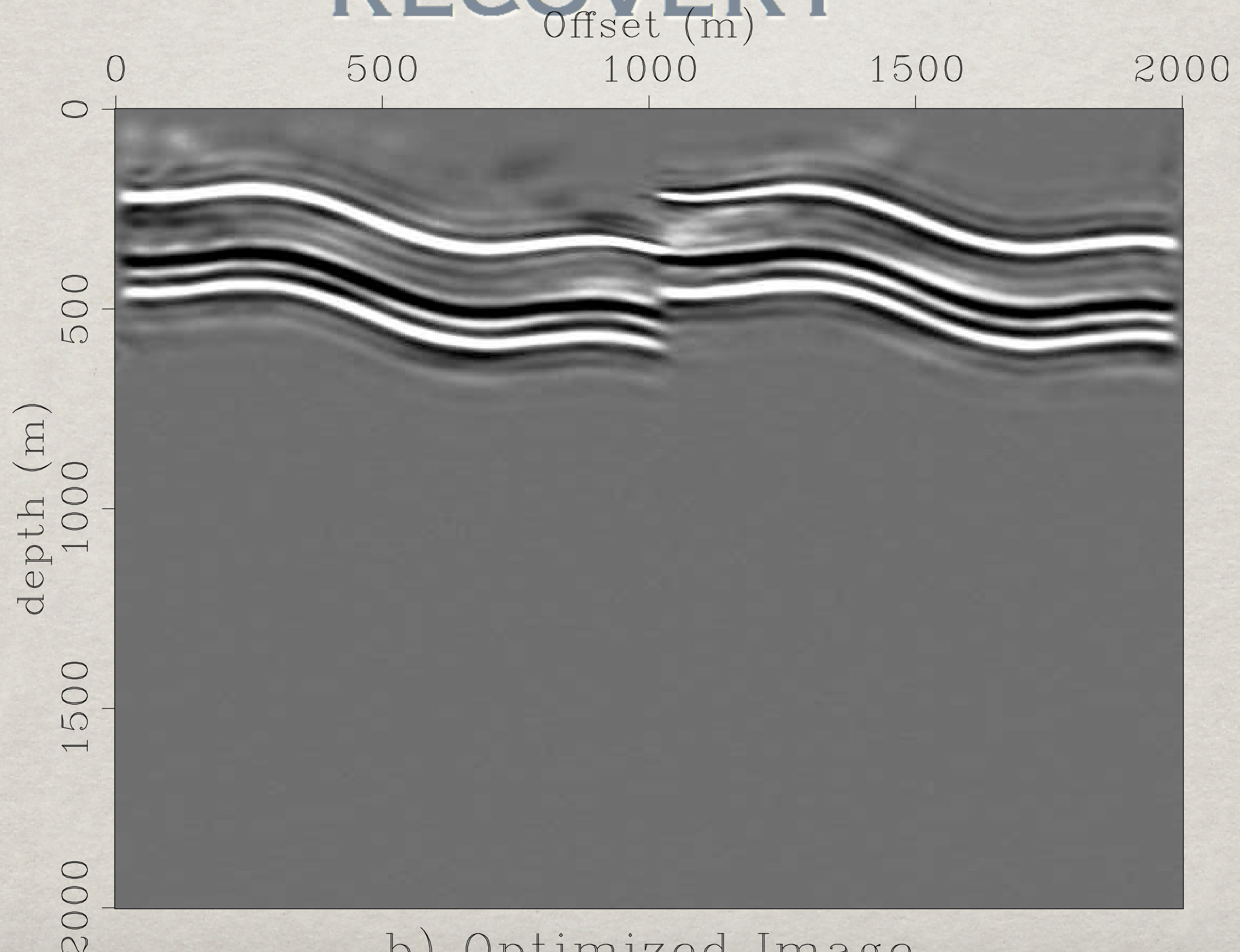
SLIM Seismic Laboratory for Imaging and Modeling




# Recovered Image (Sparsity and Continuity)

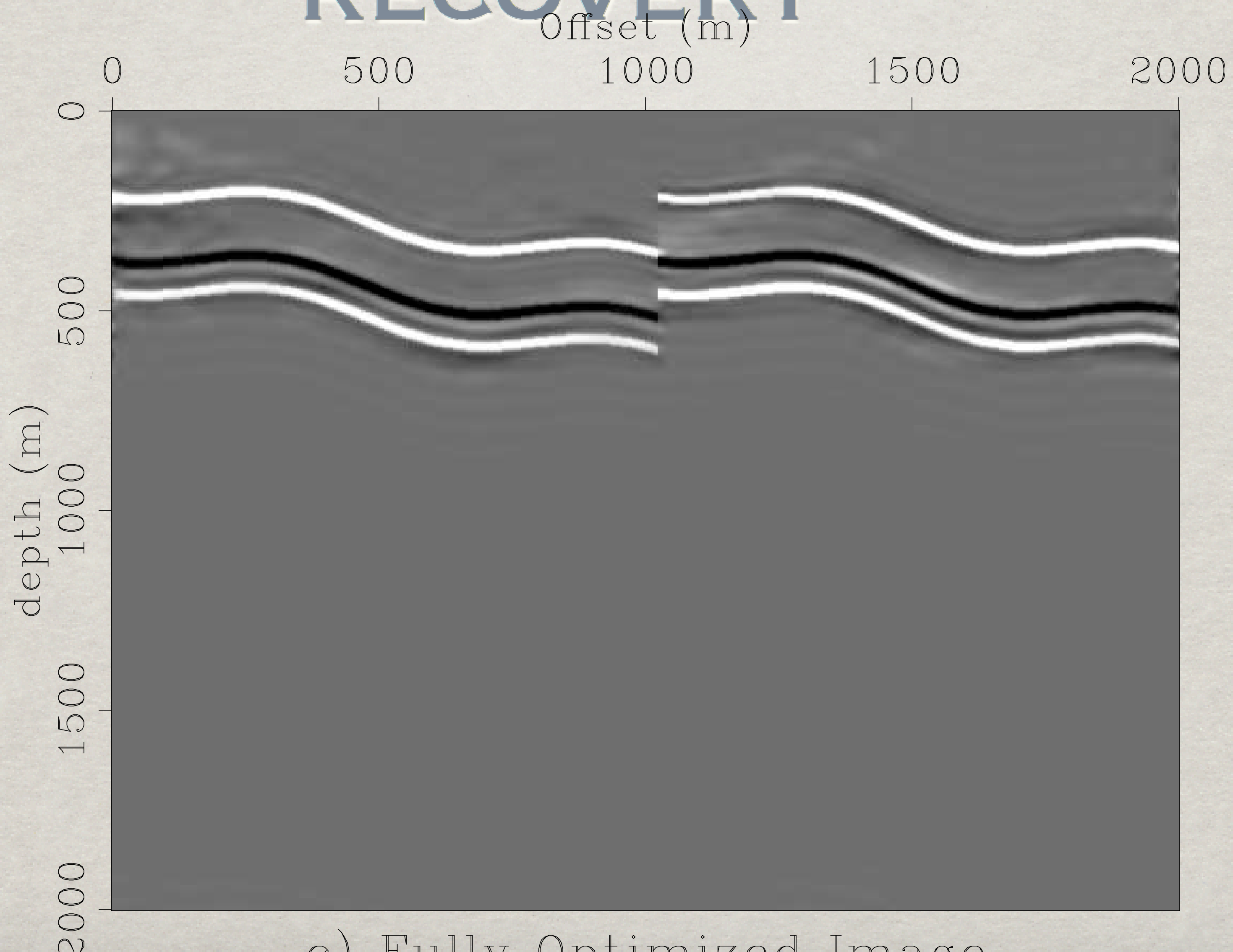
SLIM
Seismic Laboratory for Imaging and Modeling

# Image amplitude recovery



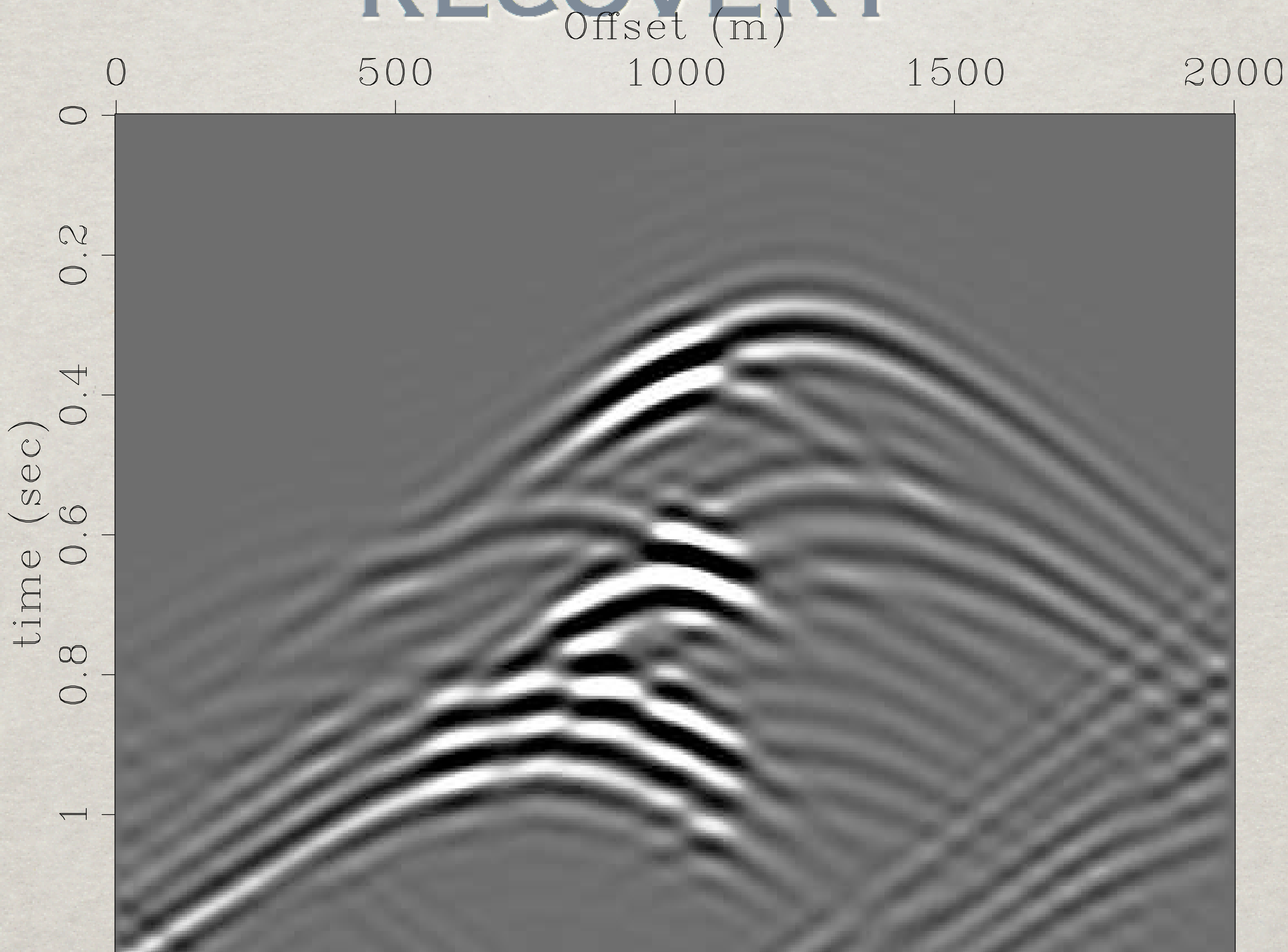

b) Optimized Image

# Image amplitude recovery



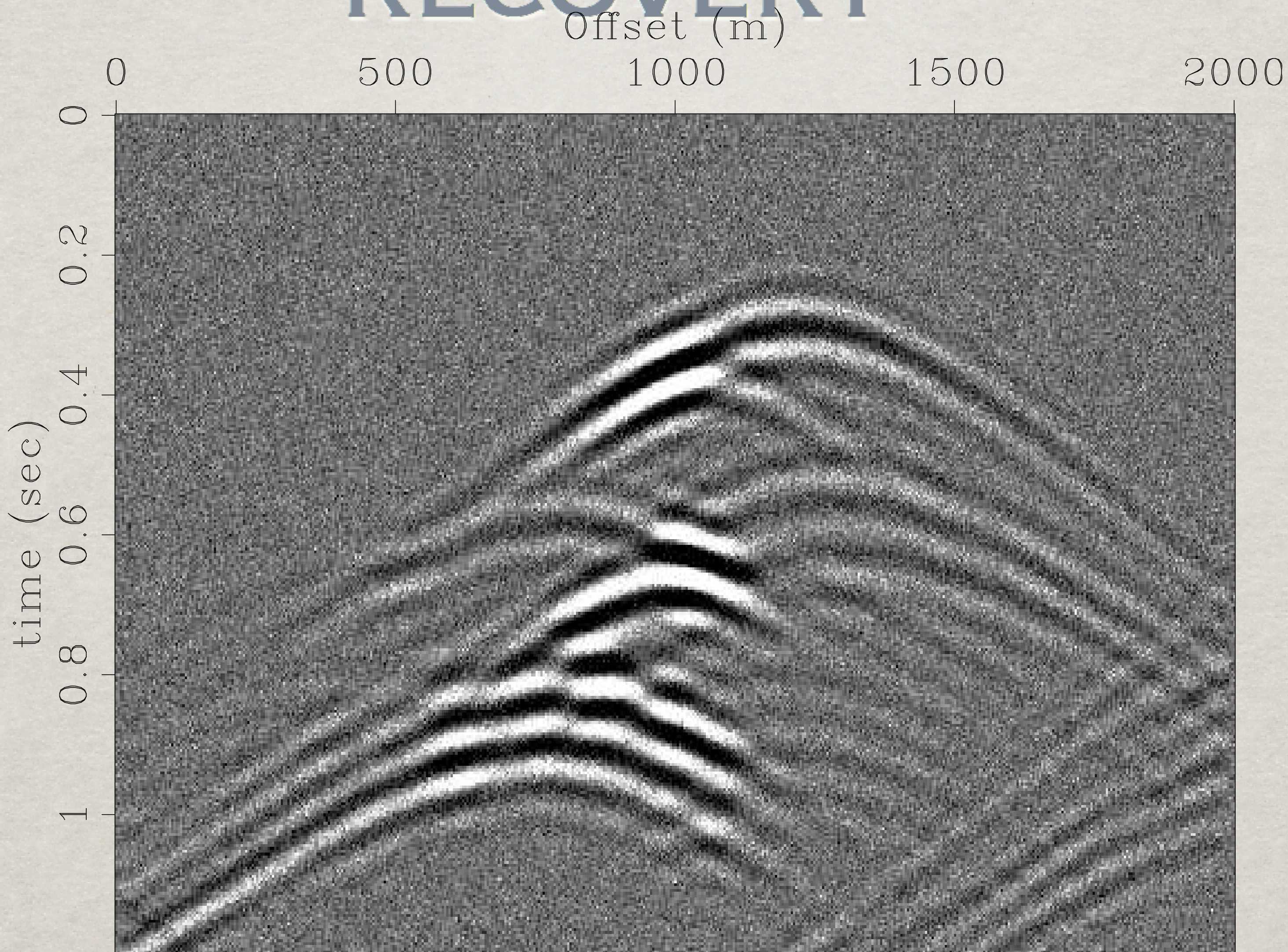

c) Fully Optimized Image

# Image amplitude recovery



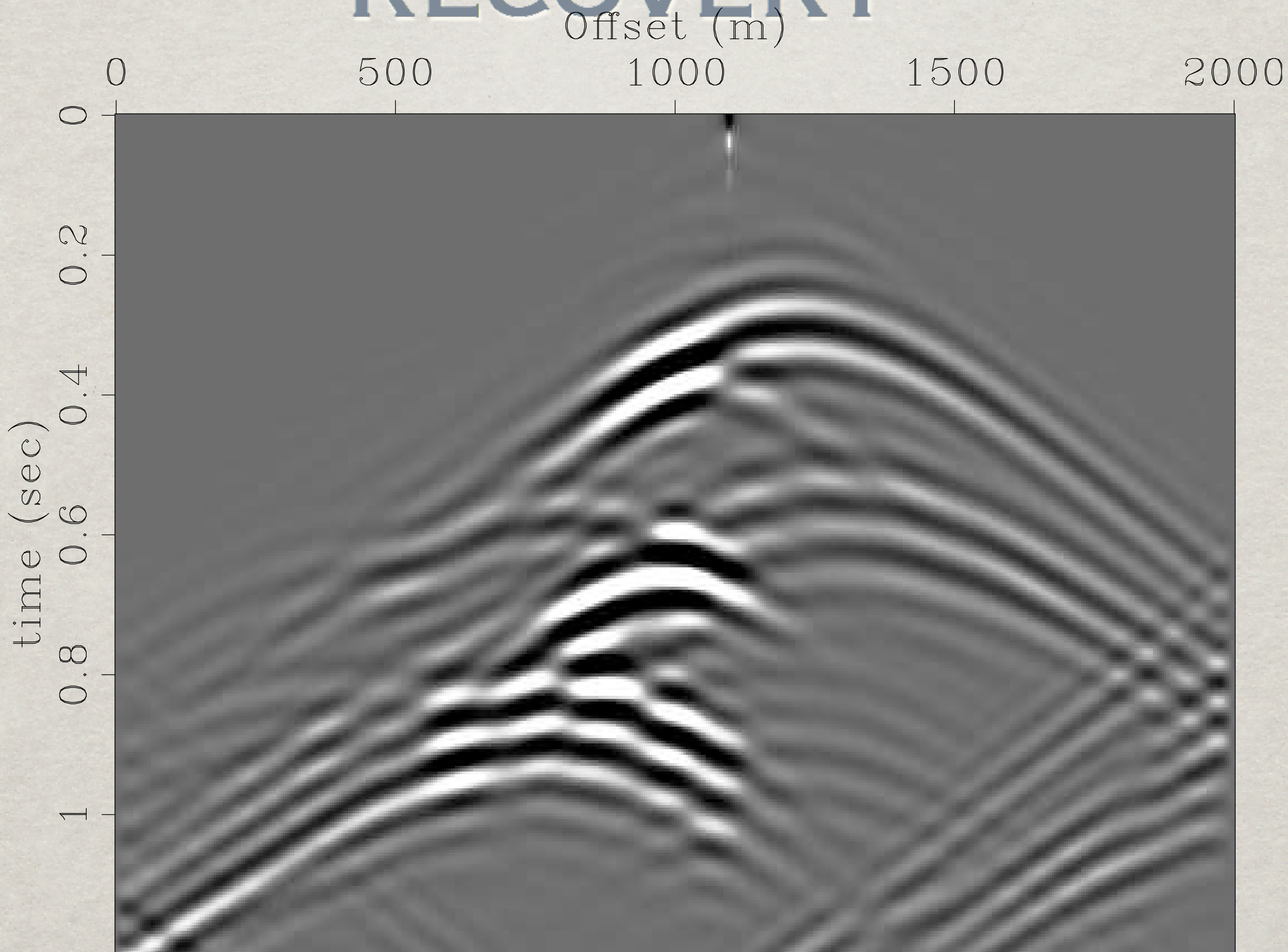

a) Synthetic Common Shot Gather

# Image amplitude recovery



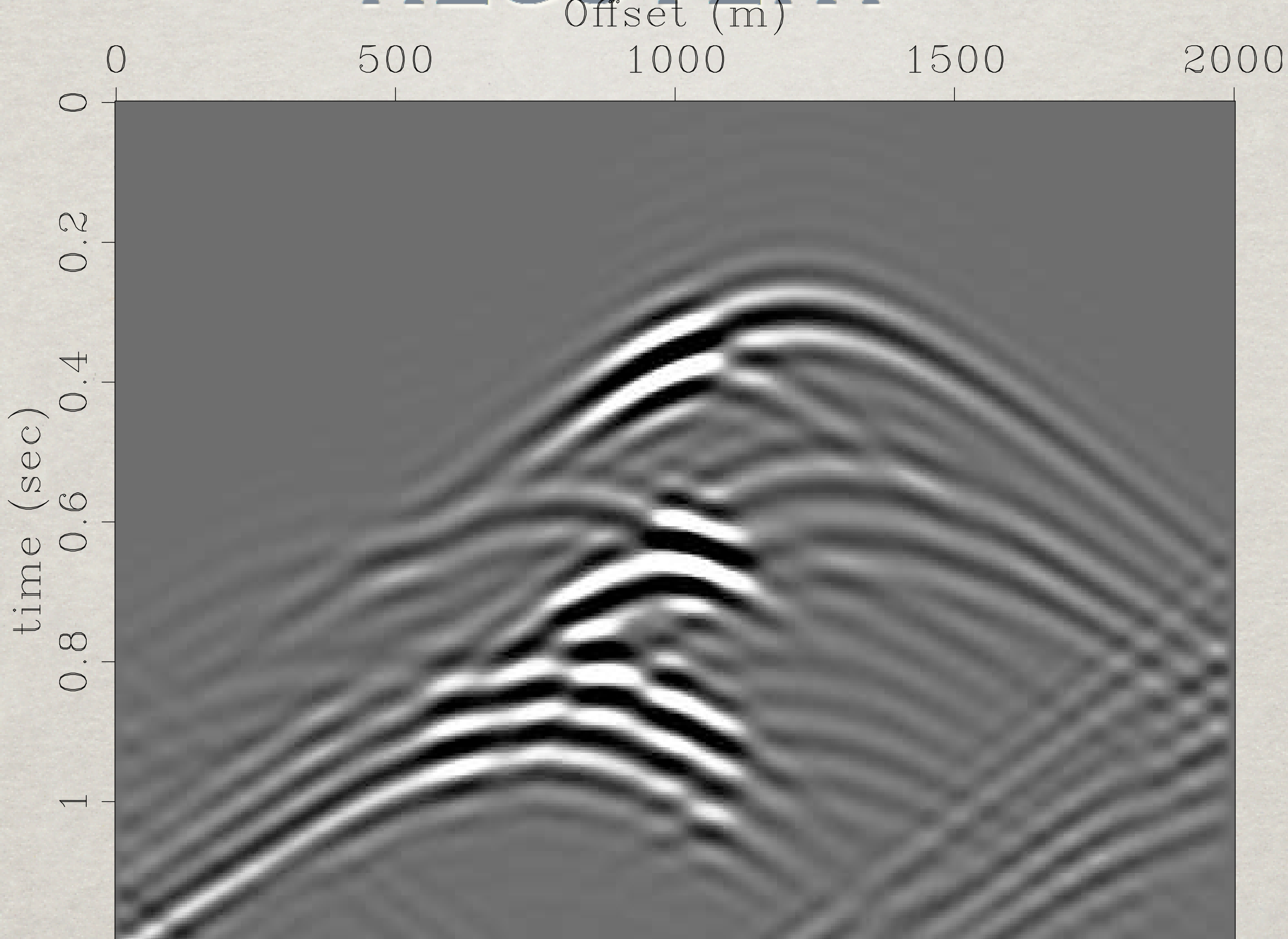

b) Noisy Sythetic

# Image amplitude recovery




b) Denoised Sythetic

# Image amplitude recovery




a) Synthetic Common Shot Gather

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- D.J. Verschuur for the synthetic dataset
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