A primer on weak recovery conditions

Context

"Sparse Solution of Underdetermined Linear Equations by Stagewise Orthogonal Matching Pursuit" David L. Donoho, Yaakov Tsaig, Iddo Drori, Jean-Luc Starck

"Sparse Nonnegative Solution of underdetermined Linear Equations by Linear Programming" David L. Donoho and Jared Tanner

"High-Dimensional Centrally-Symmetric Polytopes With Neighborliness Proportional to Dimension" David L. Donoho

Challenges

Seismic data recovery is extremely large **scale**

Strong recovery conditions are

- prohibitively expensive to calculate
- overly pessimistic

Need a framework for **large** data volumes

- feasible conditions for recovery
- qualitative tests for recovery
- quantitative test for recovery

Only hope for an approximate solution to the recovery problem ...

Sparse recovery

[Candes, Donoho etc.]

Uniform uncertainty principles (UUP):

- **strong** conditions for inverting underdetermined systems
- strong equivalence between l1 and l0
- valid for any sparsity vector $\mathbf{x}_0$
- unique solution in noise-free case
- pessimistic bounds
- unfeasible bounds for 'large' systems

Donoho proposes weak conditions under the slogan:

noiseless underdetermined problems behave like noisy well-determined problems ...

The problem

Linear system

$$\mathbf{Ax} = \mathbf{y}$$

with

$$\mathbf{A} := \mathbf{RMS}^H$$

and

- **x** the N-length sparsity vector (N>>n) with
- k non-zero entries
- **y** the n-length measurement vector

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Strong recovery conditions
[Candes et al]

Prescribe recovery conditions for the
- measurement matrix
- sparsity matrix and vector

Recovery depends on
- mutual coherence
- compression of the sparsity vector

Guarantees recovery for every possible restriction and sparsity vector

Recovery conditions impossible to compute & pessimistic

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UUP examples

Recovery conditions are
- pessimistic
- in practice much lower
- sharp for each experiment but different

Indication of a phase transition
- large systems with randomness
- UUP's worked well with randomness in the restriction and/or sparsity vector

Opens the way to study large systems in probabilistic a framework.

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Weak recovery conditions
[Donoho et al]

Approximate probabilistic framework:
- valid for **typical** sparsity vectors
- requires mixing (randomness)
- more accurate for larger systems

Mixing depends on
- randomness restriction matrix
- incoherence measurement-sparsity matrices
- randomness & compression of sparsity vector

Fast recovery possible for typical sparsity vectors when

Gaussian approximation is valid …

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Gaussian approximation

Matched filter minus sparsity vector needs to be Gaussian, i.e.,

$$\mathbf{z} = \tilde{\mathbf{x}} - \mathbf{x}_0 \quad \text{with} \quad \tilde{\mathbf{x}} = \mathbf{A}^H \mathbf{y}$$

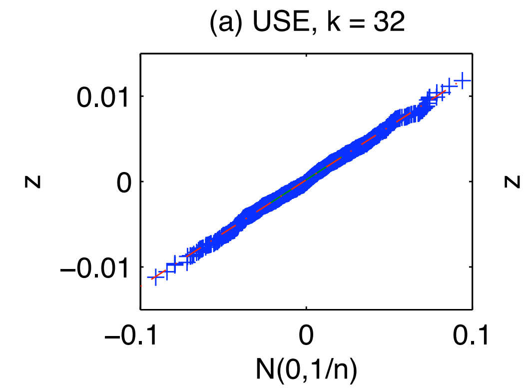
is Gaussian

- depends on the mixing
- can be tested qualitatively with QQ-plot
- related to digital communication theory
- easy to check for large systems

Example:

- Uniform Spherical Ensemble (random matrix with normalized columns)

QQ Plot



linearity corresponds to Gaussianity

Gaussian approximation

Translates in seismic situation to

- source/receiver positions distributed uniformly
- random sampling is known to reduce the adverse effects from under sampling

When valid there exists a fast approximate solution for typical recovery problems.

Exist accurate recovery conditions (phase diagrams) for medium size problems.

Allows design acquisition geometries.

Phase diagrams

Interested in two questions:

- *How many measurements does one need to take to recover typical seismic wavefields to within a prescribed accuracy?*
- *To what accuracy can one reconstruct typical seismic wavefields given a certain acquisition grid and noise level?*

Phase diagrams delineate regions for successful (white) and unsuccessful recovery(dark).

Phase diagrams

Define "measurement density"

$$\delta = n/N$$

as the ratio of the number of measurements over the length of the sparsity vector

and " sparsity density"

$$\rho = k/n.$$

the ratio of the number of non-zero entries in the sparsity vector over the number of measurements

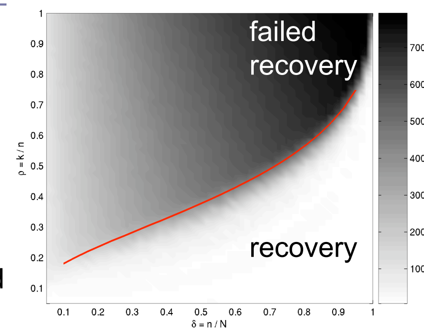
Phase diagrams tests the recovery for combinations $(\delta, \rho) \in [0,1] \times [0,1]$

Phase diagrams

l1 solver [from Donoho et al]

fully sampled rich signal

fully sampled rich signal



under sampled sparse signal

fully sampled sparse signal

In the white region

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y}$$

recovers exactly.

Phase diagrams

[Donoho et al]

White area corresponds to **successful** recovery.

Dark area corresponds to **unsuccessful** recovery.

Can be used to answer two main questions

- expensive to compute
- for large systems sharp transition
- theory applies to ideal matrices

How can these results be extended to large to very large systems?

StOMP

[from Donoho et al]

Stage-wise Orthonormal Matching Pursuit:

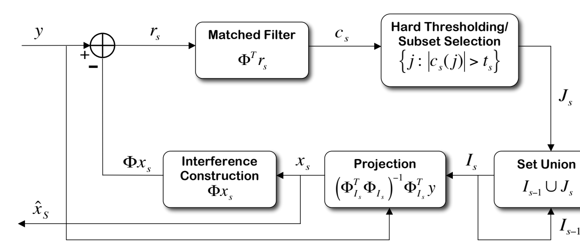
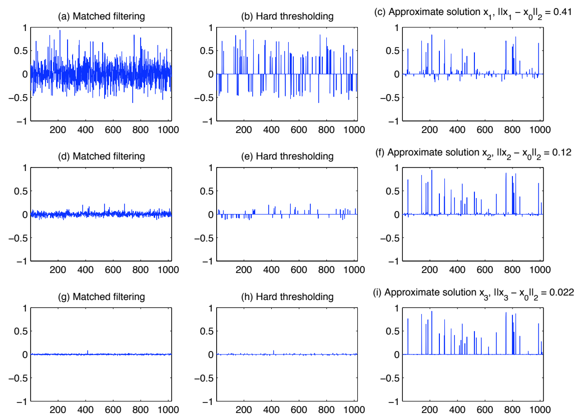


Figure 1: Schematic Representation of the StOMP algorithm.

Solves problem approximately for typical problems ...

StOMP
[from Donoho et al]



StOMP
[from Donoho et al]

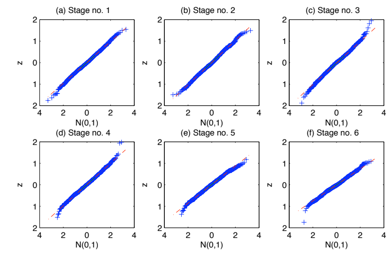


Figure 4: QQ plots comparing residual MAI with Gaussian distribution. Quantiles of residual MAI at different stages of StOMP are plotted against Gaussian quantiles. Near-linearity indicates approximate Gaussianity.

StOMP

Norm-one solvers are prohibitively expensive:

Algorithm	Dense Matrices	Fast Operators
StOMP	$S(\nu+2)nN + O(N)$	$S(\nu+2)\cdot V + O(N)$
OMP	$4n^3/3 + n^2N + O(N)$	$4n^3/3 + 2n\cdot V + O(N)$
ℓ_1 min. with PDCO	$S(2N)^3/3 + O(nN)$	$2N \cdot V + O(nN)$

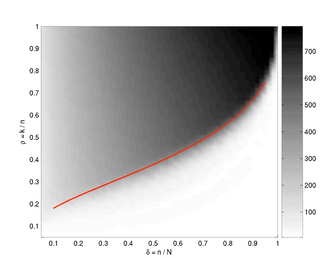
with

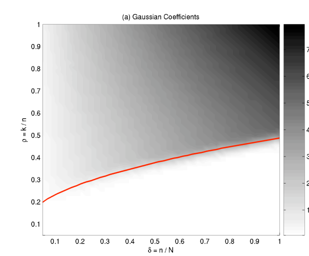
- **N length sparsity vector**
- **V=cost 1 matrix-vector multiplication**
- **S=10 the number of loops of StOMP**
- ν **number of lsqr iterations**

Fast operators and StOMP save!

StOMP versus BP

Stage-wise Orthonormal Matching Pursuit:





- **approximate**
- **good for sparse problems**
- **Order of magnitude faster**

StOMP

[from Donoho et al]

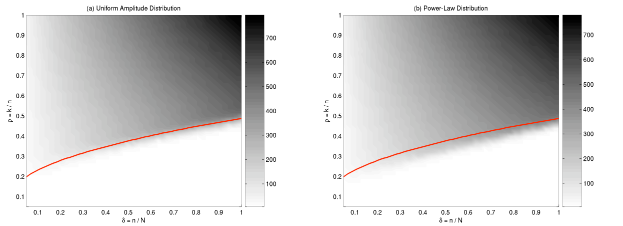


Figure 10: Phase Diagrams for CFAR thresholding when the nonzeros have nonGaussian distributions. Panel (a) uniform amplitude distribution; Panel (b) Power law distribution $x_0(j) = c/j$, $j = 1, \ldots, k$. Compare to Figure 6. Shaded attribute: number of entries where reconstruction misses by 10^{-4}.

StOMP

[from Donoho et al]

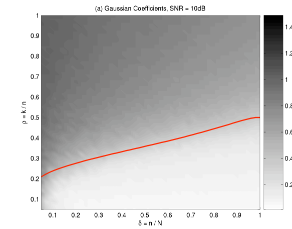
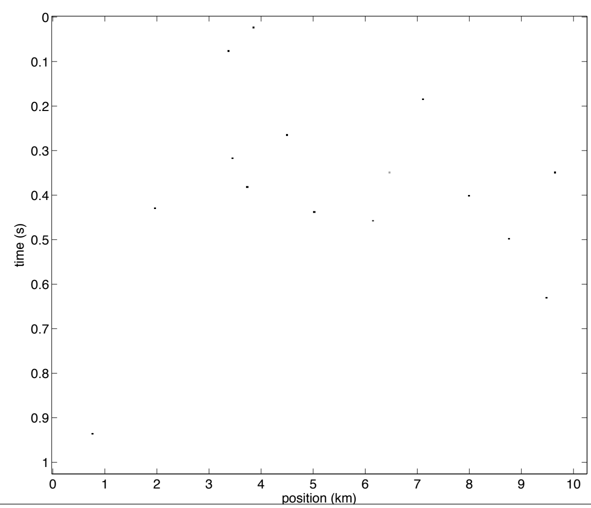
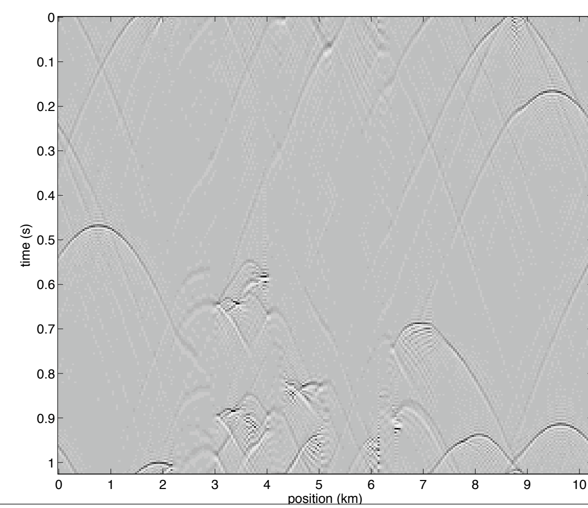


Figure 11: Performance of CFDR thresholding, noisy case. Relative ℓ_2 error as a function of indeterminacy and sparsity. Performance at signal-to-noise ratio 10. Shaded attribute gives relative ℓ_2 error of reconstruction. Signal-to-Noise ratio is $\|x_0\|_2/\|\Phi^T\xi\|_2$.

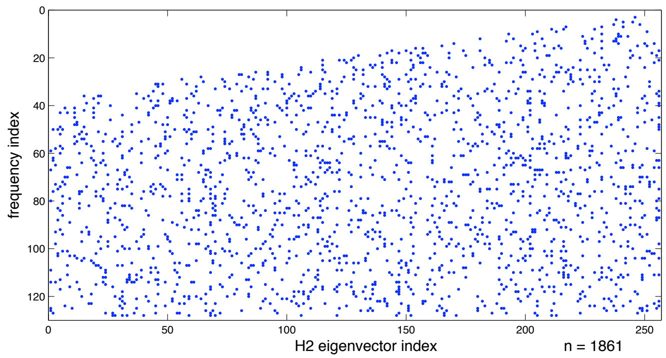
Compressed imaging



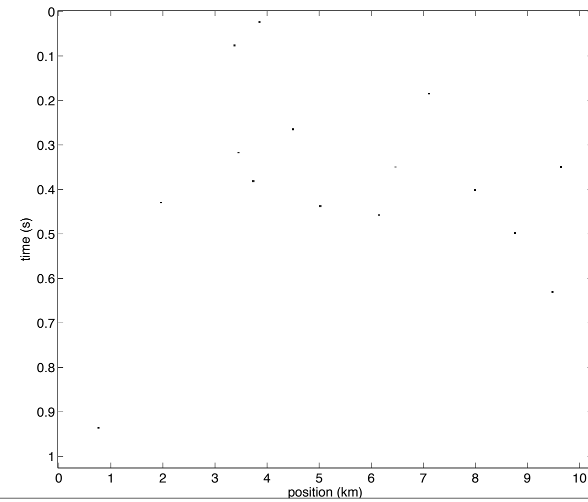
Compressed imaging



Compressed imaging



Compressed imaging



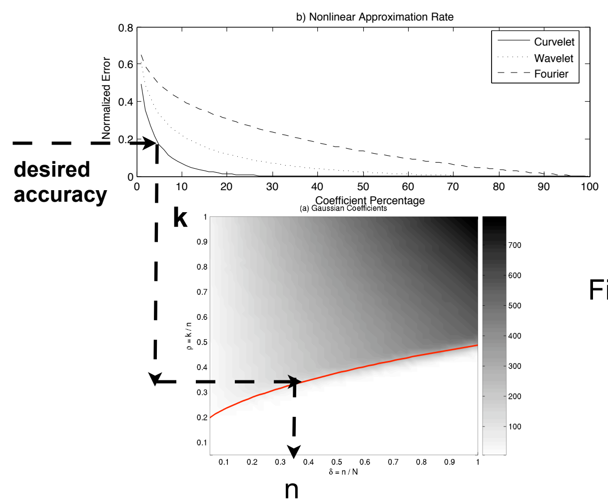
Use of phase diagram

How many measurements does one need to to take to recover typical seismic wavefields to within a prescribed accuracy?

- determine grid size of the problem N
- plots for nonlinear approximation rate => k
- phase diagram => n for which recovery is feasible

Assumes knowledge of the decay of the approximation error for a typical seismic data set.

Use of phase diagram



Fix N

Use of phase diagram

To what accuracy can one reconstruct typical seismic wavefields given a certain acquisition size, geometry and noise level?

- use N (grid size), n (# of measurements) to determine recoverable k
- use k in the NLA plot to compute the recovery error
- for noisy data use adapted phase diagram with the relative recovery error

Assumes knowledge on the phase diagram. Unattainable in practice ...

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Use of phase diagram

Analytic results for the phase transition are known for idealized matrices.

We will show empirically how far we can extend.

Remaining questions

- can analytic results be extended to other matrices?
- can we come up with qualitative and quantitative recovery conditions for large problems?
- MAI may provide perspective ...

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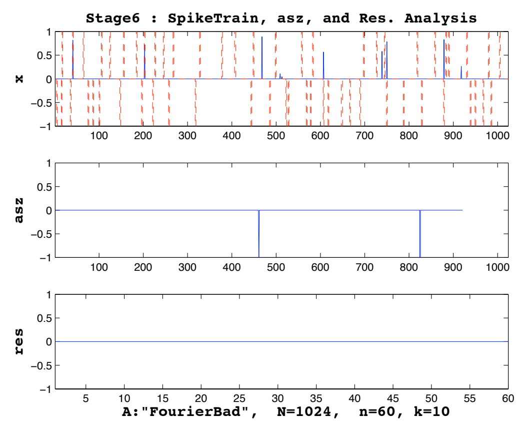

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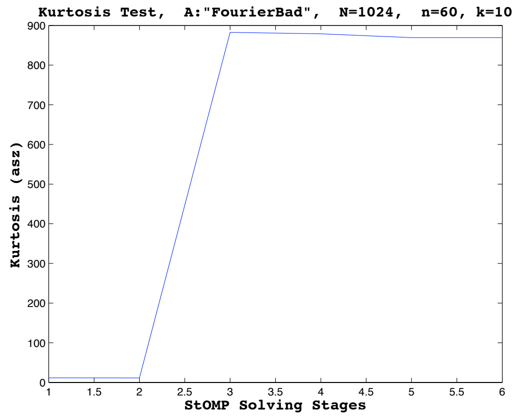

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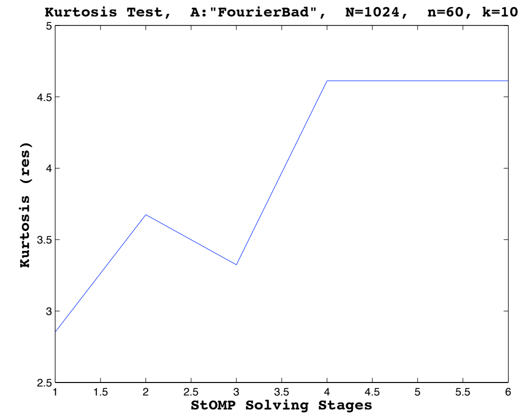

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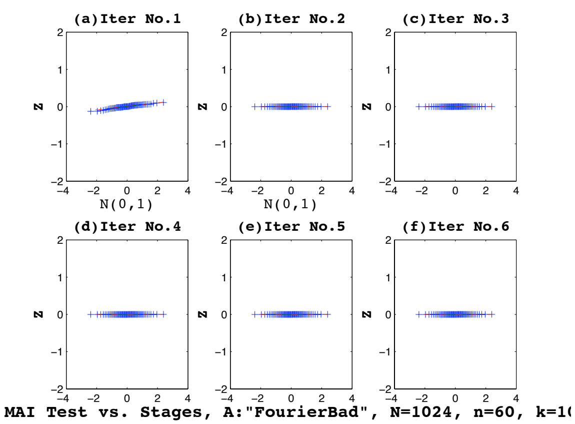

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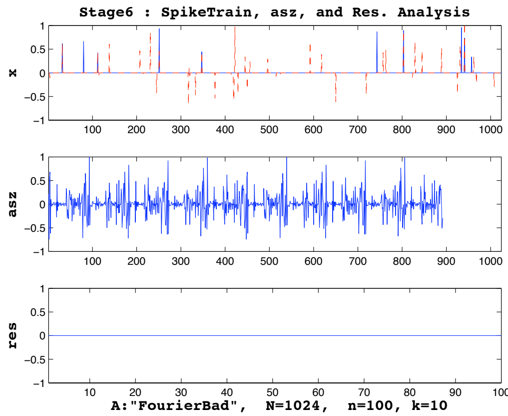

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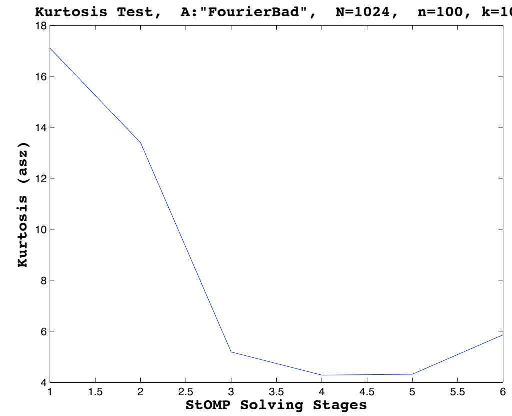

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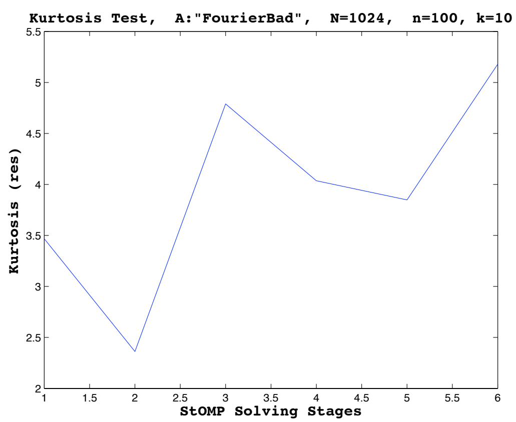

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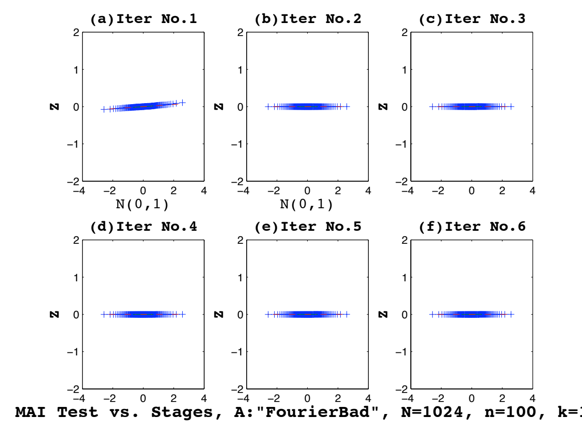

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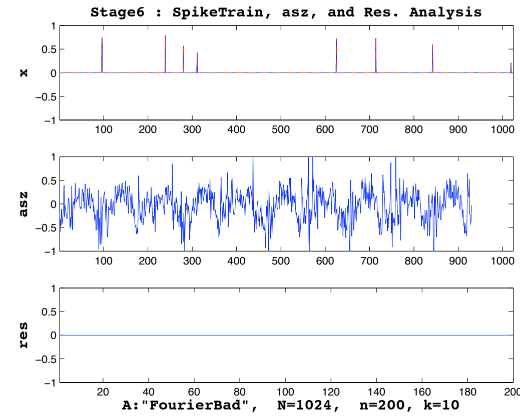

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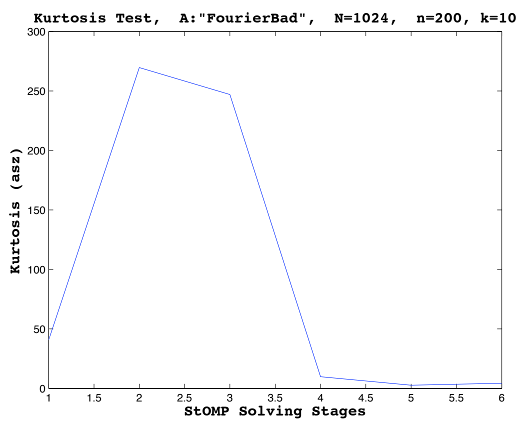

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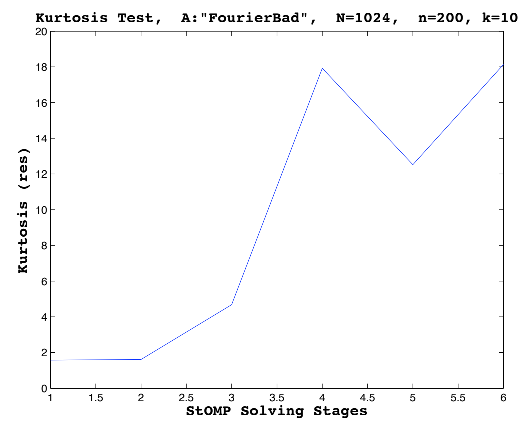

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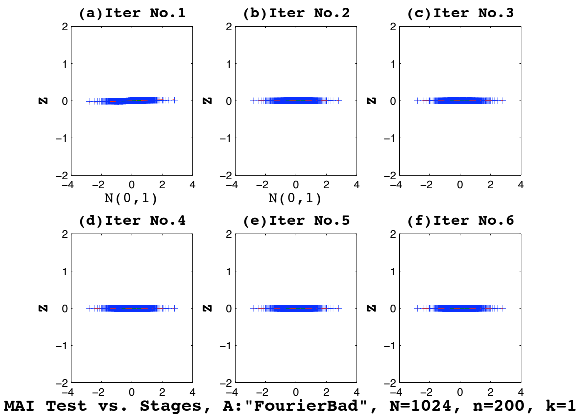

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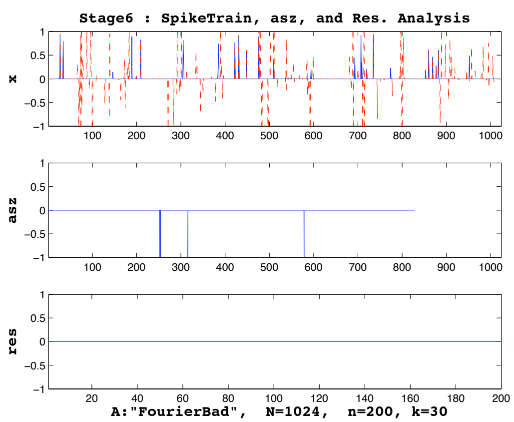

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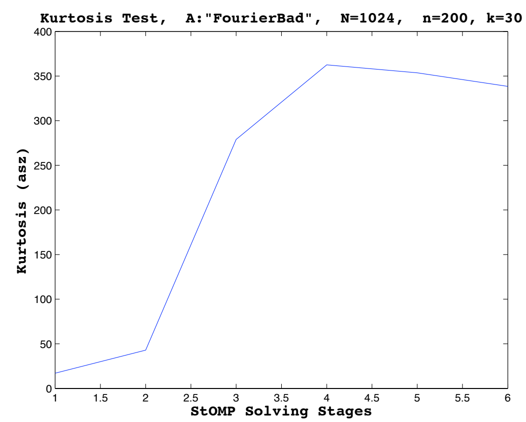

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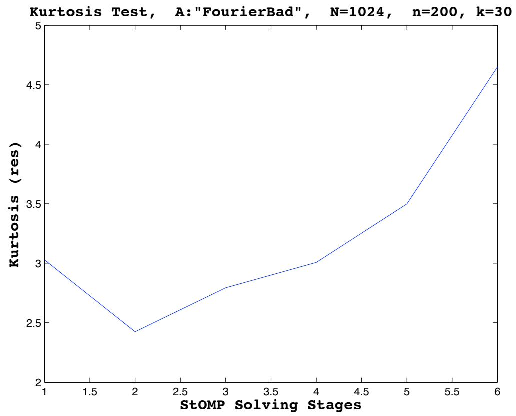
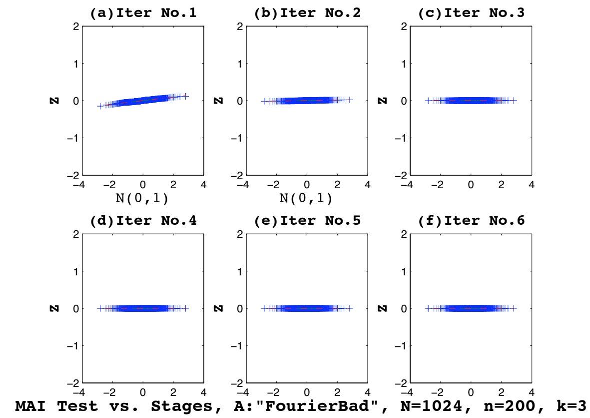
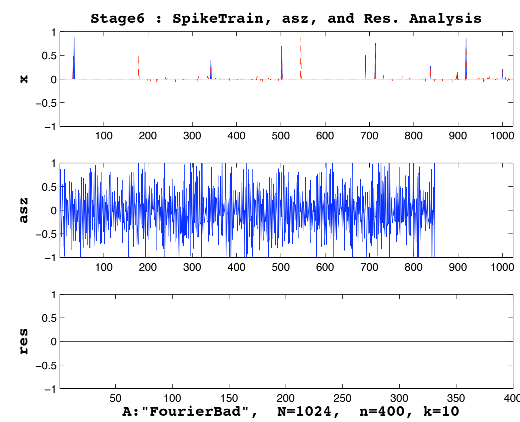
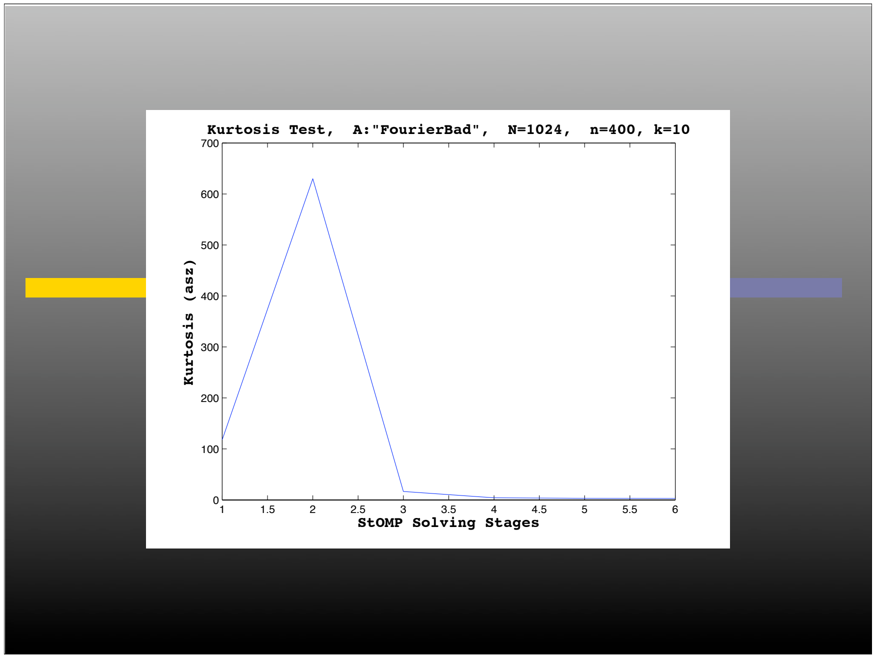

Kurtosis Test, A:"FourierBad", N=1024, n=200, k=30
Kurtosis (res)
StOMP Solving Stages


(a)Iter No.1
(b)Iter No.2
(c)Iter No.3
(d)Iter No.4
(e)Iter No.5
(f)Iter No.6
z
N(0,1)
MAI Test vs. Stages, A:"FourierBad", N=1024, n=200, k=3


Stage6 : SpikeTrain, asz, and Res. Analysis
x
asz
res
A:"FourierBad", N=1024, n=400, k=10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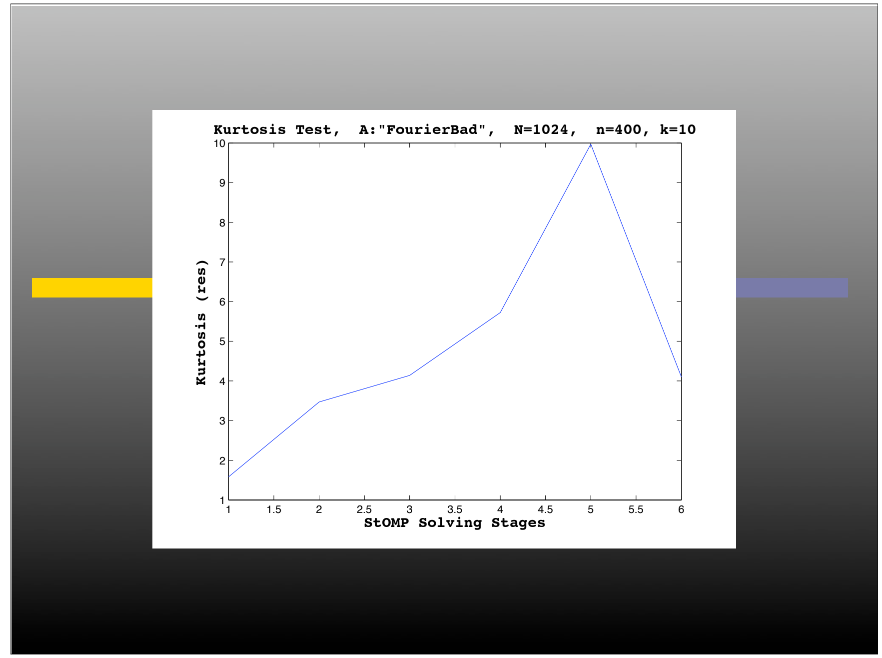
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Kurtosis (res)
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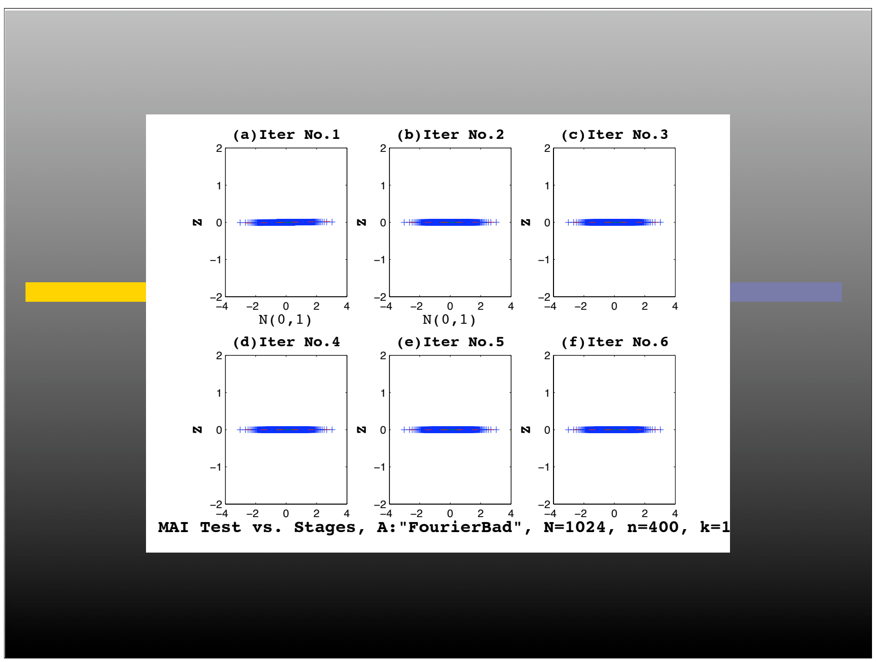
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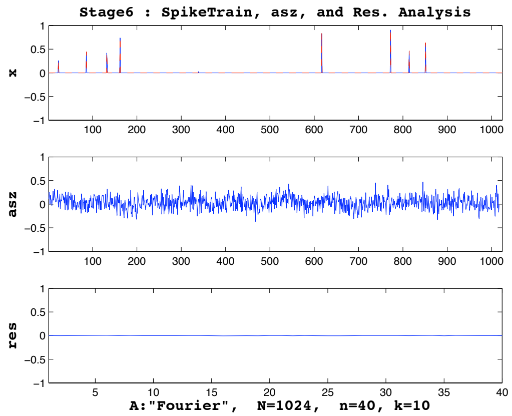
N(0,1)
MAI Test vs. Stages, A:"FourierBad", N=1024, n=400, k=1



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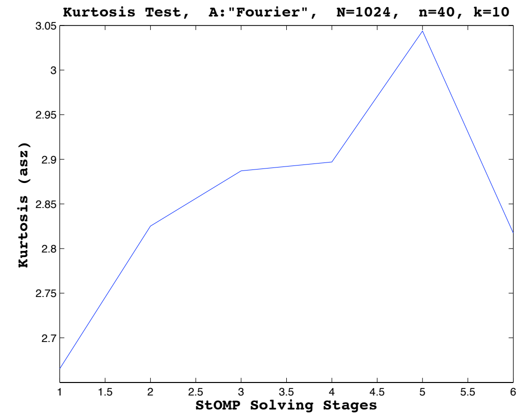
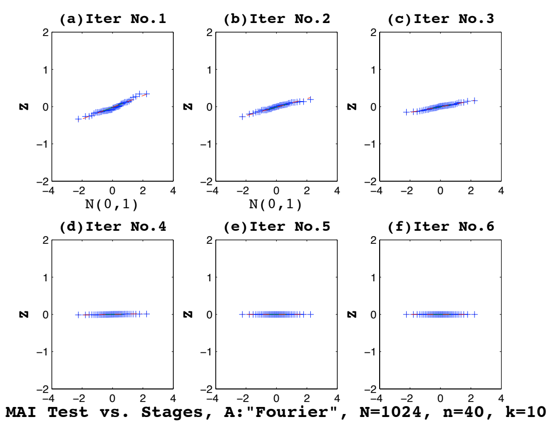
x
asz
res
A:"Fourier", N=1024, n=40, k=10



Kurtosis Test, A:"Fourier", N=1024, n=40, k=10
Kurtosis (asz)
StOMP Solving Stages



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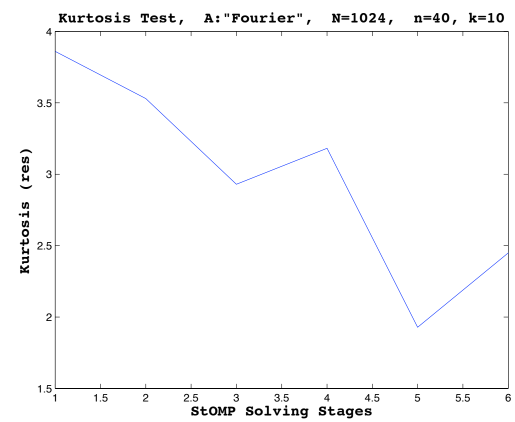
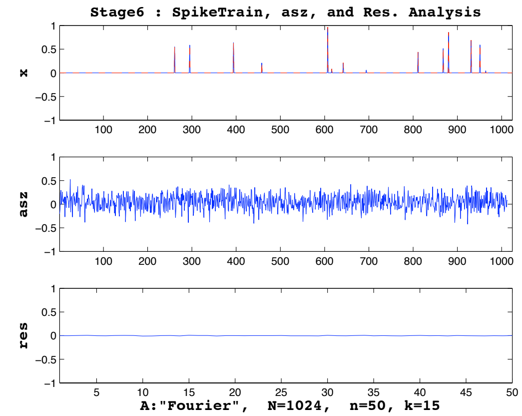
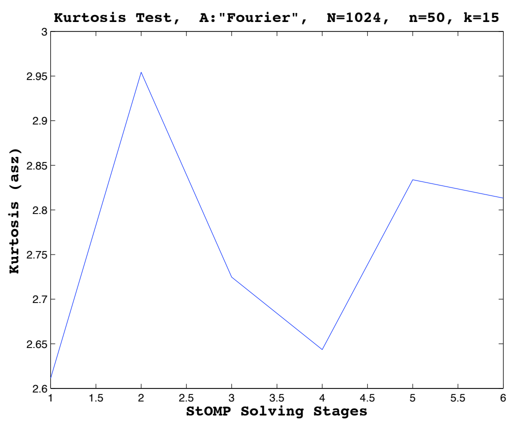
(f)Iter No.6
z
N(0,1)
MAI Test vs. Stages, A:"Fourier", N=1024, n=40, k=10



Kurtosis Test, A:"Fourier", N=1024, n=40, k=10
Kurtosis (res)
StOMP Solving Stages


Stage6 : SpikeTrain, asz, and Res. Analysis
x
asz
res
1
0.5
0
−0.5
−1
100
200
300
400
500
600
700
800
900
1000
5
10
15
20
25
30
35
40
45
50
A:"Fourier", N=1024, n=50, k=15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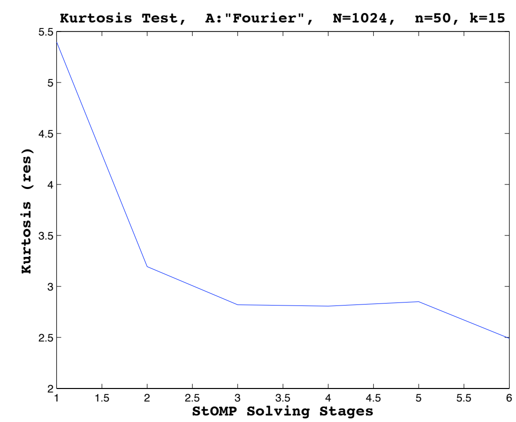
Kurtosis (asz)
3
2.95
2.9
2.85
2.8
2.75
2.7
2.65
2.6
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
StOMP Solving Stages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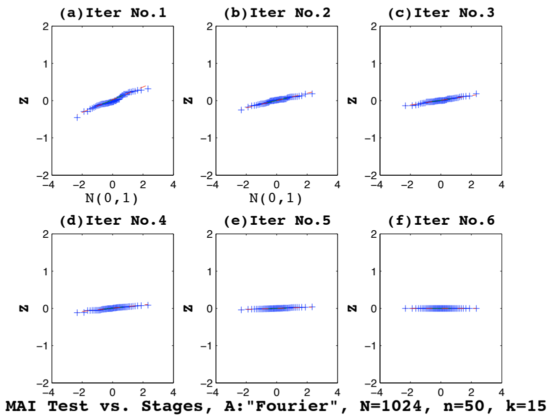
Kurtosis (res)
5.5
5
4.5
4
3.5
3
2.5
2
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
StOMP Solving Stages



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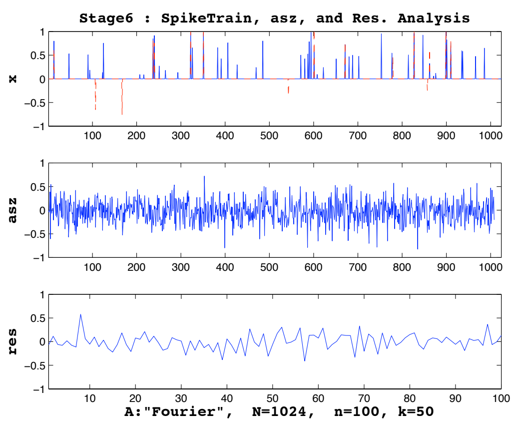

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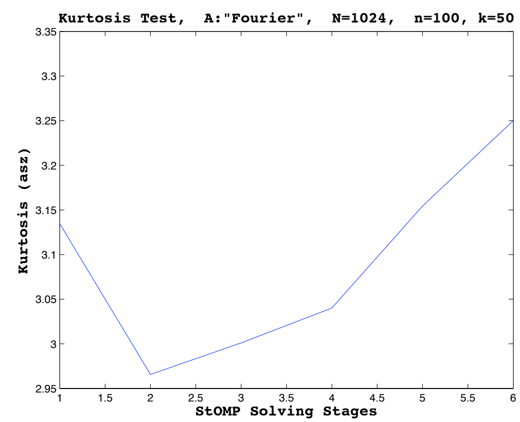

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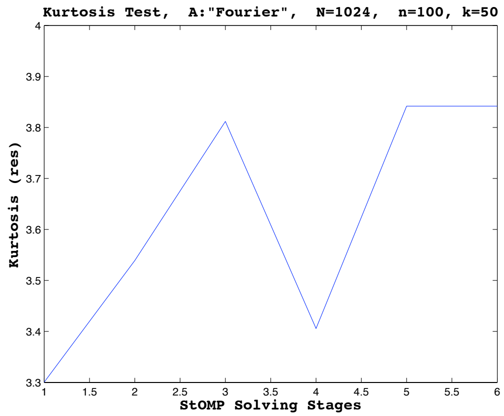
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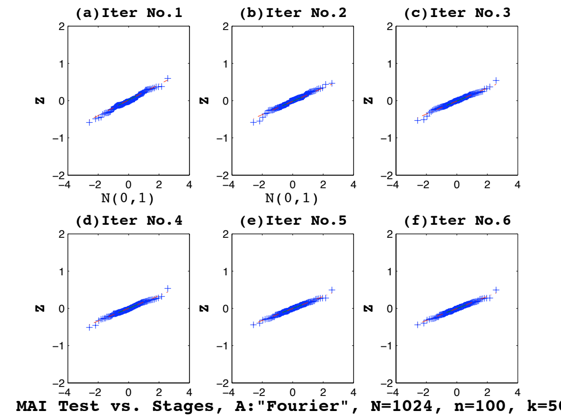
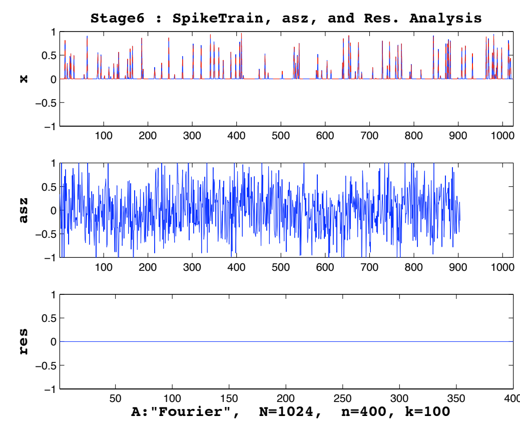
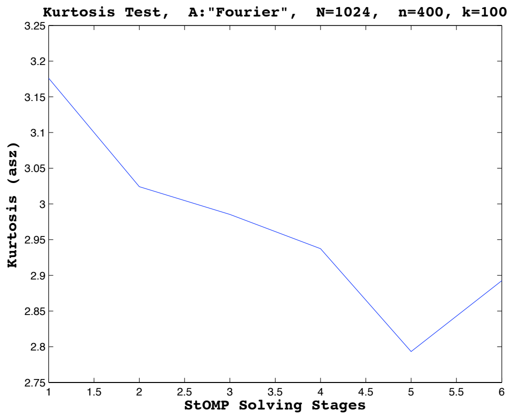
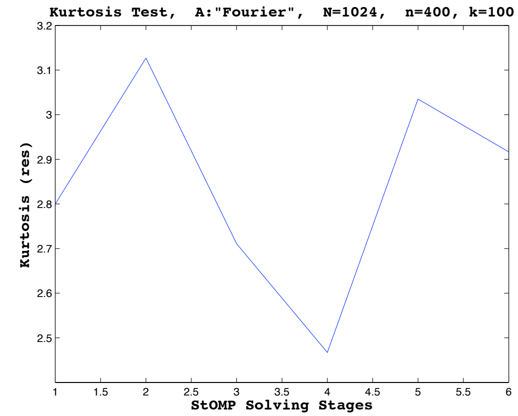
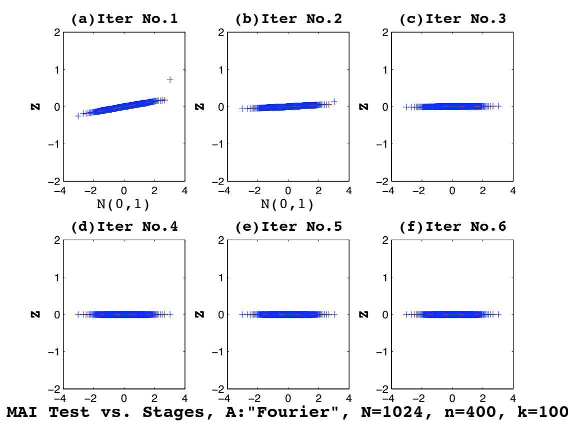

(a)Iter No.1
(b)Iter No.2
(c)Iter No.3
(d)Iter No.4
(e)Iter No.5
(f)Iter No.6
N(0,1)
z
MAI Test vs. Stages, A:"Fourier", N=1024, n=100, k=50


Stage6 : SpikeTrain, asz, and Res. Analysis
x
asz
res
A:"Fourier", N=1024, n=400, k=100







Conclusions

For the good Fourier, i.e. mixing matrix

- if kurtosis of MAI & residue are app. 3 then likely recovery
- when close to non-recoverable region then behavior difficult to predict
- if both kurtosi are far from >> 3 then no recovery likely

For the bad Fourier, i.e. no mixing matrix

- difficult to predict
- recoverable region is much smaller
- kurtosis is not a good indicator

Outlook

StOMP is orders of magnitude faster then BP

Weak recovery conditions better suited for

Test for Gaussianity and detectability may be feasible for given

- typical sparsity vector (e.g. permutation of histogram)
- typical acquisition geometry
- sparsity matrix

Findings are an extension of known matched filter arguments (randomness = good)

May lead to feasible tests for recovery.

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Outlook

Applications discussed during the meeting

- event detection (Mohhammad)
- compressed imaging (Tim)
- NFFT's (Sastry)

Extension to large systems

Relation to other solvers

Monte-Carlo sampling for the phase diagram with sparse recovery?

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