A primer on sparsity transforms: curvelets and wave atoms

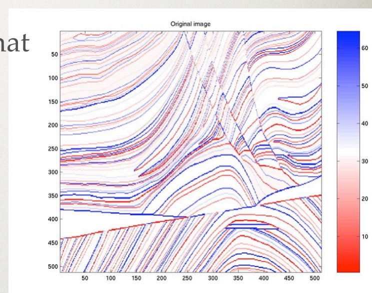
Context

- "Fast Discrete Curvelet Transforms (FDCT)" by Candes, E., Demanet, L., Donoho, D., and Ying, L., explains the curvelet transform in detail
- "Wave Atoms and Sparsity of Oscillatory Patterns" by Demanet, and D., and Ying, L., explains the wave atom transform in detail
- 3-D curvelet extension by Ying.
- Courtesy Demanet (figures from his SIAM & IPAM talks)
- Check http://www.curvelet.org/ for curvelets
- Check for wave atoms http://www.acm.caltech.edu/~demanet/software/WaveAtom.tar.gz

Our Motivation

Devise a data representation scheme that

- is non-parametric/non-adaptive
- truly exploits the 3-D continuity along wavefronts in $\mathbf{d}(r,s,t)$
- is noise resilient
- exploits redundant frames
- is $n \log n$
- deals with *intermittent* regularity!



Phase space tilings

Need a representation that is

- local in phase space (space-wave number)
- directional

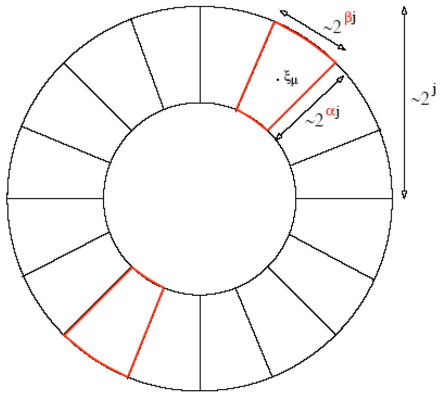
Phase space is too big ... need a tiling & sampling

Many possibilities

Need to preserve directivity of wavefronts

Phase space tilings

polar Fourier space



In red, the essential frequency support of a wave packet φ_μ.

[From Demanet '05]

Phase space tilings

Phase-space: $(x, \xi) =$ (position, frequency). Here $x \in \mathbb{R}^2$.

Covered by a family of wave packets, indexed by $\mu = (j, k_1, k_2, \ell, m)$:

$$\varphi_\mu(x) \simeq 2^{(\alpha+\beta)j/2} \cos((pR_{\theta_{j\ell}}x - k) \cdot (1,0))\varphi(D_j R_{\theta_{j\ell}}x - k)$$

for some profile $\phi(x)$ like $e^{-x^2/100}$, wave vector $p = 2^j + m2^{\alpha j}$,

$$D_j = \begin{pmatrix} 2^{\alpha j} & 0 \\ 0 & 2^{\beta j} \end{pmatrix}$$

$$\theta_{j\ell} = \ell 2^{(\beta-1)j}$$

2 parameters: α (multiscaleness) and β (aspect ratio).

[From Demanet '05]

Phase space tilings

Curvelets are a family of wave packets indexed by 4 integers $\mu = (j, k_1, k_2, \ell)$:

$$\varphi_\mu(x) = 2^{3j/4}\varphi(D_j R_{\theta_{j\ell}}x - k)$$

with

$$D_j = \begin{pmatrix} 2^{j} & 0 \\ 0 & 2^{j/2} \end{pmatrix} \quad \text{Parabolic scaling}$$

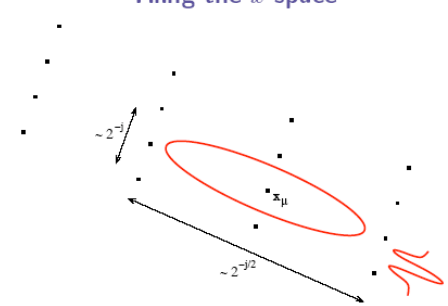
$$\theta_{j\ell} \sim \ell 2^{-j/2}$$

[From Demanet '05]

Phase space tilings

physical space

Tiling the x space

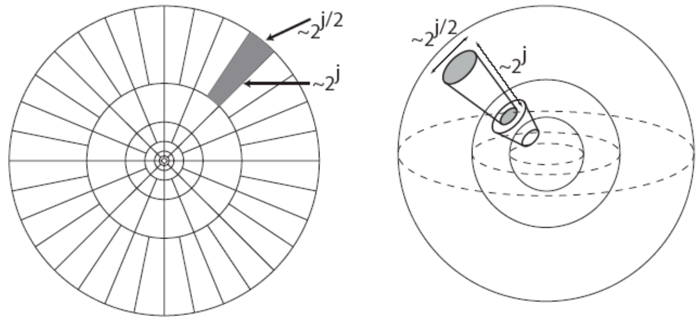


The red ellipse sketches the essential support of a curvelet ϕ_μ.

[From Demanet '05]

3-D extension

polar Fourier space



[From Demanet '05]

SLIM

Curvelet construction

Cartesian Fourier space

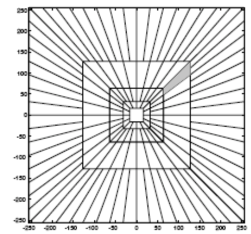


Figure 2: The figure illustrates the basic digital tiling. The windows $\tilde{U}_{j,\ell}$ smoothly localize the Fourier transform near the sheared wedges obeying the parabolic scaling. The shaded region represents one such typical wedge.

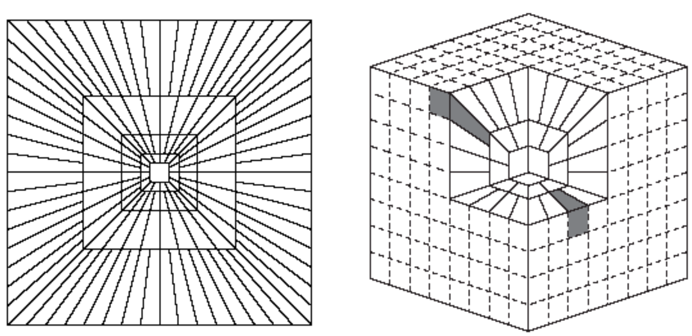
From Fast Digital Curvelet transform

SLIM

Extenstion to 3-D

Cartesian Fourier space

[courtesy Demanet '05, Ying '05]

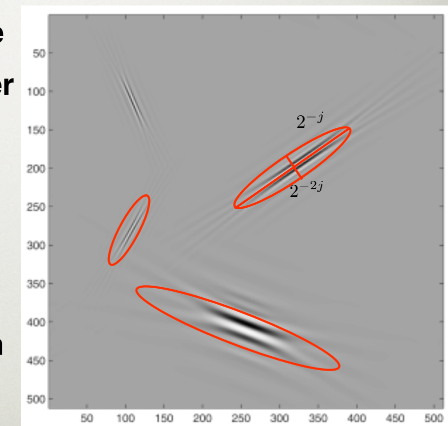


Curvelets live in a wedge in the 3 D Fourier plane...

SLIM

Why curvelets

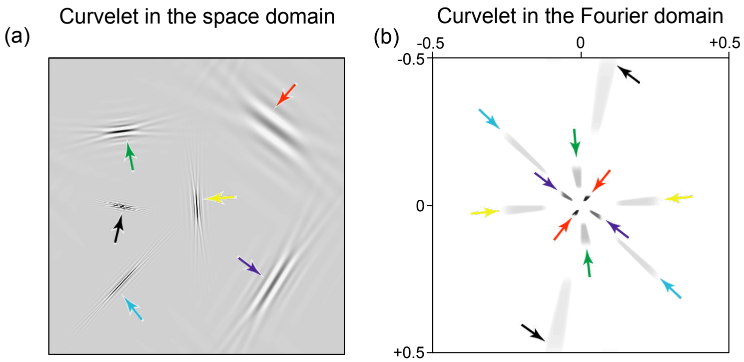
- Non-separable directional
- Local in 2 & 3-D space
- Local in 2 & 3-D Fourier
- Anisotropic
- Multiscale
- Almost orthogonal
- Tight redundant frame
- Optimal approximation rates
- Released www.curvelet.org



Curvelet tiling

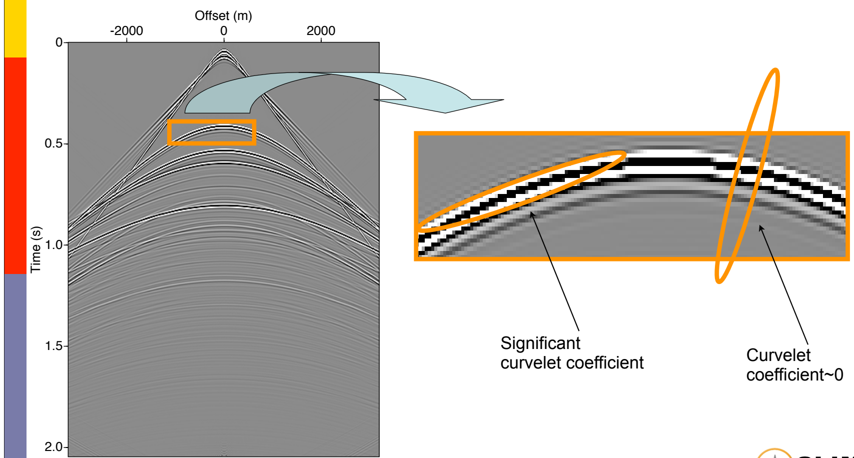
[Candes & Donoho '02-'05, Do '02, Demanet '05, Ying '05]

Partitioning example

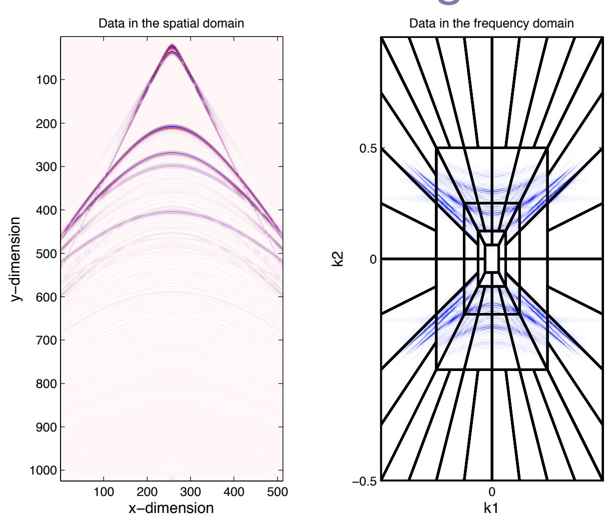


Micro-local correspondence

Curvelets & Seismic Data



Curvelet tiling



Curvelet

nonlinear approximation rate

Optimal: [Donoho, 01]

$$\|\mathbf{f} - \mathbf{f}_p^O\|_2^2 \propto p^{-2}, \quad p \to \infty$$

Fourier:

$$\|\mathbf{f} - \mathbf{f}_p^F\|_2^2 \propto p^{-1/2}, \quad p \to \infty$$

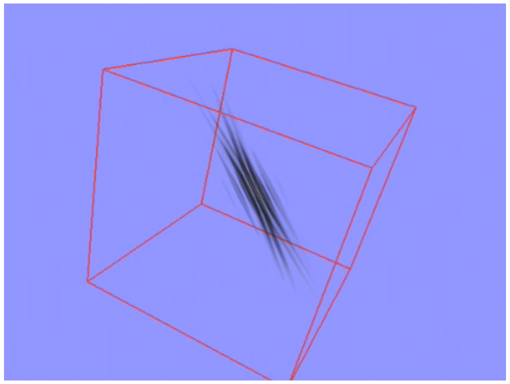
Wavelets:

$$\|\mathbf{f} - \mathbf{f}_p^W\|_2^2 \propto p^{-1}, \quad p \to \infty$$

Curvelets: [Candes & Donoho, 99]

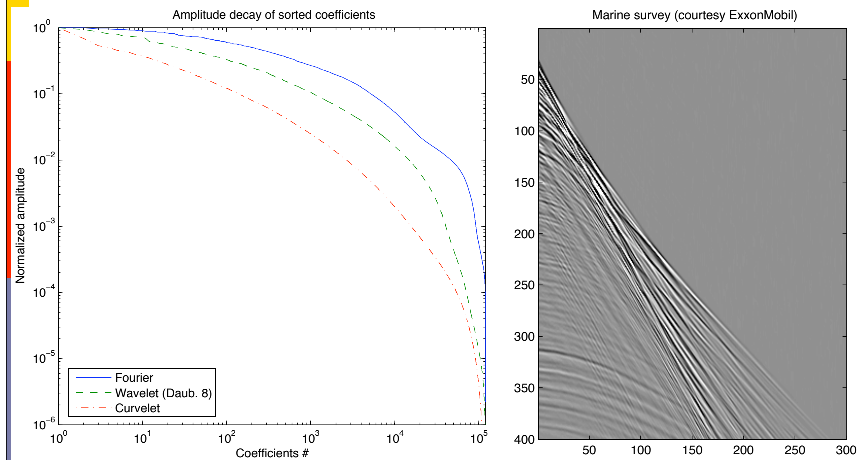
$$\|\mathbf{f} - \mathbf{f}_p^C\|_2^2 \leq C\, p^{-2}\, (\log p)^3, \quad p \to \infty$$

Extension to 3-D



Curvelets live in wedges in the 3 D Fourier plane...

Empirical approximation rates



Other choices

- Directional wavelets: $\alpha = 1,\ \beta = 1$

$$\varphi_\mu(x) = 2^j \psi(2^j R_{\theta_{j\ell}} x - k)$$

with $\psi(x) = \cos(x_1)\phi(x)$ some profile, oscillatory in one direction.

- Curvelets/contourlets: $\alpha = 1,\ \beta = 1/2$

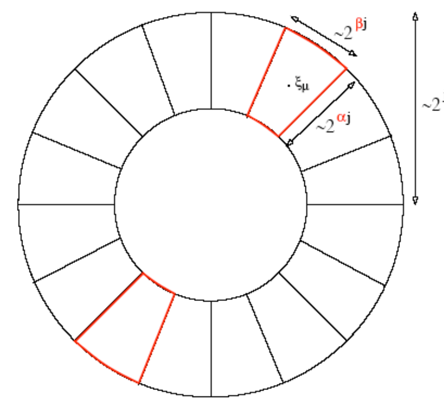
$$\varphi_\mu(x) = 2^{3j/4} \psi(\mathrm{diag}(2^j, 2^{j/2}) R_{\theta_{j\ell}} x - k)$$

with $\psi(x) = \cos(x_1)\phi(x)$.

[From Demanet '05]

Other choices

polar Fourier space



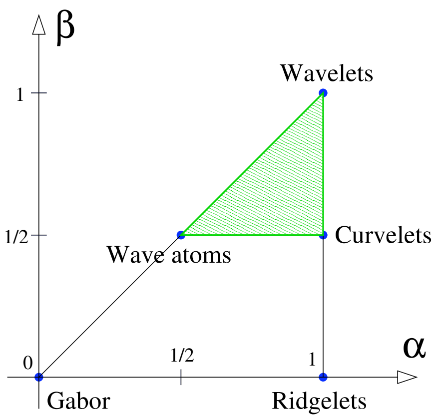
In red, the essential frequency support of a wave packet φ_μ.

[From Demanet '05]

Other choices

different aspect ratios

parabolic scaling space coordinate & # oscillations



parabolic scaling space coordinates

[From Demanet '06]

SLIM Seismic Laboratory for Imaging and Modeling

Curvelets

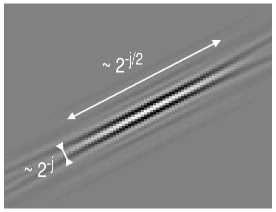
Collection of wave packets $\varphi_\mu(x)$, $x \in \mathbb{R}^2$, indexed by the quadruple of integers $\mu = (j, k_1, k_2, \ell)$.

$$\varphi_\mu(x) \simeq 2^{3j/4}\varphi(D_j R_{\theta_\ell} x - k),$$

$$D_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix},$$

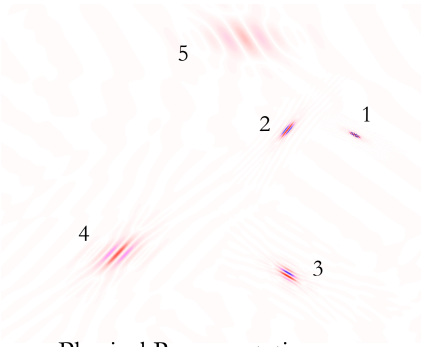
$$\theta_\ell \simeq \ell \cdot 2^{-\lfloor j/2 \rfloor}.$$

Tight frame: $f = \sum_\mu \langle f, \varphi_\mu \rangle \varphi_\mu$.

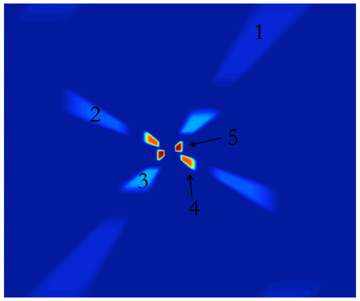


SLIM Seismic Laboratory for Imaging and Modeling

Curvelets



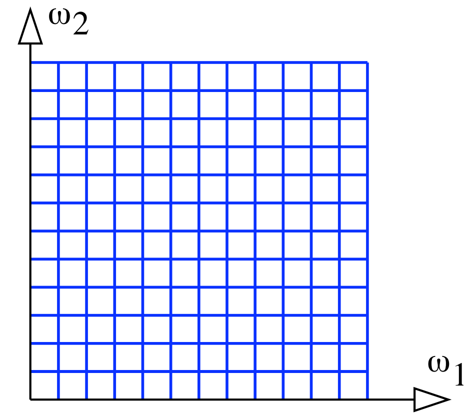
Physical Representation



Fourier Spectrum Representation

SLIM Seismic Laboratory for Imaging and Modeling

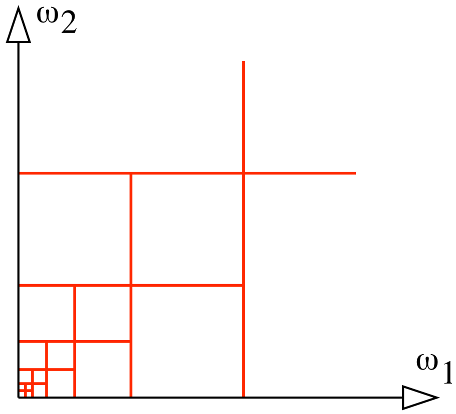
Gabor tiling of the frequency plane



[From Demanet '06]

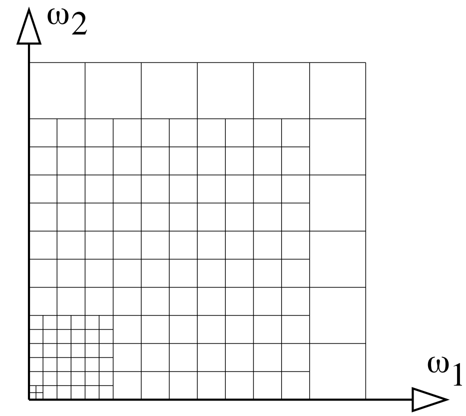
SLIM Seismic Laboratory for Imaging and Modeling

Wavelet tiling of the frequency plane



[From Demanet '06]

Wave Atom tiling of the frequency plane



[From Demanet '06]

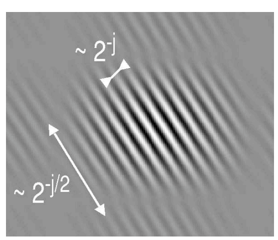
What are Wave Atoms

Collection of wave packets $\varphi_\mu(x)$, $x \in \mathbb{R}^2$, indexed by the 5-uple of integers $\mu = (j, m_1, m_2, n_1, n_2)$,

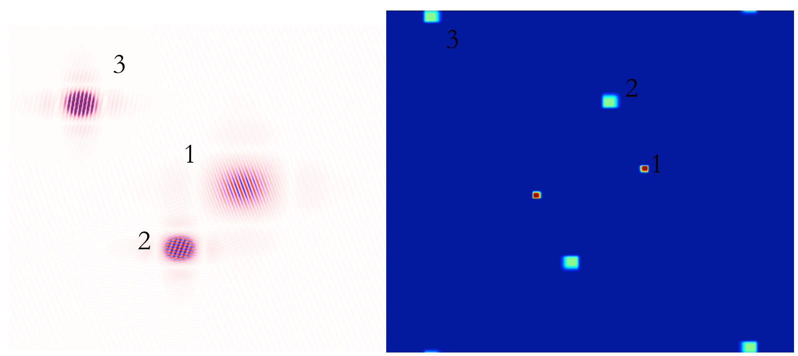
$\varphi_\mu(x) \simeq 2^j \phi(2^j x - \mathbf{n}) \cos(2^j \mathbf{m} \cdot x)$

Wave vector $\xi_\mu \simeq 2^j \mathbf{m}$, position vector $x_\mu \simeq 2^{-j} \mathbf{n}$. Note $|\mathbf{m}| \simeq 2^j$.

Tight frame: $f = \sum_\mu \langle f, \varphi_\mu \rangle \varphi_\mu$.



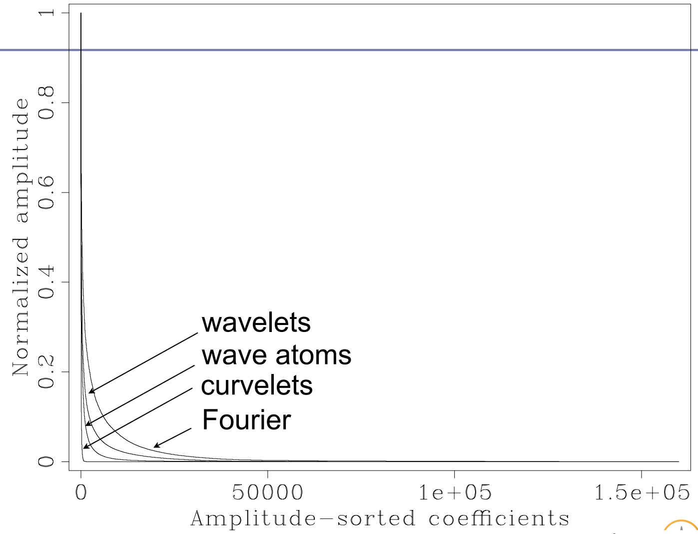
Wave Atoms



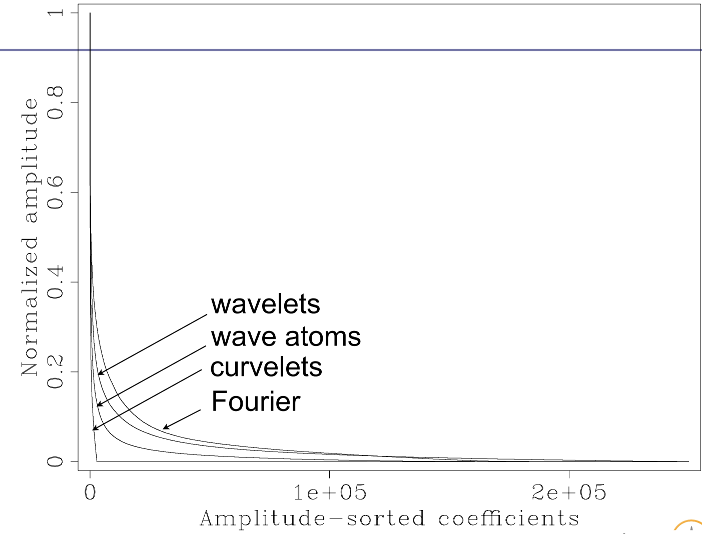
Physical Representation

Fourier Spectrum Representation

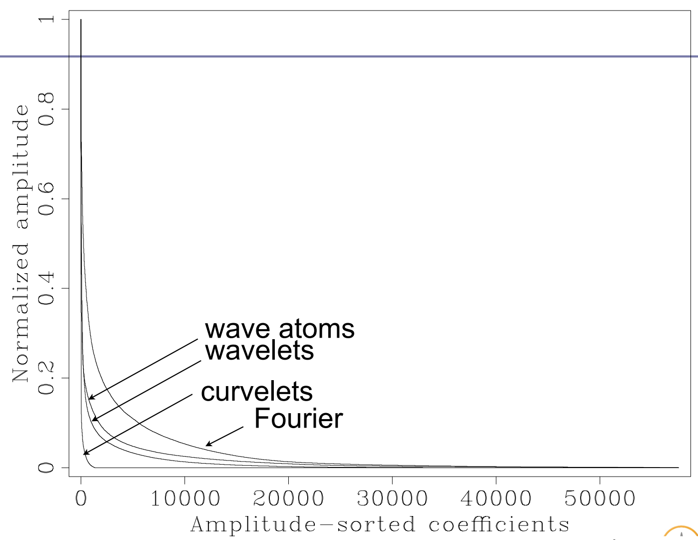
Seismic data (shot)

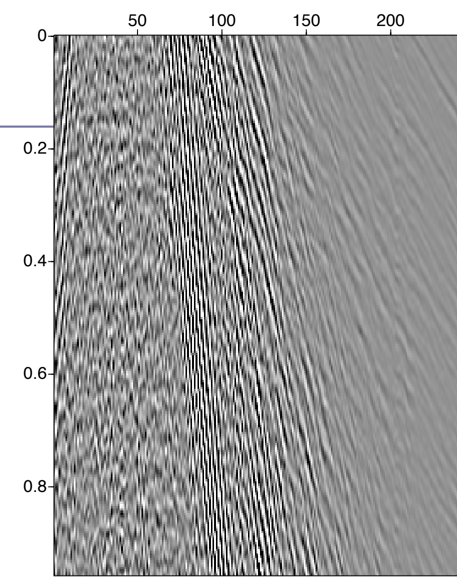


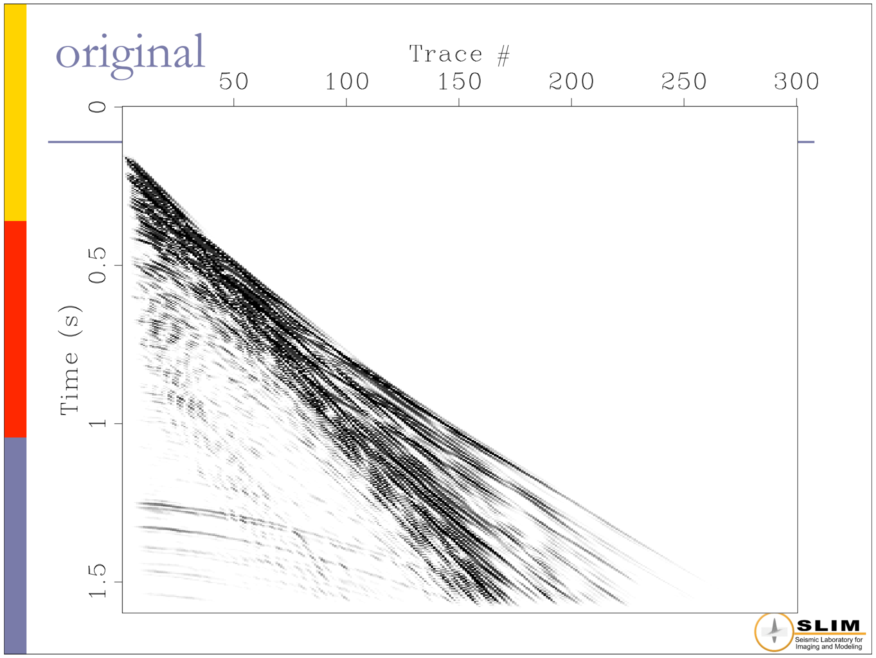
Imaged reflectivity

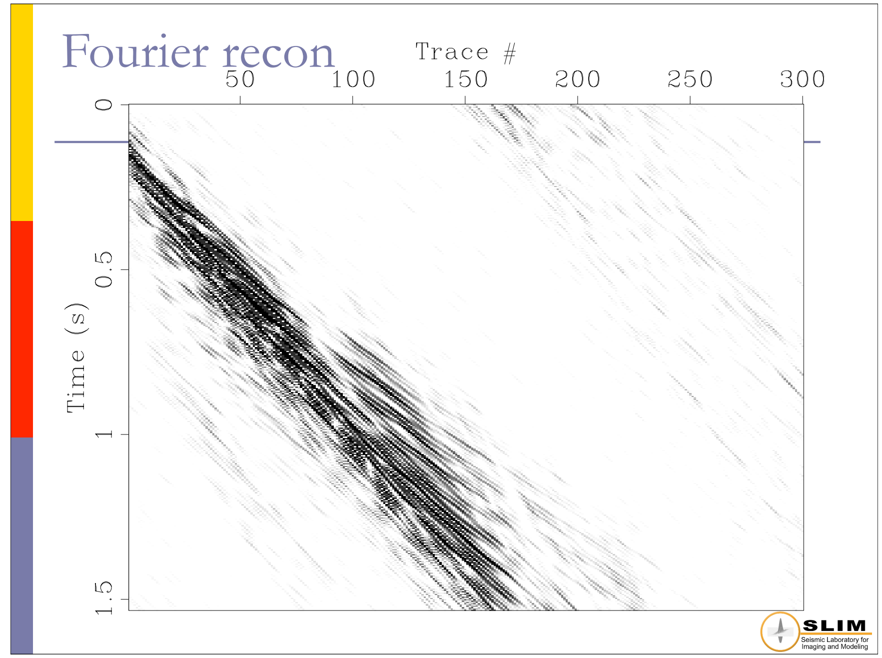


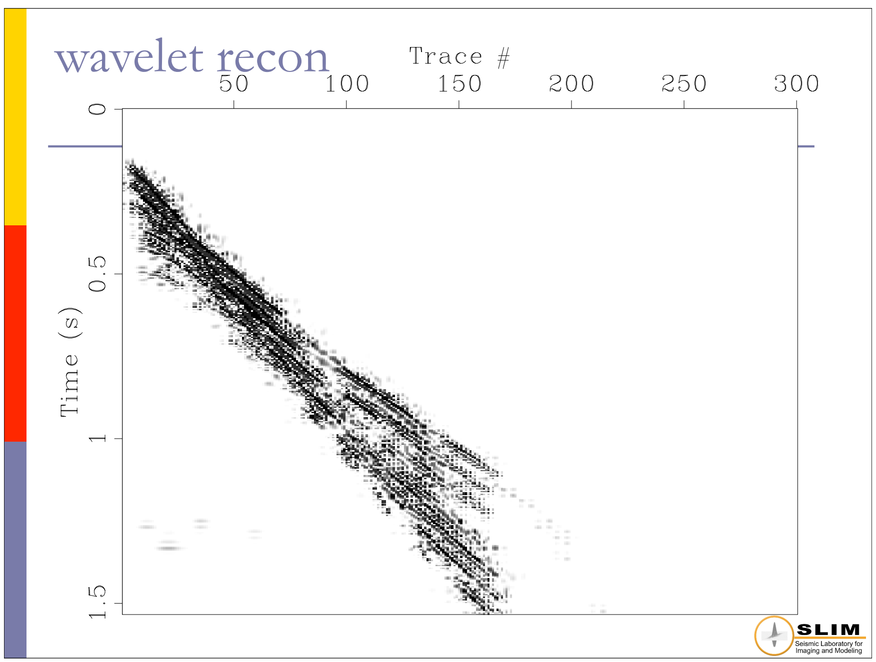
Seismic shot with GR

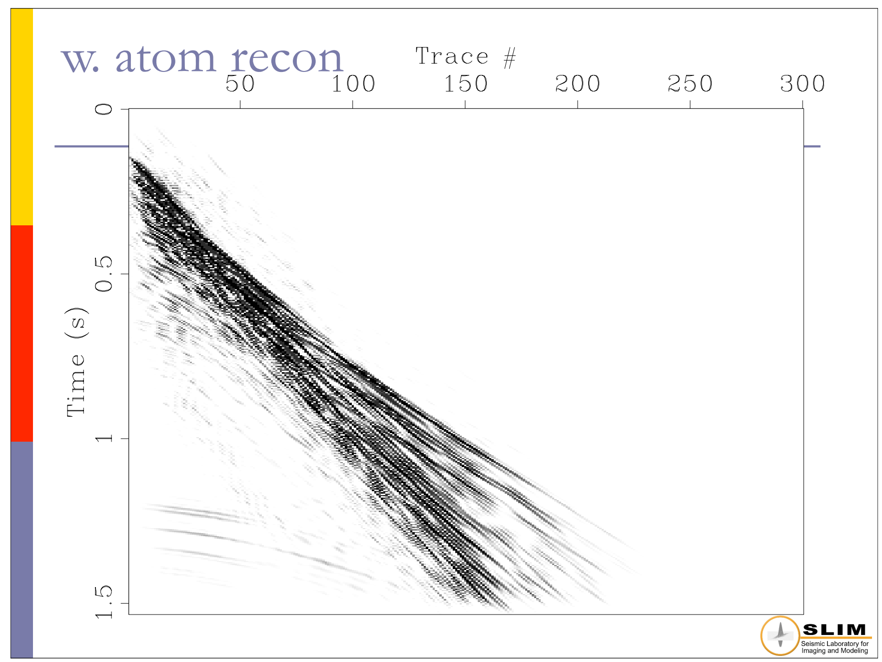


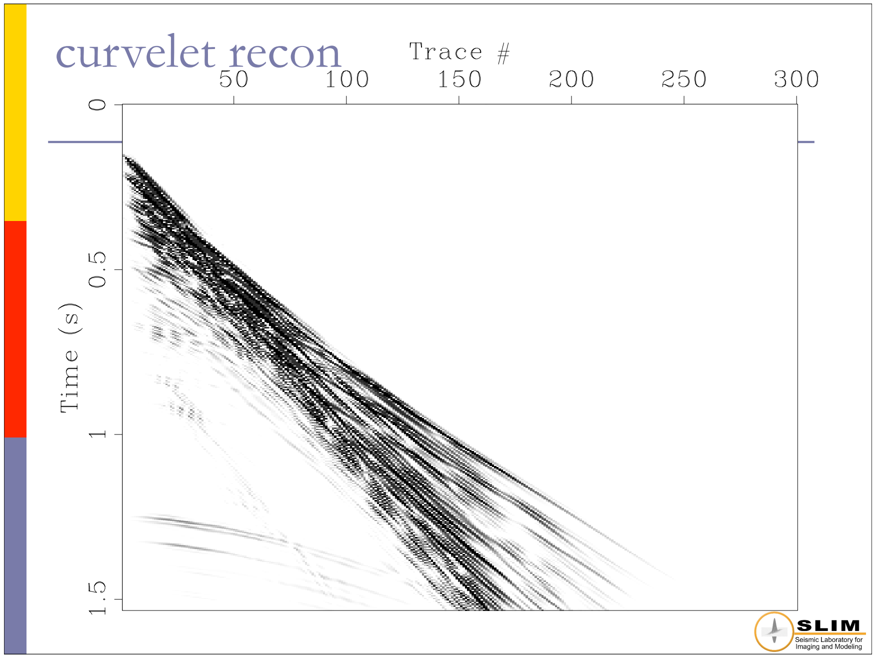


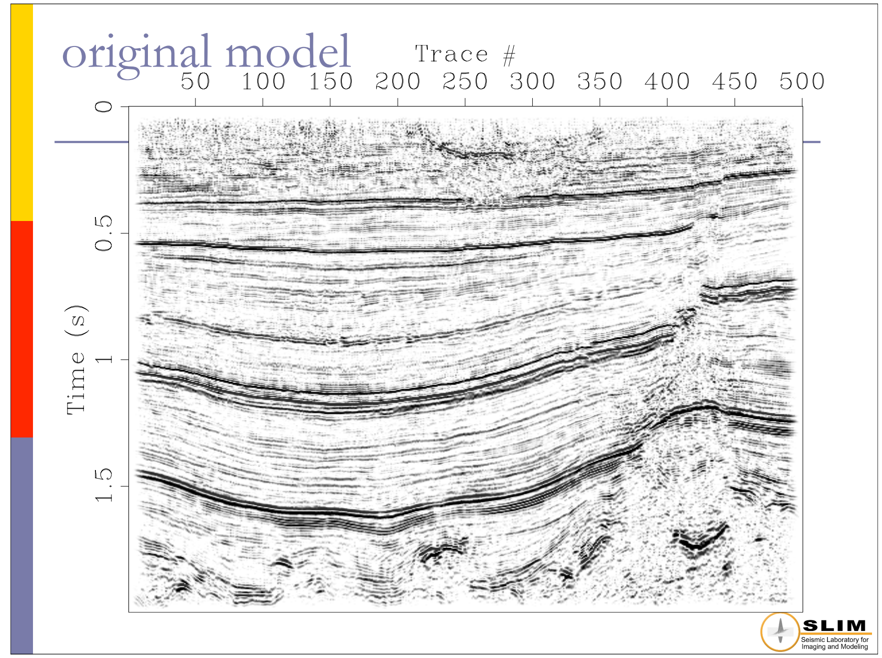

original
Trace #
50 100 150 200 250 300
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling

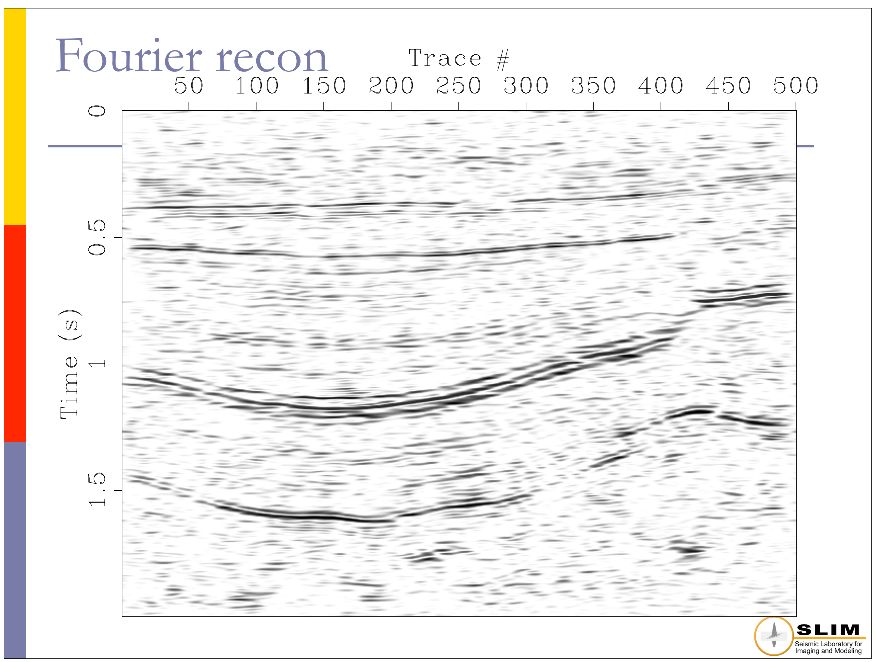

Fourier recon
Trace #
50 100 150 200 250 300
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling

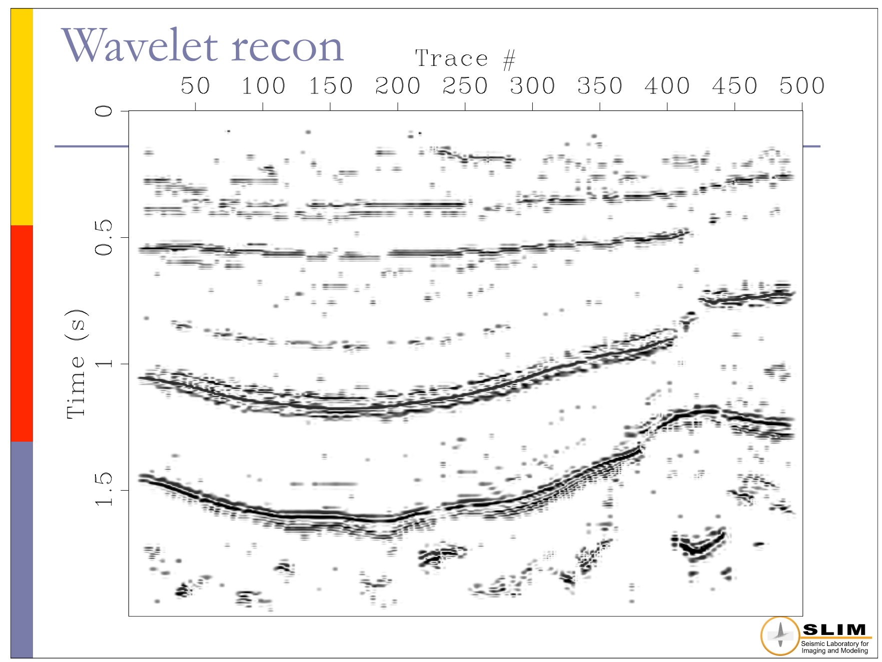

wavelet recon
Trace #
50 100 150 200 250 300
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling


w. atom recon
Trace #
50 100 150 200 250 300
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling

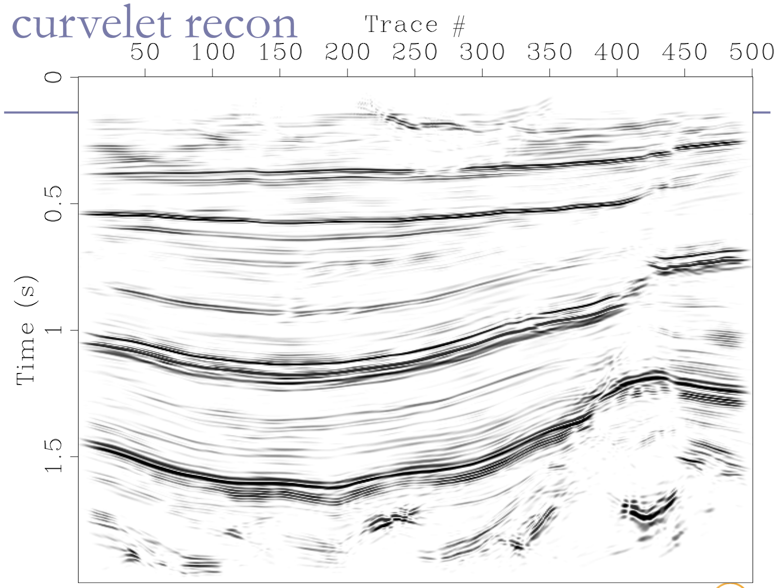

curvelet recon
Trace #
50 100 150 200 250 300
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling


original model
Trace #
50 100 150 200 250 300 350 400 450 500
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling

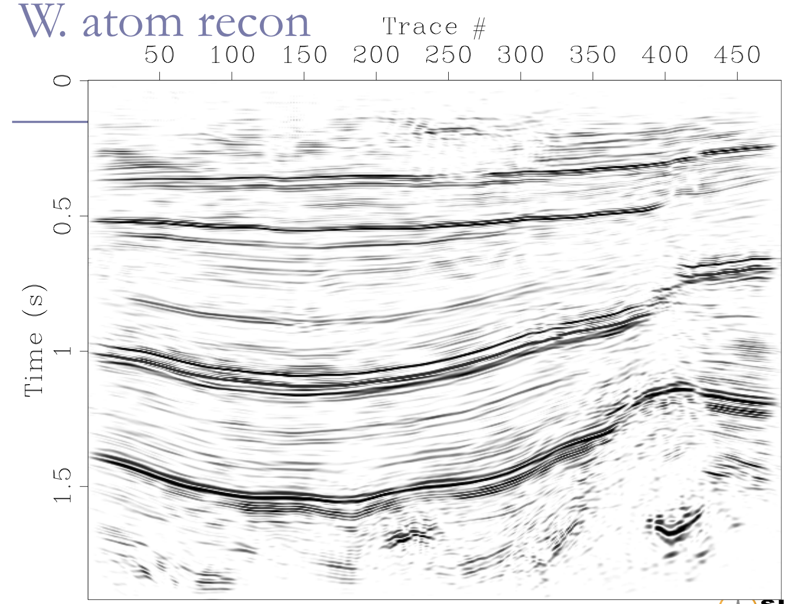

Fourier recon
Trace #
50 100 150 200 250 300 350 400 450 500
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling


Wavelet recon
Trace #
50 100 150 200 250 300 350 400 450 500
Time (s)
0 0.5 1 1.5
SLIM
Seismic Laboratory for Imaging and Modeling

curvelet recon



W. atom recon



Observations

Different tilings of phase space

Given the multiscale and multidirection behavior of seismic data and images

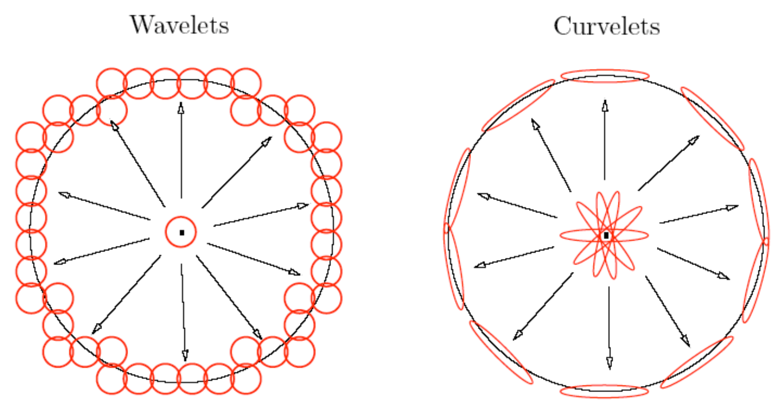
- multiscale partitioning
- multidirectional partitioning

Invariance under FIO's (high-freq. solution operators of the wave equation)

- parabolic scaling principle
- two candidates
  - curvelets
  - wave atoms

How does this work?

Wavelets do not work

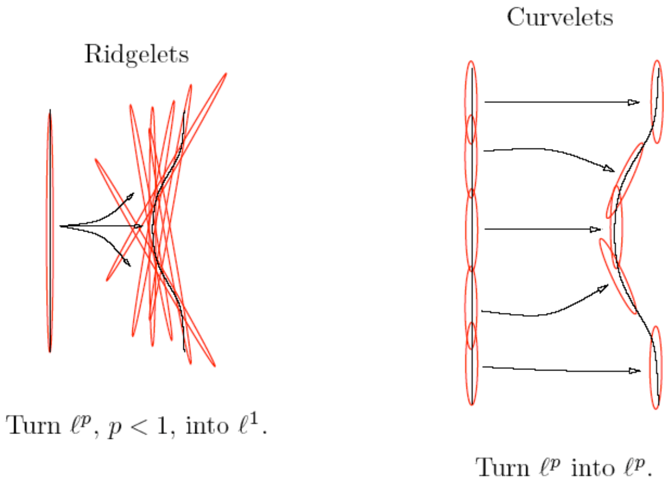


Turn ℓ^p, $p < 1$, into ℓ^1.

Turn ℓ^p into ℓ^p.

[From Demanet '05]

Ridgelets do not work

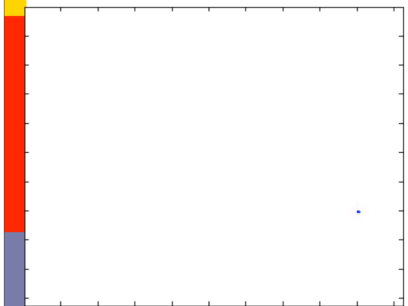


[From Demanet '05]

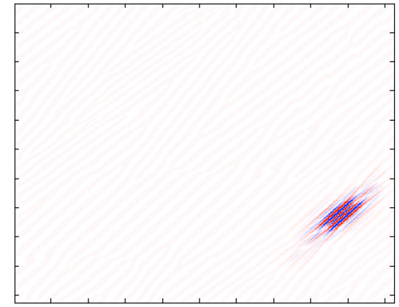
Curvelets & waves

homogeneous medium

impulse response



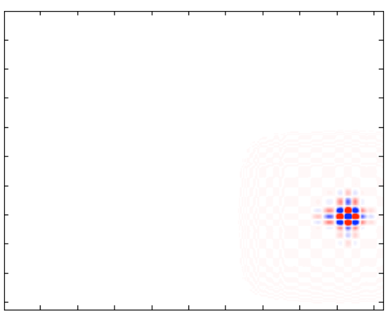
curvelet response



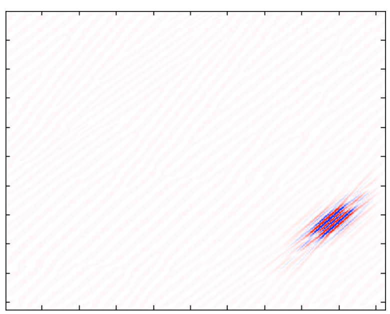
Curvelets & waves

homogeneous medium

wavelet response



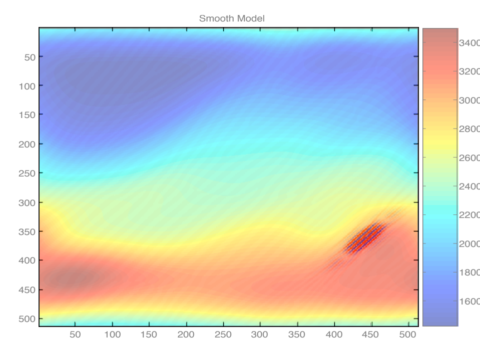
curvelet response



Curvelets & waves

heterogeneous medium

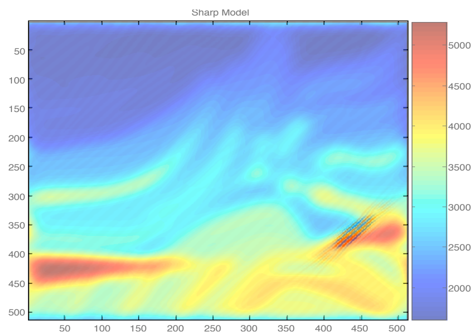
Smooth model



Curvelets & waves

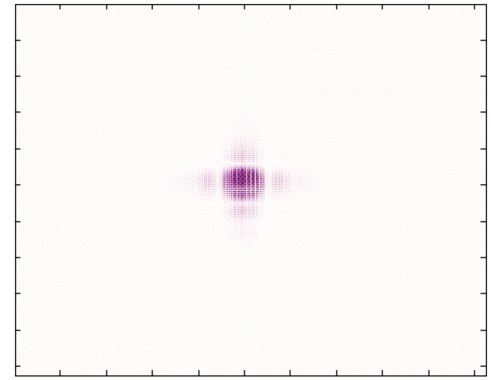
heterogeneous medium

“Hard” model



SLIM Seismic Laboratory for Imaging and Modeling

Movies for wave atoms

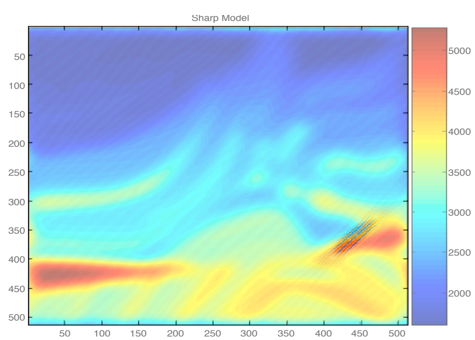


SLIM Seismic Laboratory for Imaging and Modeling

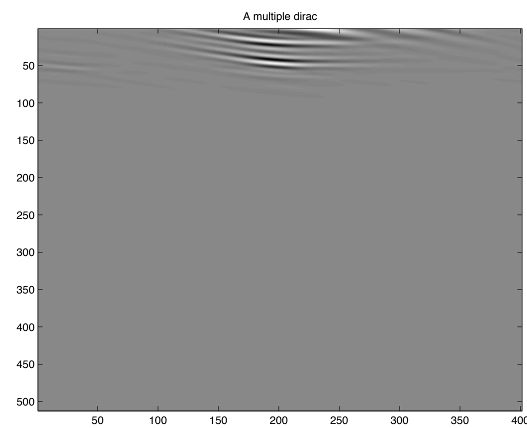
Curvelets & Hessian

[Demanet ‘05, H & M, 03-05]

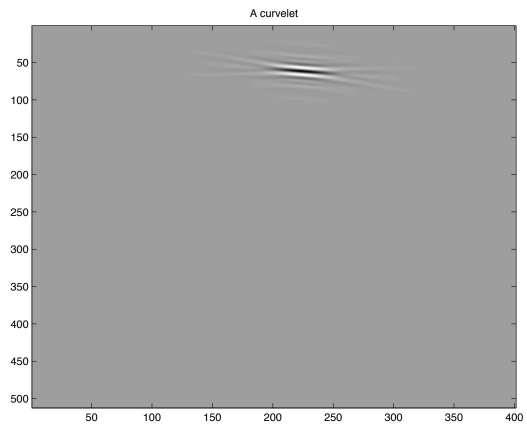
Hessian $\mathrm{K}^H\mathbf{K}$ for “Hard” model

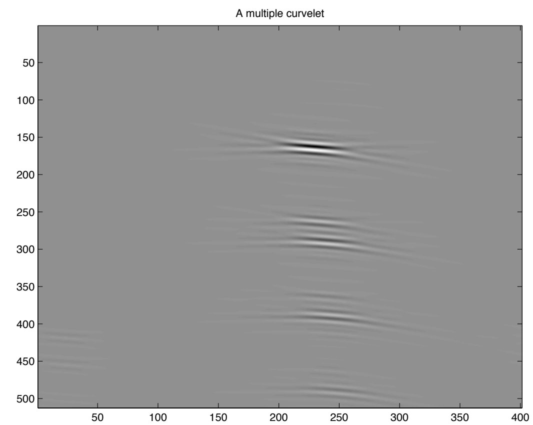


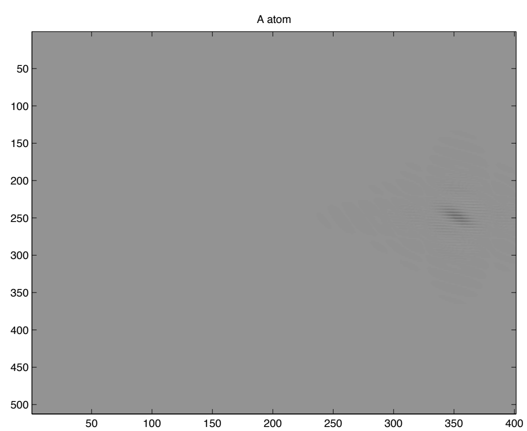
SLIM Seismic Laboratory for Imaging and Modeling

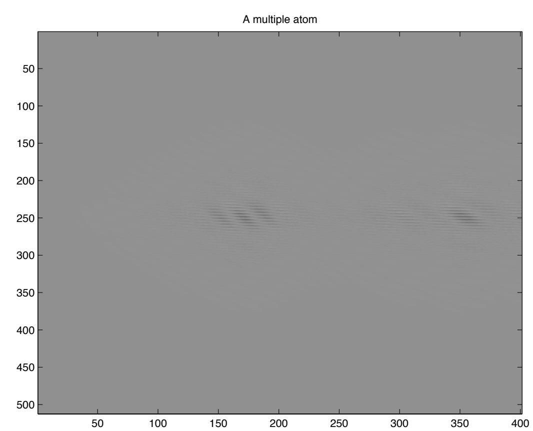


SLIM Seismic Laboratory for Imaging and Modeling


A curvelet


A multiple curvelet

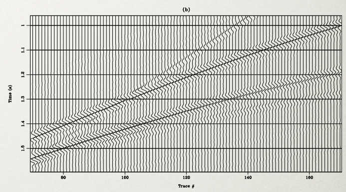

A atom

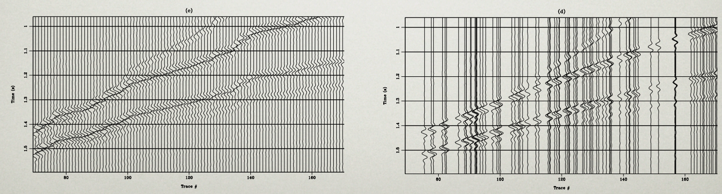

A multiple atom

Fast Discrete Curvelet Transforms (FDCT)

- FDCT's assume regularly sampled data

Casting irregularly sampled data to regular grid destroyed continuity along wavefronts





[Hennenfent & Herrmann, 2005]

NFDCT

Curvelet transform for unstructured data

FDCT's

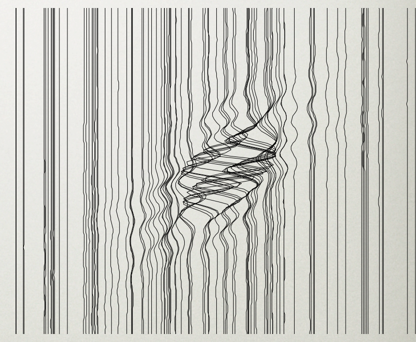
$$\mathbf{C} := \mathbf{T}\mathbf{B}_s\mathbf{B}_t$$

FFT's

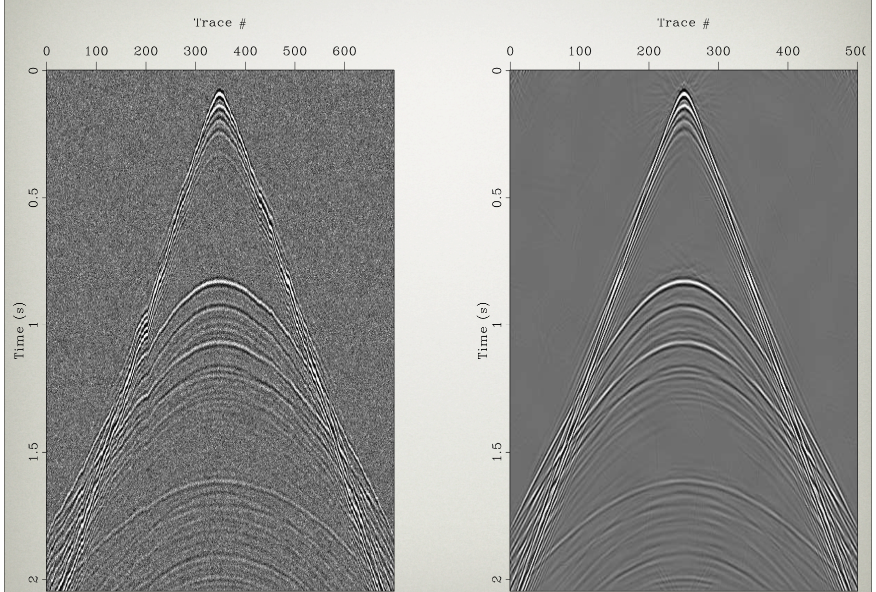
NFDCT

$$\mathbf{C}_N := \mathbf{T}\breve{\mathbf{B}}_s\mathbf{B}_t$$

NFFT *FFT*



[Hennenfent & Herrmann, 2005]



Observations

Curvelets are sparse on distinct **high-frequency** wavefronts

Wave atoms are sparse on **oscillatory** wavefronts

Both redundant with fast relative decay

Wave atom simpler to implement & less redundant.

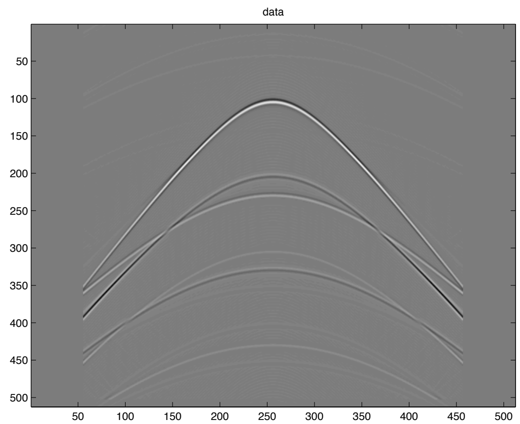
Both can unstructured.

Which one will perform better?

- seismic data holds middle between high-freq. and "single" frequency
- subsurface contains distinct singularities

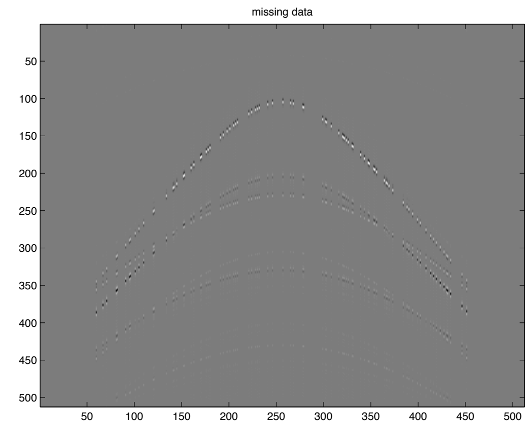
Both invariant under FIO's/migration ...

Data



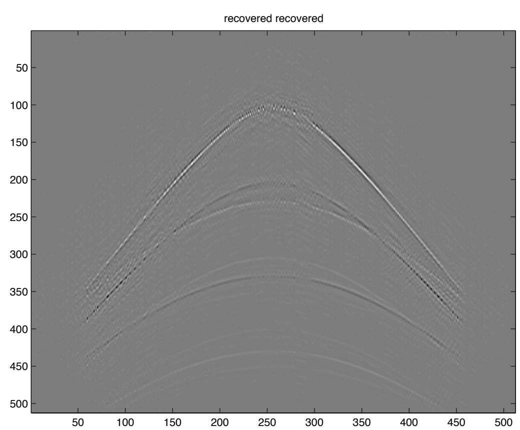
SLIM Seismic Laboratory for Imaging and Modeling

80 % missing data



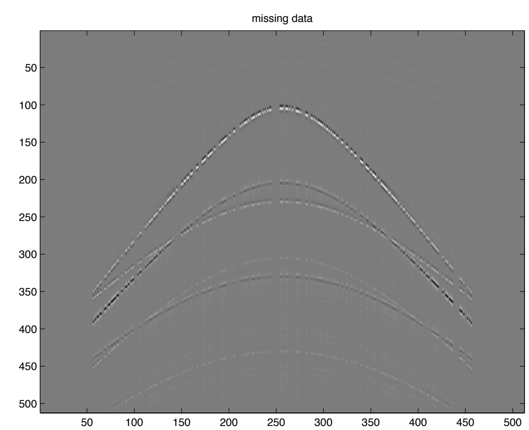
SLIM Seismic Laboratory for Imaging and Modeling

recovered data



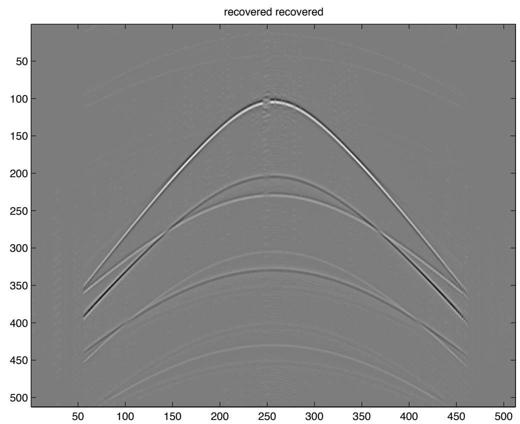
SLIM Seismic Laboratory for Imaging and Modeling

50 % missing data



SLIM Seismic Laboratory for Imaging and Modeling

recovered data



SLIM
Seismic Laboratory for Imaging and Modeling