A dual formulation for time-domain wavefield reconstruction inversion

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PDE-constrained optimization

$$\inf_{m,u} \frac{1}{2}||d - R\,u||^2 \quad \text{s.t.} \quad A(m)u = S^* f$$

d	:	data	$A(m)$	:	wave equation
f	:	source	R	:	receiver interp.
m	:	model	S	:	source interp.
u	:	wavefield			

PDE-constrained optimization (all-at-once full-space)

$$\mathcal{L}(m,u,v) = \frac{1}{2}||d - R\,u||^2 + \langle v, S^* f - A(m)\,u \rangle$$

[Haber, E., and Ascher, U. M., 2001; Biros, G., and Ghattas, O., 2005; Grote et al, 2011]

Pros:
- ✓ **simultaneous** update of model, wavefield, multiplier
- ✓ **cheap** evaluation and gradient computation
- ✓ sparse Hessian

Cons:
- ✗ **storage of fields and multipliers**
  (for each source and time/frequency sample)
- ✗ **updates** computationally demanding

PDE-constrained optimization (reduced space)

$$J(m) = \frac{1}{2}||d - F(m)\, S^* f||^2 \qquad F(m) = RA(m)^{-1}$$

forward map

[Tarantola, A., ‘84; Haber, E., et al, 2000; Epanomeritakis, I., et al, 2008]

Pros:
- ✓ low(er) **storage** requirements

Cons:
- ✗ highly **non-convex** (needs good starting model)
- ✗ requires exact **PDE solutions**
- ✗ dense Hessian

PDE-constrained optimization (penalty method)

$$J_\lambda(m,u) = \frac{1}{2}||d - R\,u||^2 + \frac{\lambda^2}{2}\underbrace{||S^* f - A(m)\,u||^2}_{\text{PDE penalty}}$$

[van den Berg, P. M., and Kleinman, R. E., 1997; van Leeuwen, T. and Herrmann, F. J., 2013]

PDE-constrained optimization (wavefield reconstruction inversion)

$$\bar{J}_\lambda(m) = \frac{1}{2}||d - R\,\bar{u}||^2 + \frac{\lambda^2}{2}||S^* f - A(m)\,\bar{u}||^2$$

$$\begin{bmatrix} R \\ \lambda A(m) \end{bmatrix} \bar{u} = \begin{bmatrix} d \\ \lambda S^* f \end{bmatrix}$$

augmented wave equation

[van Leeuwen, T. and Herrmann, F. J., 2013]

Pros:
- ✓ no simultaneous field **storage** for all sources
- ✓ sparse Gauss-Newton Hessian

Cons:
- ✗ continuation strategy on λ
- ✗ need **augmented PDE solution:**
  - **frequency** domain: proper preconditioning?
  - **time** domain: no explicit time-marching scheme

Attainable model extension

Question: feasible model extension?

- convenient handling of slack variables
- feasible computation of augmented PDE (e.g., can be solved by time-marching)

Wavefield Reconstruction Inversion (time domain approximation)

$$J_\lambda(m) = \frac{1}{2}||d - F(m)\,\bar{q}||^2 + \frac{\lambda^2}{2}||S^* f - \bar{q}||^2$$

$$\begin{bmatrix} F(m) \\ \lambda I \end{bmatrix} \bar{q} = \begin{bmatrix} d \\ \lambda S^* f \end{bmatrix}$$

augmented wave equation

[Wang et al, 2016]

$\bar{q}$ ~ generalized source
(see also [Huang et al, 2018])

First order Taylor expansion wrt $1/\lambda^2$:

$$\bar{q} \approx S^* f + \frac{1}{\lambda^2} F(m)^* (d - F(m) S^* f), \quad \lambda \to +\infty$$

Denoising reformulation of WRI

$$\inf_{m,u} \underbrace{\frac{1}{2}||S^* f - A(m)u||^2}_{\text{PDE misfit}} \quad \text{s.t.} \quad \underbrace{||d - R\,u|| \leq \varepsilon}_{\text{data constraint}}$$

[Wang, R., and Herrmann, F. J., 2017]

Dual formulation of WRI - Lagrangian

$$\inf_{m,u} \frac{1}{2} \underbrace{||S^* f - A(m)u||^2}_{\text{PDE misfit}} \quad \text{s.t.} \quad \underbrace{||d - R\,u|| \leq \varepsilon}_{\text{data constraint}}$$

Saddle-point problem

$$\inf_{m,u} \sup_{y} \mathcal{L}(m,u,y)$$

$$\mathcal{L}(m,u,y) = \frac{1}{2}||S^* f - A(m)u||^2 + \langle y, d - R\,u \rangle - \varepsilon ||y||$$

Dual formulation of WRI – Augmented wave equation

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Solving for u ...

$$A(m)\bar{u} = S^* f + F(m)^* y$$

augmented wave equation

Data interfere with the system!

Saddle-point problem

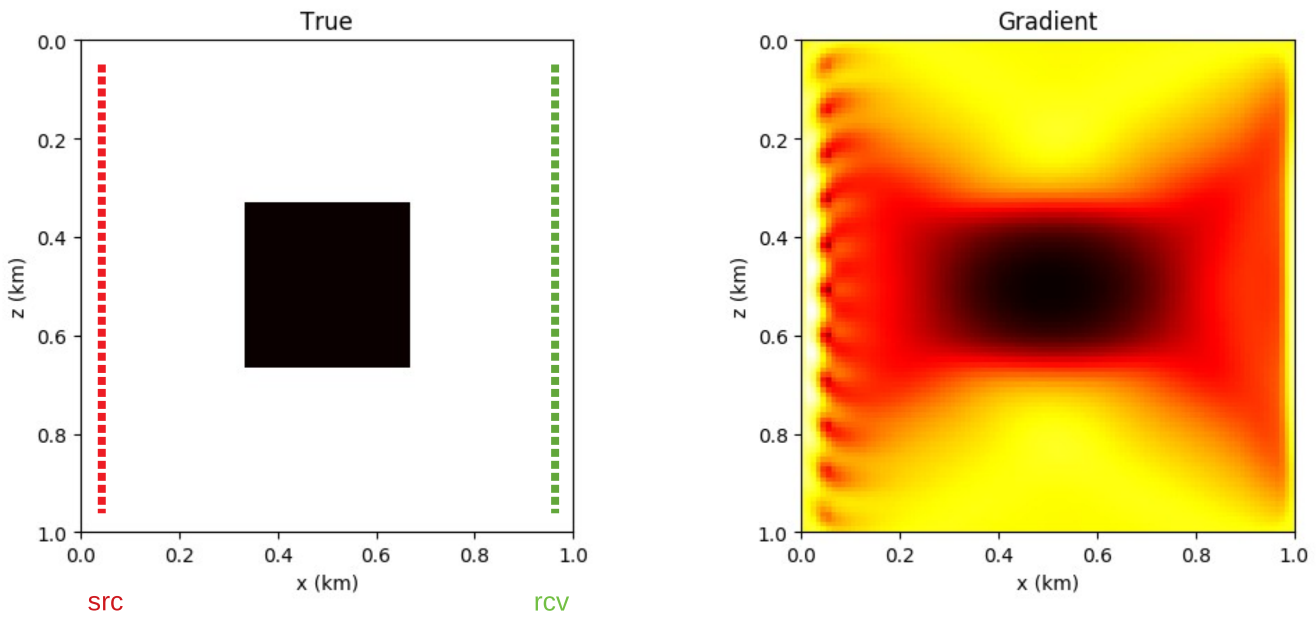
$$\inf_{m,u} \sup_{y} \mathcal{L}(m,u,y) \geq \inf_{m} \sup_{y} \inf_{u} \mathcal{L}(m,u,y)$$

$$\mathcal{L}(m,u,y) = \frac{1}{2}||S^* f - A(m)u||^2 + \langle y, d - Ru \rangle - \varepsilon ||y||$$

Dual formulation of WRI – Augmented wave equation

Transmission data problem:

- 1 km x 1 km model
- background velocity: 2000 m/s, inclusion 2100 m/s
- 11 sources (10 Hz peak), 51 receivers
- y = background data residual



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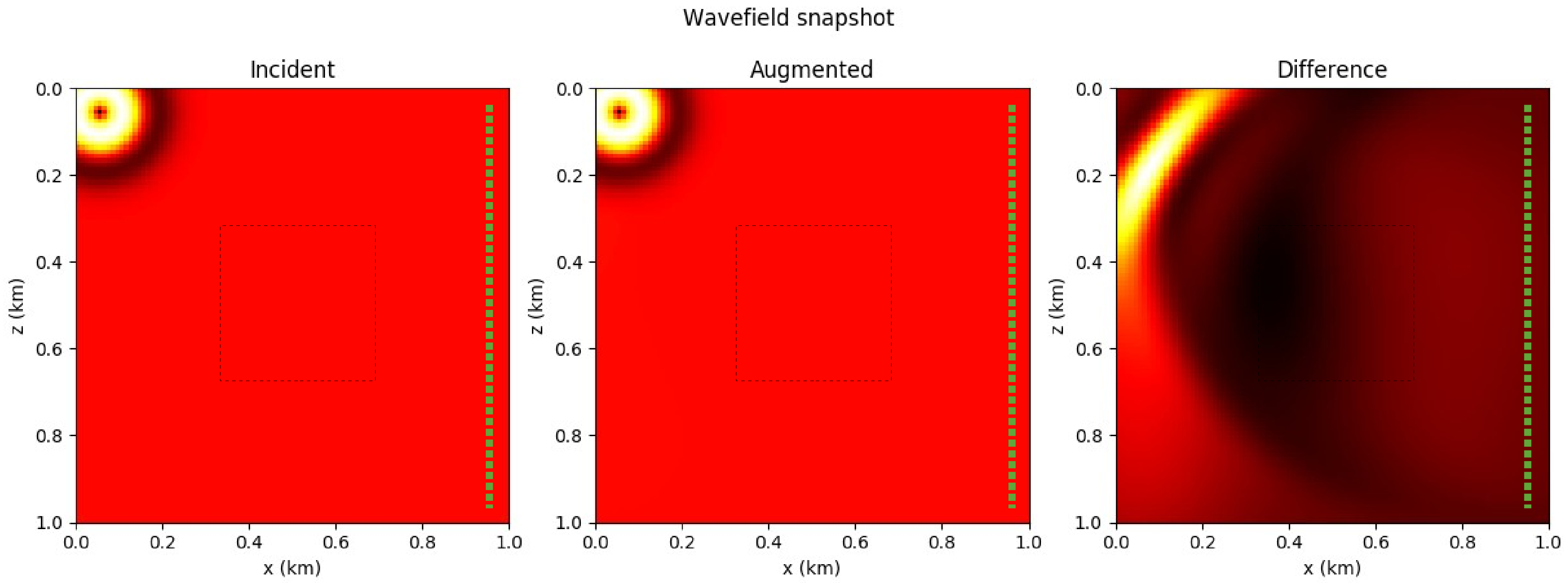

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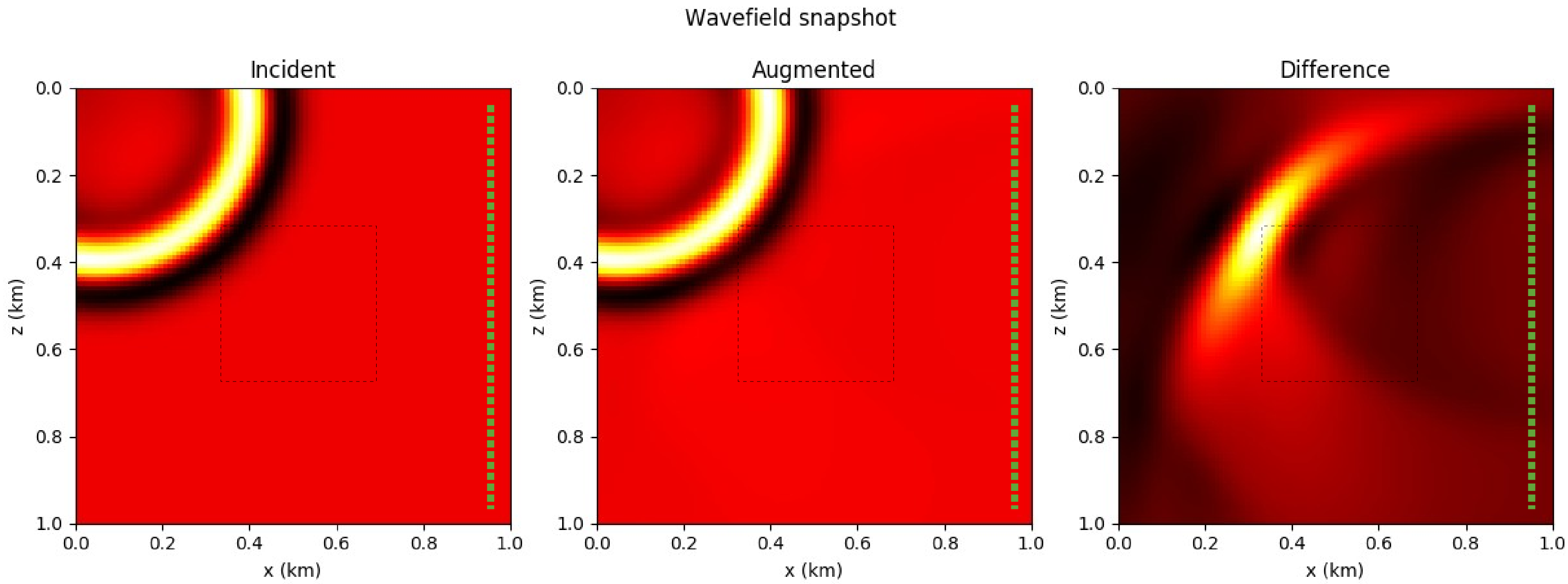

Dual formulation of WRI – Augmented wave equation

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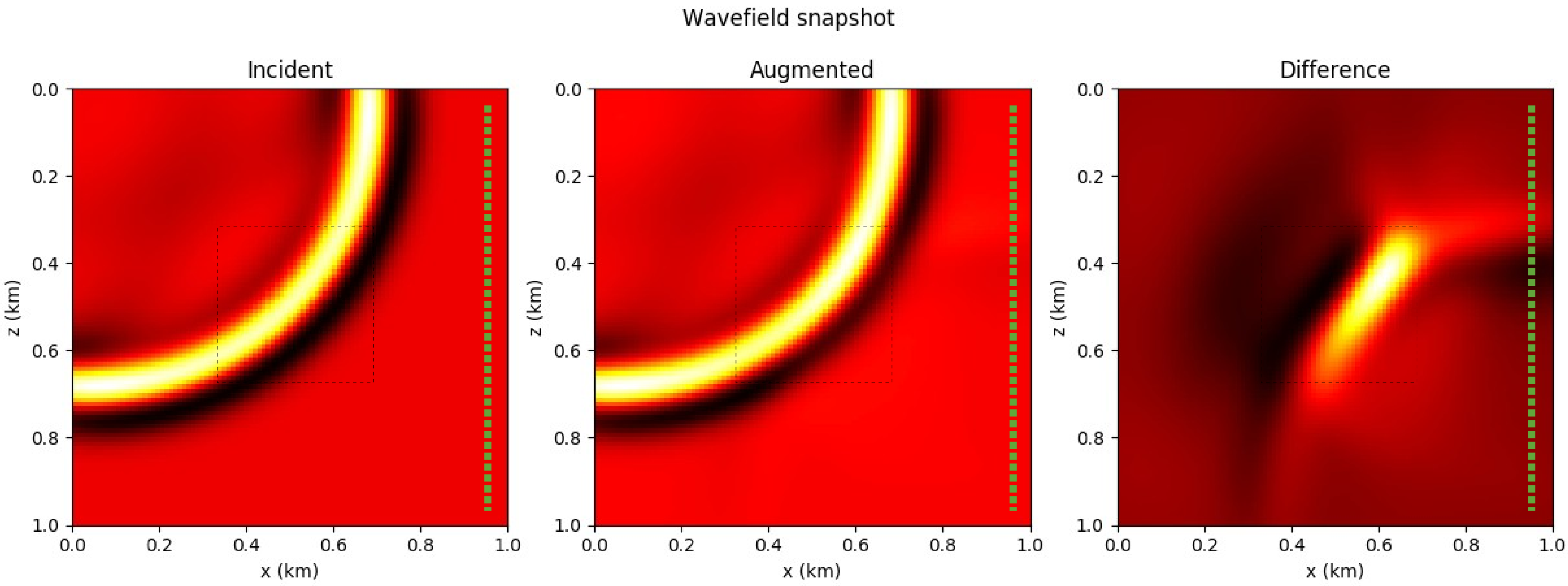


Dual formulation of WRI – Objective and gradients

Dual saddle-point formulation

$$\inf_m \sup_y \bar{\mathcal{L}}(m,y)$$

$$\bar{\mathcal{L}}(m,y) = -\frac{1}{2}||F(m)^* y||^2 + \langle y, d - F(m)S^* f\rangle - \varepsilon||y||$$

Gradients

$$\nabla_m \bar{\mathcal{L}} = -\mathrm{Jac}(m, S^* f + F(m)^* y)^* y$$

~ conventional FWI gradient

$$\nabla_y \bar{\mathcal{L}} = d - F(m)(S^* f + F(m)^* y)$$

generalized-source data residual

$$\mathrm{Jac}(m,q) = \frac{\mathrm{d}}{\mathrm{d}m}(\cdot \mapsto F(\cdot)\, q)\Big|_m$$

Dual formulation of WRI (recap)

Dual saddle-point formulation

$$\inf_m \sup_y \bar{\mathcal{L}}(m,y)$$

$$\bar{\mathcal{L}}(m,y) = -\frac{1}{2}||F(m)^* y||^2 + \langle y, d - F(m)S^* f\rangle - \varepsilon||y||$$

Obtained **model extension** along **data space**:

$$\bar{\mathcal{L}} : M \times D \to \mathbb{R}$$

- ✓ amenable to **time-domain** methods
- ✓ extra variable **storage is affordable**
- ✓ **no continuation strategy** needed for relaxation parameter

- ✗ extra **time complexity** (2x PDE solutions, wrt FWI)
- ✗ dense Hessian
- ✗ need to figure out best **optimization strategy** (work in progress ...)

Dual formulation of WRI – Dual variable scaling

Dual saddle-point formulation

$$\inf_m \sup_y \bar{\mathcal{L}}(m,y)$$

$$\bar{\mathcal{L}}(m,y) = -\frac{1}{2}||F(m)^* y||^2 + \langle y, d - F(m)S^* f\rangle - \varepsilon||y||$$

Preconditioning/**Scaling issue** in Lagrangian/augmented wave equation:

$$A(m)\bar{u} = S^* f + \boxed{\alpha} F(m)^* y$$

Adaptive strategy?

Dual formulation of WRI – Dual variable scaling

Dual saddle-point formulation

$$\inf_m \sup_y \bar{\mathcal{L}}(m, y)$$

$$\bar{\mathcal{L}}(m, y) = -\frac{1}{2}||F(m)^* y||^2 + \langle y, \underbrace{d - F(m)S^* f}_{\text{r: data residual}} \rangle - \varepsilon ||y||$$

Solving for the scaling parameter

$$\bar{\bar{\mathcal{L}}}(m, y, \alpha) := \bar{\mathcal{L}}(m, \alpha y)$$

$$\bar{\bar{\bar{\mathcal{L}}}}(m, y) := \sup_\alpha \bar{\bar{\mathcal{L}}}(m, y, \alpha) = \bar{\bar{\mathcal{L}}}(m, y, \alpha(m, y))$$

$$\alpha(m, y) = \begin{cases} \dfrac{\langle y, r \rangle - \varepsilon \mathrm{sign}(\langle y, r \rangle)||y||}{||F(m)^* y||^2}, & |\langle y, r \rangle| \geq \varepsilon ||y|| \\ 0, & \text{otherwise} \end{cases}$$

Dual formulation of WRI (scaled)

Dual saddle-point formulation (scaled)

$$\inf_m \sup_y \bar{\bar{\mathcal{L}}}(m,y)$$

$$\bar{\bar{\mathcal{L}}}(m,y) = \begin{cases} \frac{1}{2}(|\langle \hat{y}, r\rangle| - \varepsilon||\hat{y}||)^2, & |\langle \hat{y}, r\rangle| \geq \varepsilon||\hat{y}|| \\ 0, & \text{otherwise} \end{cases} \quad \left(\hat{y} = \frac{y}{||F(m)^* y||}\right)$$

Spurious minima

Diving-wave example:

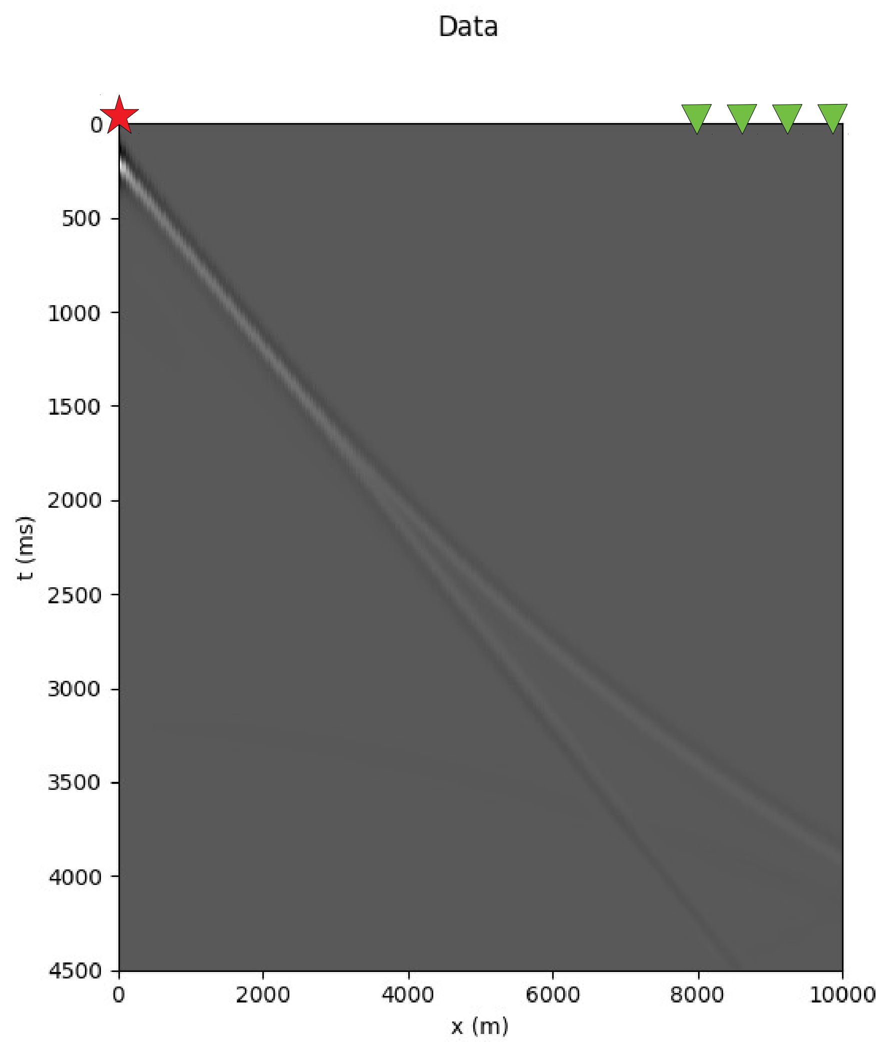
- 5 km x 10 km model
- linear velocity, varying slope:

$$v(x,z) = v_0 + \beta z$$

$$v_0 = 2\,\mathrm{km/s}, \quad 0.5 \leq \beta \leq 1$$

- single source (5 Hz peak), 21 receivers

➢ Plot misfit landscape as a function of slope



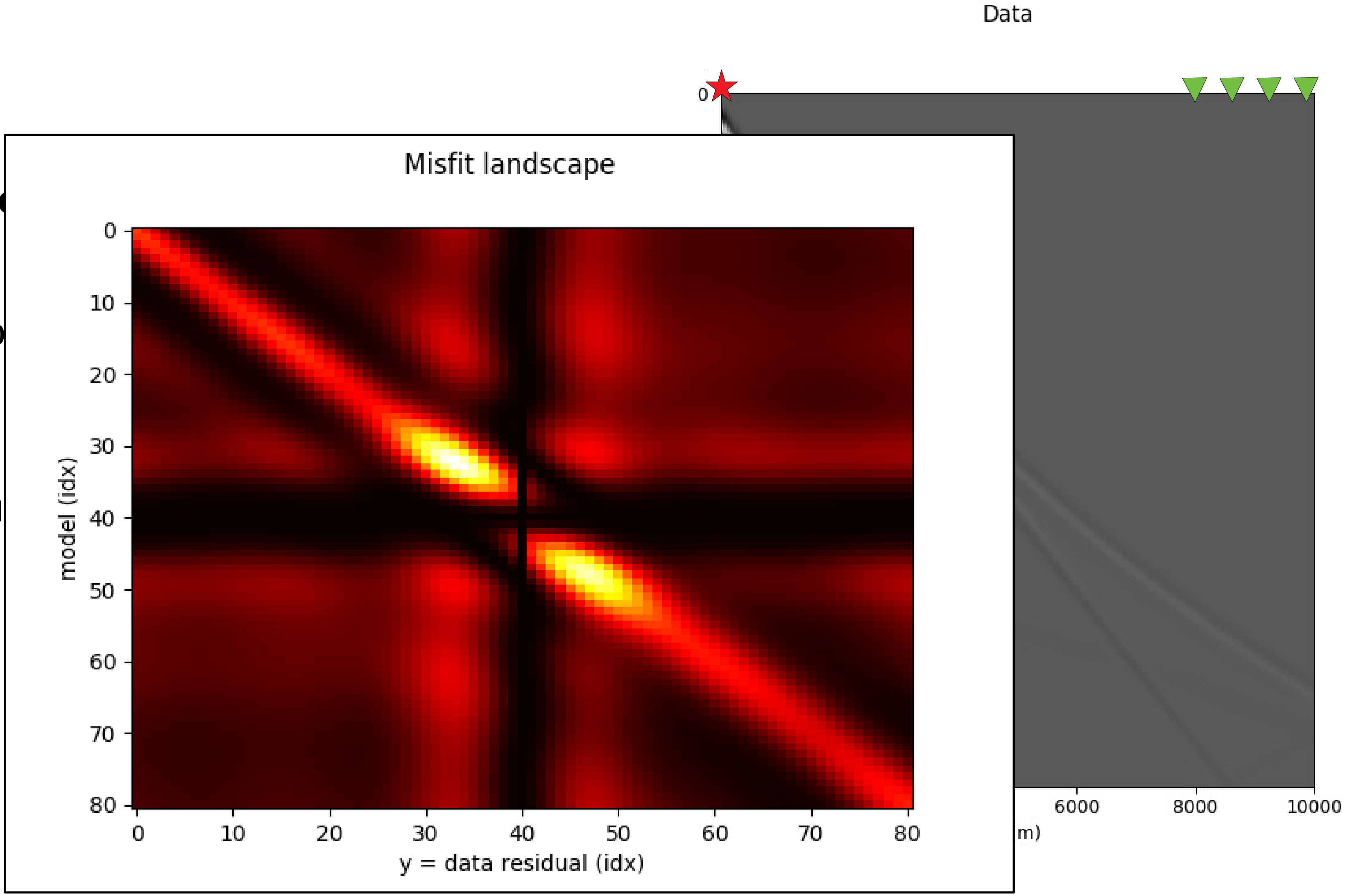
Spurious minima

Diving-wav

- 5 km x 10
- linear velo

- single sou

➢ Plot misfit



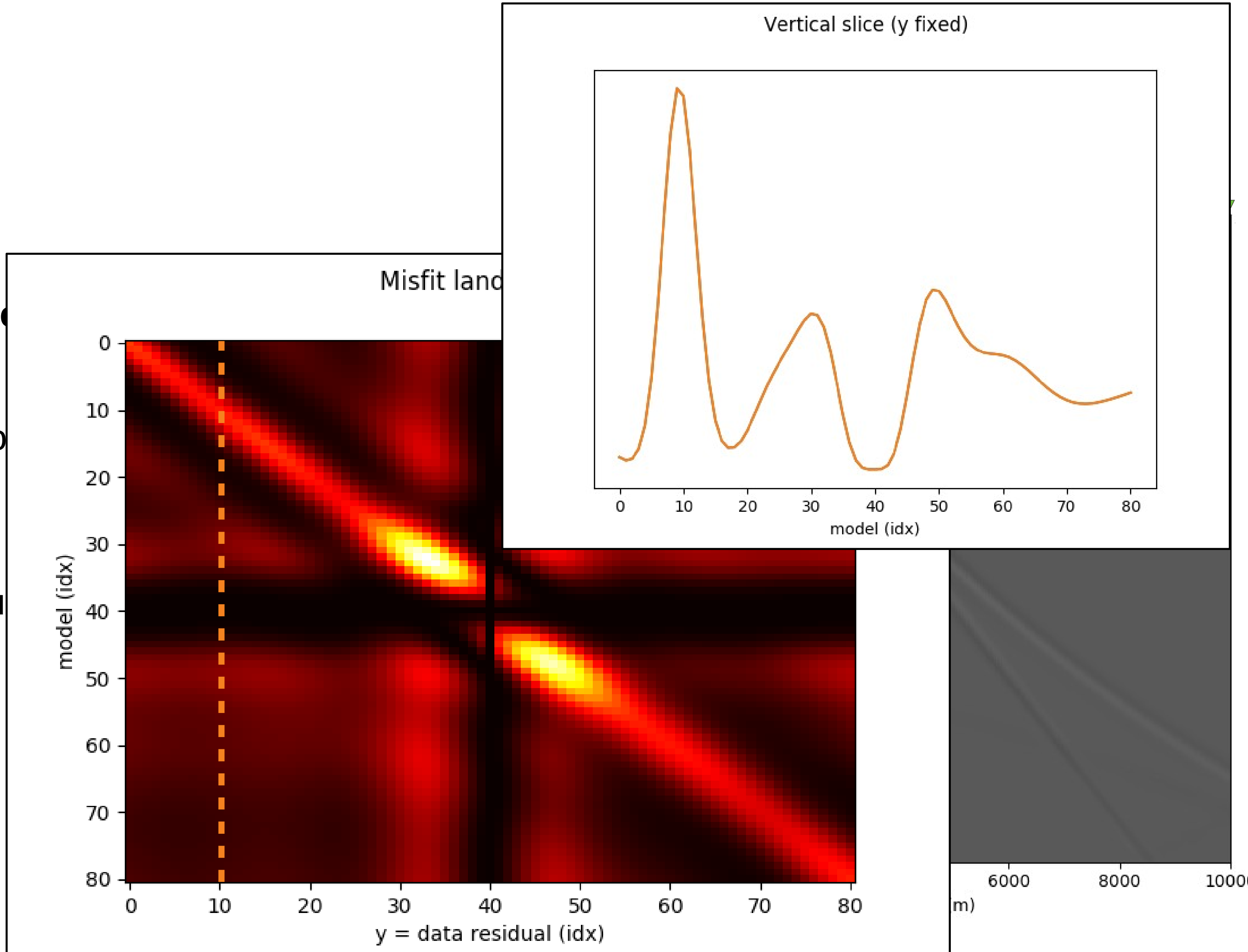
Spurious minima

Diving-wav

- 5 km x 10
- linear velo

- single sou

➢ Plot misfit



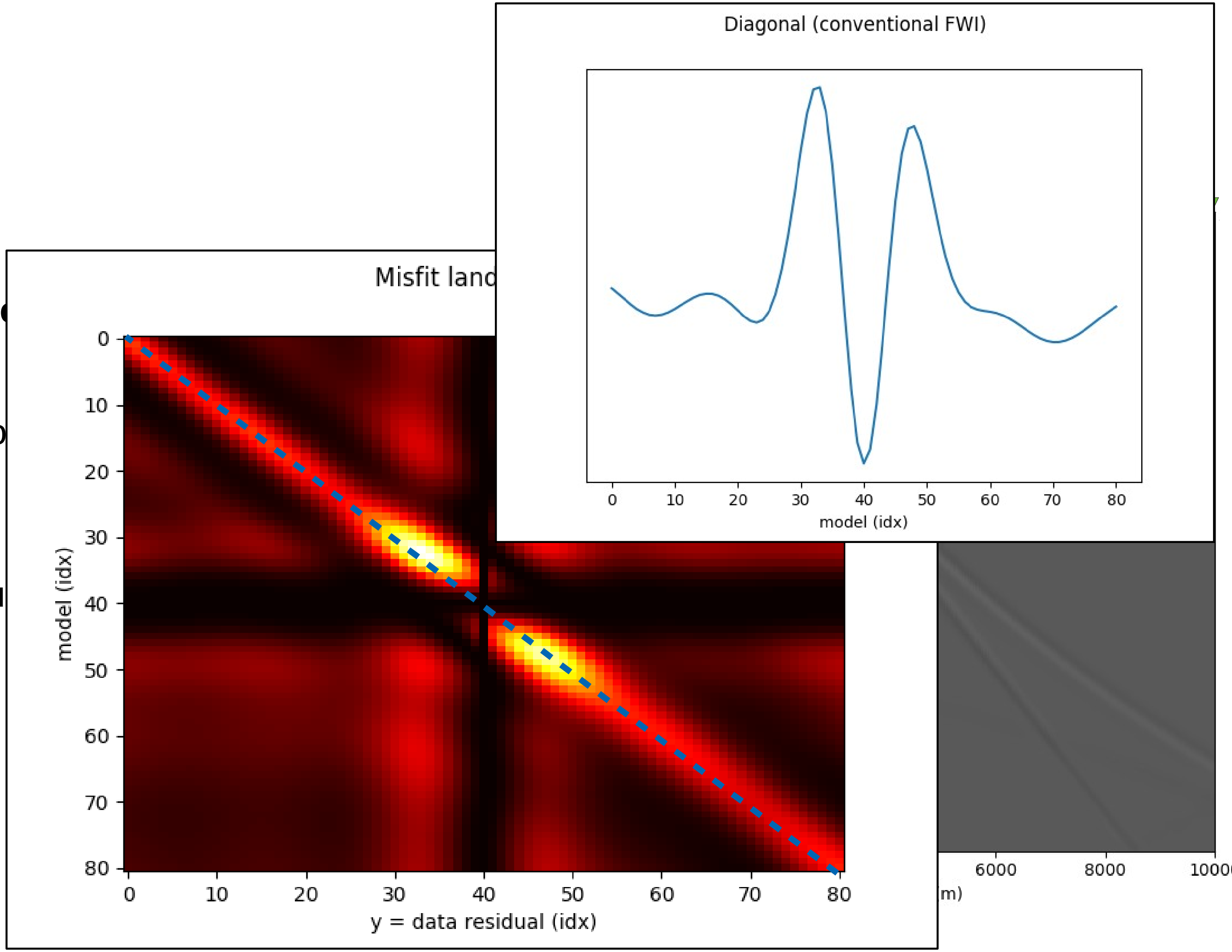
Spurious minima

Diving-wav

- 5 km x 10
- linear velo

- single sou

➢ Plot misfit



Reconstruction algorithm (sketch)

Alternating gradient descent

```
Given  m_0, y_0 = d - F(m_0)S^* f

for n = 1:Niter

    compute gradient wrt m:     ∇_{m_n} L̄̄

    update m (minimize):        m_n ← m_n - β∇_{m_n} L̄̄

    compute gradient wrt y:     ∇_{y_n} L̄̄

    update y (maximize):        y_n ← y_n + γ∇_{y_n} L̄̄
```

Given $m_0, y_0 = d - F(m_0)S^* f$

for n = 1:Niter

- compute gradient wrt m: $\nabla_{m_n} \bar{\bar{\mathcal{L}}}$
- update m (minimize): $m_n \leftarrow m_n - \beta \nabla_{m_n} \bar{\bar{\mathcal{L}}}$
- compute gradient wrt y: $\nabla_{y_n} \bar{\bar{\mathcal{L}}}$
- update y (maximize): $y_n \leftarrow y_n + \gamma \nabla_{y_n} \bar{\bar{\mathcal{L}}}$

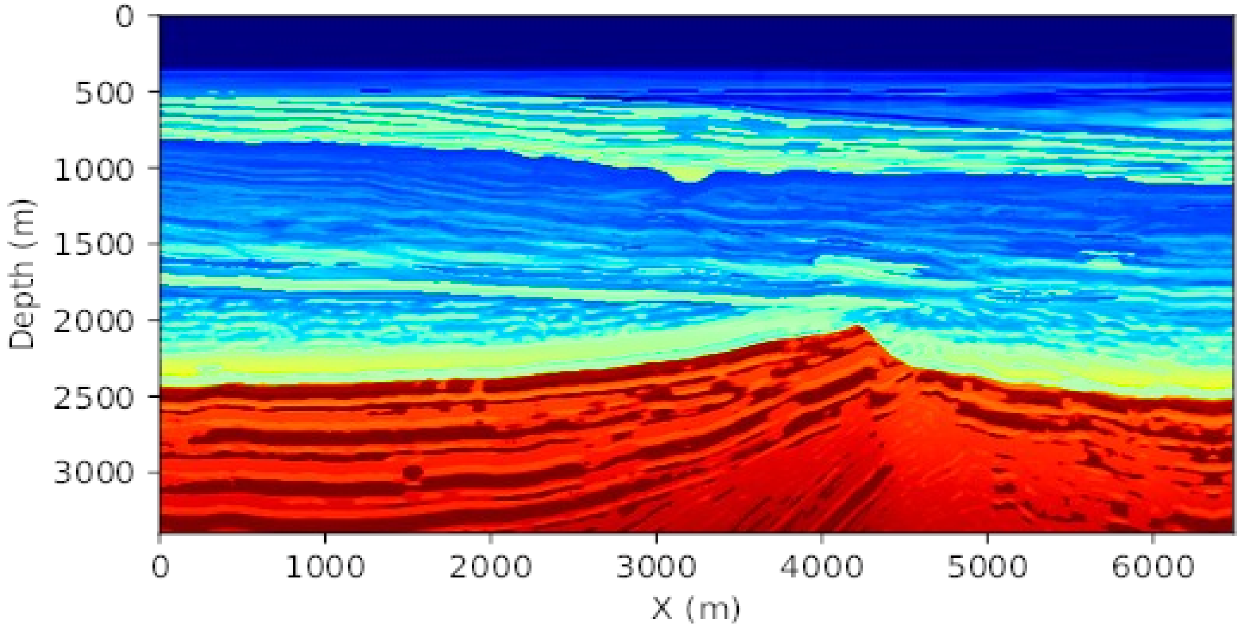
BG Compass (reflections/diving waves) **[preliminary results]**

BG compass:

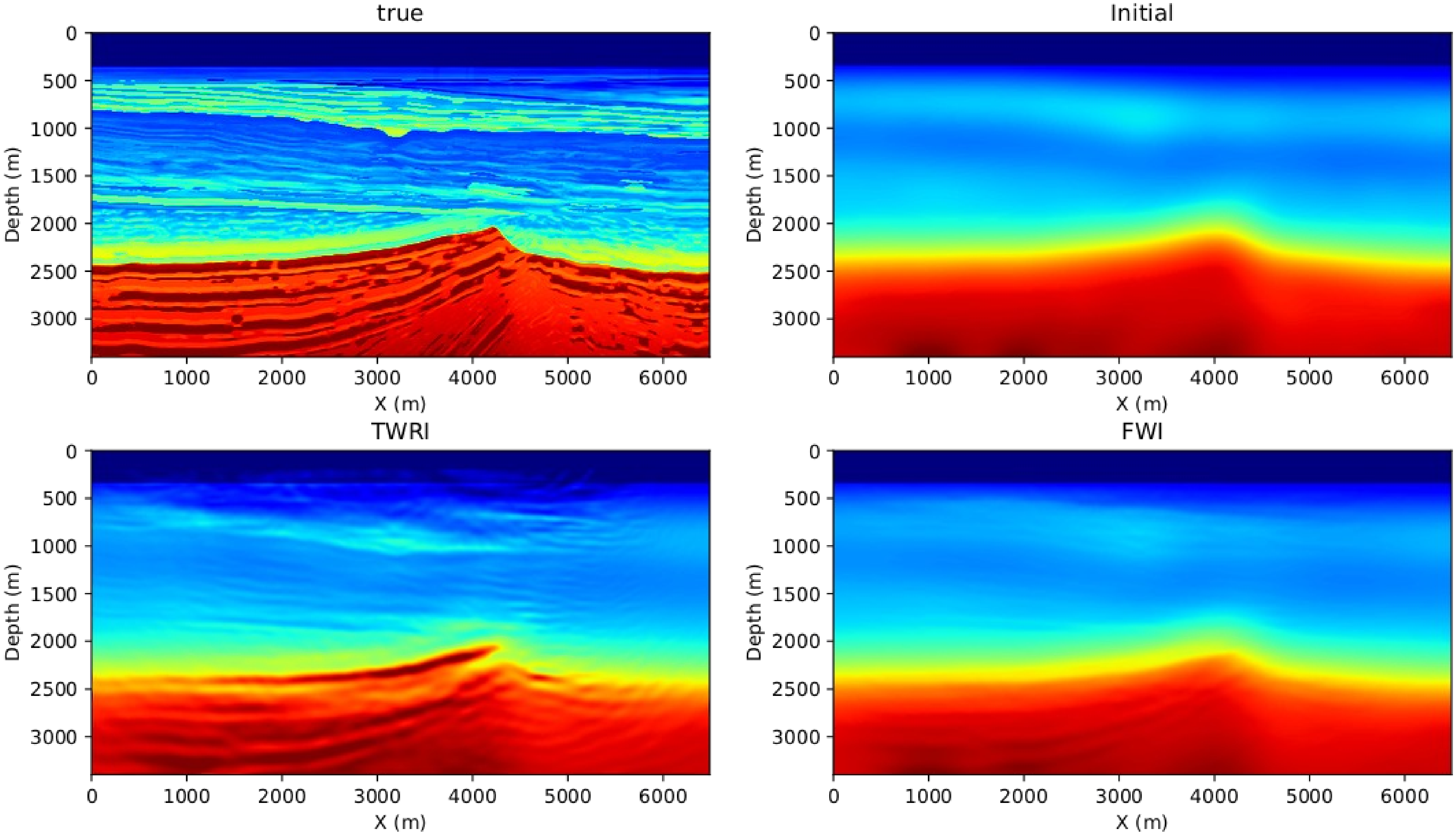
- ~ 3 km x 6 km model
- 25 sources (10 Hz peak), 251 receivers
- *challenging* for conventional FWI!

Optimization setup:

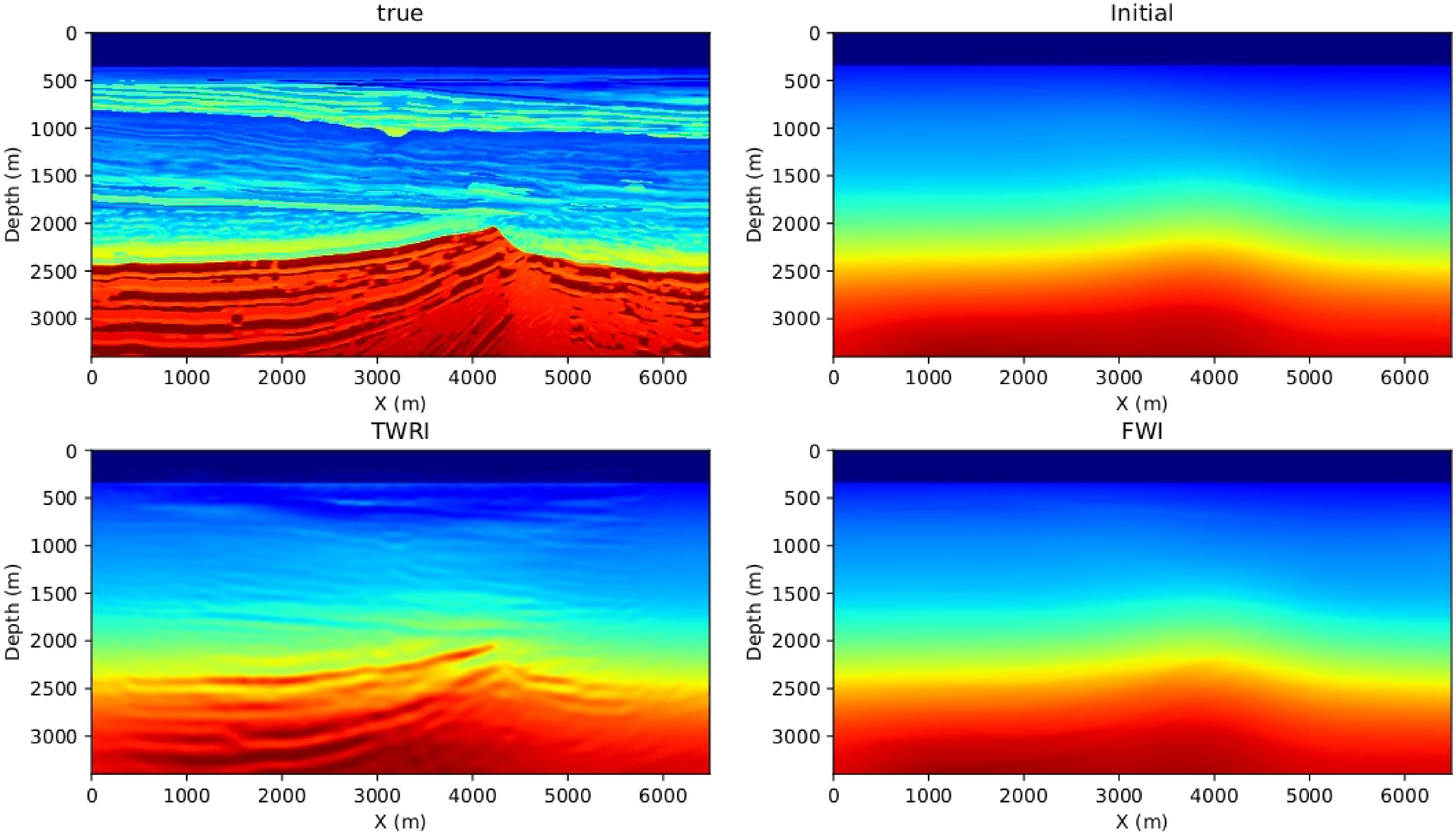
- *gradient descent* (fixed step length)
- just *10 iterations*



BG Compass **[preliminary results]**



BG Compass (worse background) **[preliminary results]**



Conclusions and road ahead

Presented a **reconstruction algorithm** potentially apt to large **3D** problems:

- based on **model extension** along data dimension: **robust to spurious minima**
- **storage** of slack variables is **affordable** (unlike full-space methods)
- computationally advantageous:
  PDEs can be solved in time domain with **explicit schemes** (unlike classical WRI)

To do:

- study of **optimization strategy**
- implement **constraints**

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