Wavefield Reconstruction Inversion – Reaping the Benefits from Extending the Search Space

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Wavefield Reconstruction Inversion – Reaping the Benefits from Extending the Search Space & **Imposing Asymmetric Convex Constraints**

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John “Ernie” Esser (May 19, 1980 – March 8, 2015)

Motivation

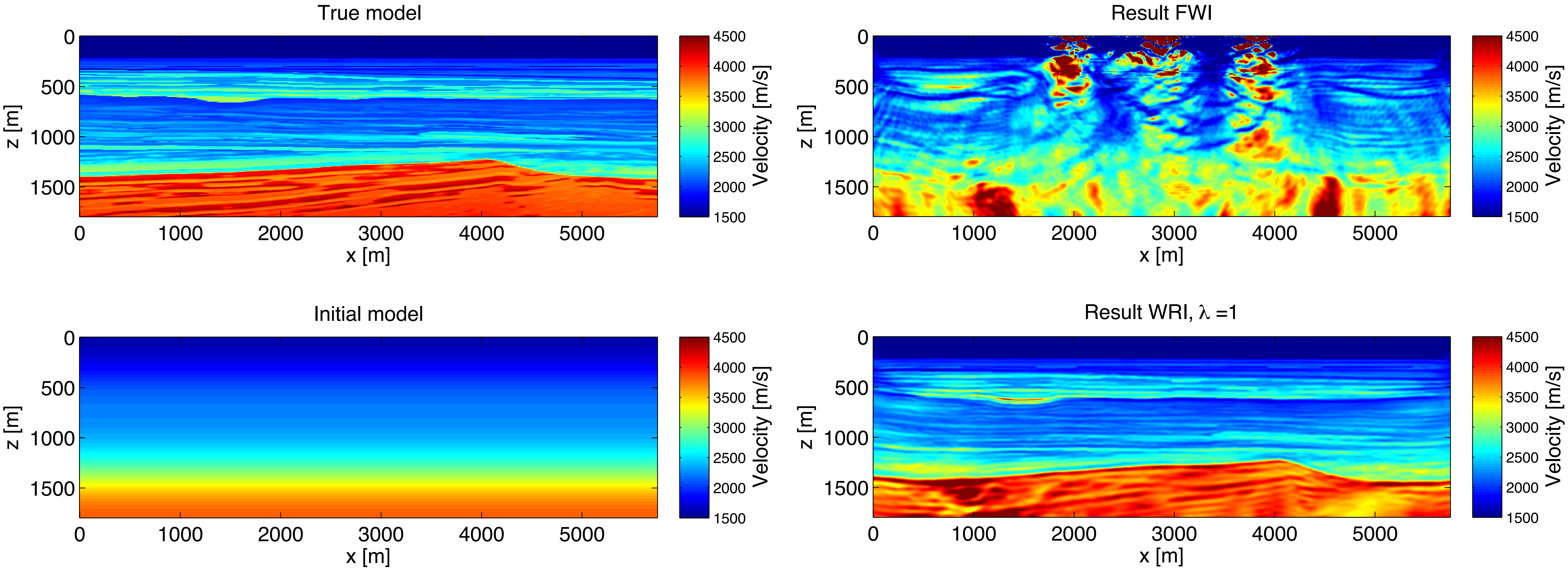
Wave-equation based inversions suffer from local minima (cycle skips)

- for poor starting models
- especially detrimental for high-contrast & high-velocity unconformities (salt & basalt)

Borrow ideas from

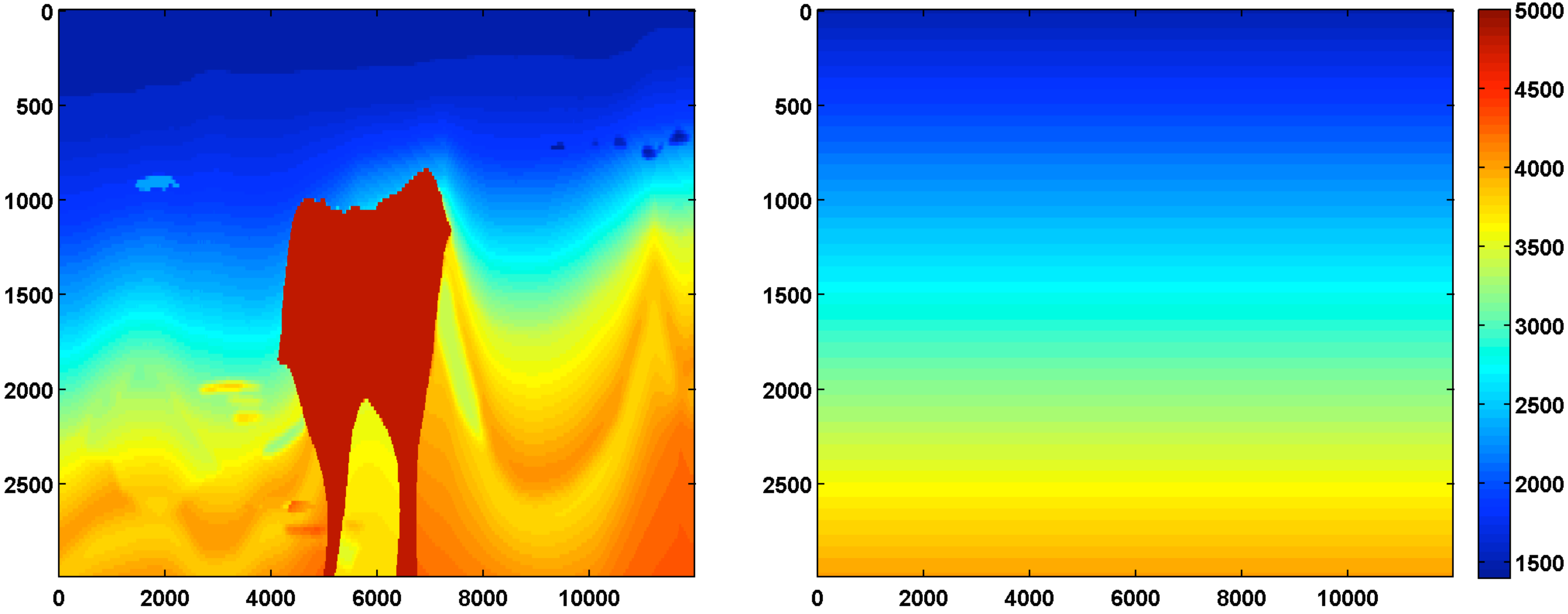
- wave-equation based inversions w/ extensions
- edge-preserving regularization in image processing & compressive sensing
- hinge-loss functions in machine learning
- continuation strategies from (convex) constrained optimization

Wavefield Reconstruction Inversion (WRI)
– poor starting model



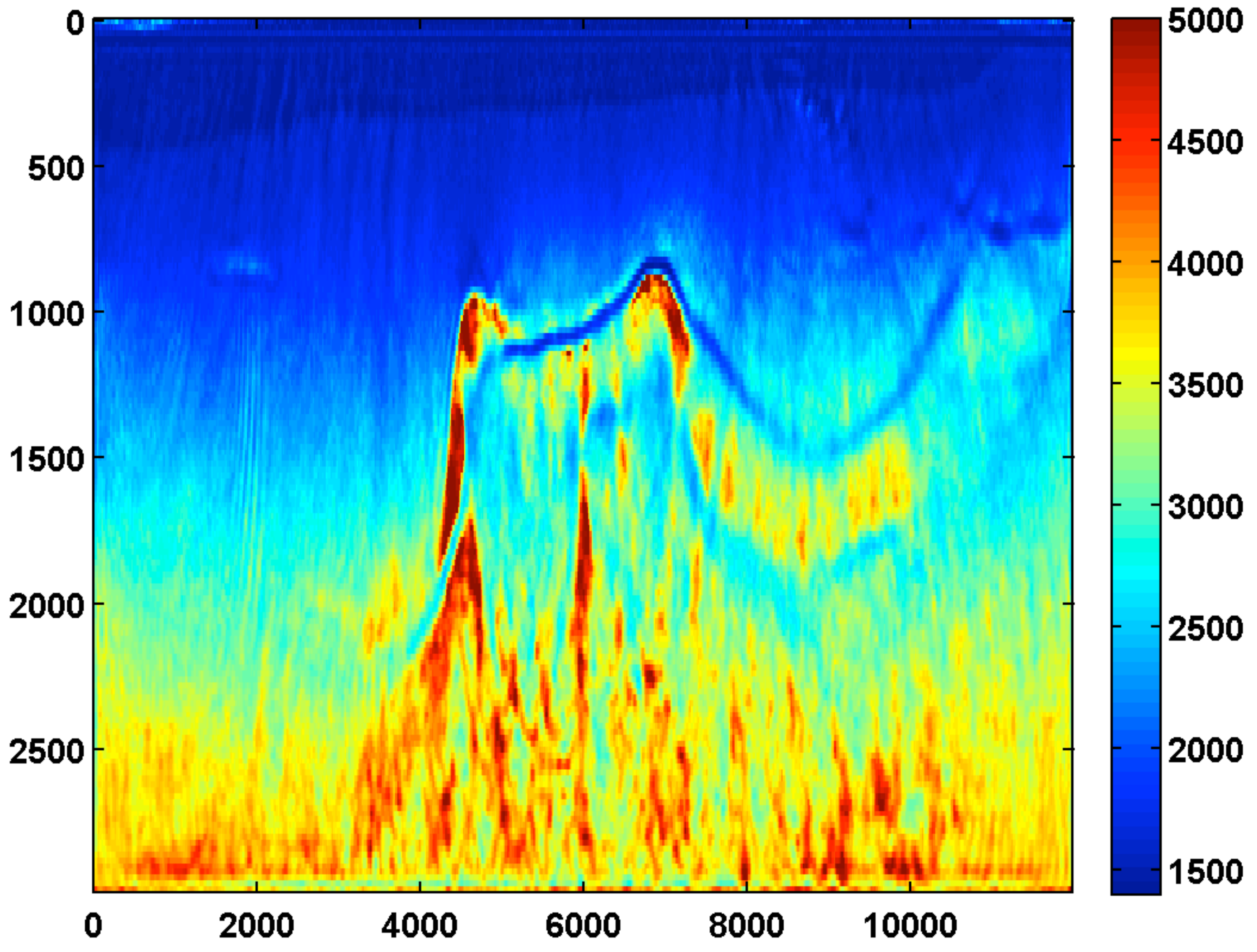
Example from **[Peters et al. 2013]**

Waveform inversion – poor starting model

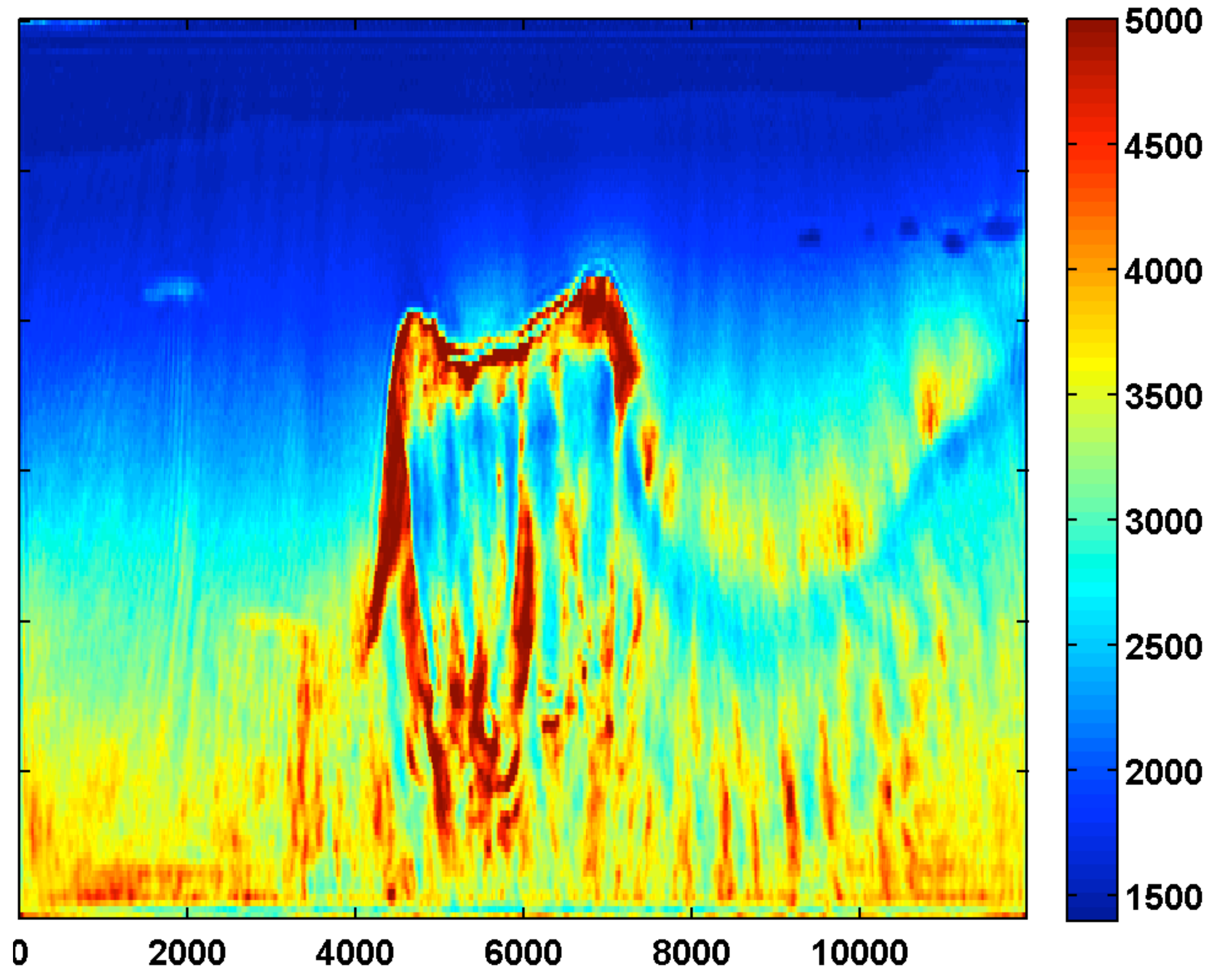


WRI results w/o TV

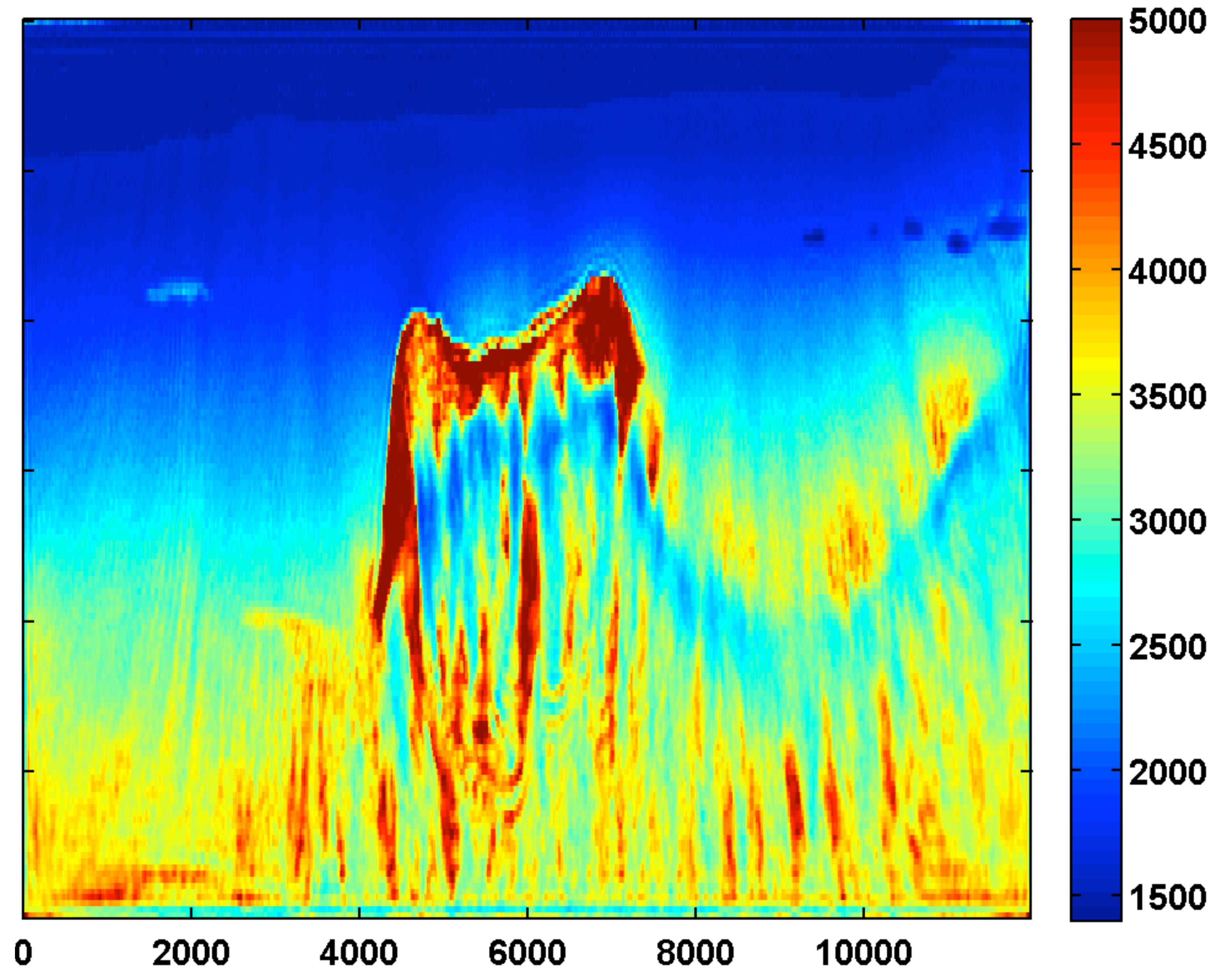
after one cycle through the frequencies



after two cycles through the frequencies

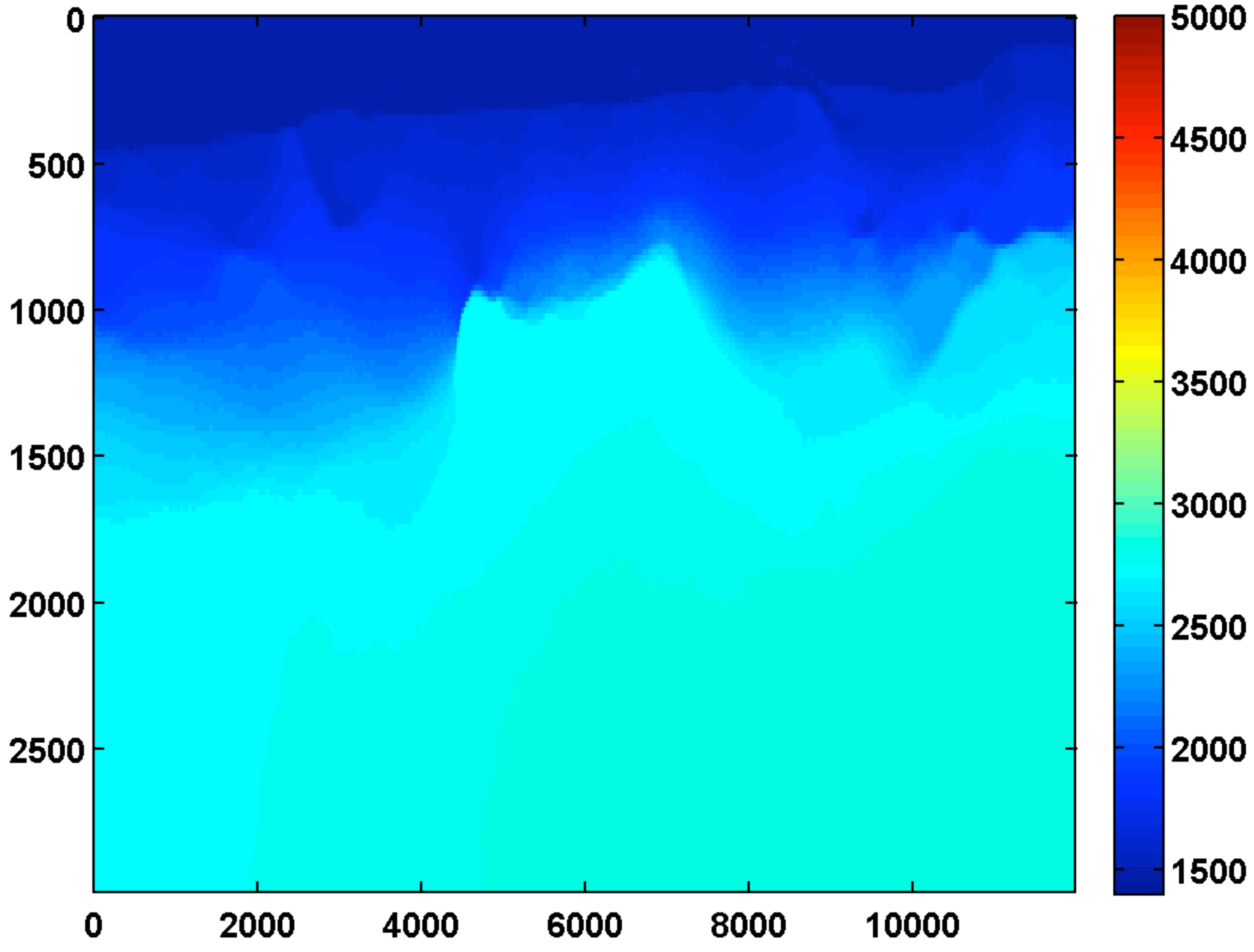


after three cycles through the frequencies

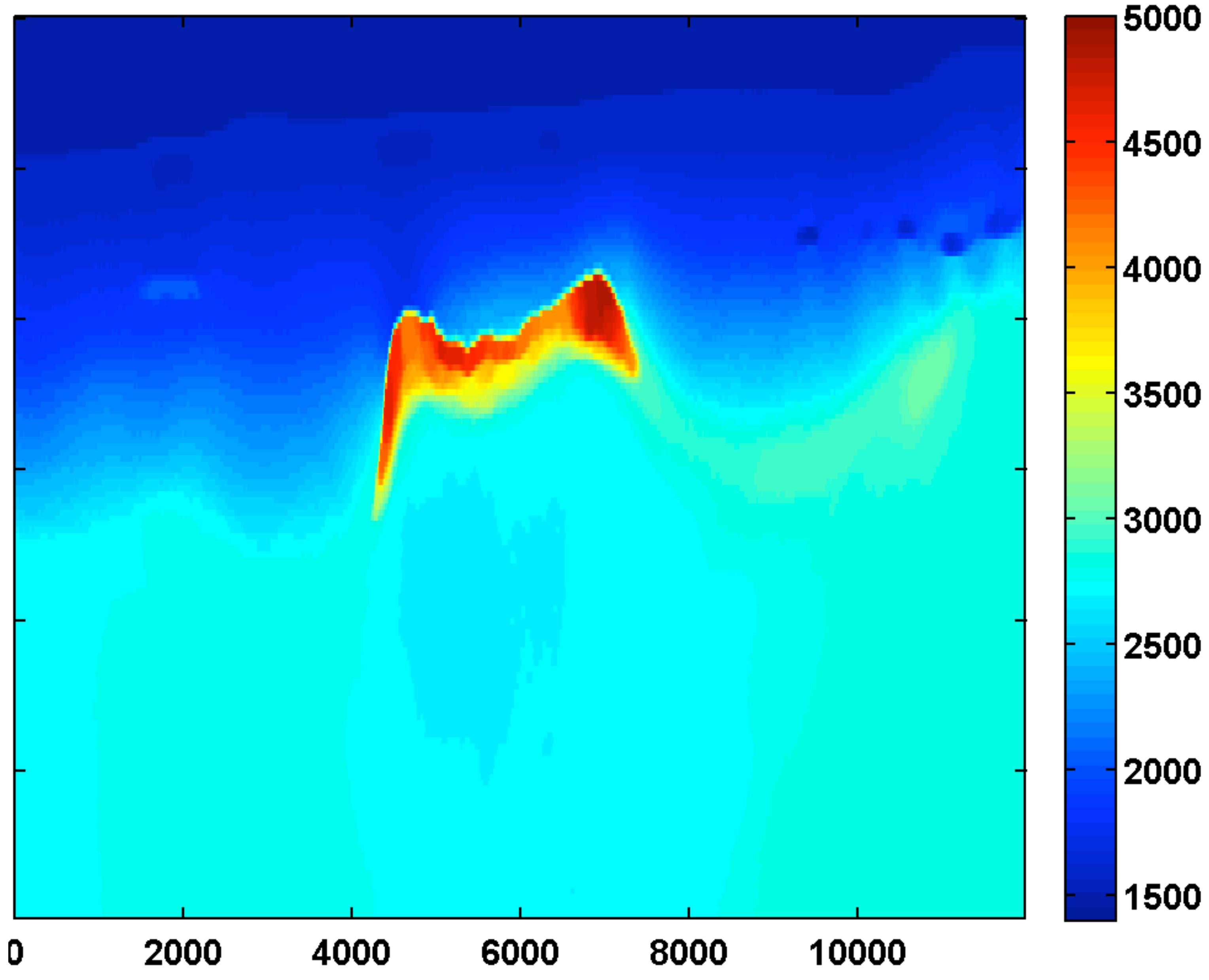


Results w/ TV

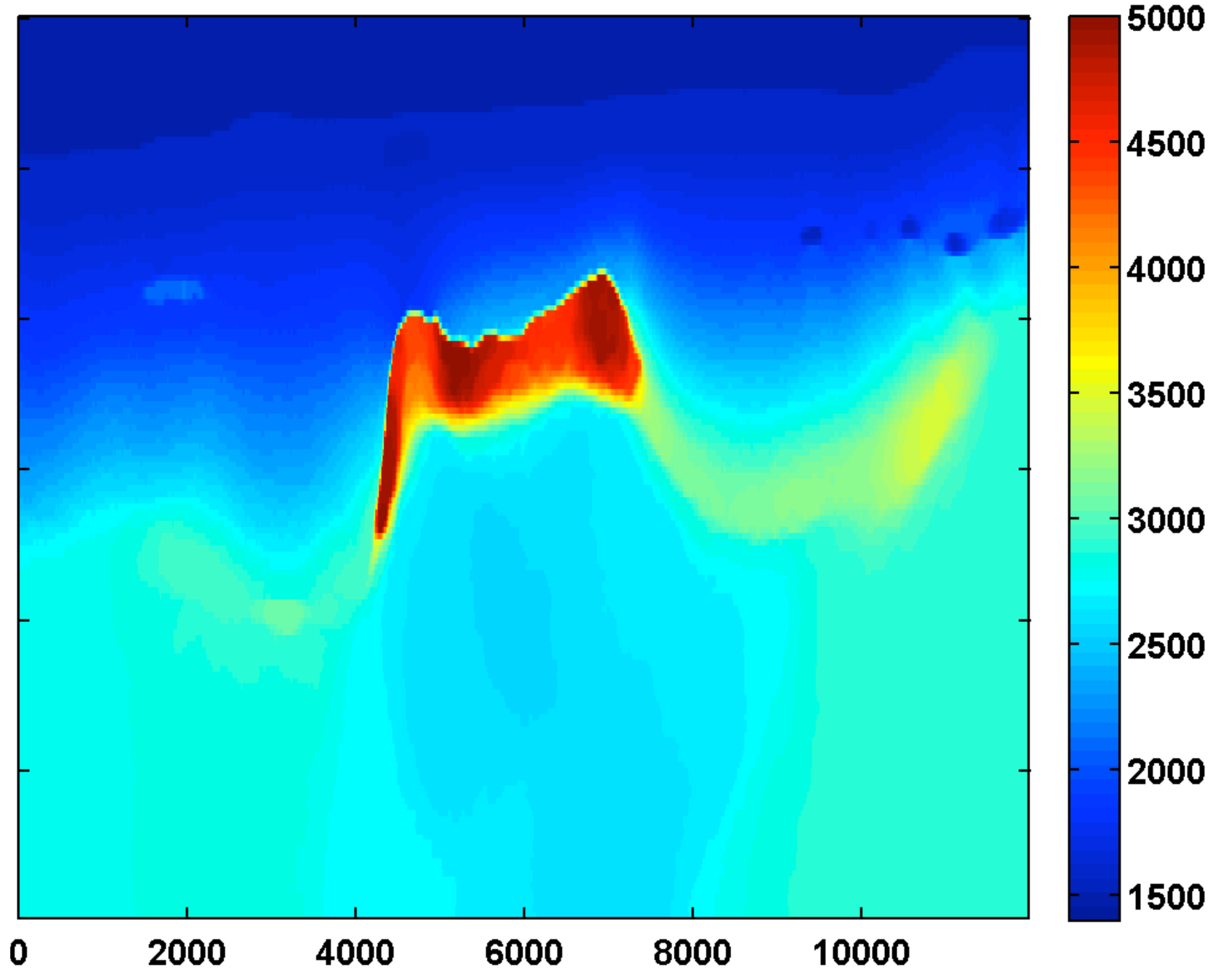
after one cycle through the frequencies



after two cycles through the frequencies



after three cycles through the frequencies



Strategy

Extend the search space

- “less” nonlinear (bi-convex)
- ensures data fit & avoids cycle skips

“Squeeze” the extension by

- enforcing the wave equation to compute model updates
- imposing *asymmetric* convex constraints that encode “rudimentary” properties of the geology
- relaxing the convex constraints while stressing the physics

[Heinkenschloss, ’98 , Haber, ’00]

Wavefield-reconstruction Inversion – WRI

Replace PDE-constrained formulation for FWI:

simulated data — simulated wavefield

$$\min_{\mathbf{m},\mathbf{u}} \sum_{sv} \frac{1}{2} \| P\mathbf{u}_{sv} - \mathbf{d}_{sv} \|^2 \quad \text{such that} \quad A_v(\mathbf{m})\mathbf{u}_{sv} = \mathbf{q}_{sv}$$

observed data — Helmholtz equation — source

[Bertsekas, '96; Wright, '00; van Leeuwen & FJH, '13, '15]

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WRI

by the penalty formulation

$$\min_{\mathbf{m},\mathbf{u}} \sum_{sv} \frac{1}{2}\|P\mathbf{u}_{sv} - \mathbf{d}_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(\mathbf{m})\mathbf{u}_{sv} - \mathbf{q}_{sv}\|^2$$

and solve at the n$^{\text{th}}$ iteration for proxy wavefields

$$\bar{\mathbf{u}}_{sv} = \arg\min_{\mathbf{u}_{sv}} \frac{1}{2}\|P\mathbf{u}_{sv} - \mathbf{d}_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(\mathbf{m}^n)\mathbf{u}_{sv} - \mathbf{q}_{sv}\|^2$$

followed by computing the gradient for the model

$$\mathbf{g}^n = \sum_{sv} \mathrm{Re}\left\{\lambda^2\omega_v^2\, \mathrm{diag}(\bar{\mathbf{u}}_{sv})^* \left(A_v(\mathbf{m}^n)\bar{\mathbf{u}}_{sv} - \mathbf{q}_{sv}\right)\right\}$$

Ernie Esser, Lluís Guasch, Tristan van Leeuwen, Aleksandr Y. Aravkin, and Felix J. Herrmann, "**Total variation regularization strategies in full waveform inversion for improving robustness to noise, limited data and poor initializations**". 2015.

WRI

and reduced diagonal Gauss-Newton Hessian

$$H^n_{sv} \approx \sum_{sv} \mathrm{Re}\left\{\lambda^2 \omega_v^4 \,\mathrm{diag}(\bar{\mathbf{u}}_{sv}(\mathbf{m}^n))^* \,\mathrm{diag}(\bar{\mathbf{u}}_{sv}(\mathbf{m}^n))\right\}$$

to minimize the reduced objective

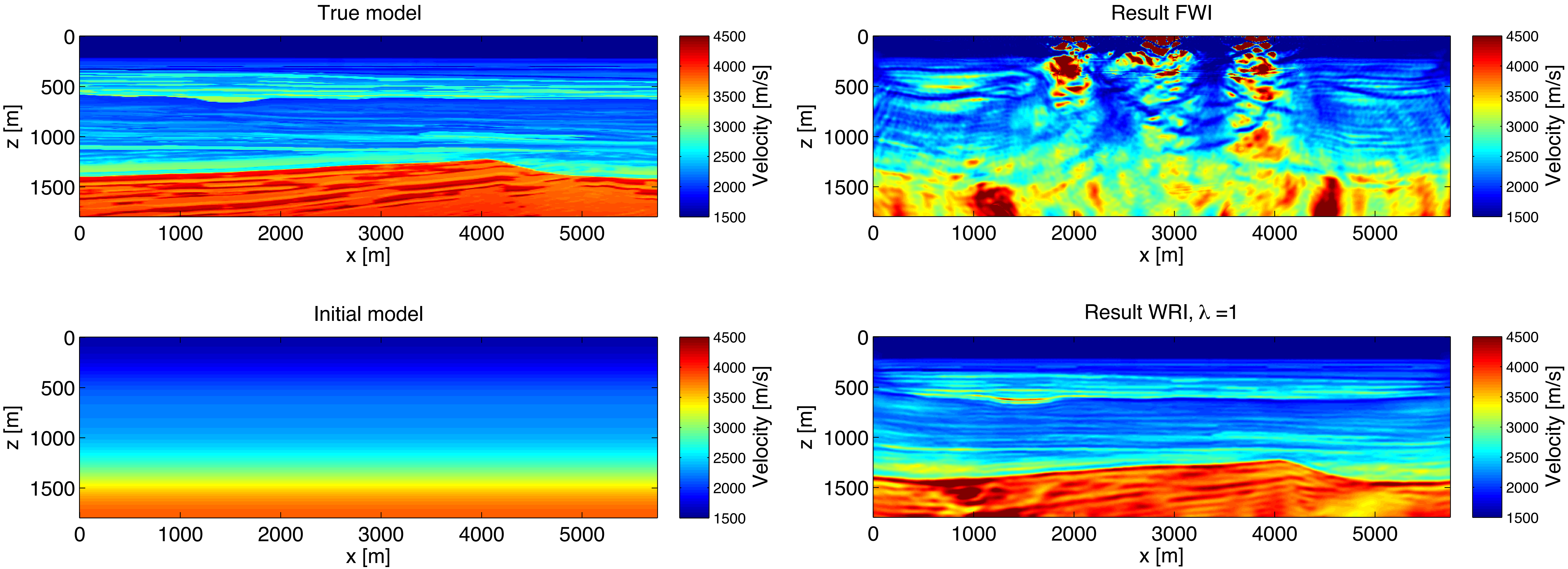
$$\Phi(\mathbf{m}) = \sum_{sv} \frac{1}{2}\|P\bar{\mathbf{u}}_{sv}(\mathbf{m}) - \mathbf{d}_{sv}\|^2 + \frac{\lambda^2}{2}\|A_v(\mathbf{m})\bar{\mathbf{u}}_{sv}(\mathbf{m}) - \mathbf{q}_{sv}\|^2$$

via scaled gradient descents [Bertsekas '99]

$$\Delta\mathbf{m} = \arg\min_{\Delta\mathbf{m}\in\mathbb{R}^N} \Delta\mathbf{m}^T\mathbf{g}^n + \frac{1}{2}\Delta\mathbf{m}^T H^n \Delta\mathbf{m} + c_n \Delta\mathbf{m}^T\Delta\mathbf{m}$$

$$\mathbf{m}^{n+1} = \mathbf{m}^n + \Delta\mathbf{m} \text{ with } c_n \geq 0$$

Waveform inversion – poor starting model



Example from **[Peters et al. 2013]**

Including convex constraints

Wave-equation based inversions call for regularization, e.g. via convex constraints

$$\Delta \mathbf{m} = \underset{\Delta \mathbf{m} \in R^N}{\arg\min}\, \Delta \mathbf{m}^T \mathbf{g}^n + \frac{1}{2} \Delta \mathbf{m}^T H^n \Delta \mathbf{m} + c_n \Delta \mathbf{m}^T \Delta \mathbf{m}$$

$$\text{such that} \quad \mathbf{m}^n + \Delta \mathbf{m} \in C$$

- guarantees $\mathbf{m}^{n+1} \in C$
- more difficult to compute
- feasible if it is easy to project onto
- naive projections $\mathbf{m}^{m+1} = \Pi_C \left(\mathbf{m}^n - (H^n)^{-1} \mathbf{g}^n \right)$ are not guaranteed to converge [Bertsekas ’99]

Bound constraints
– via scaled gradient projections

For strictly positive diagonal Gauss-Newton Hessians:

$$\Delta\mathbf{m} = \arg\min_{\Delta\mathbf{m}} \Delta\mathbf{m}^T\mathbf{g}^n + \frac{1}{2}\Delta\mathbf{m}^T(H^n + c_n\mathrm{I})\Delta\mathbf{m}$$
$$\text{subject to } \mathbf{m}_i^n + \Delta\mathbf{m}_i \in [B_i^l, B_i^u], \;\; i = 1\cdots N$$

for which there exists a closed form solution

$$\Delta\mathbf{m}_i = \max\left(B_i^l - \mathbf{m}_i^n, \min\left(B_i^u - \mathbf{m}_i^n, -[(H^n + c_n I)^{-1}\mathbf{g}^n]_i\right)\right)$$

that is computationally affordable.

Total-variation regularization
– w/ bound constraints

Promote models w/ sharp boundaries via

$$\mathbf{m}^{n+1} = \mathbf{m}^n + \Delta\mathbf{m} \quad \text{subject to} \quad \mathbf{m}^{n+1} \in C_{\text{box}} \cap C_{\text{TV}}$$

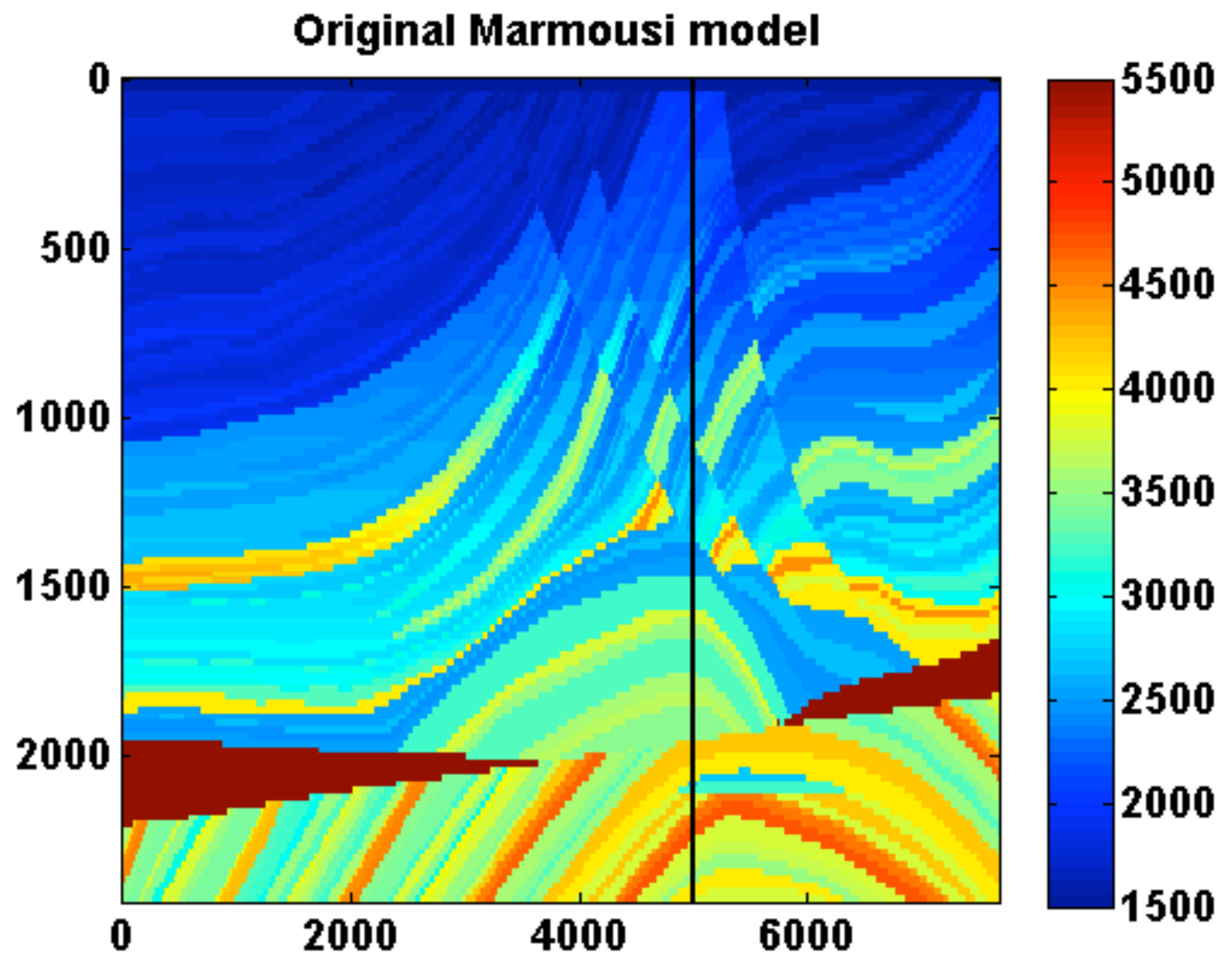
where $C_{\text{TV}} = \{\mathbf{m} : \|\mathbf{m}\|_{\text{TV}} \leq \tau\}$ and

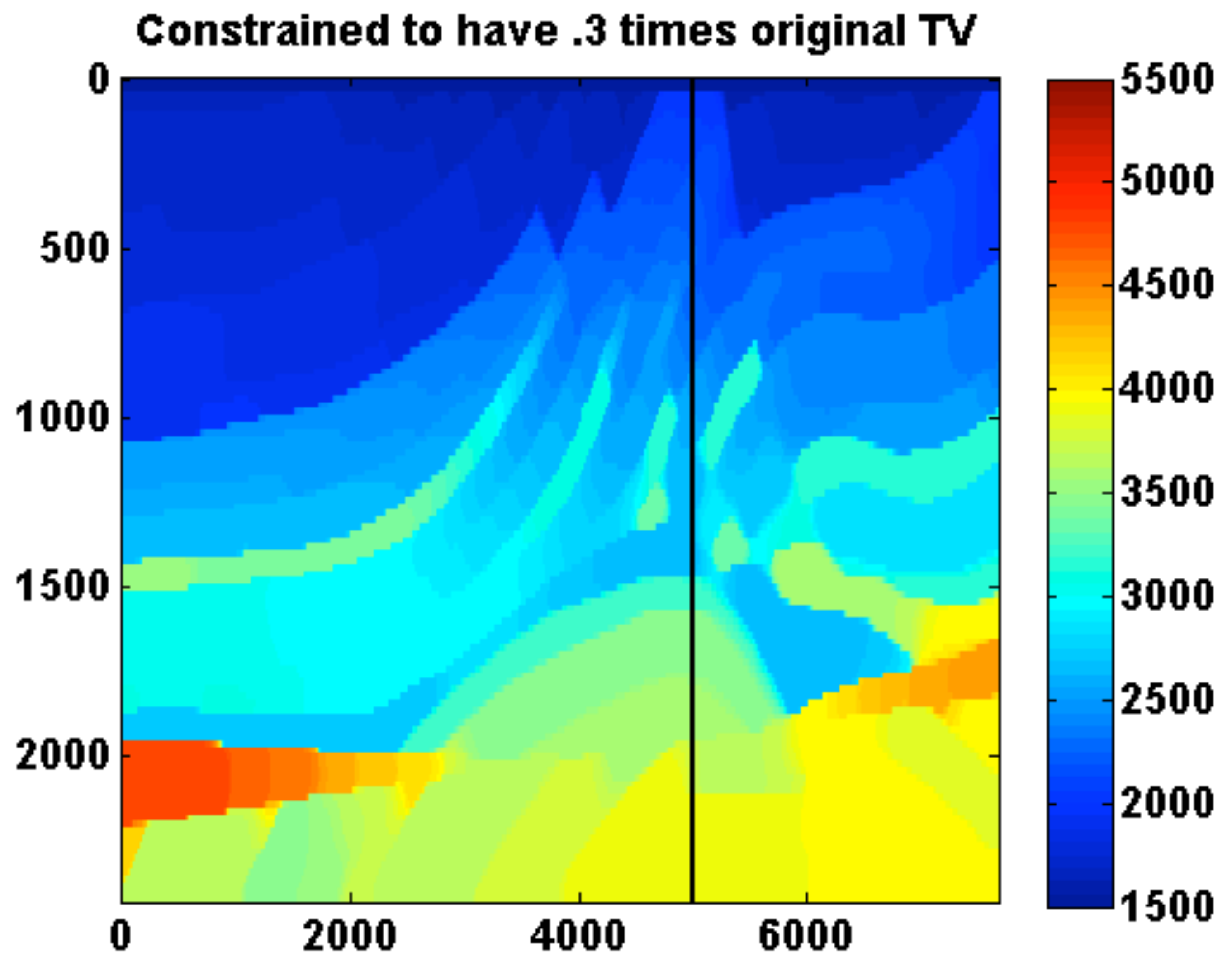
$$\begin{aligned} \|\mathbf{m}\|_{TV} &= \frac{1}{h}\sum_{ij}\sqrt{(m_{i+1,j} - m_{i,j})^2 + (m_{i,j+1} - m_{i,j})^2} \\ &= \sum_{ij}\frac{1}{h}\left\|\begin{bmatrix}(m_{i,j+1} - m_{i,j}) \\ (m_{i+1,j} - m_{i,j})\end{bmatrix}\right\| \\ &= \|D\mathbf{m}\|_{1,2} := \sum_{l=1}^{N}\|(D\mathbf{m})_l\| \ . \end{aligned}$$

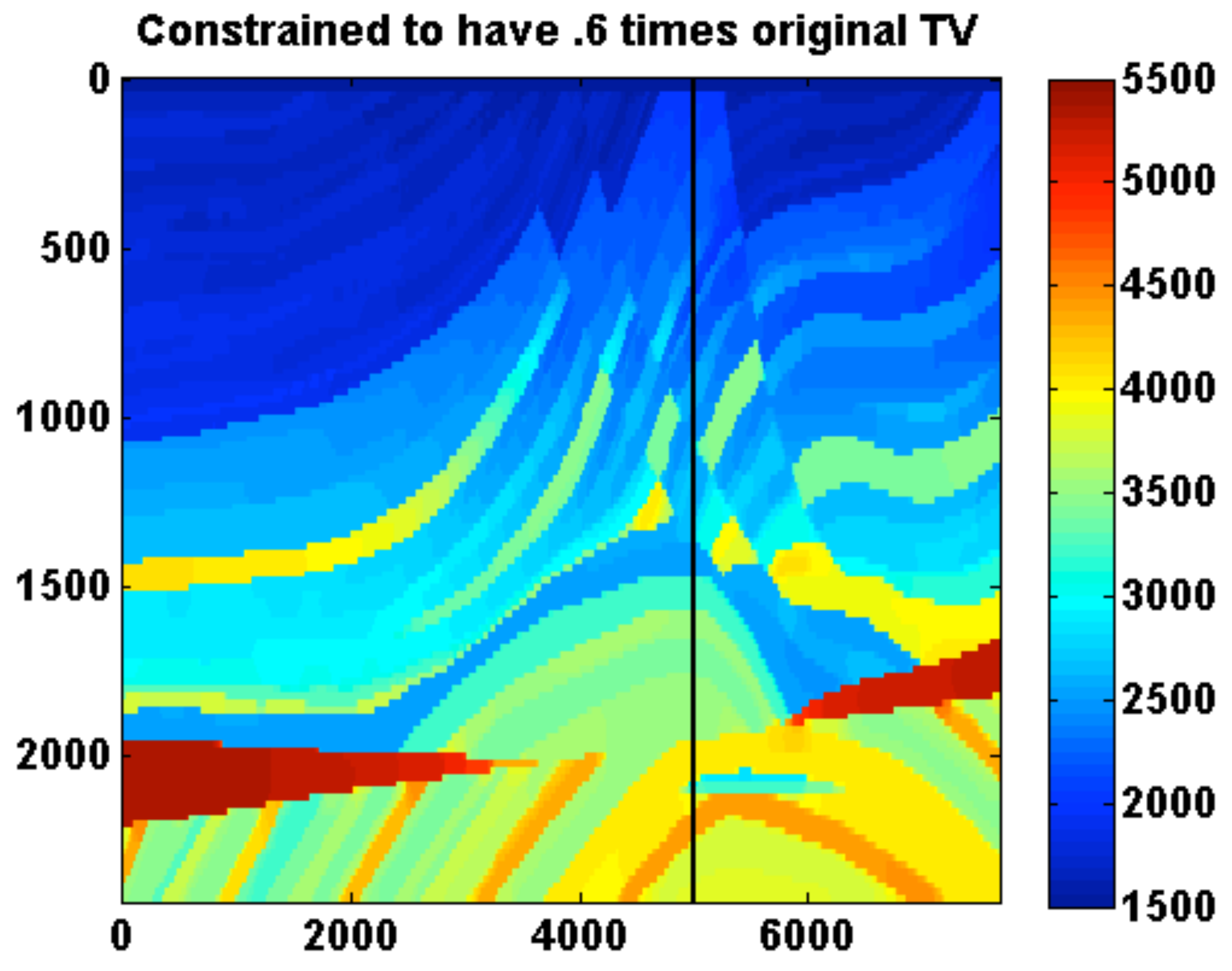
Projections onto convex sets

$v_{\min} = 1500,\ v_{\max} = 5500,$ and $\tau = \{0.3\tau_0,\ 0.6\tau_0\}$

$$\Pi_C(\mathbf{m}_0) = \arg\min_{\mathbf{m}} \frac{1}{2}\|\mathbf{m} - \mathbf{m}_0\|^2 \quad \text{subject to} \quad \mathbf{m}_i \in [B_i^l, B_i^u] \text{ and } \|\mathbf{m}\|_{TV} \leq \tau$$







Proposed algorithm

Solve

$$\underset{\mathbf{m}}{\text{minimize}}\,\Phi(\mathbf{m}) \quad \text{subject to} \quad \mathbf{m}^{n+1} \in C_{\text{box}} \cap C_{\text{TV}}$$

by iterating

$$\Delta\mathbf{m} = \underset{\Delta\mathbf{m}}{\arg\min}\, \Delta\mathbf{m}^T \mathbf{g}^n + \frac{1}{2}\Delta\mathbf{m}^T (H^n + c_n \mathrm{I})\Delta\mathbf{m}$$

$$\text{subject to} \quad \mathbf{m}_i^n + \Delta\mathbf{m}_i \in [B_i^l, B_i^u] \text{ and } \|\mathbf{m}^n \Delta\mathbf{m}\|_{TV} \leq \tau$$

$$\mathbf{m}^{n+1} = \mathbf{m}^n + \Delta\mathbf{m}$$

Solving the convex subproblems

Find saddle point of

$$\mathcal{L}(\Delta\mathbf{m},\mathbf{p}) = \Delta\mathbf{m}^T\mathbf{g}^n + \frac{1}{2}\Delta\mathbf{m}^T(H^n + c_n\mathrm{I})\Delta\mathbf{m} + g_B(\mathbf{m}^n + \Delta\mathbf{m})$$
$$+\,\mathbf{p}^T D(\mathbf{m}^n + \Delta\mathbf{m}) - \tau\|\mathbf{p}\|_{\infty,2}$$

with indicator functions for

Bound constraint

$$g_B(\mathbf{m}) = \begin{cases} 0 & \text{if } m_i \in [B_i^l, B_i^u] \\ \infty & \text{otherwise} \end{cases}$$

TV-norm constraint

$$\sup_{\mathbf{p}} +\mathbf{p}^T D(\mathbf{m}^n + \Delta\mathbf{m}) - \tau\|\mathbf{p}\|_{\infty,2}$$
$$= \begin{cases} 0 & \text{if } \|D(\mathbf{m}^n + \Delta\mathbf{m})\|_{1,2} \le \tau \\ \infty & \text{otherwise} \end{cases}$$

Ernie Esser, Xiaoqun Zhang, and Tony F. Chan. A General Frame- work for a Class of First Order Primal-Dual Algorithms for Convex Optimization in Imaging Science. SIAM Journal on Imaging Sciences, 3(4):1015–1046, 2010.

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Iterations
– primal dual hybrid gradient (PDHG)

projection onto TV ball

$$\mathbf{p}^{k+1} = \mathbf{p}^k + \delta D(\mathbf{m}^n + \Delta\mathbf{m}^k) - \Pi_{\|\cdot\|_{1,2}\leq\tau\delta}(\mathbf{p}^k + \delta D(\mathbf{m}^n + \Delta\mathbf{m}^k))$$

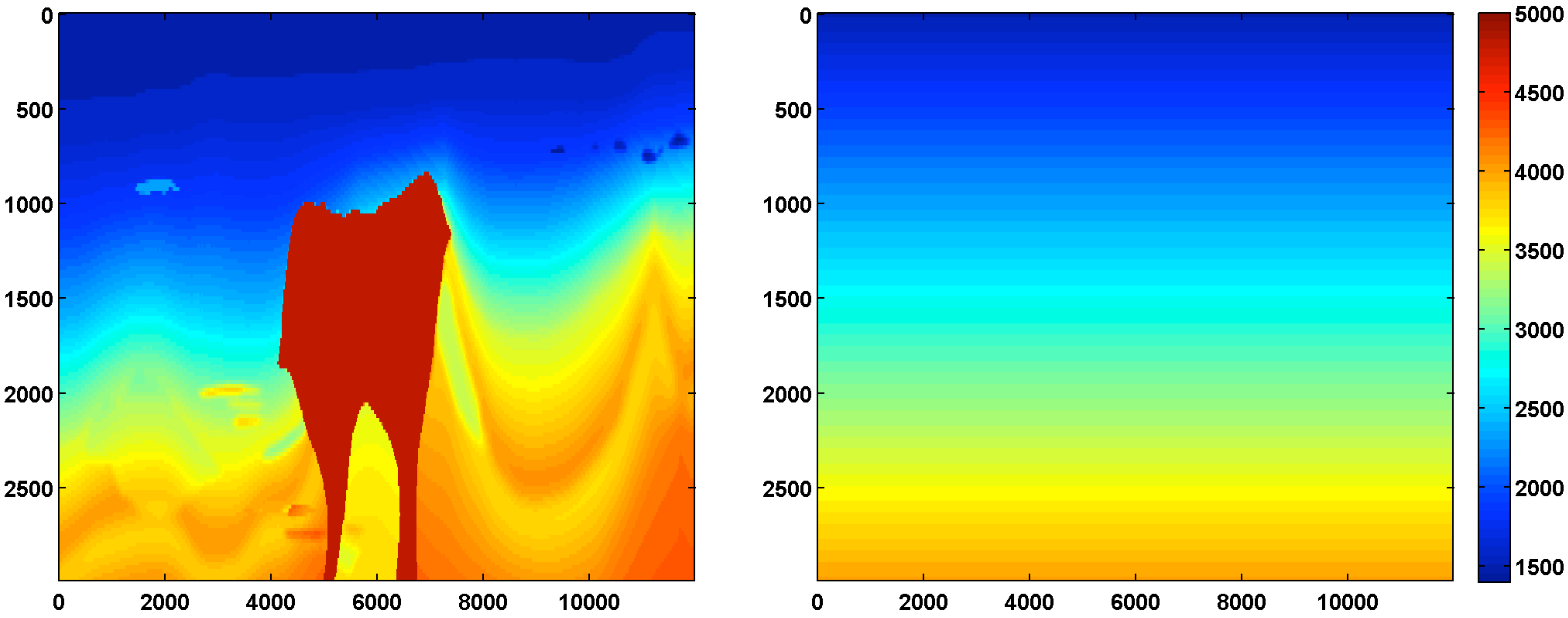
$$\Delta\mathbf{m}_i^{k+1} = \max\left((B_i^l - \mathbf{m}_i^n), B_i\right)$$

$$B_i = \min\left((B_i^u - \mathbf{m}_i^n), [(H^n + (c_n + \frac{1}{\alpha})\mathrm{I})^{-1}(-\mathbf{g}^n + \frac{\Delta\mathbf{m}^k}{\alpha} - D^T(2\mathbf{p}^{k+1} - \mathbf{p}^k)]_i\right)$$

for steplengths $\alpha\delta \leq \frac{1}{\|D^T D\|}$ and $\alpha = \frac{1}{\max(H^n + c_n \mathrm{I})}$

- do not involve solutions of (data-augmented) wave equations
- allows for data-dependent stepsizes

True velocity & poor starting model

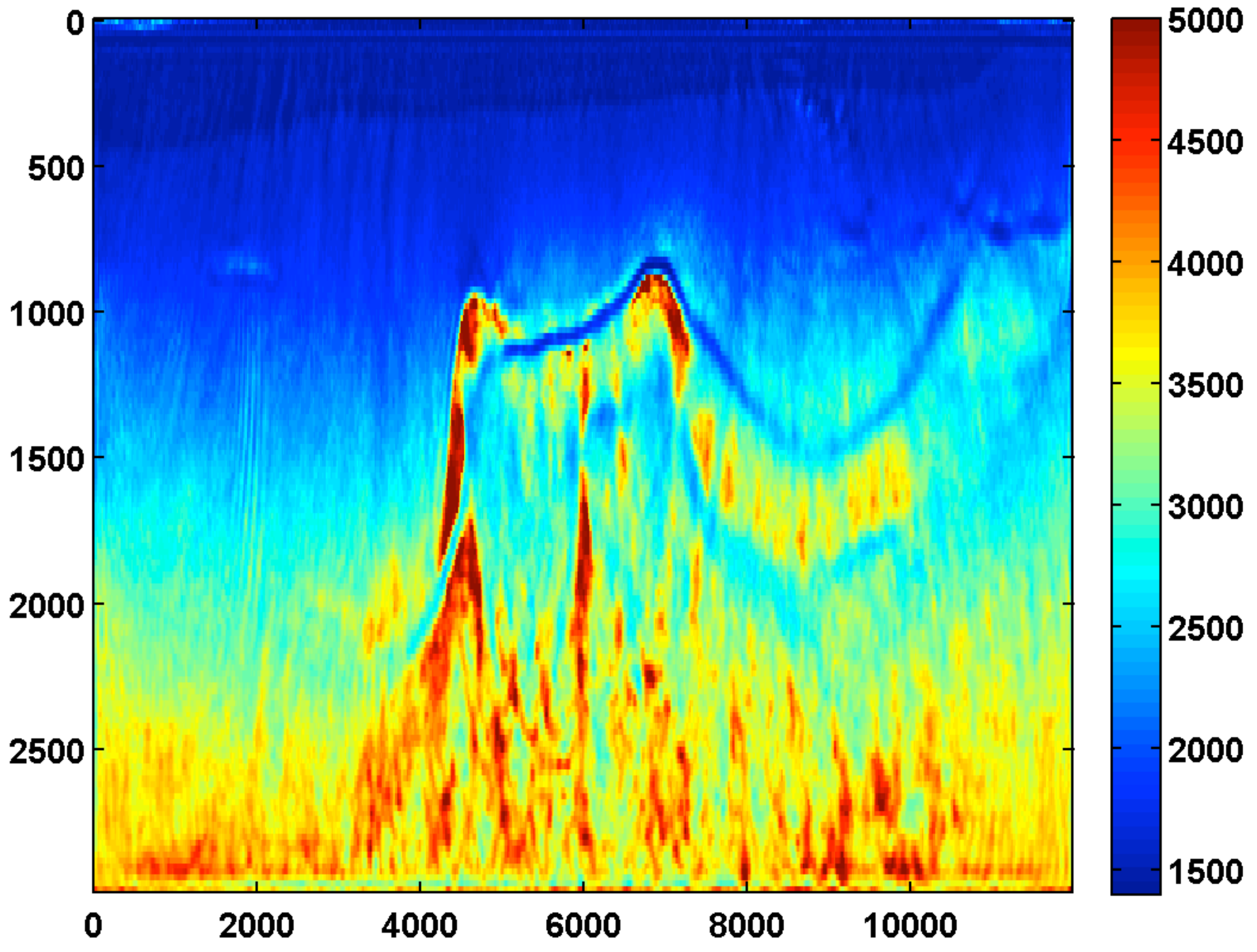


BP model

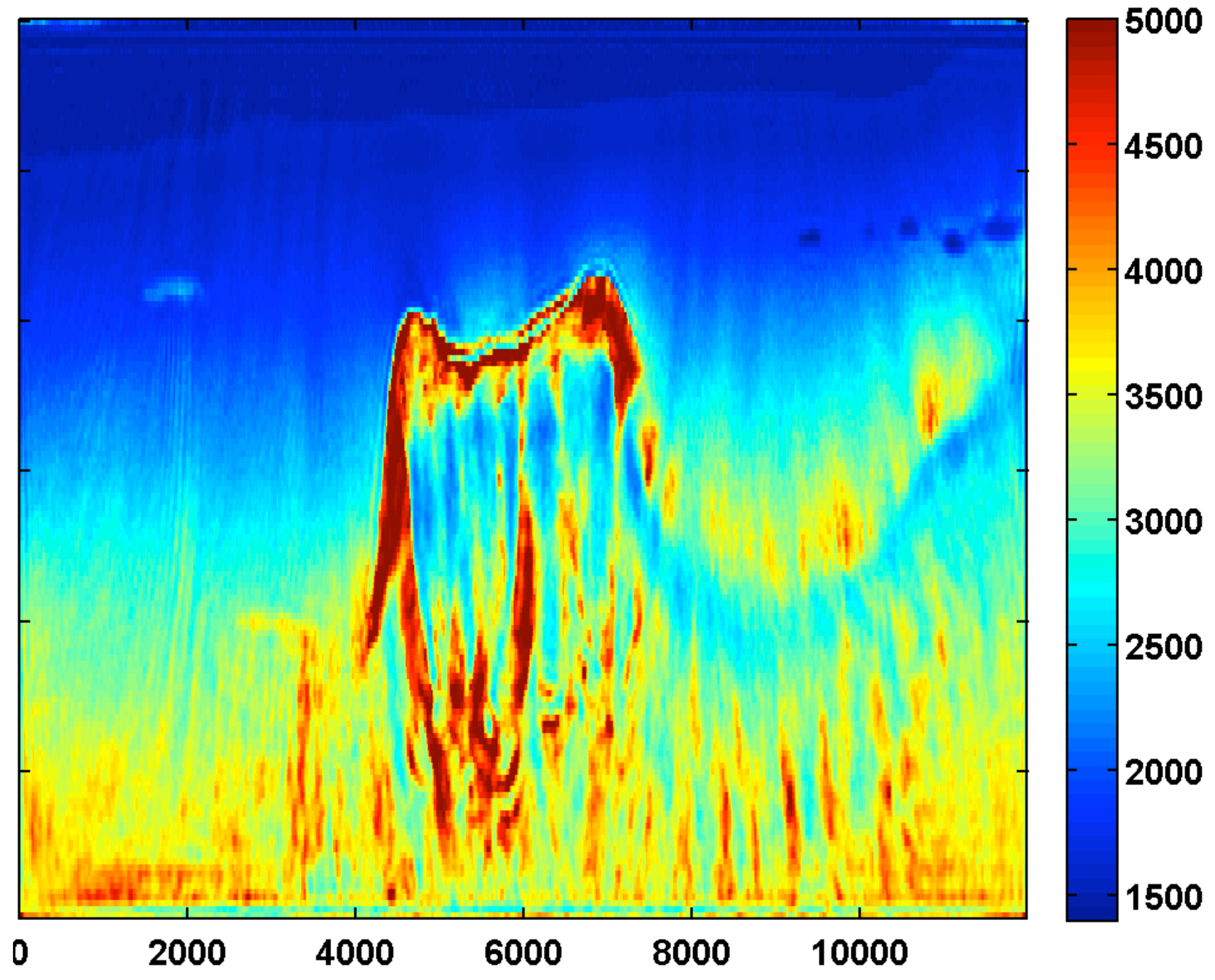
- number of sources: 126
- number of receivers: 299
- frequency continuation over 3-20Hz in overlapping batches of 2
- maximum number of outer iterations per frequency batch: 25
- maximum number of inner iterations for convex subproblems: 2000
- known Ricker wavelet sources with 15Hz peak frequency
- two simultaneous shots with Gaussian weights w/ redraws
- no added noise

Results w/o TV

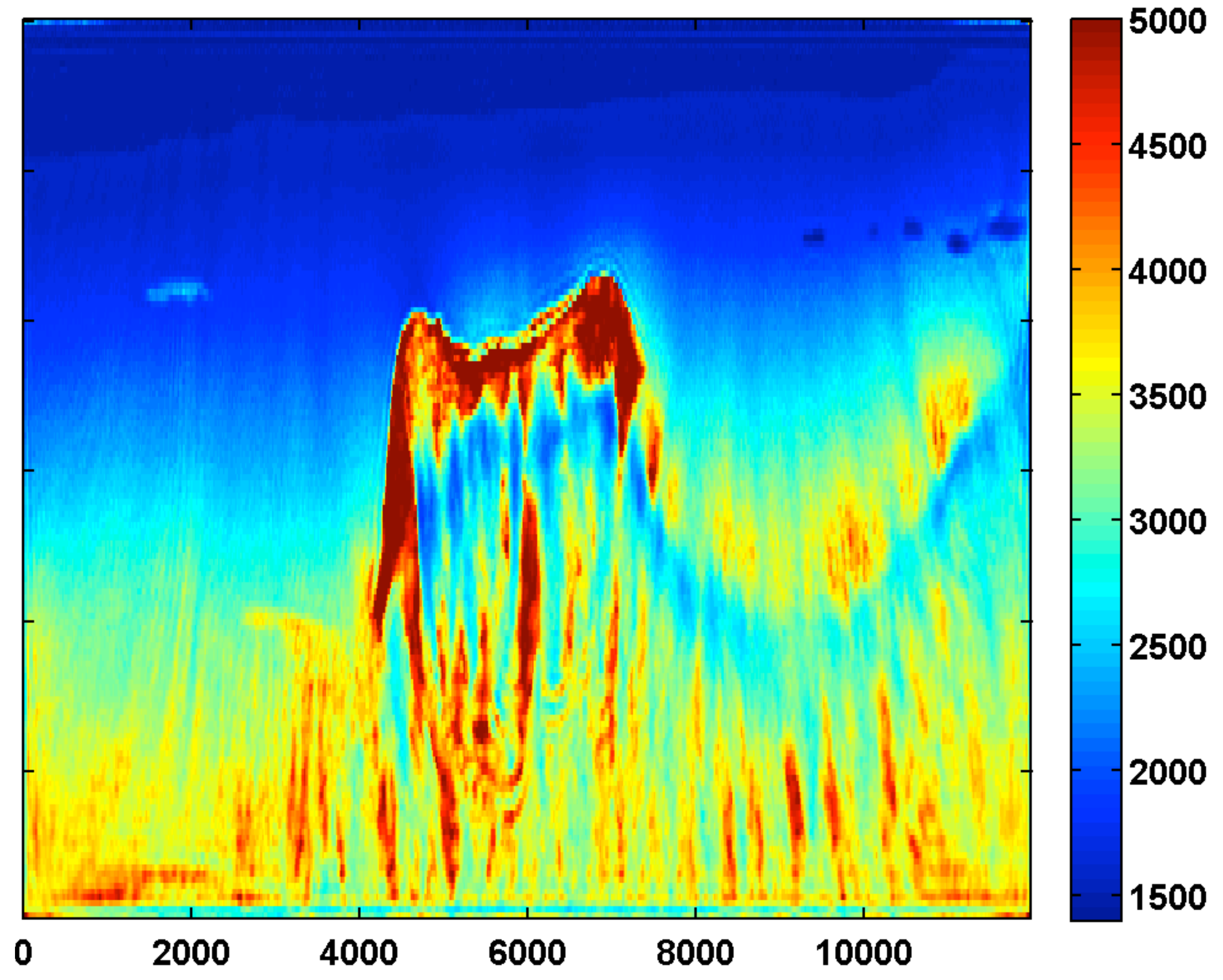
after one cycle through the frequencies



after two cycles through the frequencies

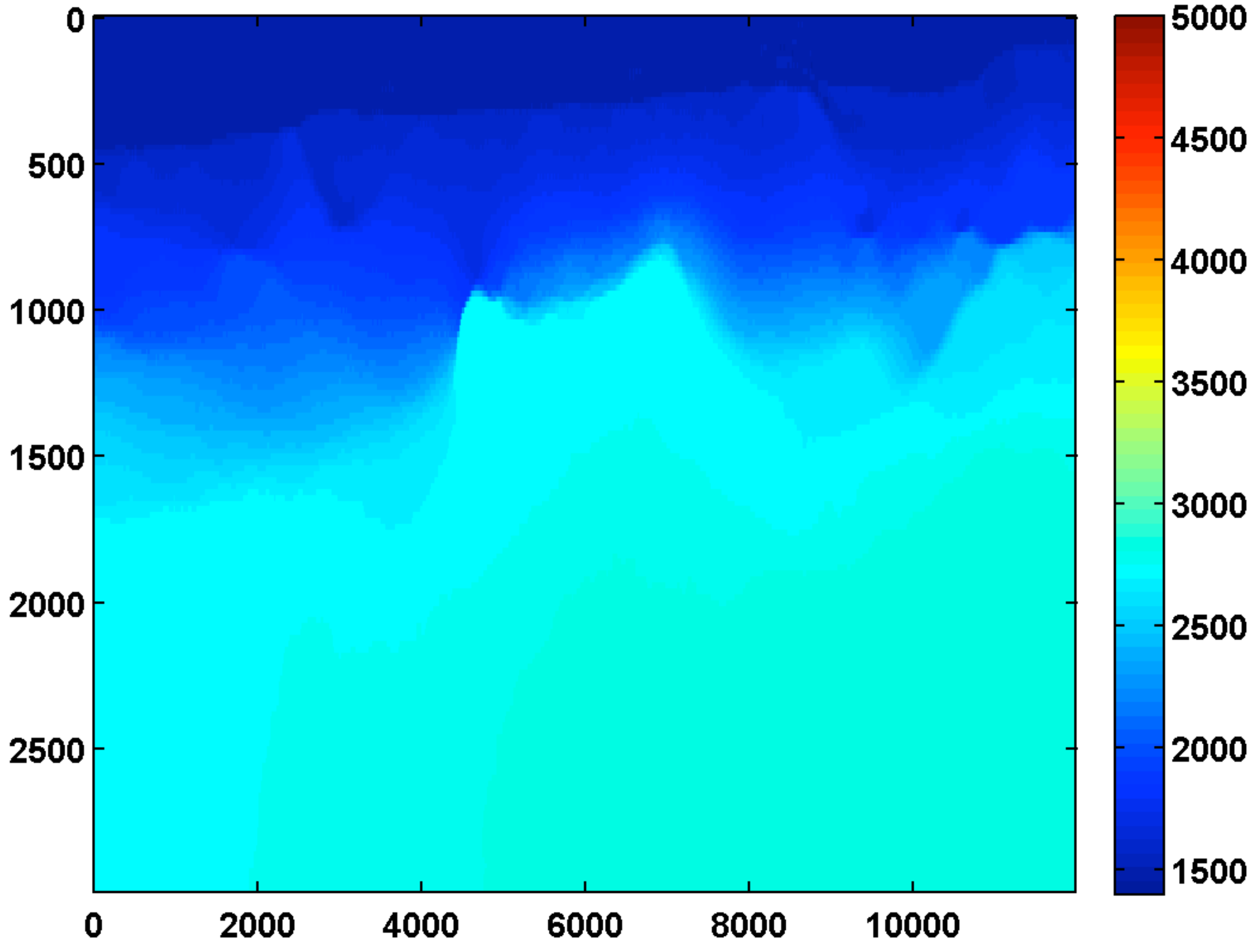


after three cycles through the frequencies

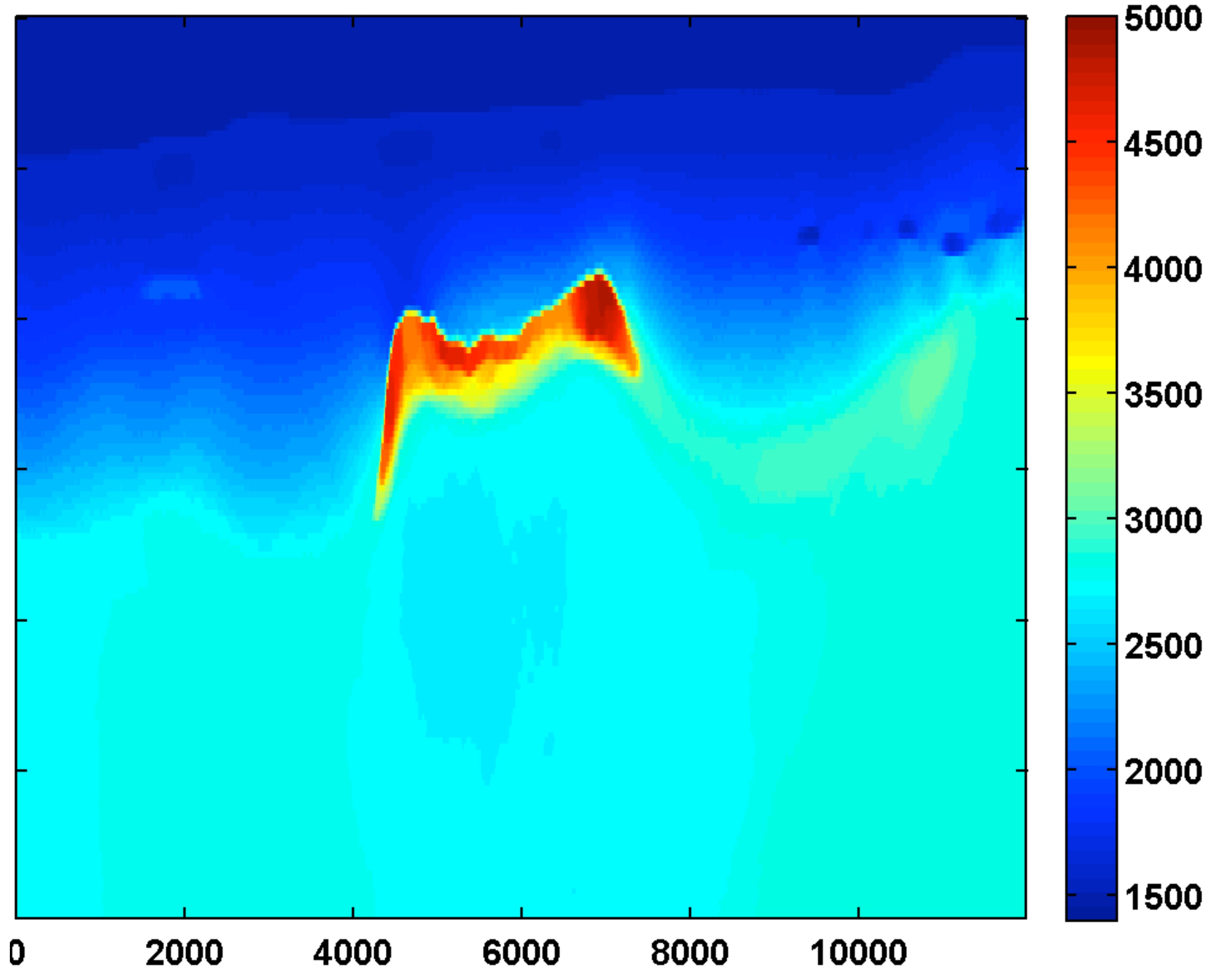


Results w/ TV

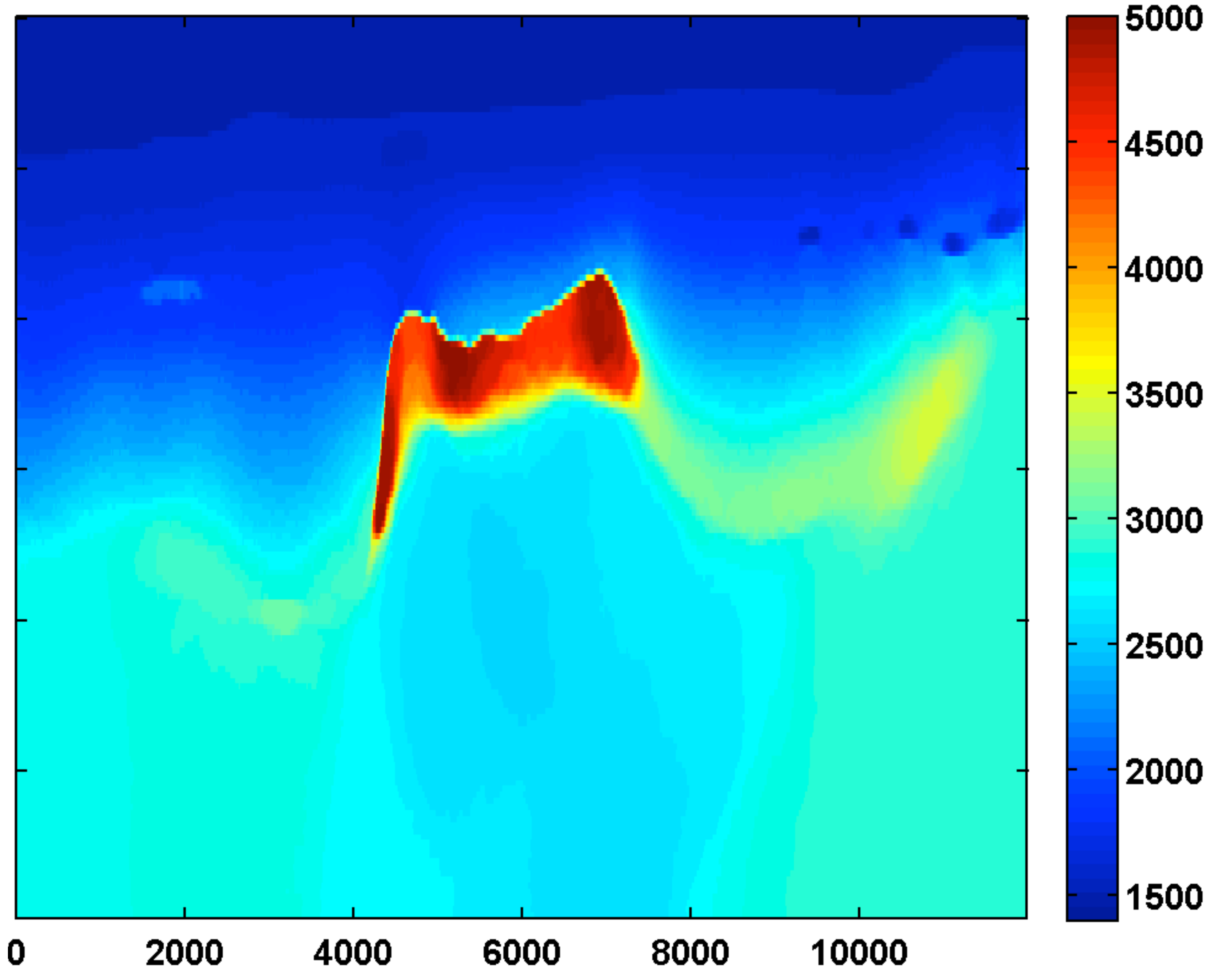
after one cycle through the frequencies



after two cycles through the frequencies



after three cycles through the frequencies



Joint work w/ Sub Salt Solutions Ltd. Patent filed GB1509337.0

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Hinge loss
– one-sided TV constraint

Mitigate erroneous velocity model updates by using the fact that

- vertical slowness profiles tend to decrease w/ depth
- makes it less probable that velocities jump down along the vertical

Mathematically expressed as the one-norm of a hinge-loss function

$$\|\max(0, D_z\mathbf{m})\|_1 \leq \xi$$

- for ξ small slowness is unlikely to step up
- extended to a weighted directional gradient
- combined w/ omni-directional TV and bound constraints

Ernie Esser, Lluís Guasch, Tristan van Leeuwen, Aleksandr Y. Aravkin, and Felix J. Herrmann, "Total variation regularization strategies in full waveform inversion for improving robustness to noise, limited data and poor initializations". 2015.

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Scaled-gradient projections
– w/ convex total-variation, box, & hinge-loss constraints

Solve for given $\bar{\mathbf{u}}_\lambda$

$$\min_{\mathbf{m}} \phi(\mathbf{m}, \bar{\mathbf{u}}_\lambda) \quad \text{subject to} \quad \begin{cases} m_i \in [B_1, B_2] \\ \|\mathbf{m}\|_{\mathrm{TV}} \leq \tau \\ \|\mathbf{m}\|_{\mathrm{Hinge}} \leq \xi \end{cases}$$

with

$$\|\mathbf{m}\|_{TV} = \sum_{ij} \frac{1}{h} \left\| \begin{bmatrix} (m_{i,j+1} - m_{i,j}) \\ (m_{i+1,j} - m_{i,j}) \end{bmatrix} \right\|$$

and

$$\|\mathbf{m}\|_{\mathrm{Hinge}} = \|\max\left(0, D_z\mathbf{m}\right)\|_1$$

Proposed algorithm

Solve

$$\underset{\mathbf{m}}{\text{minimize}}\, \Phi(\mathbf{m}) \quad \text{subject to} \quad \mathbf{m}^{n+1} \in C_{\text{box}} \cap C_{\text{TV}} \cap C_{\text{Hinge}}$$

by iterating

$$\begin{aligned}
\mathbf{p}_1^{k+1} &= \mathbf{p}_1^k + \delta D(\mathbf{m}^n + \Delta\mathbf{m}^k) - \Pi_{\|\cdot\|_{1,2} \leq \tau\delta}(\mathbf{p}_1^k + \delta D(\mathbf{m}^n + \Delta\mathbf{m}^k)) \\
\mathbf{p}_2^{k+1} &= \mathbf{p}_2^k + \delta D_z(\mathbf{m}^n + \Delta\mathbf{m}^k) - \Pi_{\|\max(0,\cdot)\|_1 \leq \xi\delta}(\mathbf{p}_2^k + \delta D_z(\mathbf{m}^n + \Delta\mathbf{m}^k)) \\
B_i &= \min\left((B_i^u - \mathbf{m}_i^n), [(H^n + (c_n + \frac{1}{\alpha})\mathrm{I})^{-1}(-\mathbf{g}^n + \frac{\Delta\mathbf{m}^k}{\alpha} - D^T(2\mathbf{p}_1^{k+1} - \mathbf{p}_1^k) - D_z^T(2\mathbf{p}_2^{k+1} - \mathbf{p}_2^k))]_i\right) \\
\Delta\mathbf{m}_i^{k+1} &= \max\left((B_i^l - \mathbf{m}_i^n), B_i\right)
\end{aligned}$$

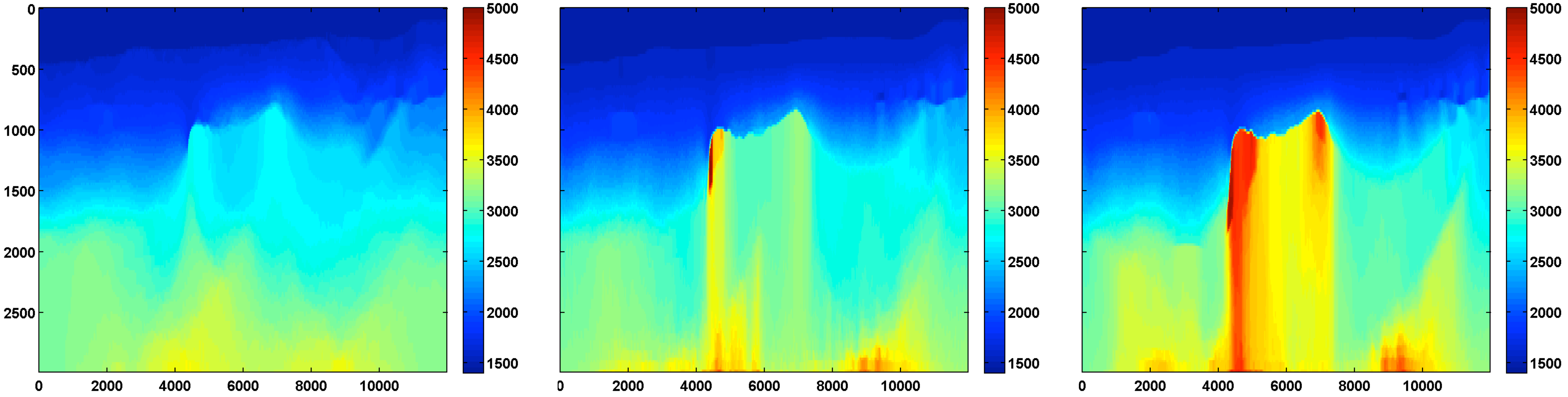
Results w/ hinge loss continuation

$$\frac{\xi}{\xi_{\text{true}}} = \{.01, .05, .10\}$$

after one cycle through the frequencies

after two cycles through the frequencies

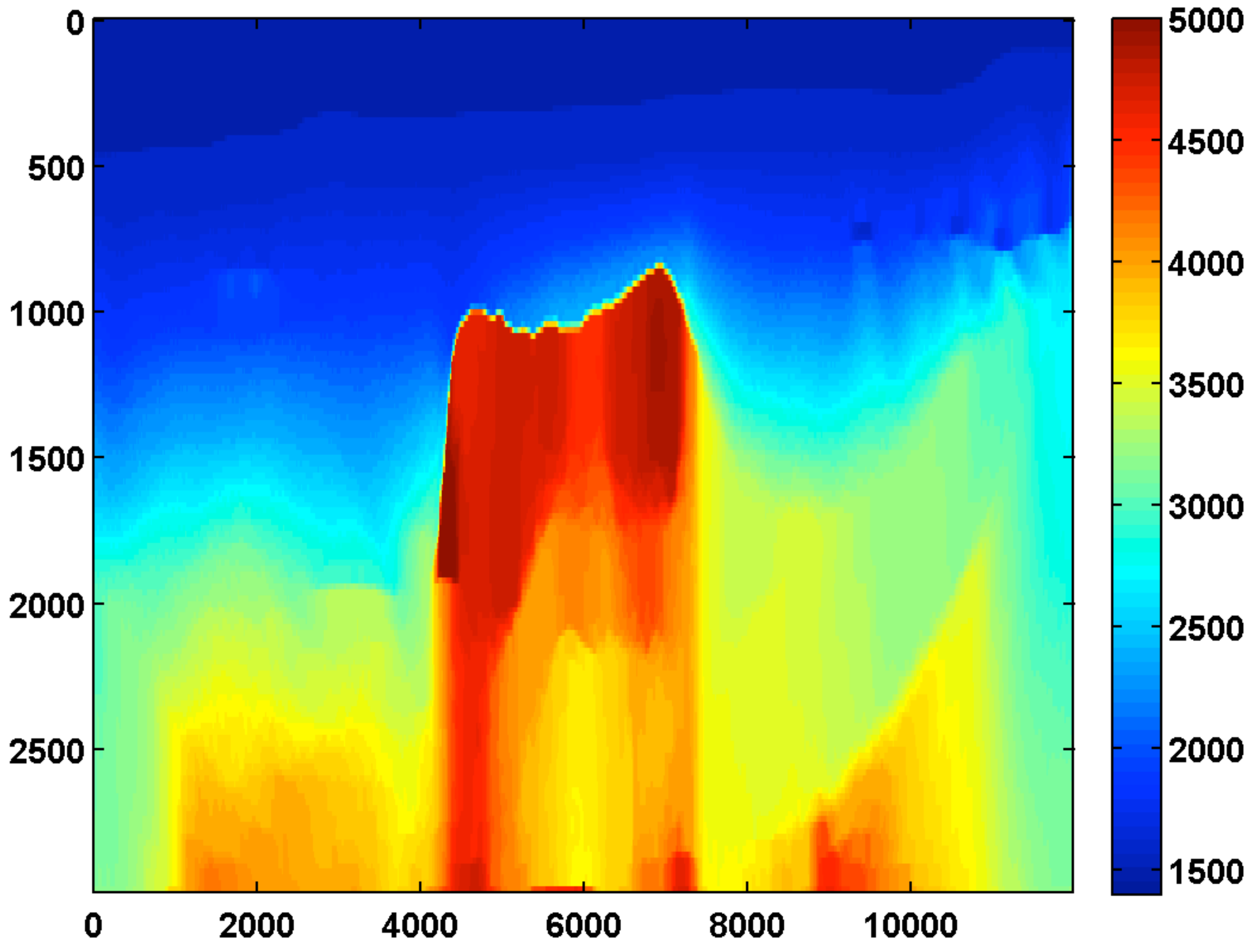
after three cycles through the frequencies



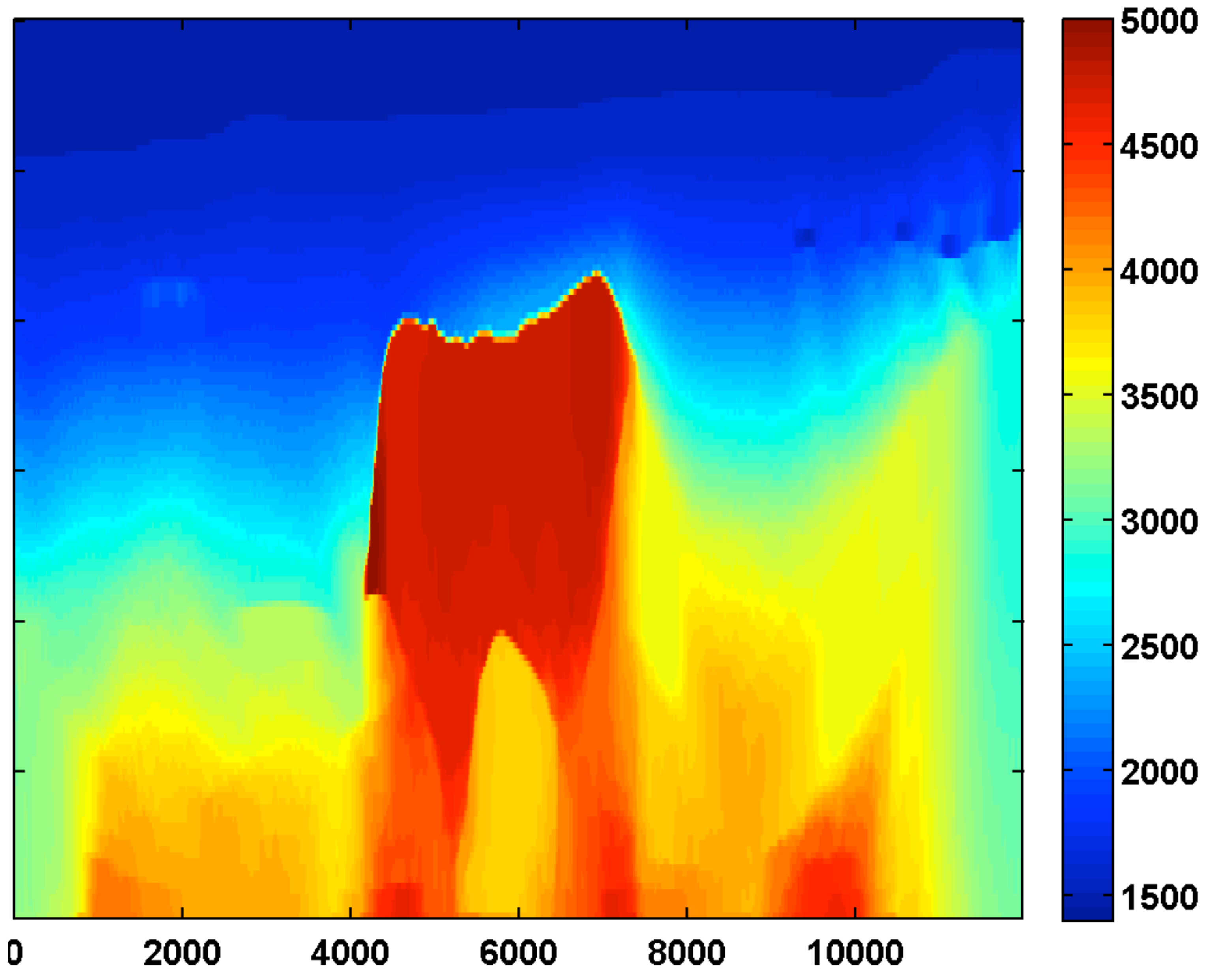
Results w/ hinge loss continuation

$$\frac{\xi}{\xi_{\text{true}}} = \{.15, .20, .25\}$$

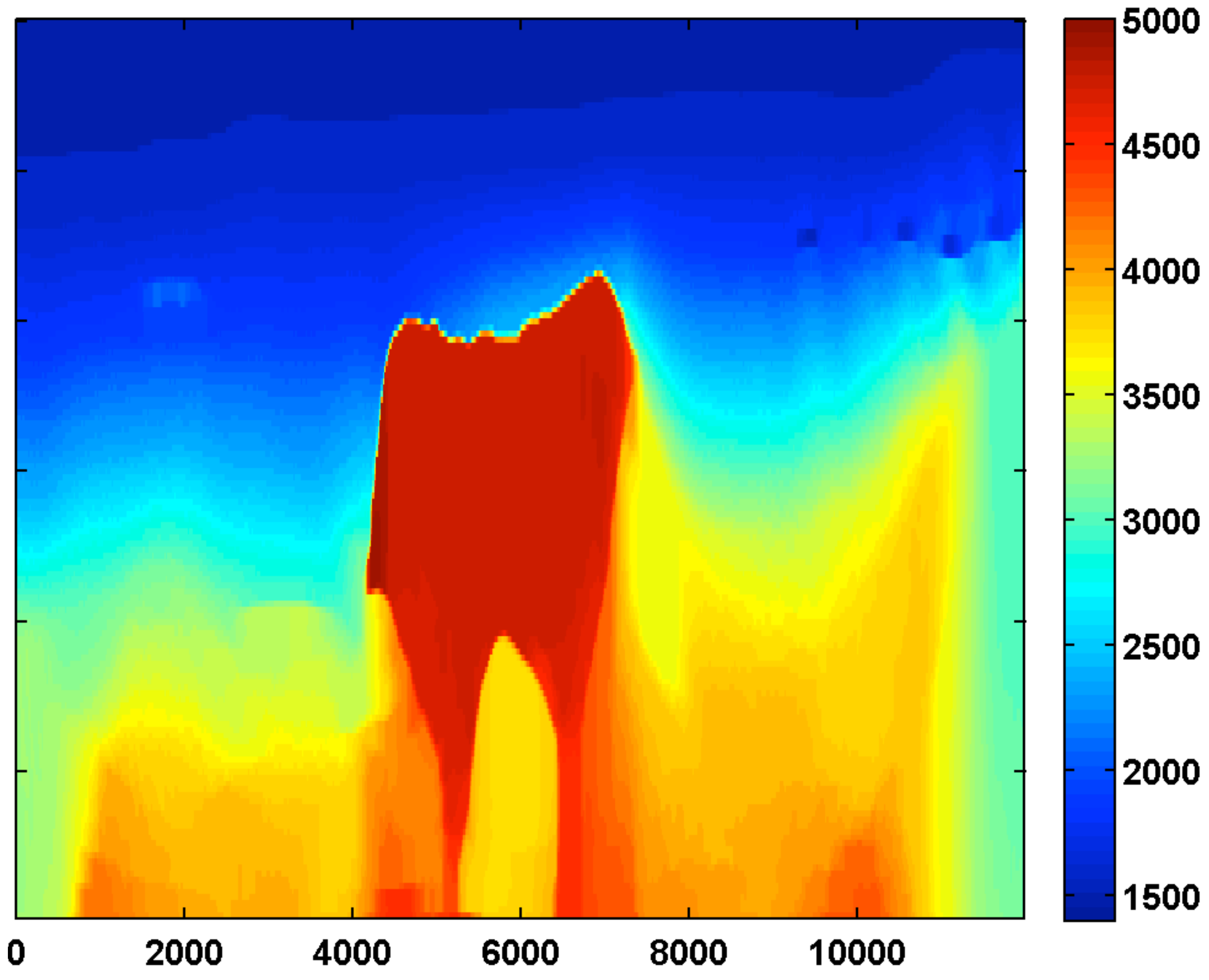
after four cycles through the frequencies



after five cycles through the frequencies

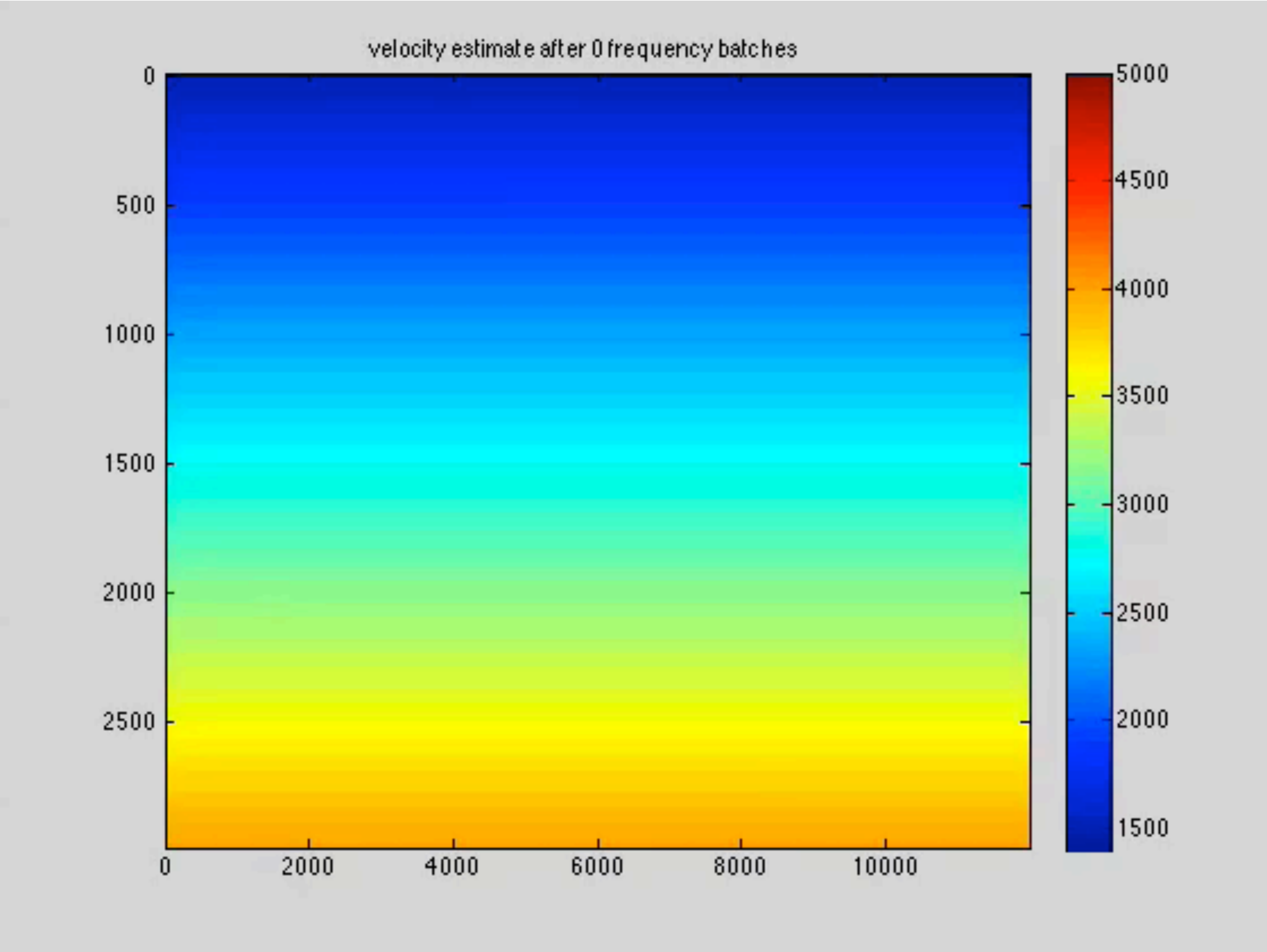


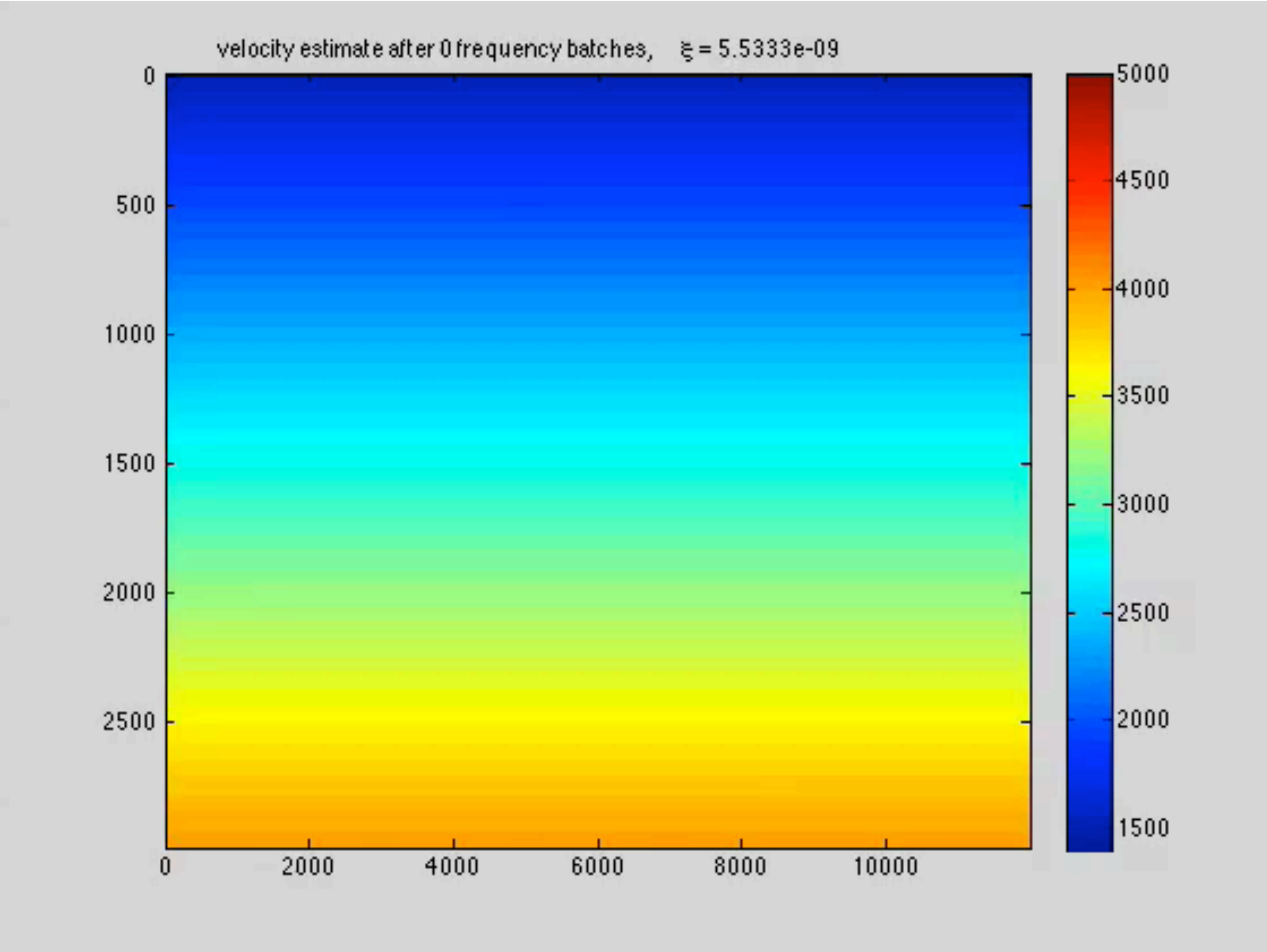
after six cycles through the frequencies



WRI

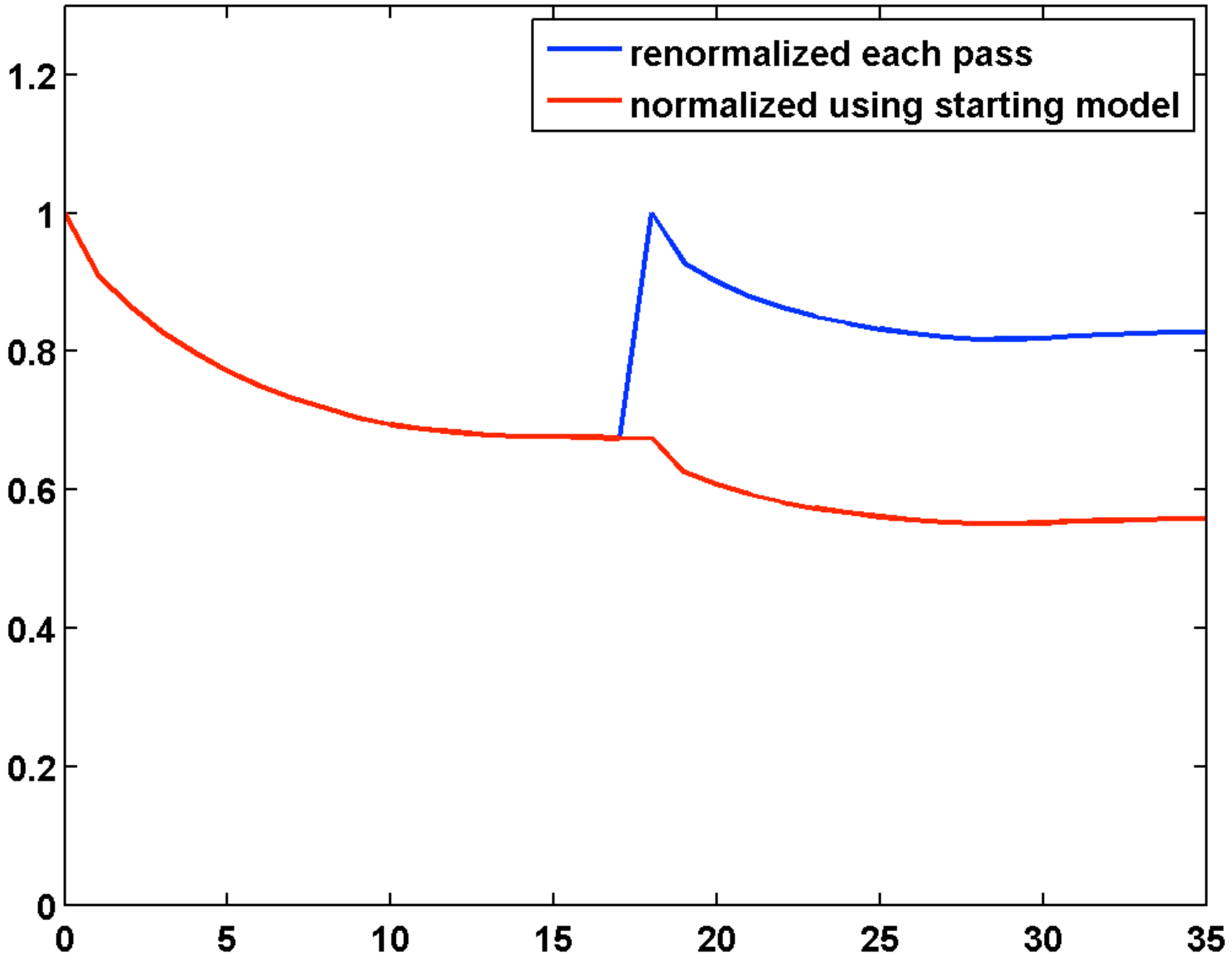
w/ or w/o TV-norm & hinge-loss projections & poor starting model



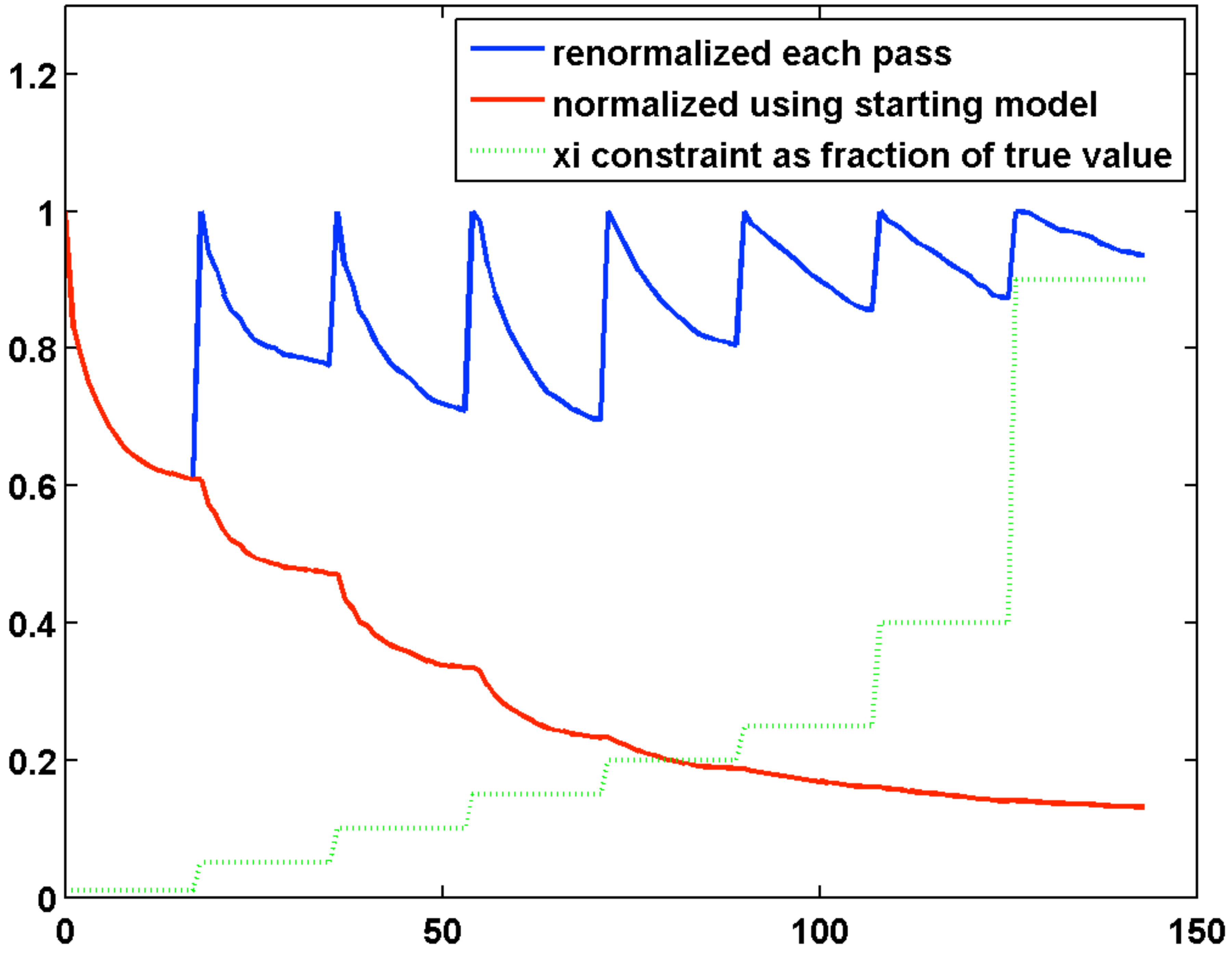


Relative model errors

w/o TV

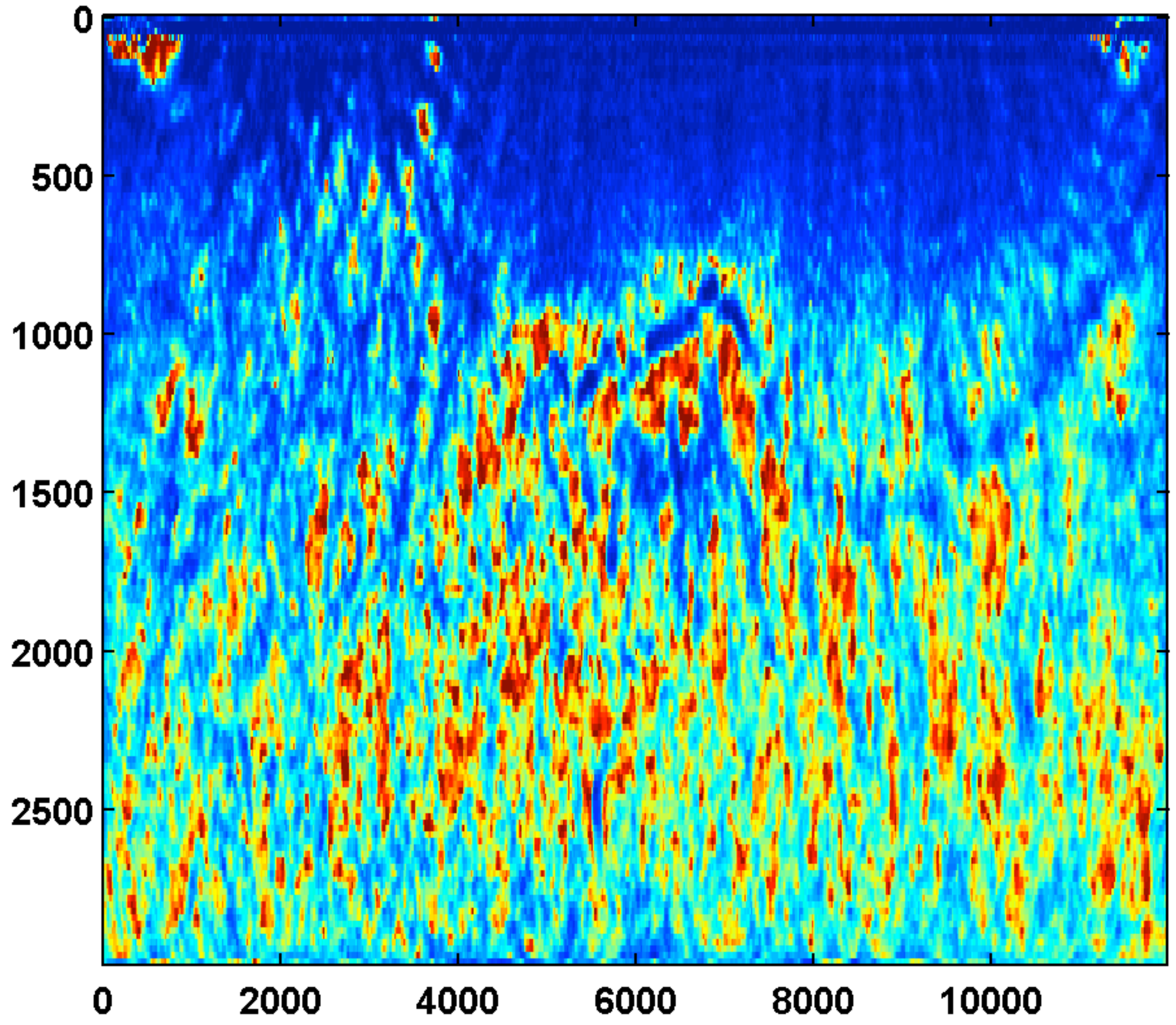


w/ TV & hinge

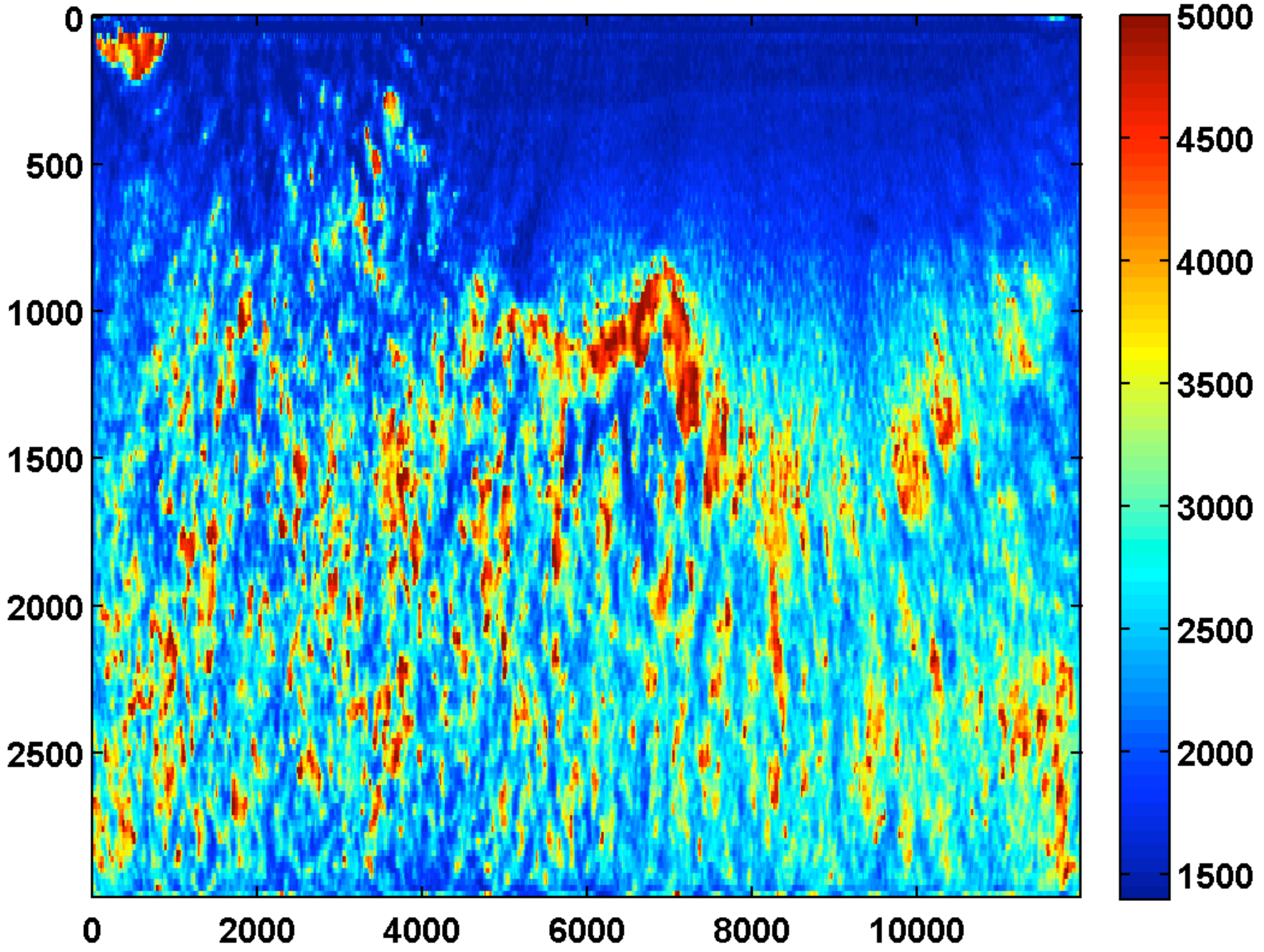


Adjoint-state w/o TV

After one cycle through the frequencies



After two cycles through the frequencies



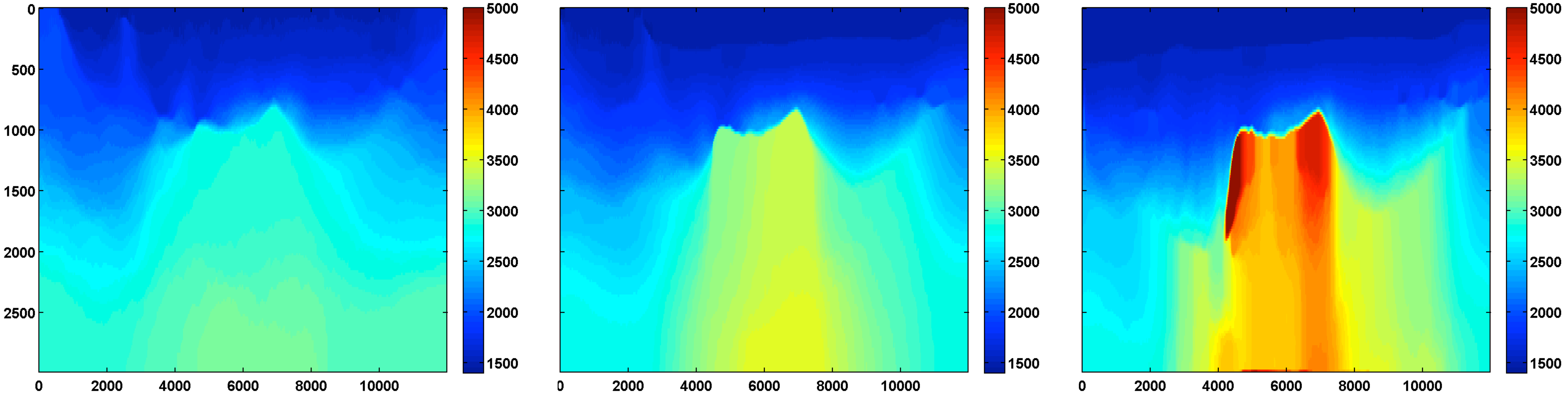
Adjoint-state w/ hinge loss continuation

$$\frac{\xi}{\xi_{\text{true}}} = \{.01, .05, .10\}$$

after one cycle through the frequencies

after two cycles through the frequencies

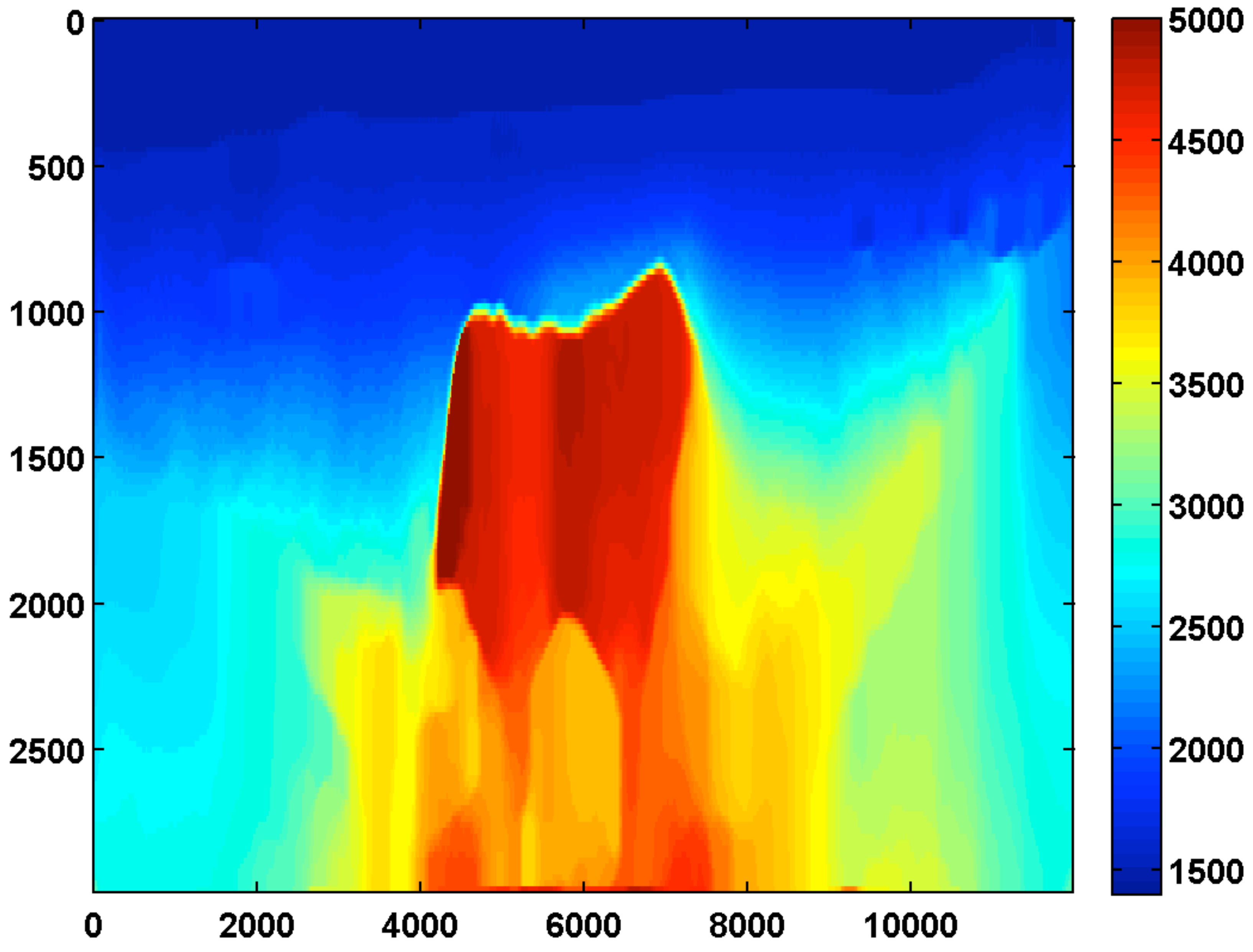
after three cycles through the frequencies



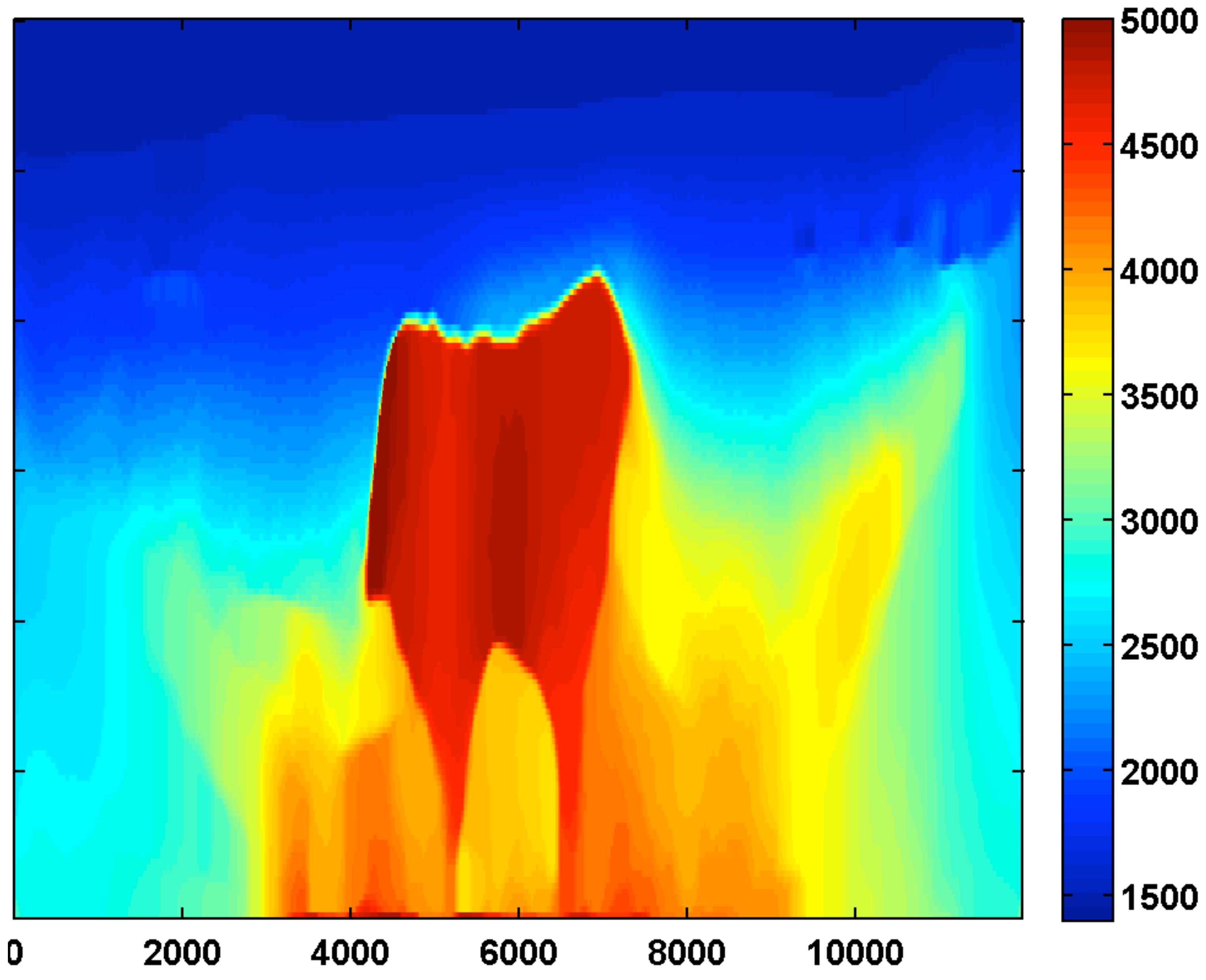
Adjoint-state w/ hinge loss continuation

$$\frac{\xi}{\xi_{\text{true}}} = \{.15, .20, .25\}$$

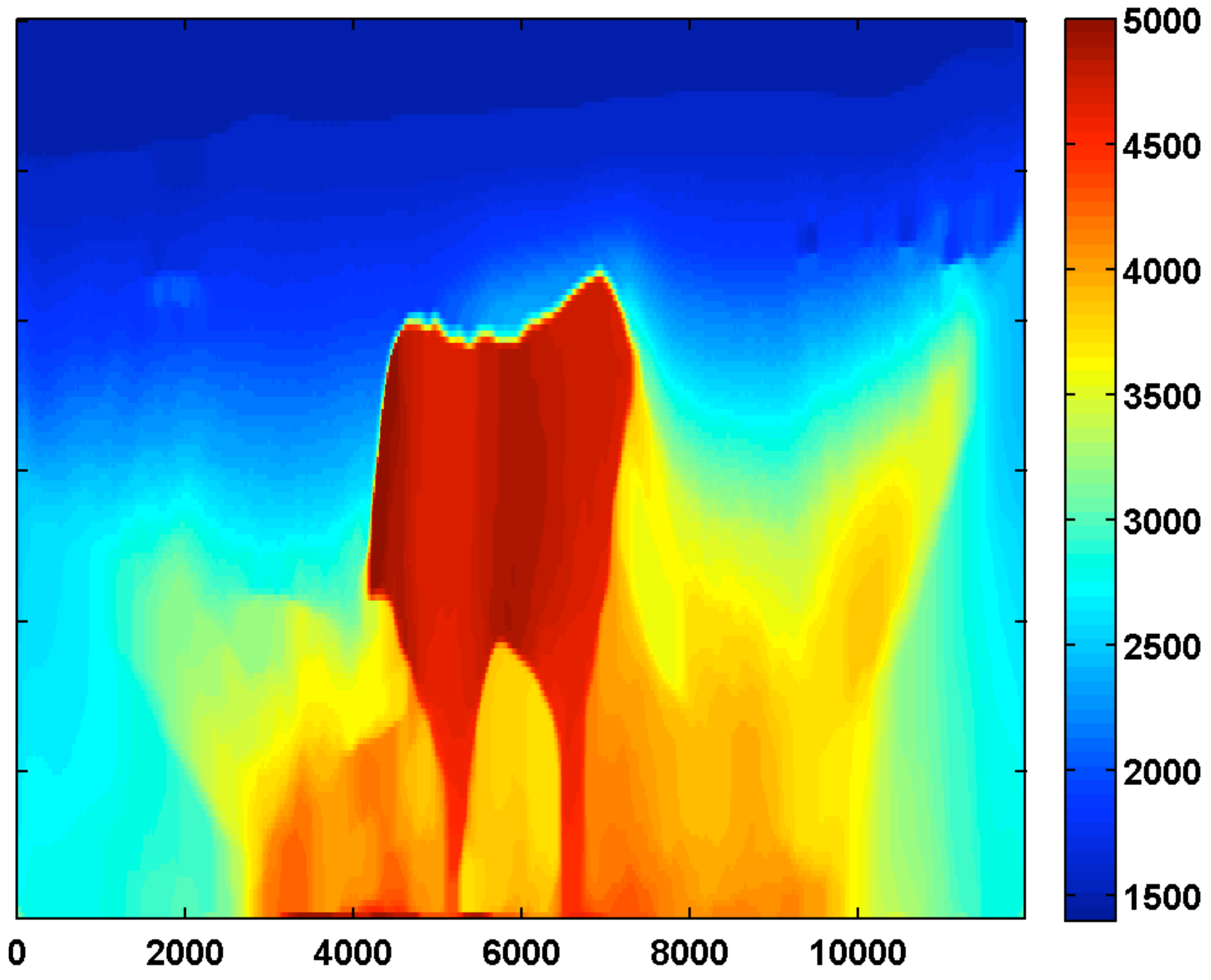
after four cycles through the frequencies



after five cycles through the frequencies

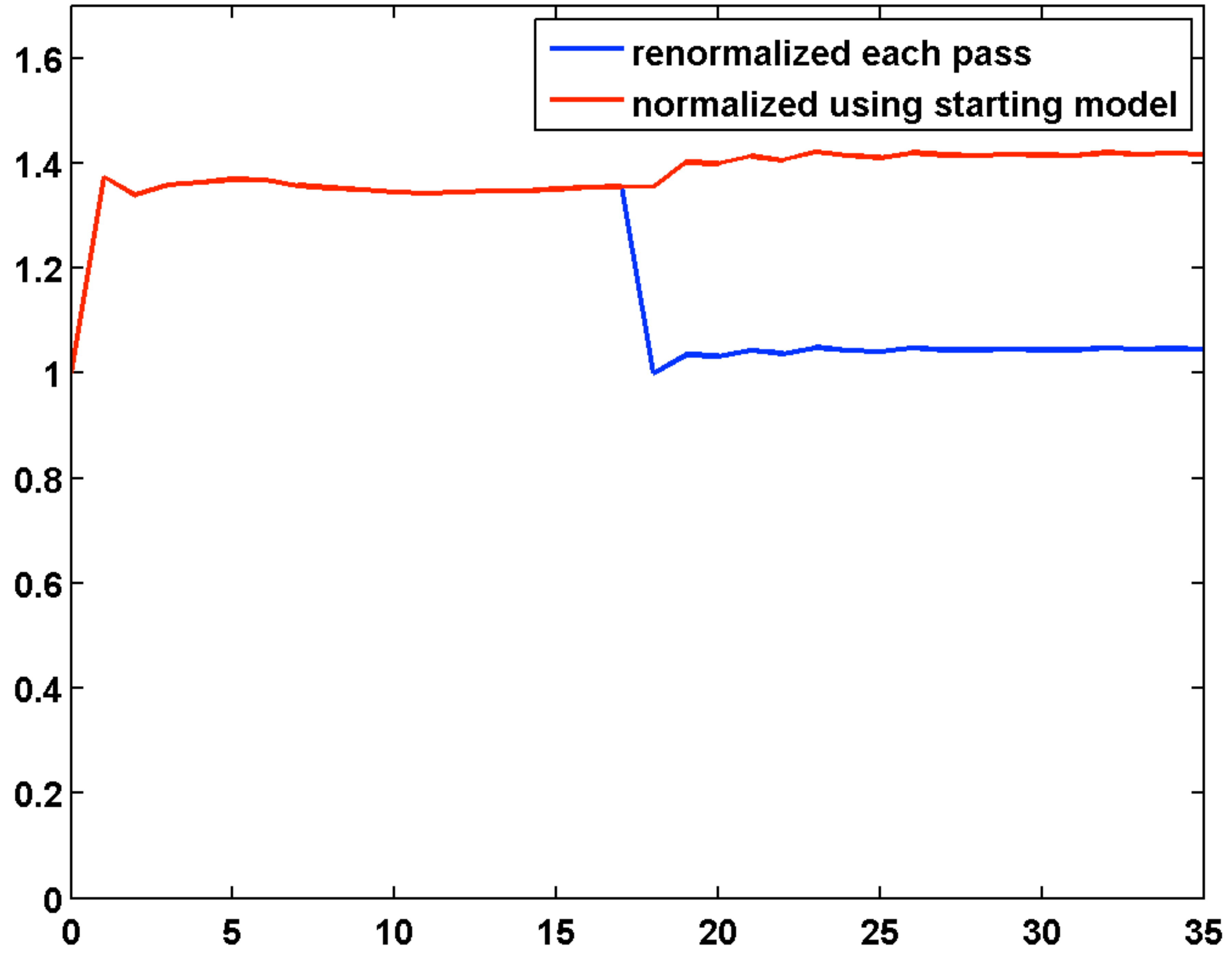


after six cycles through the frequencies

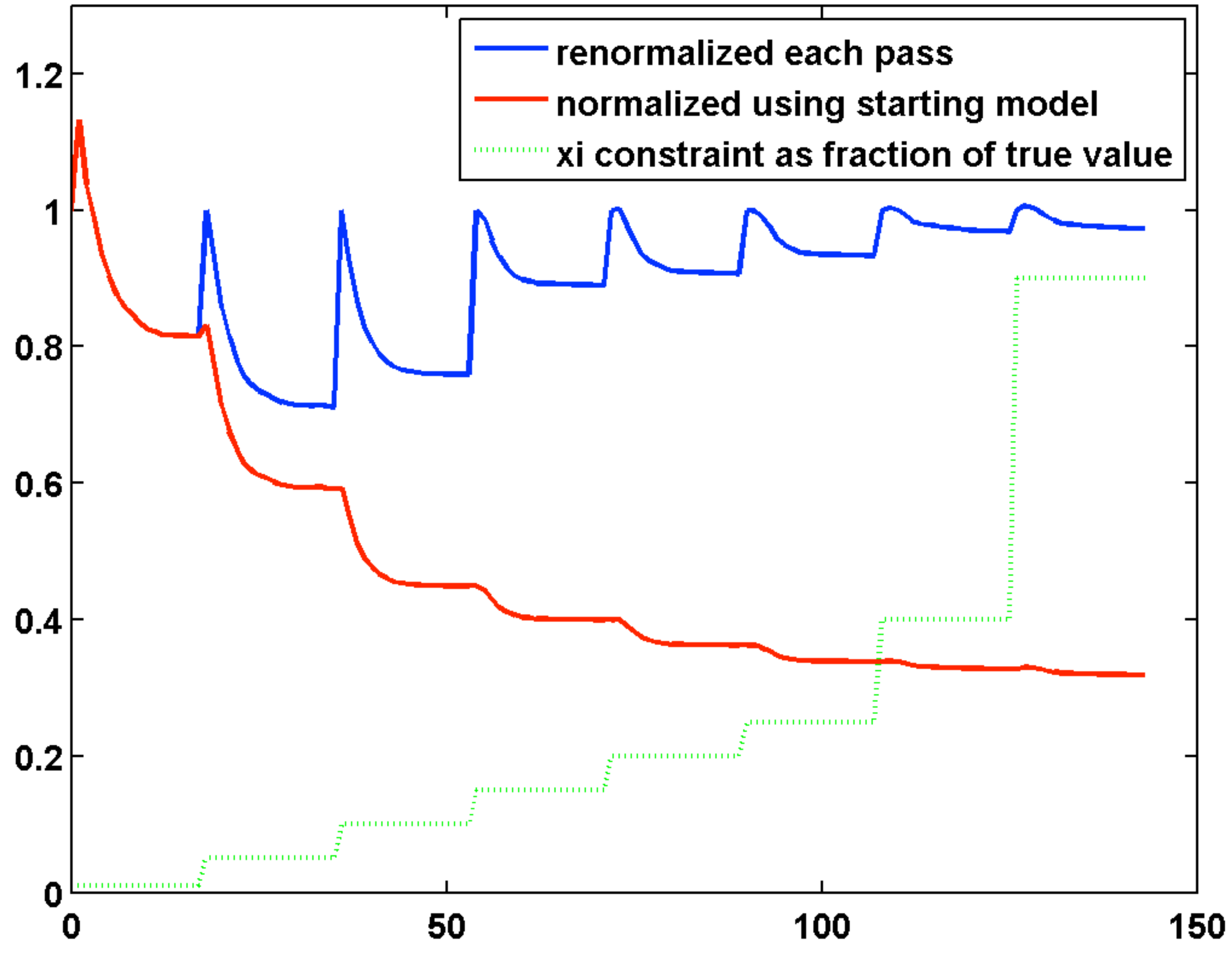


Relative model errors

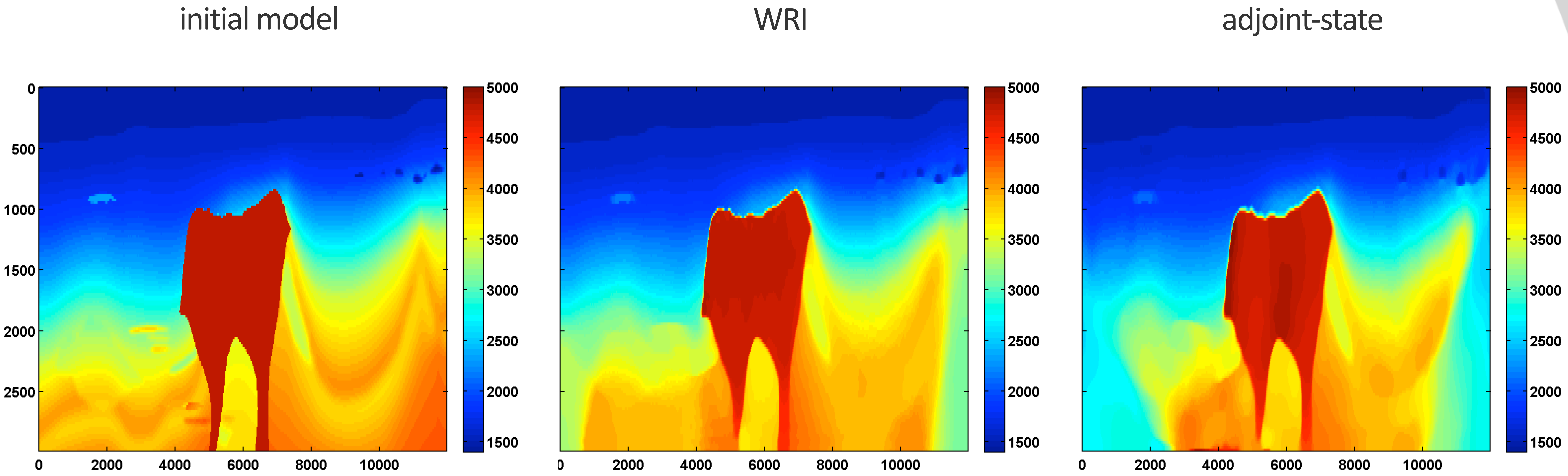
w/o TV



w/ TV & hinge



WRI vs adjoint-state



Conclusions

New method for regularizing wave-equation based inversion benefits from

- combination of convex constraints
- multiple frequency sweeps w/ warm starts & relaxing of the constraints
- a hinge-loss function, which plays a critical role

Works for both WRI & adjoint-state FWI

Development of automatic continuation strategies for relaxing the constraints is ongoing.

Candidate for “automatic” salt flooding...

SLIM

Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

SINBAD

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