A denoising formulation of Full-Waveform Inversion

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Outline

1. Motivation

2. Formulation of the proposed method

3. Algorithm development

4. Numerical tests

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Motivation

Noises in observed data consist of:

modeling error

- spatial and spectral discretization errors
- inaccurate PDE modeling, boundary reflections, multiples
- source estimation error, unknown & interfering sources
- trace truncation error

acquisition error

- timing error
- receiver location error
- measurement noise

interior of the domain

boundary

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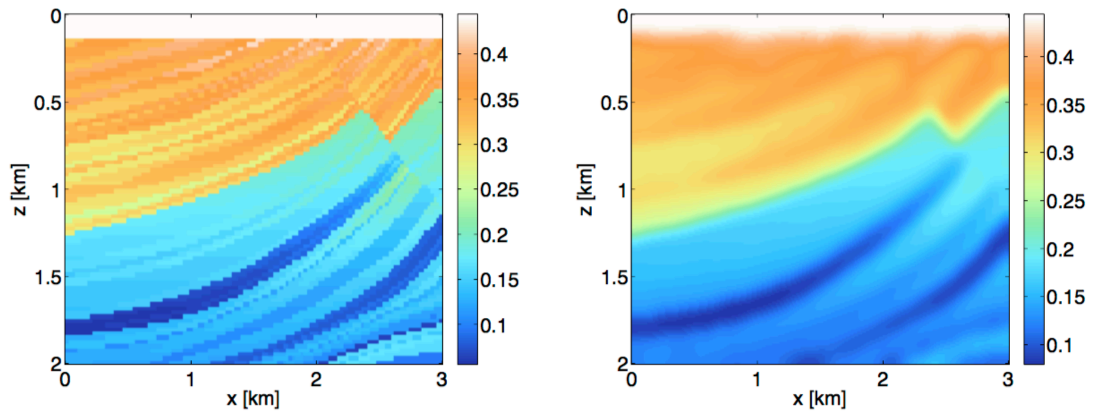
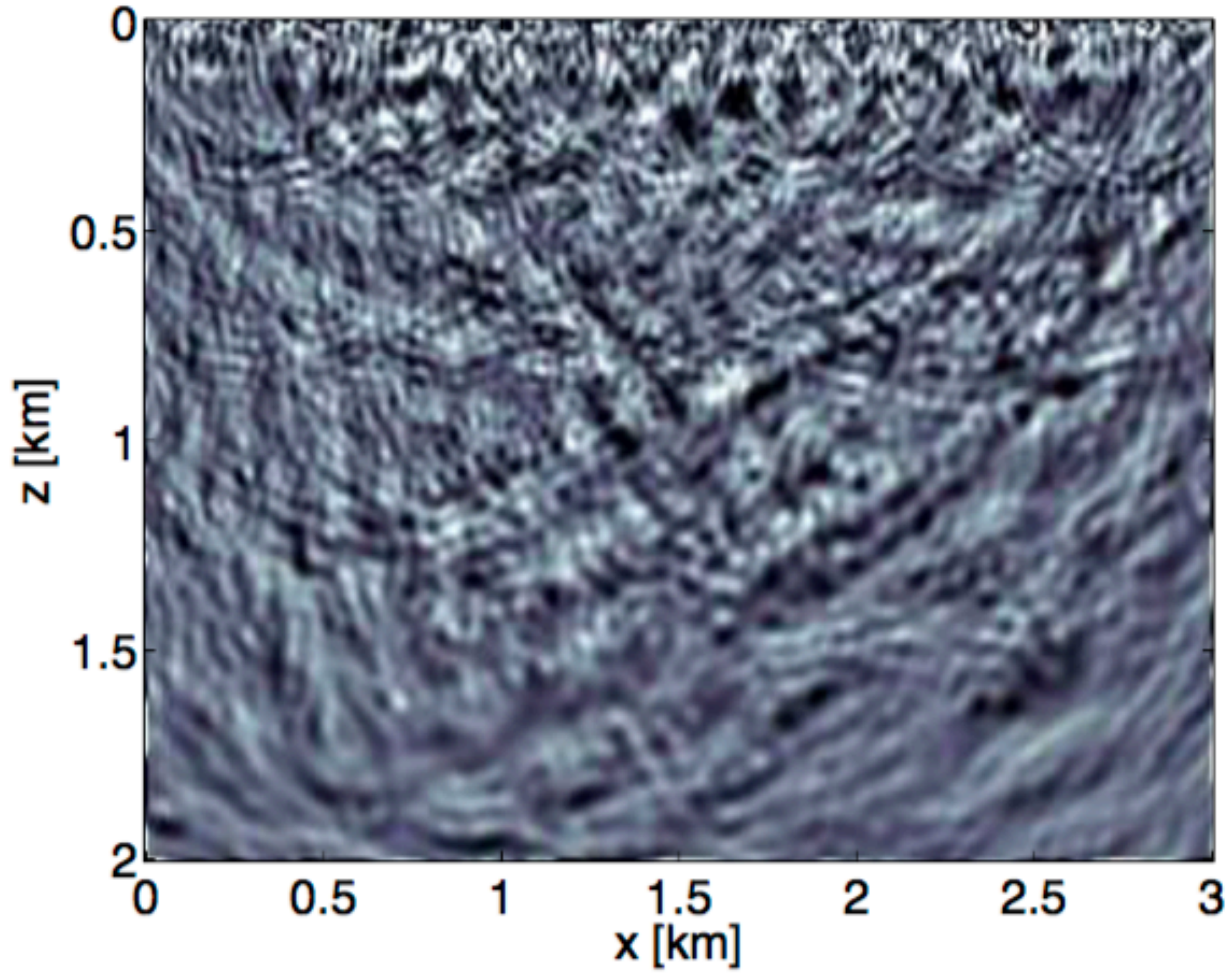


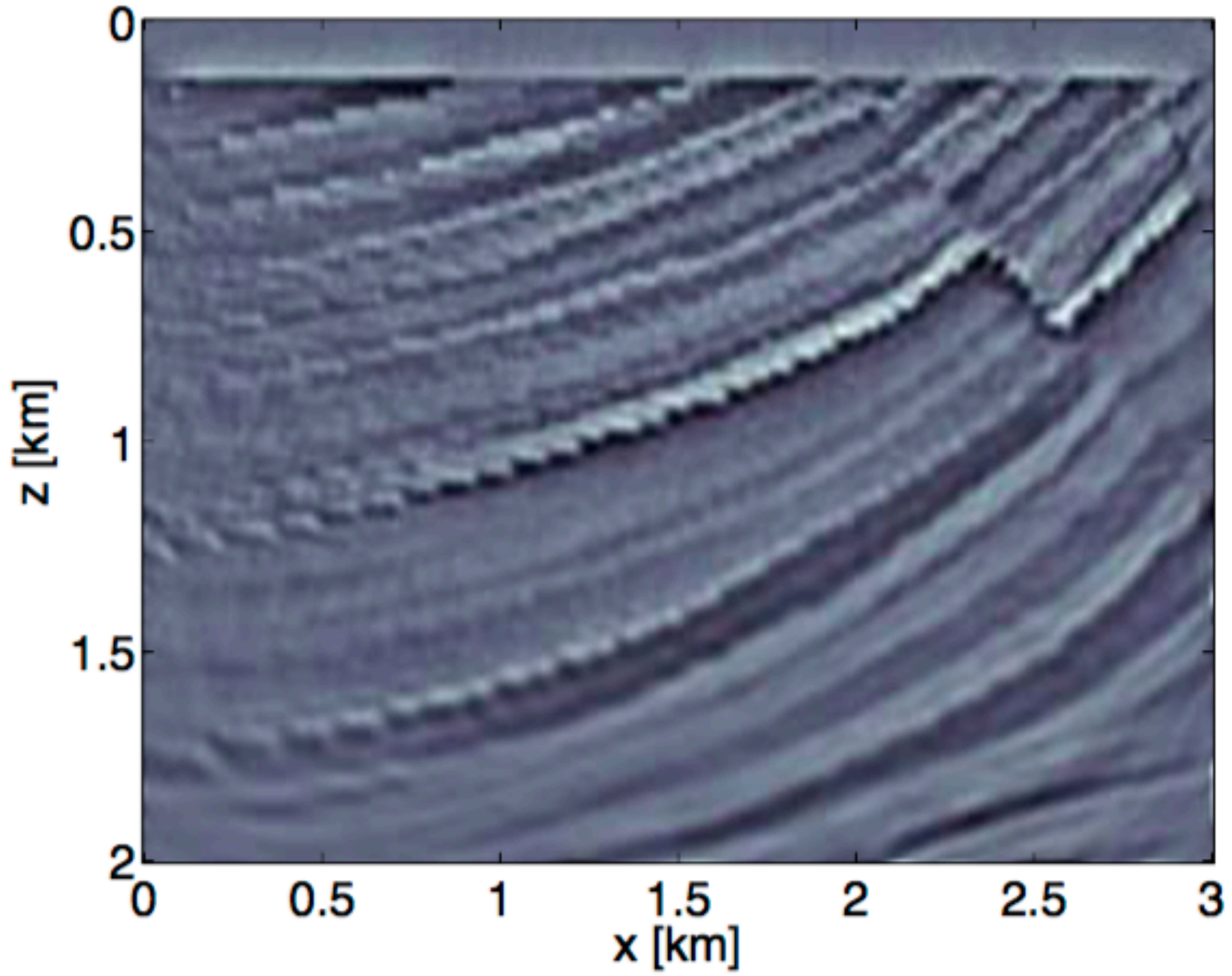
Motivation—the Failures of FWI

When measurement noise is ``spiky"



(a)

Model misfit for FWI



(b)

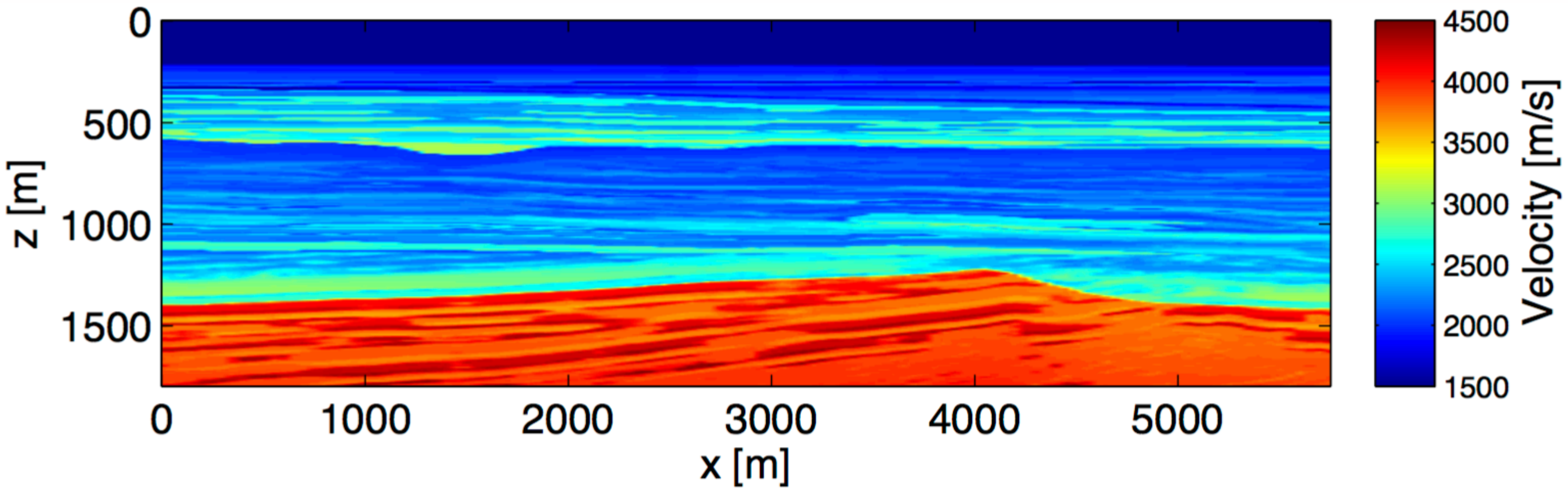
Model misfit for inversion with Student's t penalty

[Peters, A., & Herrmann, F J, 2014]

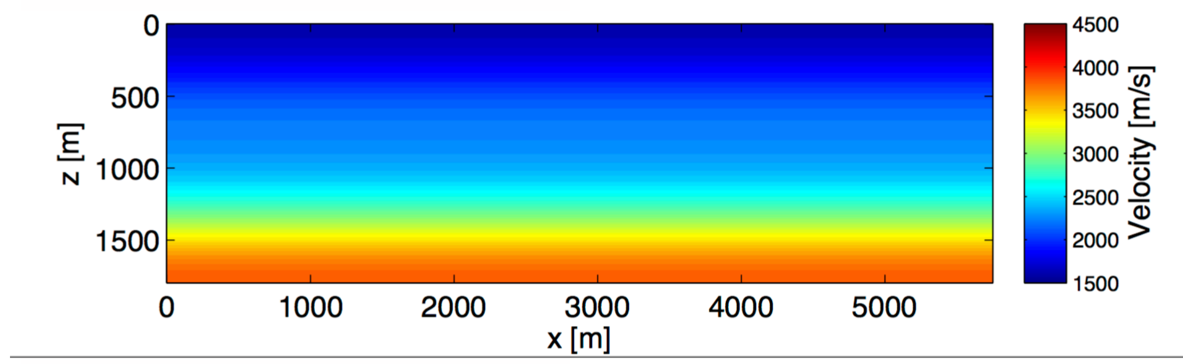
Motivation—the Failures of FWI

When water velocity is wrong

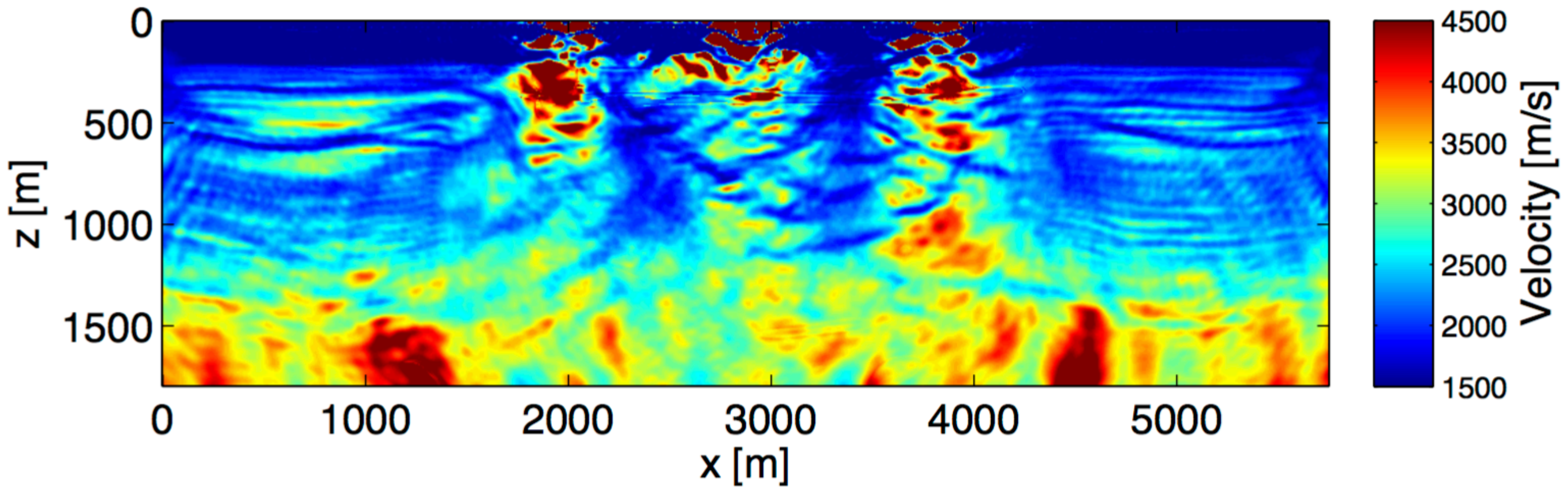
True model



Initial



FWI inversion



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FWI & its relaxation

FWI requires strict satisfaction of the PDE

$$\min_{m, u_i, i=1,\ldots,n_s} \sum_{i}^{n_s} \|P_{\Omega_i} u_i - d_i\|_2^2$$

$$\text{subject to } A(m)u_i = q_i, i = 1, \ldots, n_s$$

P_{Ω_i} : restriction operator

q_i : ith source

A : Time stepping or Helmholtz operator

d_i : Observed data for the ith source

u_i : wavefield associated to the ith source

- Implicitly assumed that noise is Gaussian distributed along sources and receivers
- Neglects modeling errors
- Cannot accommodate prior information of noise level
- Becomes problematic when water velocity is wrong

FWI & its relaxation

Direct relaxation of PDE constraint

$$\min_{m, u_i, i=1,\ldots,n_s} \sum_i \|P_{\Omega_i} u_i - d_i\|_2$$

$$\text{subject to } \|A(m)u_i - q_i\|_2 \leq \epsilon, i = 1, \ldots, n_s$$

FWI & its relaxation

Direct relaxation of PDE constraint

$$\min_{m, u_i, i=1,\dots,n_s} \sum_i \|P_{\Omega_i} u_i - d_i\|_2$$

Hard to choose!

$$\text{subject to } \|A(m)u_i - q_i\|_2 \leq \epsilon, i = 1, \dots, n_s$$

FWI & its relaxation

Direct relaxation of PDE constraint

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Hard to choose!

Flip the objective and the constraint

$$\min_{m,u_i,i=1,\ldots,n_s} \|A(m)u_i - q_i\|_2^2$$

$$\text{subject to } \|P_{\Omega_i} u_i - d_i\|_2 \leq \epsilon_i, i = 1, \ldots, n_s$$

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$$\min_{m,u_i,i=1,\dots,n_s} \|A(m)u_i - q_i\|_2^2$$

Noise level

$$\text{subject to } \|P_{\Omega_i} u_i - d_i\|_2 \leq \epsilon_i, i = 1, \dots, n_s$$

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Noise level

$$\text{subject to } \|P_{\Omega_i} u_i - d_i\|_2 \leq \epsilon_i, i = 1, \dots, n_s$$

Decomposing the wavefields variable

$$u_i = P_{\Omega_i^c}^T P_{\Omega_i^c} u_i + P_{\Omega_i}^T P_{\Omega_i} u_i$$

Boundary part

Interior part

FWI & its relaxation

Direct relaxation of PDE constraint

$$\min_{m,u_i,i=1,\dots,n_s} \sum_i \|P_{\Omega_i} u_i - d_i\|_2$$

Hard to choose!

$$\text{subject to } \|A(m)u_i - q_i\|_2 \le \epsilon, i = 1, \dots, n_s$$

Flip the objective and the constraint

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Noise level

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Decomposing the wavefields variable

$$u_i = P_{\Omega_i^c}^T \underbrace{P_{\Omega_i^c} u_i}_{v_i} + P_{\Omega_i}^T \underbrace{P_{\Omega_i} u_i}_{b_i}$$

Interior part

Boundary part

FWI & its relaxation

Direct relaxation of PDE constraint

$$\min_{m,u_i,i=1,\ldots,n_s} \sum_i \|P_{\Omega_i} u_i - d_i\|_2$$

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$$\min_{m,u_i,i=1,\ldots,n_s} \|A(m)u_i - q_i\|_2^2$$

$$\text{subject to } \|P_{\Omega_i} u_i - d_i\|_2 \leq \epsilon_i, i = 1, \ldots, n_s$$

Noise level

Decomposing the wavefields variable

$$u_i = \underbrace{P_{\Omega_i^c}^T \overbrace{P_{\Omega_i^c} u_i}^{v_i}}_{\text{Interior part}} + \underbrace{P_{\Omega_i}^T \overbrace{P_{\Omega_i} u_i}^{b_i}}_{\text{Boundary part}}$$

Interior part

Boundary part

$$\min_{m,b_i,v_i,i=1,\ldots,n_s} \|A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i) - q_i\|_2^2$$

$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i, i = 1, \ldots, n_s$$

The denoising formulation (FWIDN)

We call it the denoising formulation of FWI

$$\min_{m, b_i, v_i, i=1,\ldots,n_s} \|A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i) - q_i\|_2^2$$

$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i, i = 1, \ldots, n_s$$

Pros:

- allows noise levels ϵ_i to vary with sources, and allows $\epsilon_i = 0$
- ensures reasonable PDE fidelity while preventing overfit
- all pros of WRI

Cons: algorithmically & computationally more demanding

FWI-DN – a more general form

Weighted/preconditioned least-squares objective

$$\min_{m,b_i,v_i,i=1,\dots,n_s} \|\mathcal{D}_z(A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i) - q_i)\|_2^2$$
$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i, i = 1, \dots, n_s$$

- $\mathcal{D}_z$ reshapes PDE misfit distribution
- Impose looser PDE constraint at shallow part where the model is "noisier"
- Examples of $\mathcal{D}_z$: linear depth weighting, two-level depth weighting

$$\mathcal{D}_z f(x,z) = z f(x,z) \qquad \mathcal{D}_z f(x,z) = \chi_{z<z_0} f(x,z) + 2\chi_{z\geq z_0} f(x,z)$$

Solving FWI-DN

Strategy: alternatively update m and $b_i, i = 1, ..., n_s$

At iteration k,

1. fix m^k, solve for $b_i^{k+1}, i = 1, ..., n_s$ from

$$(b_i^{k+1}, v_i^{k+1}) = \arg \min_{b_i, v_i, i=1,\dots,n_s} \|\mathcal{D}_z(A(m^k)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i) - q_i)\|_2^2 \quad (P_d)$$
$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i, i = 1, ..., n_s$$

2. for fixed $b_i^{k+1}, i = 1, ..., n_s$, update m^k by solving T steps of

$$\min_{m, v_i, i=1,\dots,n_s} \sum_{i=1}^{n_s} \|\mathcal{D}_z(A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i^{k+1}) - q_i)\|_2^2 \quad (P_m)$$

Solving for (P_d) — a denoising step

(P_d) Fix m^k, solve for b_i^{k+1} from

$$(b_i^{k+1}, v_i^{k+1}) = \arg\min_{b_i, v_i} \|\mathcal{D}_z(A(m^k)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i) - q_i)\|_2^2$$

$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i,$$

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$$\text{subject to } \|b_i - d_i\|_2 \leq \epsilon_i,$$

The Lagrangian dual of (P_d) is

$$\max_{\lambda \geq 0} \phi(\lambda)$$

where

$$\phi(\lambda) = \min_{u_i} \|\mathcal{D}_z(A(m)u_i - q_i)\|_2^2 + \lambda\|P_{\Omega_i} u_i - d_i\|_2^2 - \lambda\epsilon_i$$

Strong duality principle [More, 1993] guarantees primal & dual optimality agree

Solving for (P_d) — a denoising step

(P_d) Fix m^k, solve for b_i^{k+1} from

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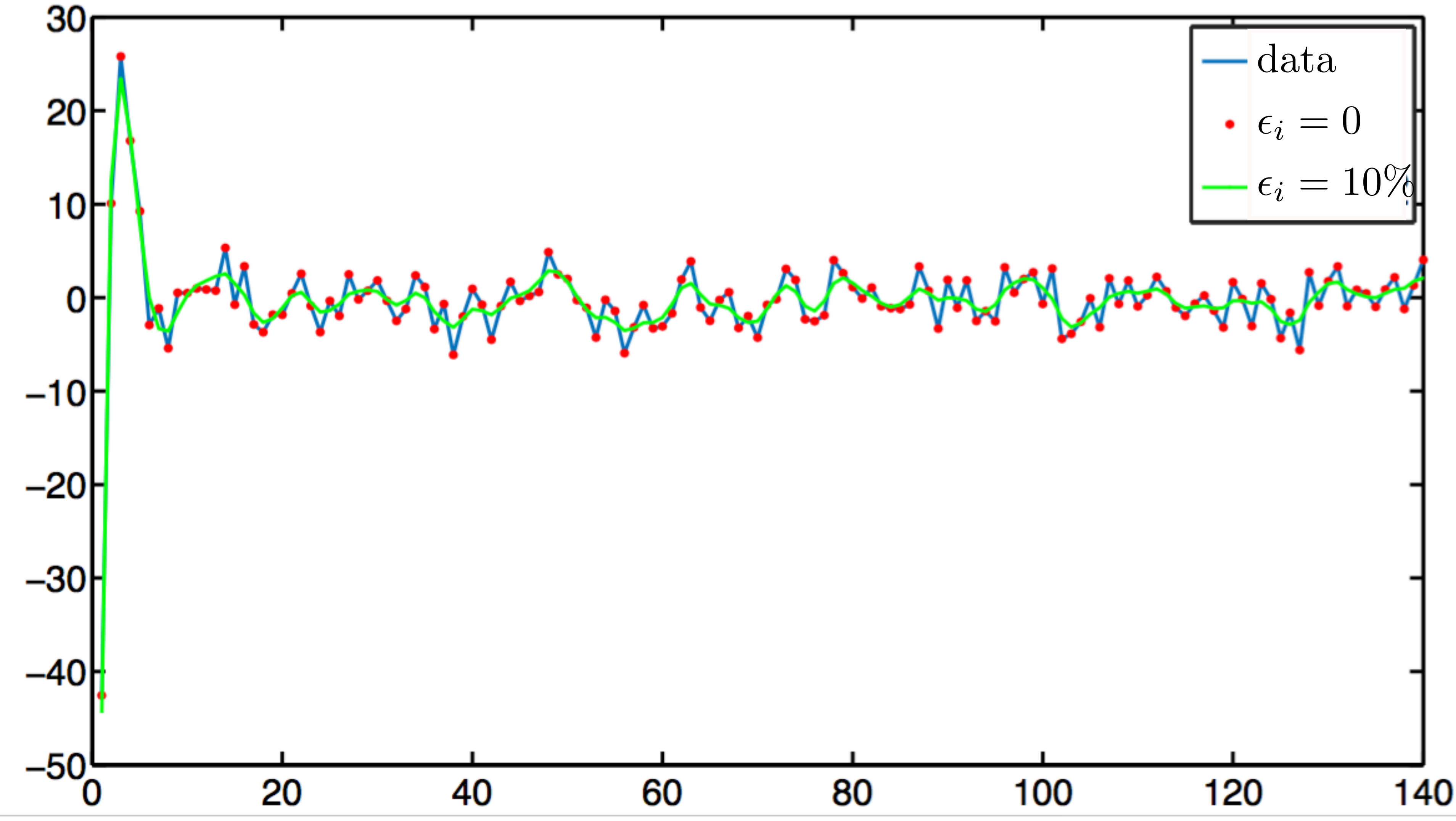

Denoising effect of (P_d)

The Lagrangian dual of (P_d) is

$$\max_{\lambda \ge 0} \phi(\lambda)$$

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Strong duality principle [More, 1993] guarantees primal & dual optimality agree

Solving for (P_d) — a denoising step

$\phi(\lambda)$ has closed-form gradient & Hessian

$$\phi'(\lambda) = \|P_{\Omega_i}\bar{u}_i(\lambda) - d_i\|_2^2 - \epsilon_i$$

$$\phi''(\lambda) = -2(P_{\Omega_i}\bar{u}_i(\lambda) - d_i)^T P_{\Omega_i} C^{-1} P_{\Omega_i}^T (P_{\Omega_i}\bar{u}_i - d_i)$$

where

$$C = A(m)^T \mathcal{D}_z^T \mathcal{D}_z A(m) + \lambda P_{\Omega_i}^T P_{\Omega_i} \quad \bar{u}_i(\lambda) = \begin{bmatrix} \mathcal{D}_z(A(m^k)) \\ \sqrt{\lambda} P_{\Omega_i} \end{bmatrix}^\dagger \begin{bmatrix} \mathcal{D}_z(q_i) \\ \sqrt{\lambda} d_i \end{bmatrix}$$

Newton steps for λ

$$\lambda^{k+1} = \lambda^k - \phi'(\lambda)/\phi''(\lambda)$$

After finding the minimizer λ^*, the primal optimizers are

$$v_i^{k+1} = P_{\Omega_i^c}\bar{u}_i(\lambda^*), \quad v_i^{k+1} = P_{\Omega_i}\bar{u}_i(\lambda^*)$$

Solving for (P_m)

For fixed $b_i^{k+1}, i = 1, ..., n_s$, update m^k by solving T steps of

$$(P_m) \quad \min_{m, v_i, i=1,\dots,n_s} \sum_{i=1}^{n_s} \|\mathcal{D}_z(A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i^{k+1}) - q_i)\|_2^2$$

Solving for (P_m)

For fixed $b_i^{k+1}, i = 1, ..., n_s$, update m^k by solving T steps of

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Solve by variable projection

[Aravkin and van Leeuwen, 2012]

Solving for (P_m)

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Solve by variable projection

[Aravkin and van Leeuwen, 2012]

$$\min_{m, v_i, i=1,\ldots,n_s} \sum_{i=1}^{n_s} \|\mathcal{D}_z(A(m)(P_{\Omega_i^c}^T v_i + P_{\Omega_i}^T b_i^{k+1}) - q_i)\|_2^2 \equiv f(m, v_1, \ldots v_{n_s})$$

$$\Longleftrightarrow \min_{m, v_i, i=1,\ldots,n_s} f(m, v_1, ..., v_{n_s}) = \min_m f(m, \bar{v}_1, ..., \bar{v}_{n_s}) \qquad (\bar{v}_1, ..., \bar{v}_{n_s}) = \arg\min_{v_1,\ldots,v_{n_s}} f(m, v_1, ..., v_{n_s})$$

Algorithm and Complexity

Inputs: $m_0, d_i, q_i, \quad i = 1, ..., n_s$, T, K

For $\omega = \omega_1, ..., \omega_n$ do

solve (P_d) using T iterations of Newton updates on λ

performs K gradient or L-BFGS updates on m towards the minimizer of (P_m)

Endfor

On average, 1 update of m requires: 2 PDE solves for FWI

2 PDE solves for WRI

3-5 PDE solves for FWI-DN

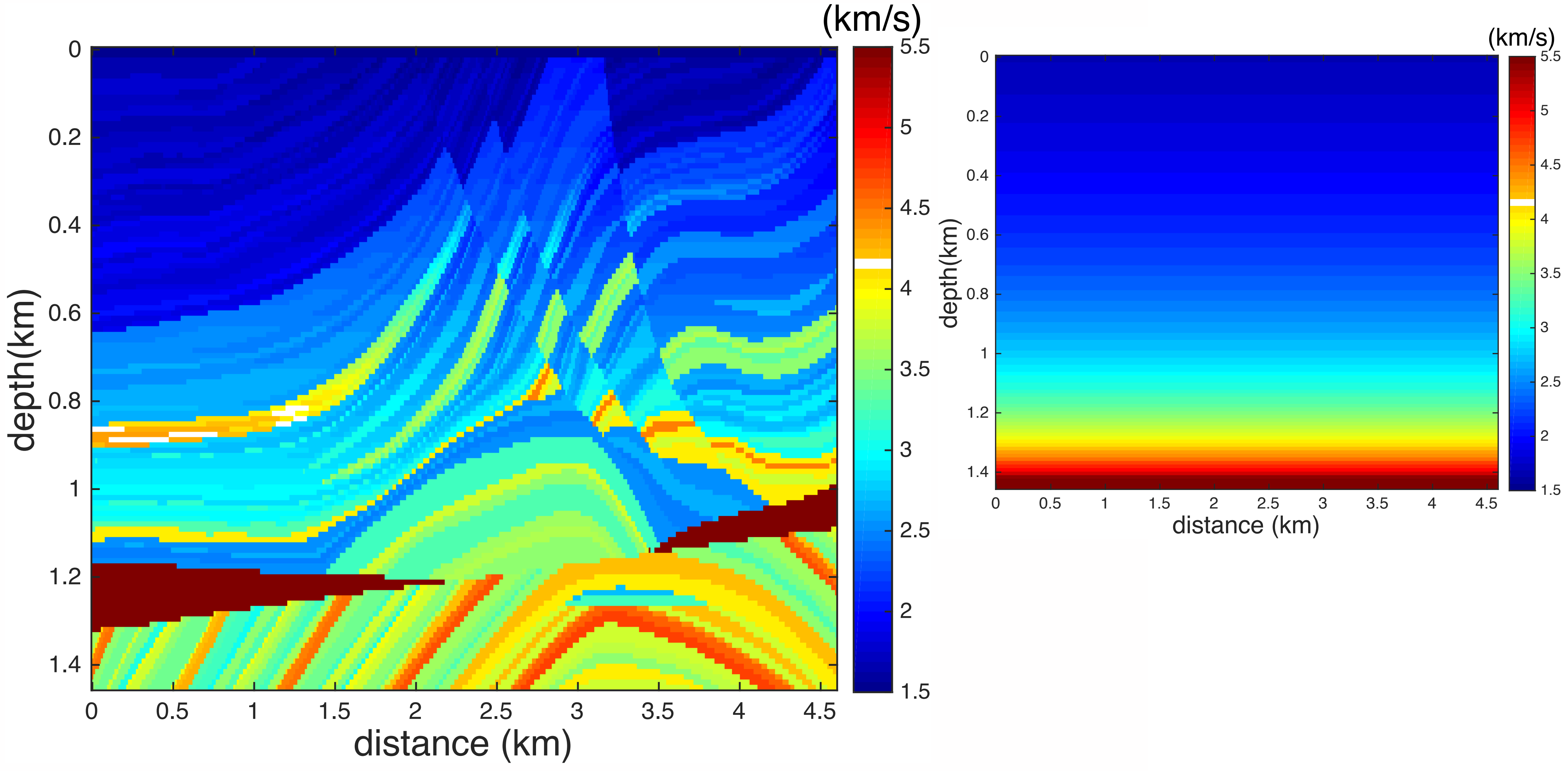
Case Study

Test 1: robustness under non-uniform noise along sources

Test 1

- Frequency continuation using batches [3,3.5][3.5,4]....[14.5,15]Hz
- source spacing : 240m
- receiver spacing : 48m
- SNR=0 for low frequency data 3-10Hz
- SNR=25dB for high frequency data 10-15Hz
- Linear depth weighting
- noise level $\epsilon_{s_j} = 3\epsilon_{s_i}$ where

$$i = 1, ..., \lfloor n_s/2 \rfloor, j = \lfloor n_s/2 \rfloor + 1, ..., n_s$$



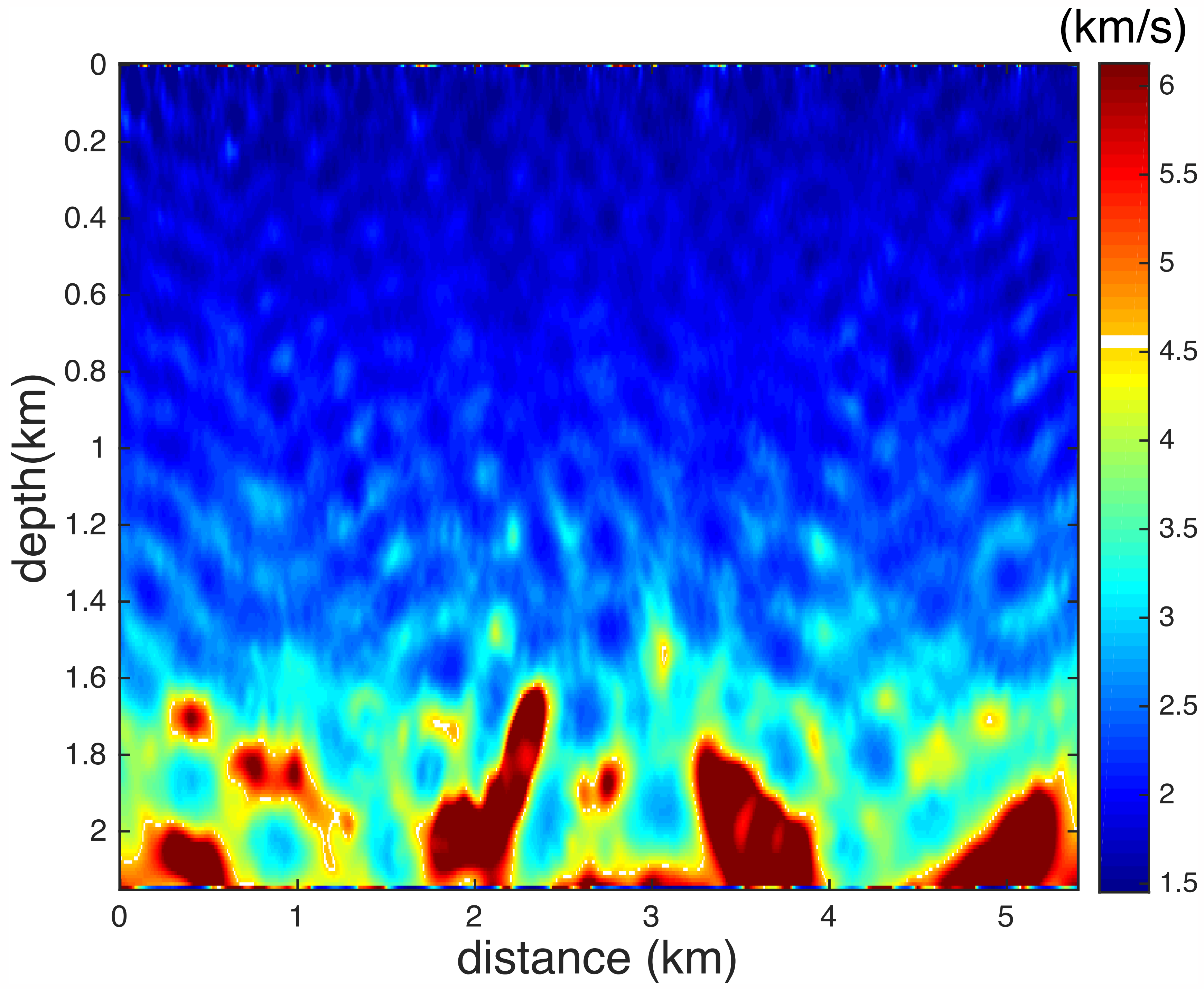
Test 1

Method for comparison: weighted FWI

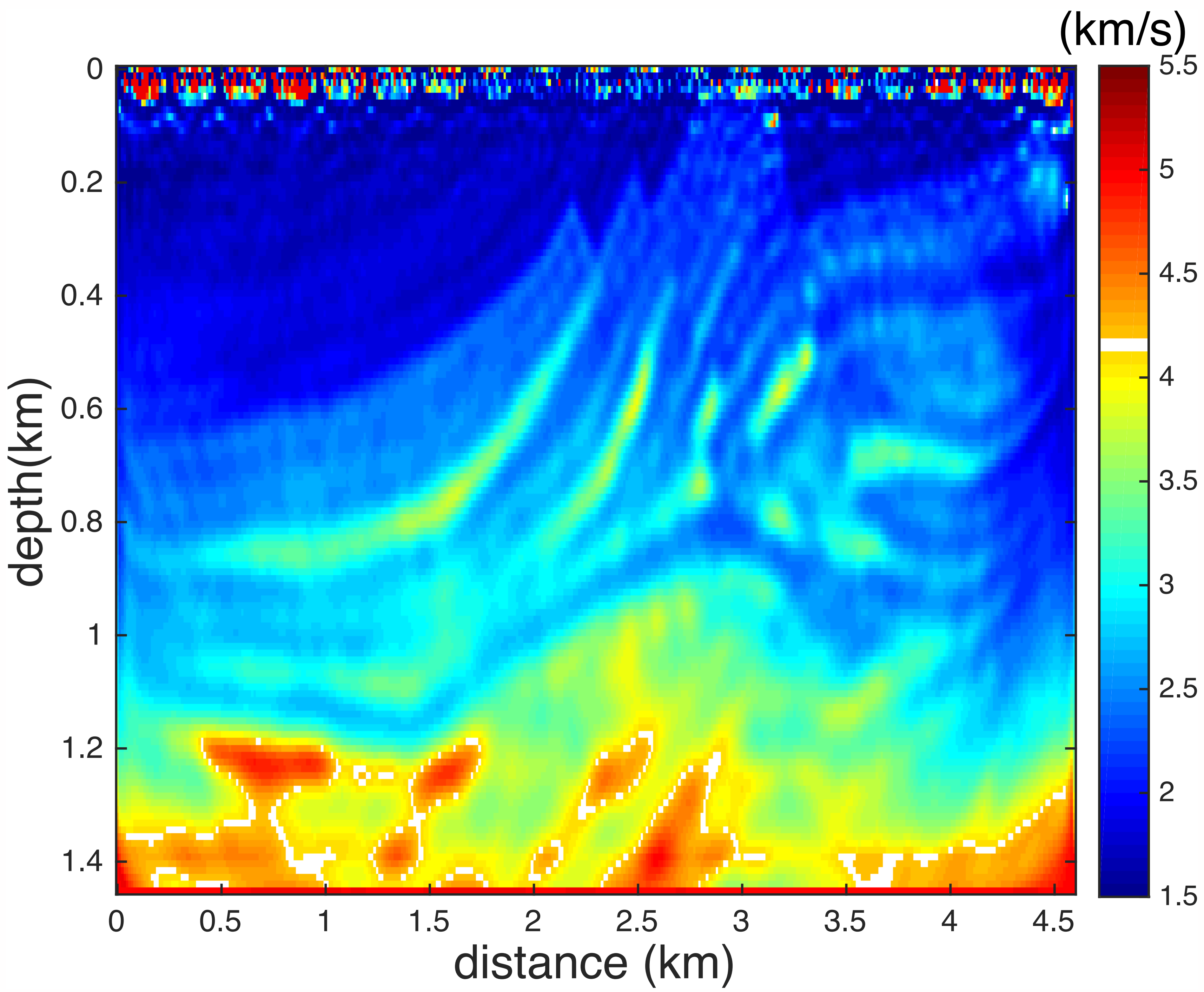
$$\min_{m} \sum_{i \in N_1} 9\|P_{\Omega_i} A^{-1}(m) q_i - d_i\|_2^2 + \min_{m} \sum_{i \in N_2} \|P_{\Omega_i} A^{-1}(m) q_i - d_i\|_2^2$$

where $N_1 = \{1, ..., \lfloor \frac{n_s}{2} \rfloor\}, N_2 = \{\lfloor \frac{n_s}{2} \rfloor + 1, ..., n_s\}$

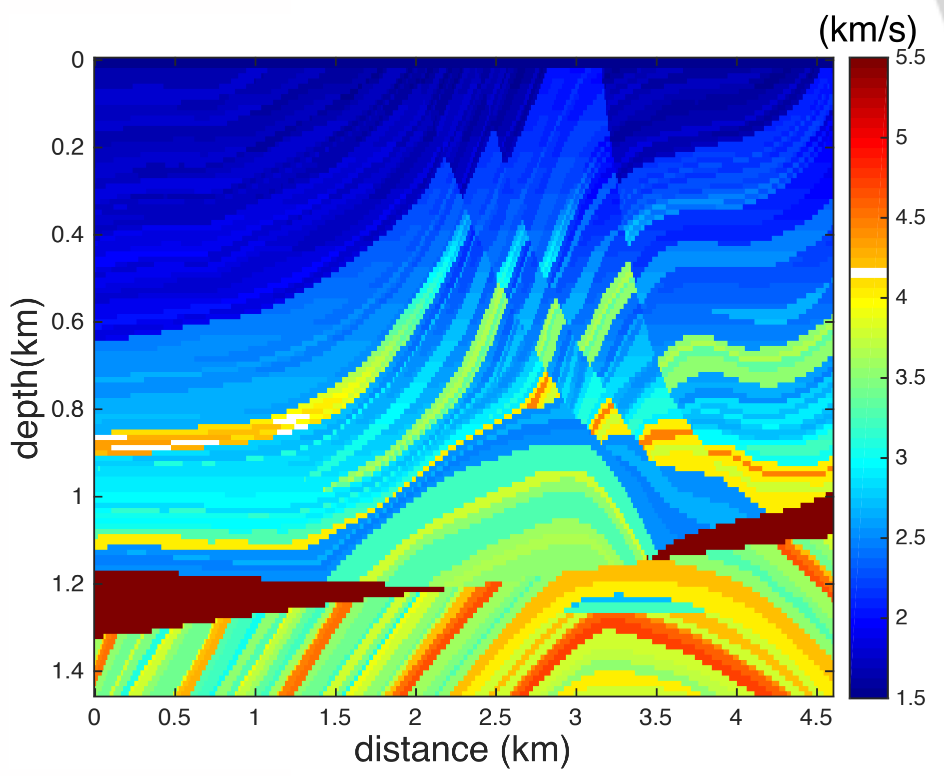
Test 1



Inverted model w/ weighted FWI



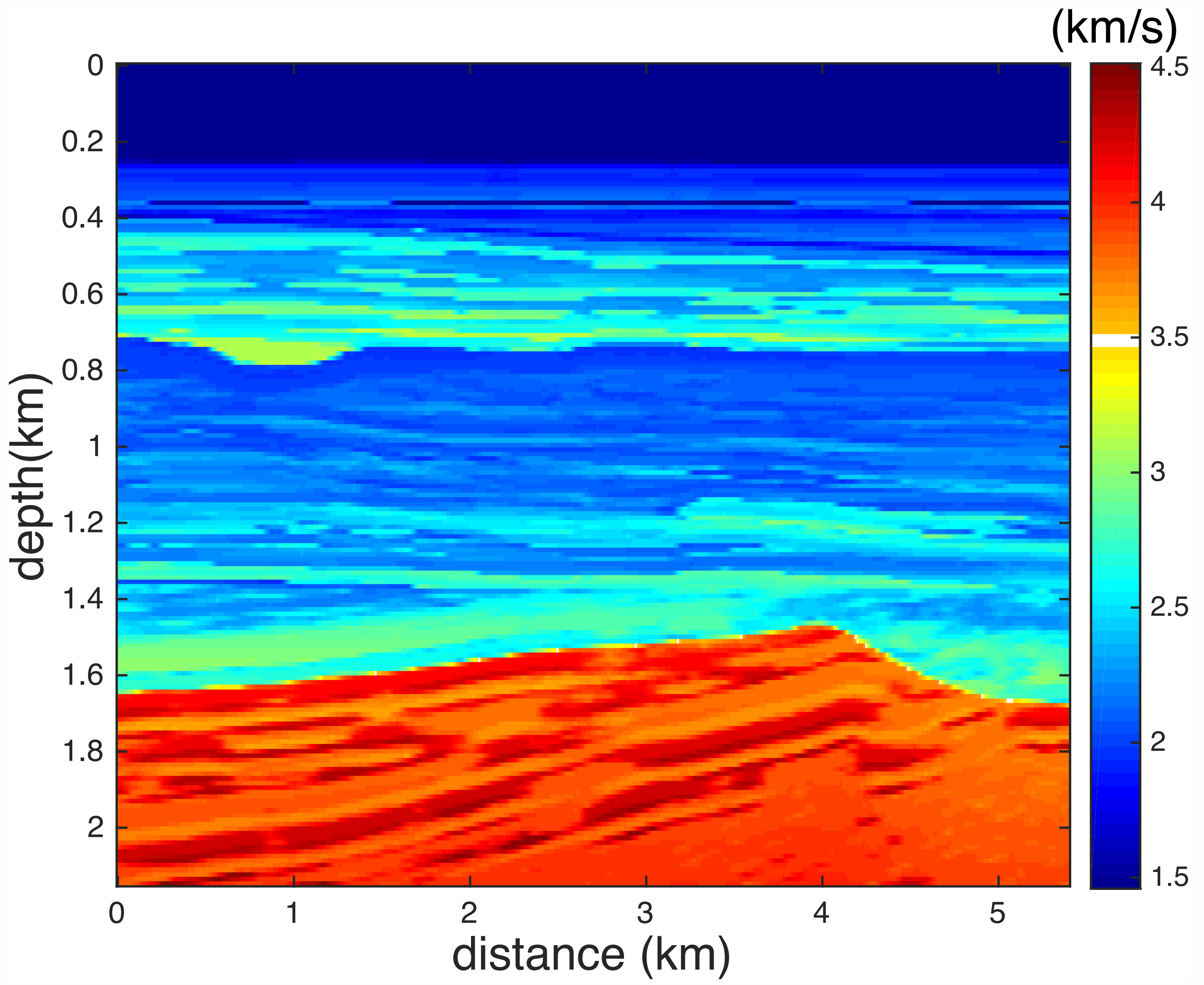
Inverted model w/ FWI-DN

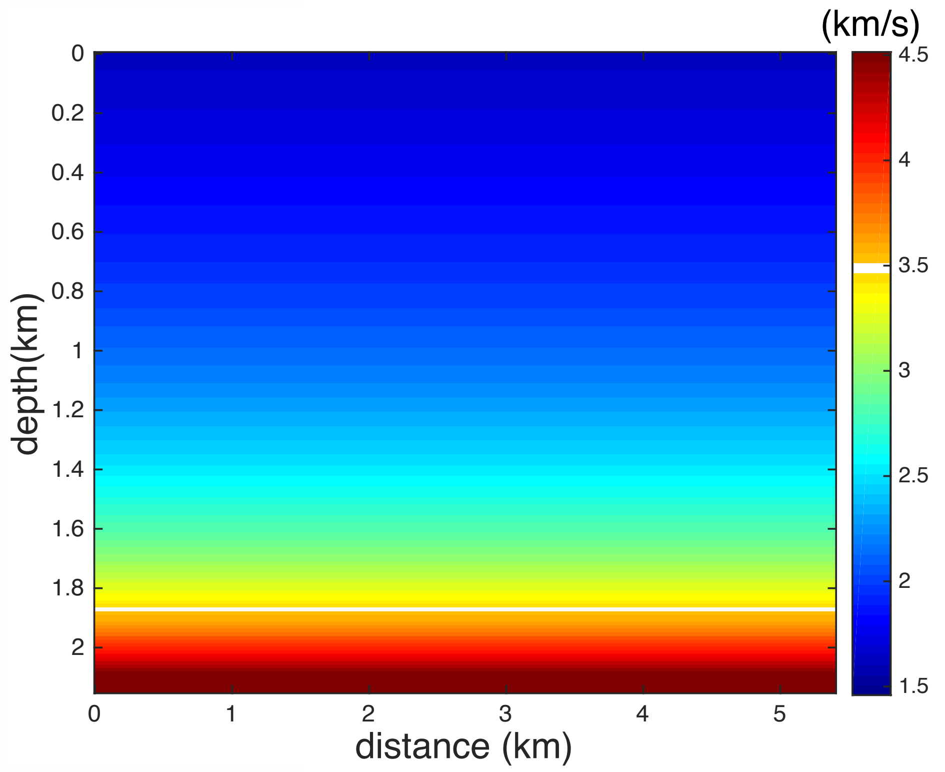


Test 2: robustness under modeling error

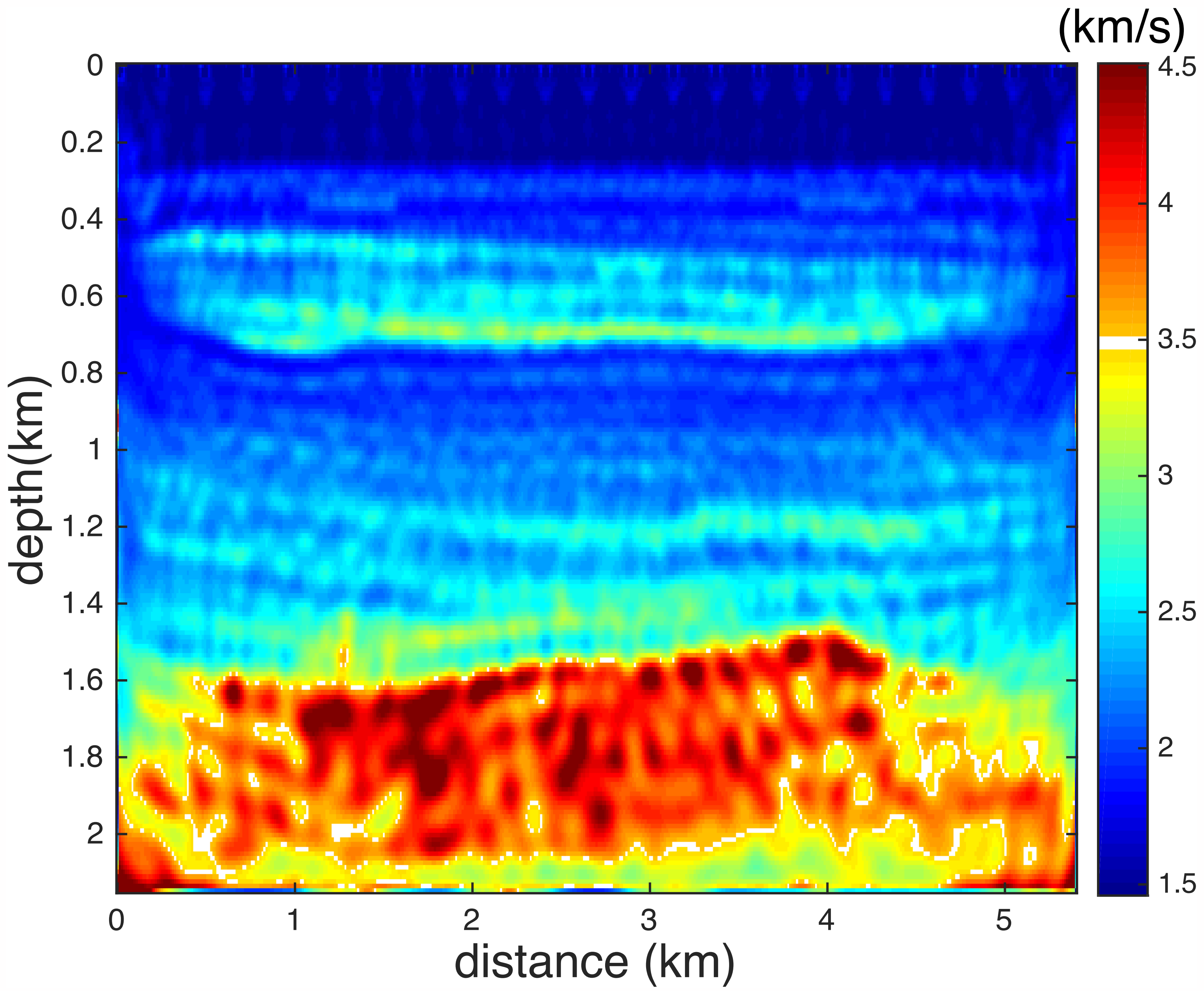
Test 2

- Frequency continuation using batches [3,3.5][3.5,4]....[14.5,15]Hz
- source spacing : 240m
- receiver spacing : 48m
- source depth: 12m
- True source q: ricker wavelet at 10Hz
- Source used for inversion: 0.8q
- Linear depth weighting

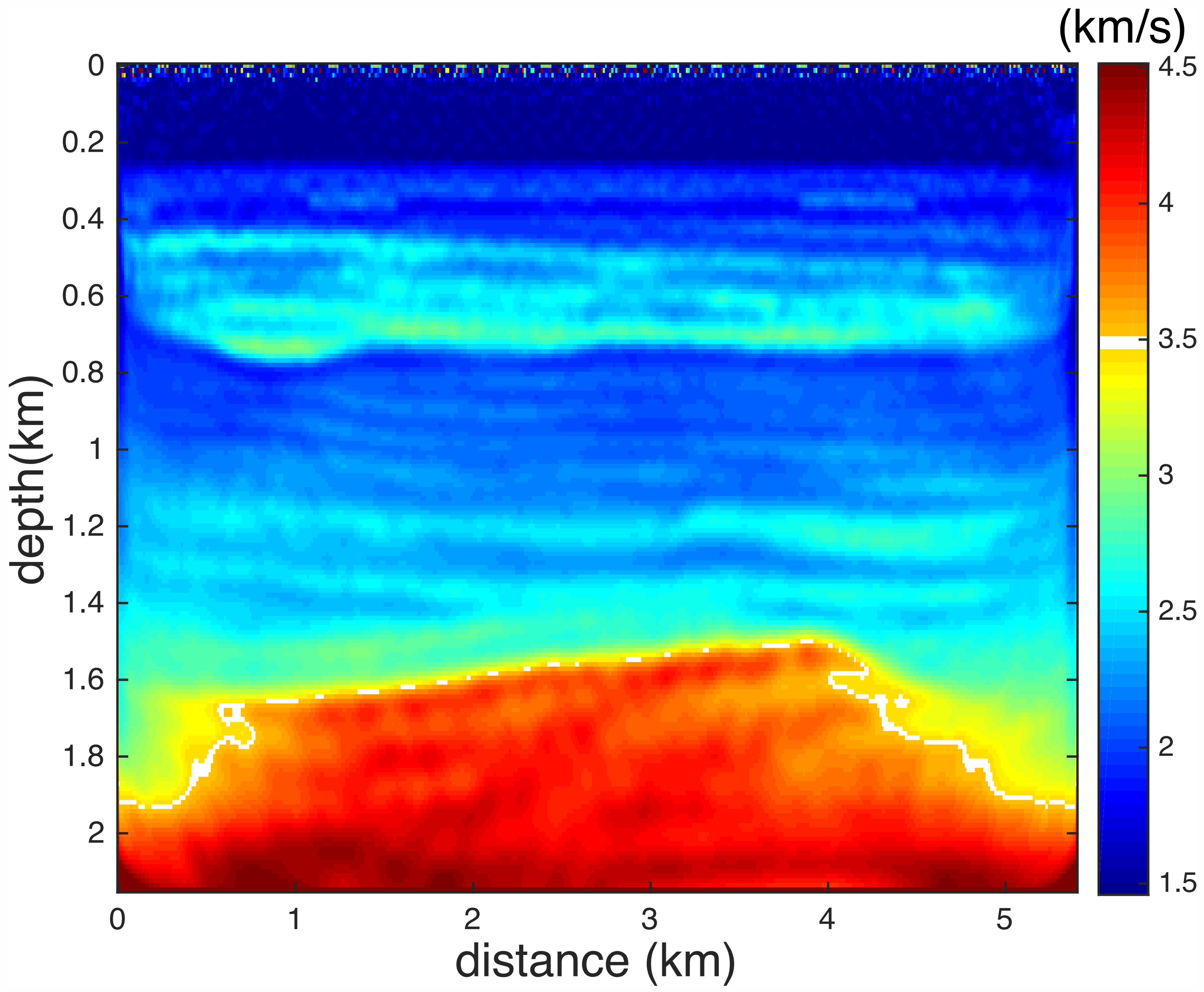




Test 2



Inverted model w/ FWI



Inverted model w/ FWI-DN with $\epsilon = 0$

Conclusion

- We proposed a denoising version of FWI
- We observed weighted/preconditioned PDE misfits dramatically increase robustness to modeling error
- The formulation makes incorporating prior knowledge of noise level convenient w/o increasing too much of complexity

Acknowledgements

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