Frequency down extrapolation with TV norm minimization

Rongrong Wang & Felix J. Herrmann

SLIM
University of British Columbia

Outline

1. Motivation and extrapolation workflow

2. Theoretical understanding of L1 minimization

3. TV minimization to enhance stability of L1

4. Numerical results

Motivation

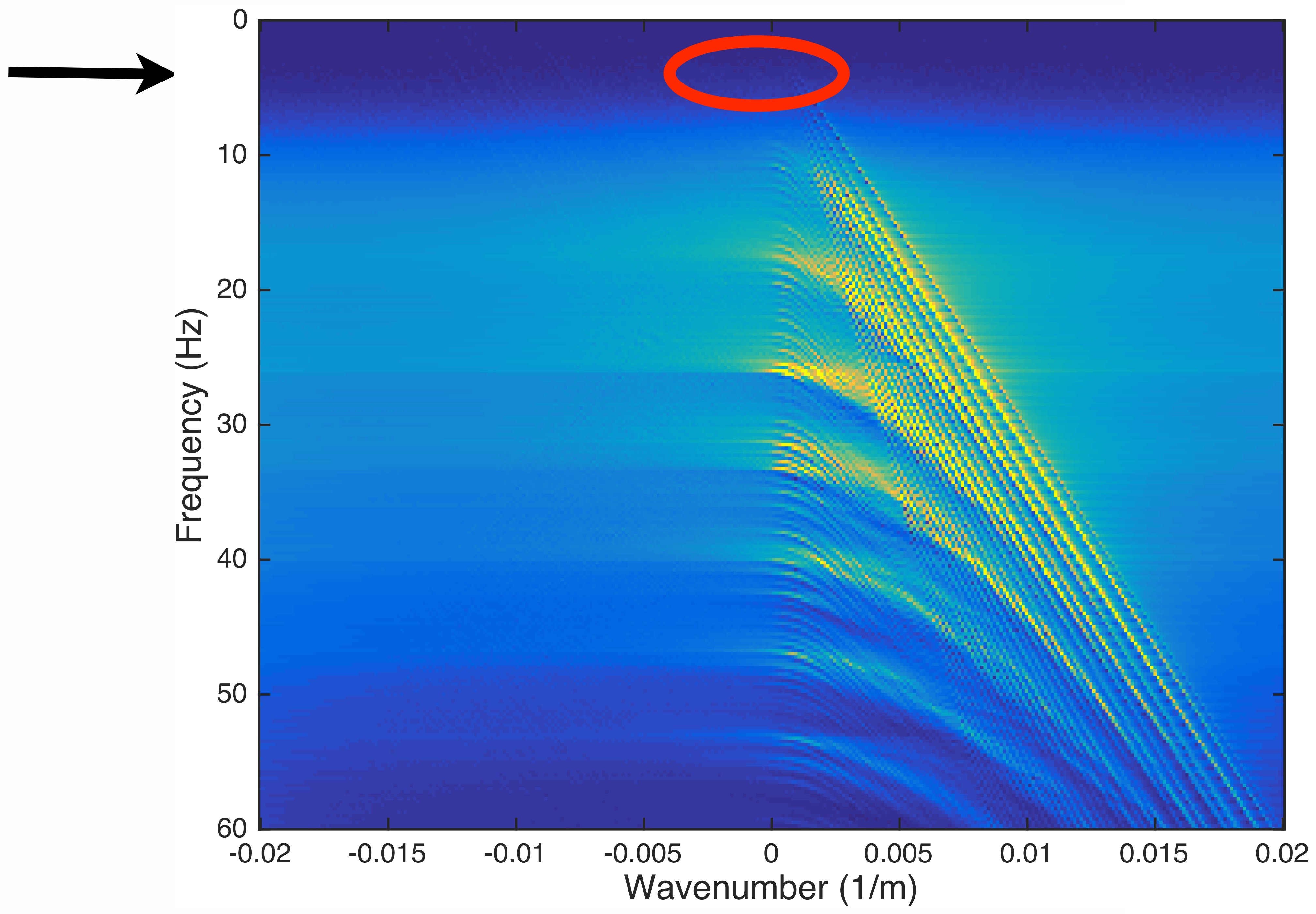
Challenges in FWI:

- High frequency data introduces abundant local minima.
- Field data lacks
  - low frequencies
  - or low frequencies are noisy

Our goal:

use mid-band data to extrapolate towards low frequencies

A shot gather in the f-k domain Chevron (2014)



Convolutional model

Near offset trace

$$\text{Trace} \longleftarrow \mathbf{d}(t) = \overset{\substack{\text{Source} \\ \uparrow}}{\mathbf{w}(t)} * \mathbf{r}(t) \longrightarrow \text{Reflectivity series}$$

Assume

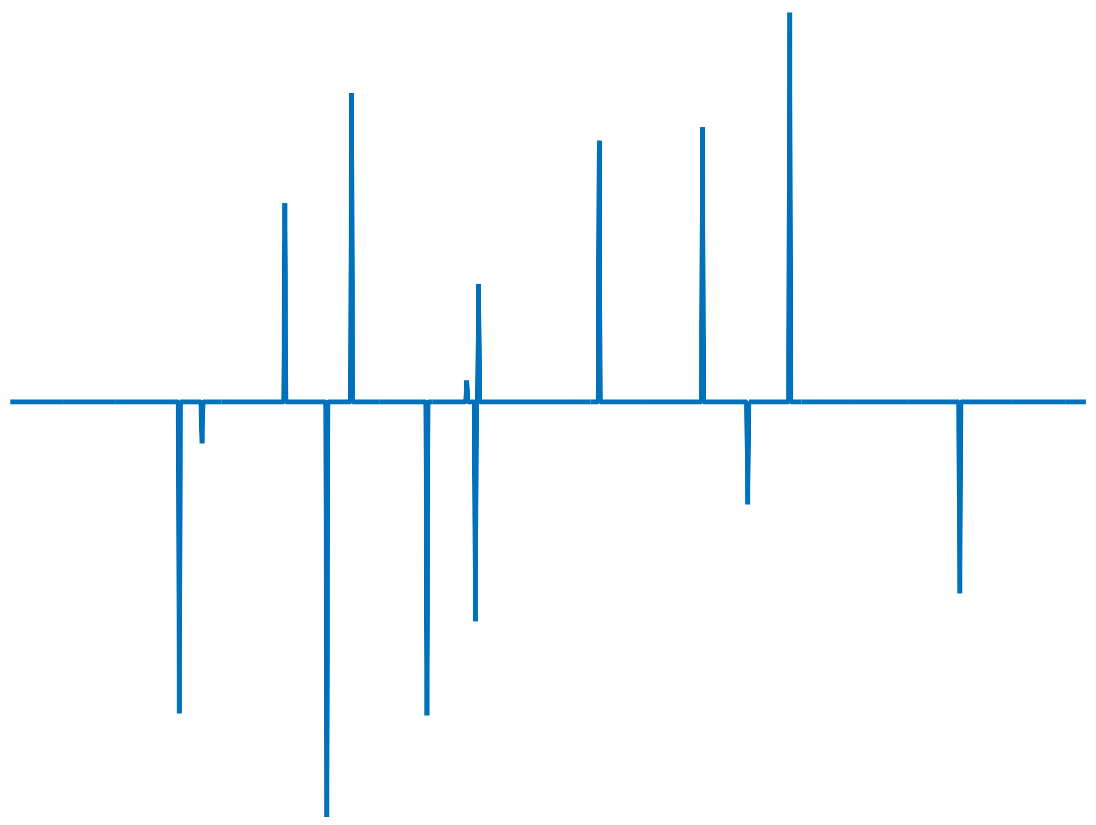
$$\mathbf{r}(t) = \sum_{i=1}^{s} a_i \delta_{t_i}(t)$$

$$\hat{\mathbf{r}}(\omega) = \sum_{i=1}^{s} a_i e^{\pi i t_i \omega}$$

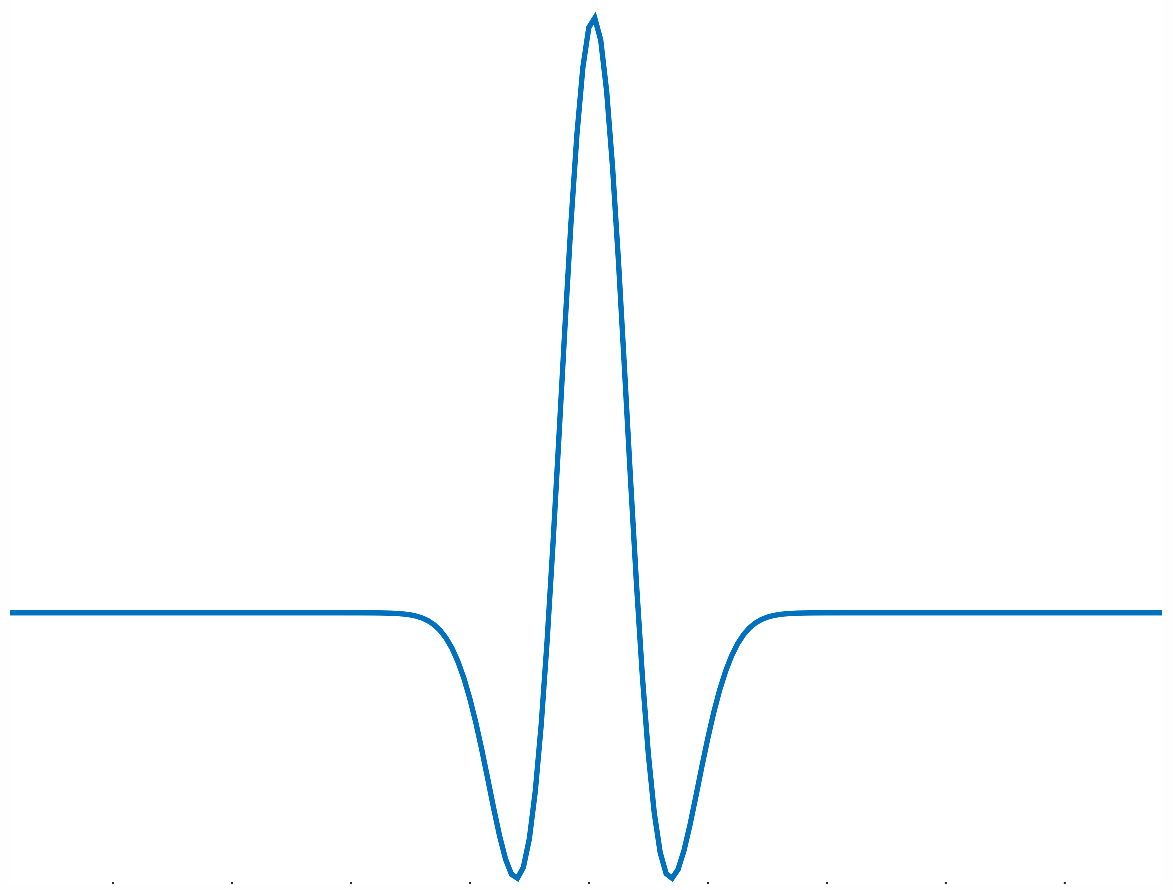
No assumptions on the wavelet

Workflow

Time domain reflectivity

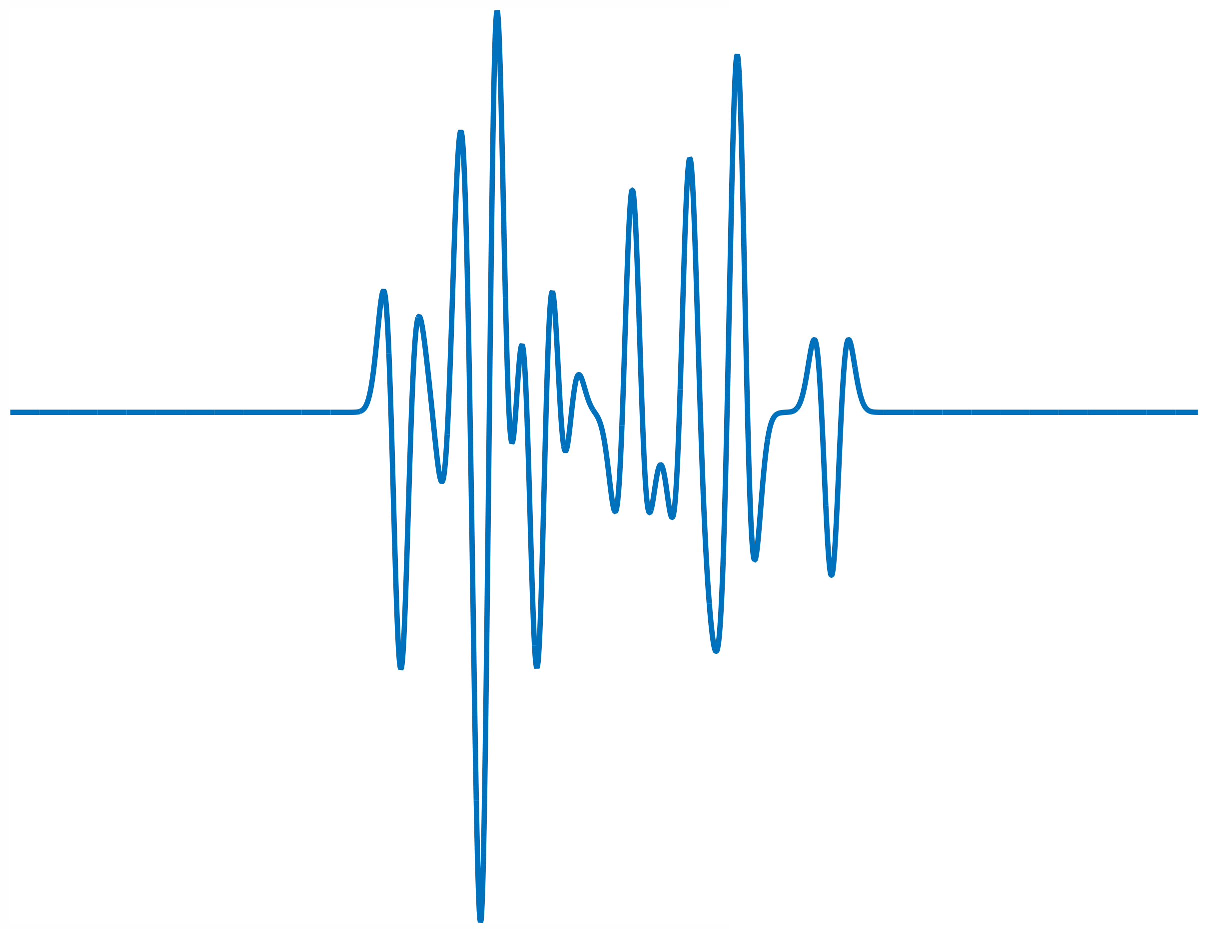


*

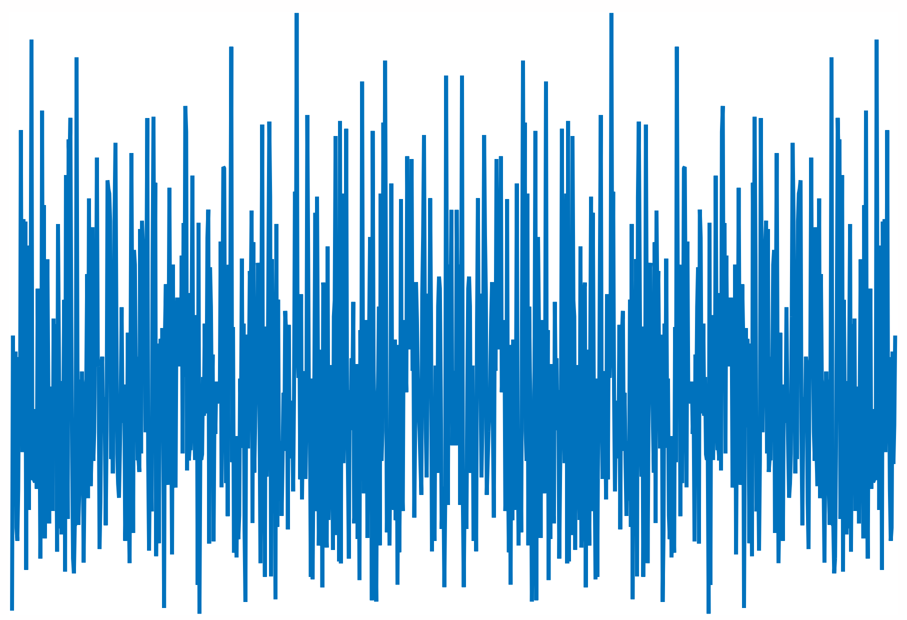


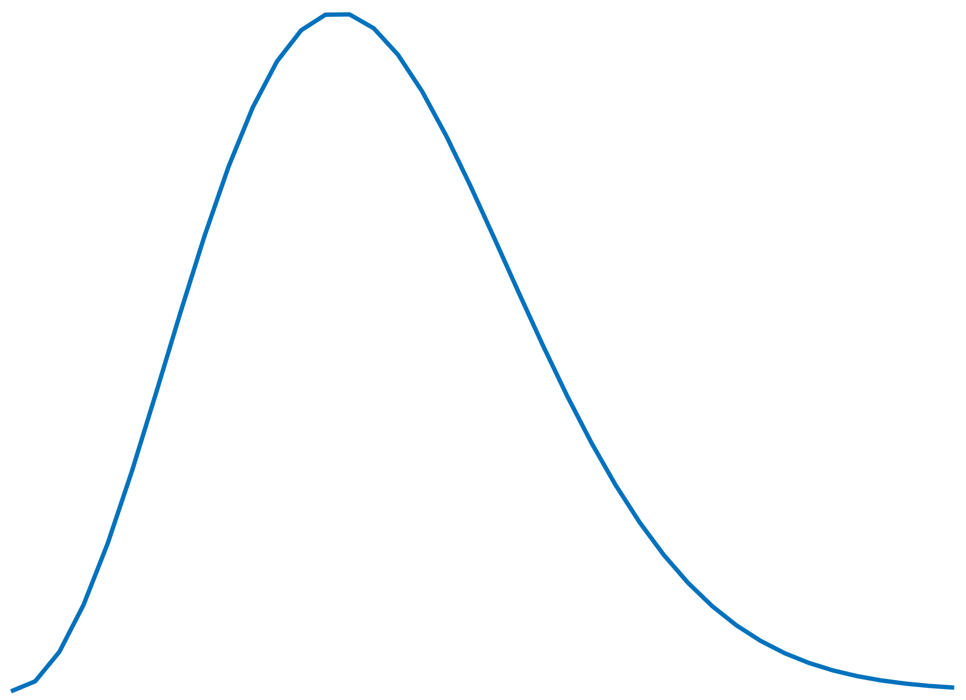
=

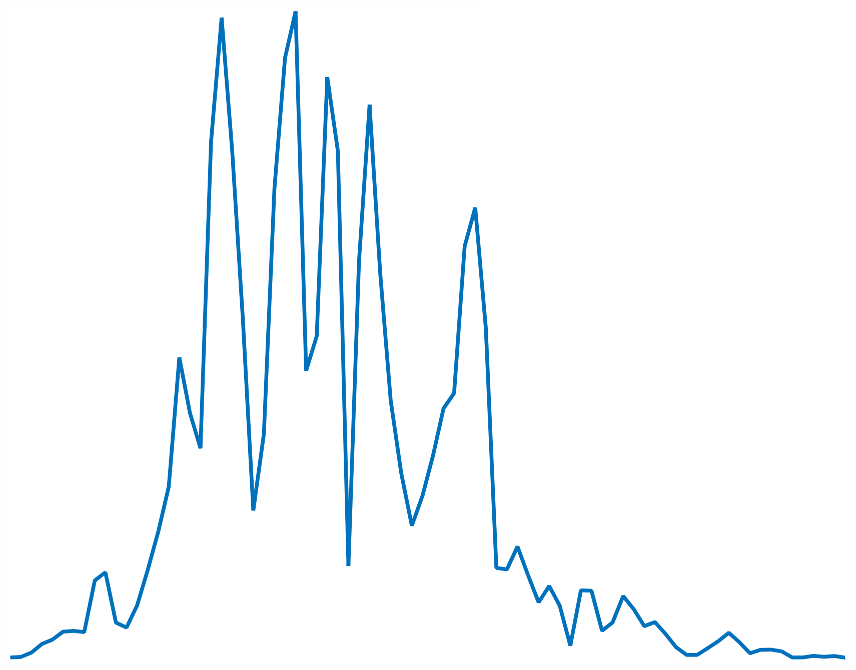
Noise-free data



Spectrum







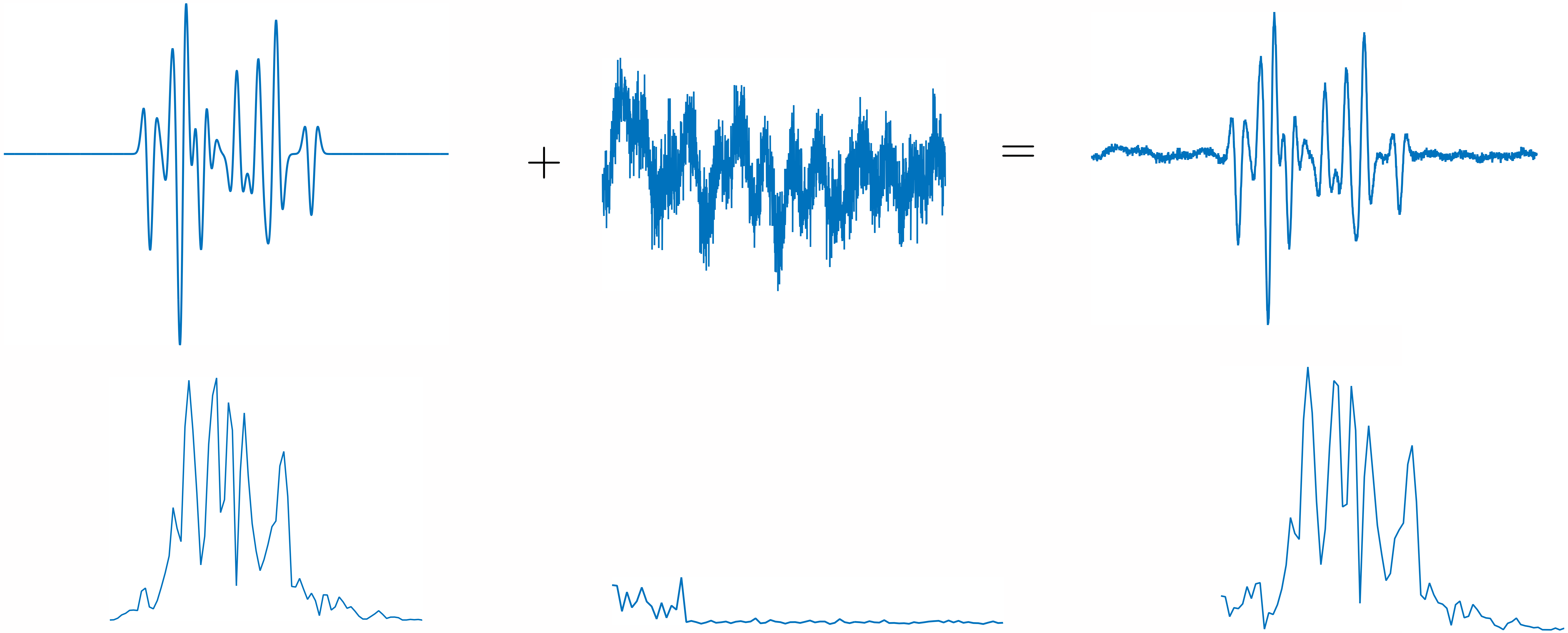
Workflow

Ideal time trace

Contamination

Observed data

Spectrum



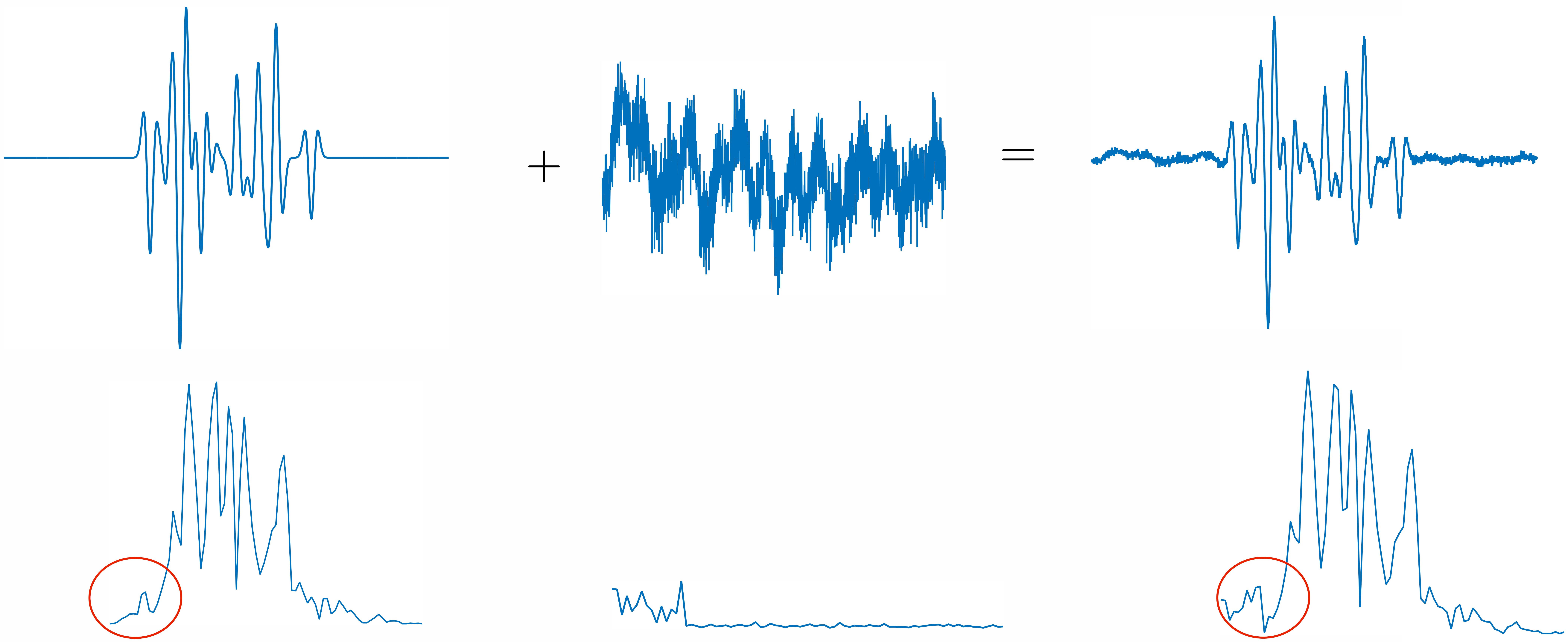
Workflow

Ideal time trace

Contamination

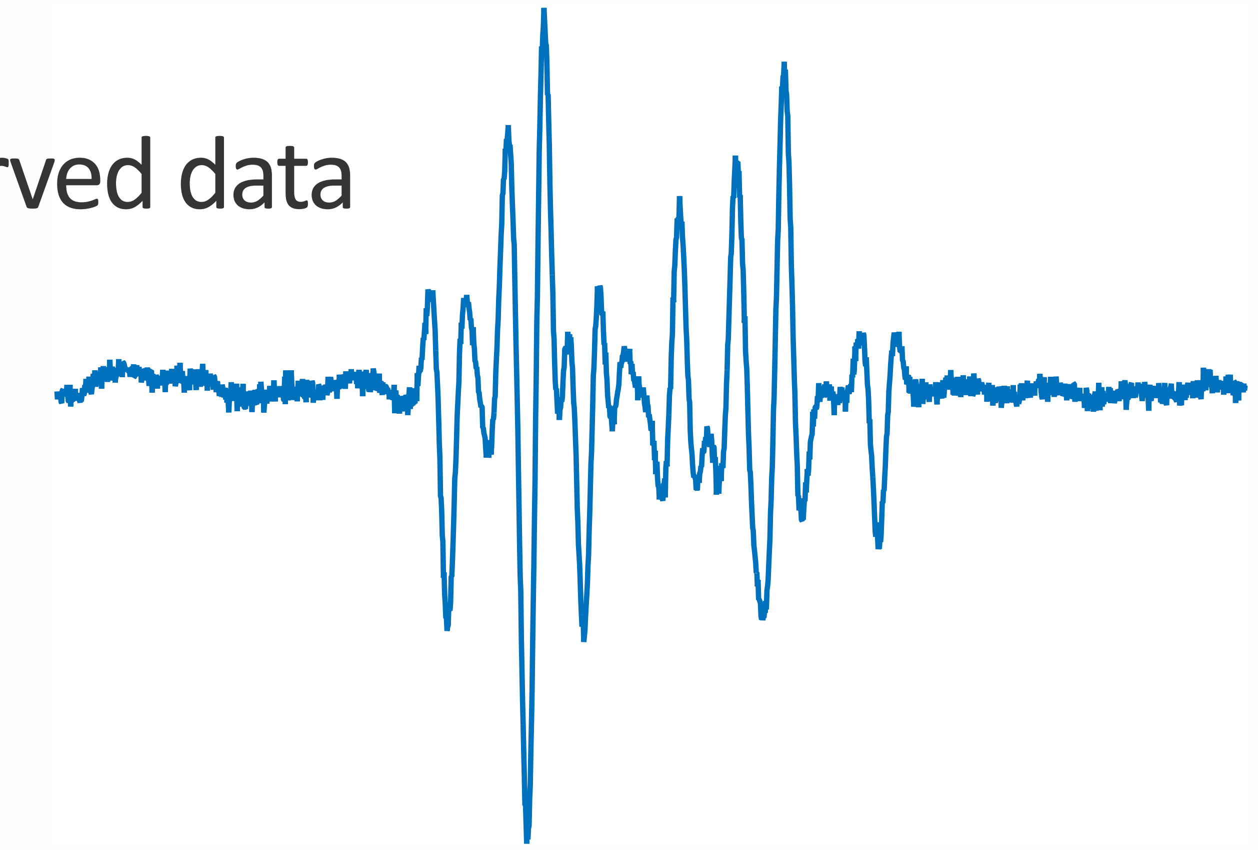
Observed data

Spectrum

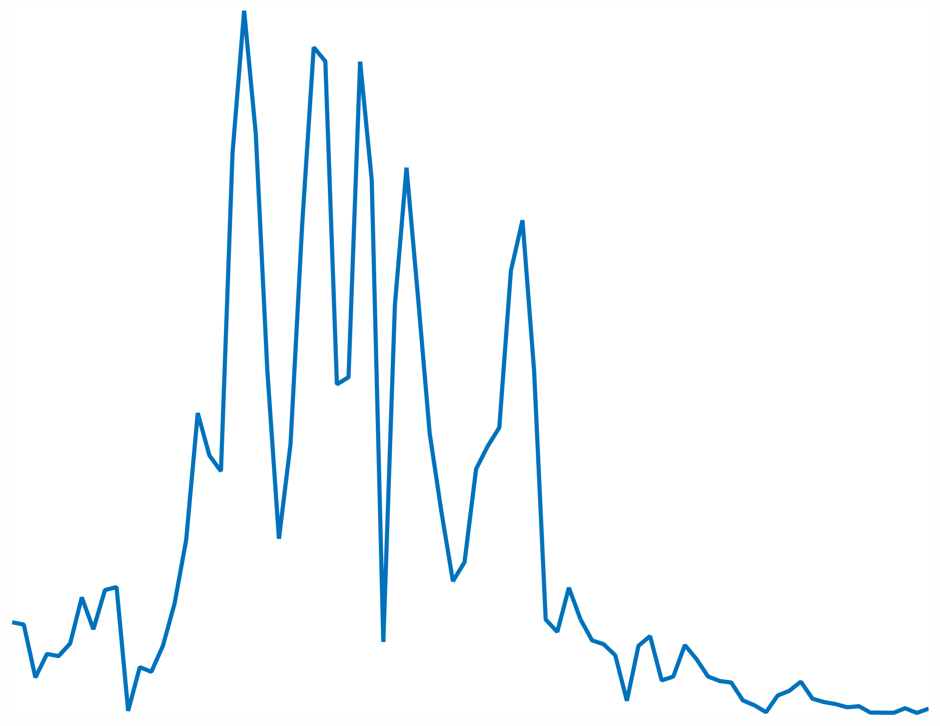


Workflow

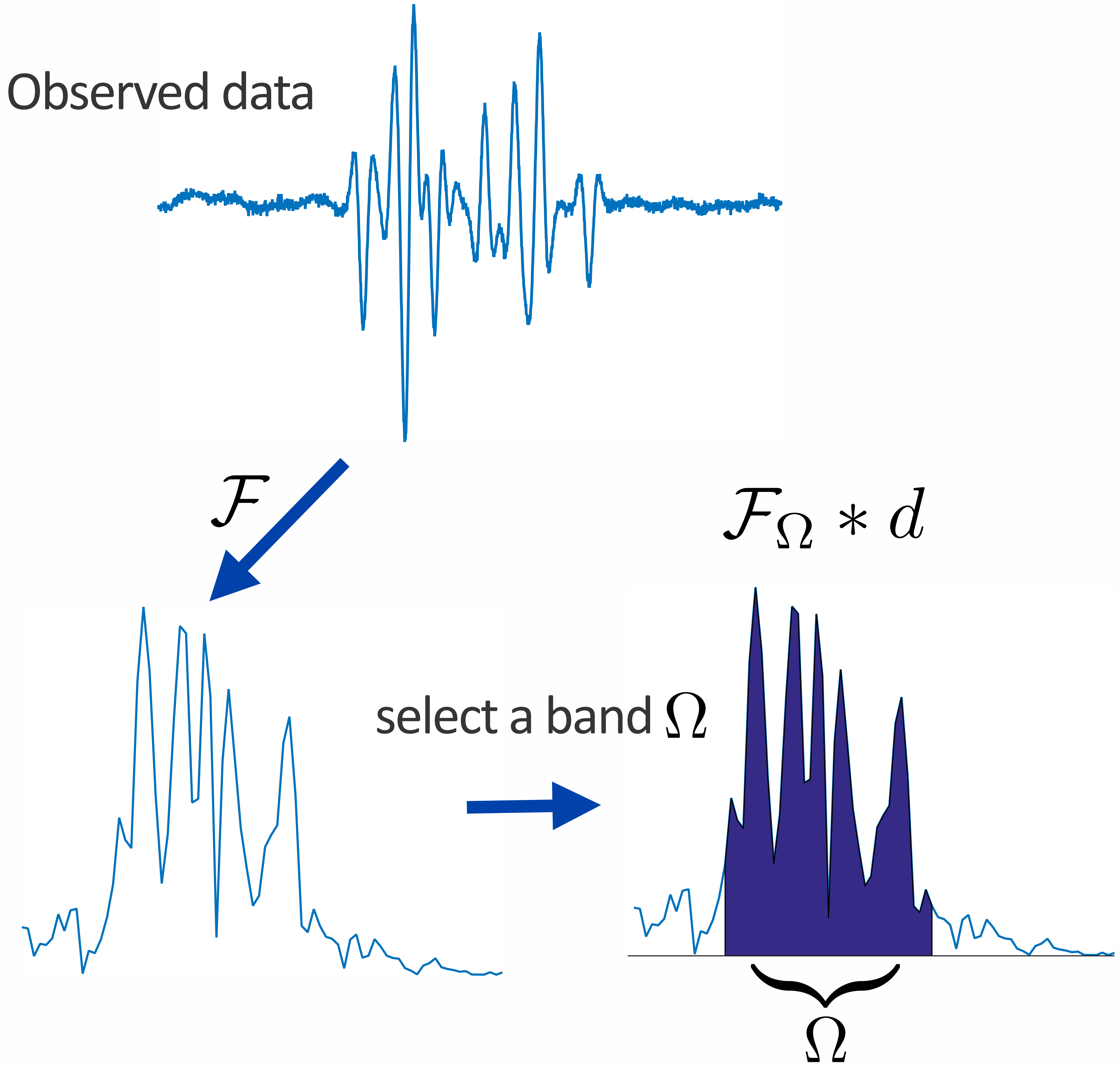
Observed data



$\mathcal{F}$

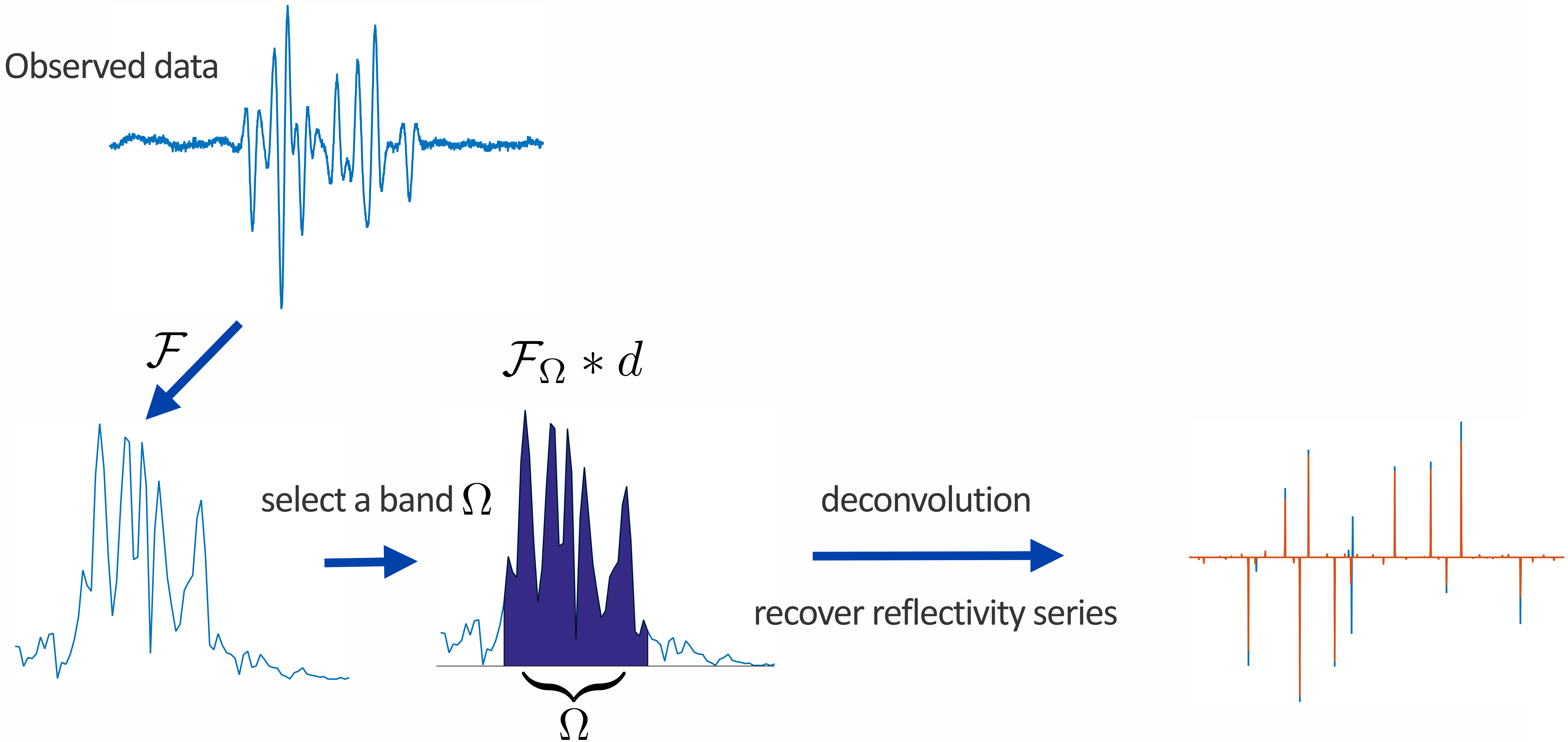


Workflow



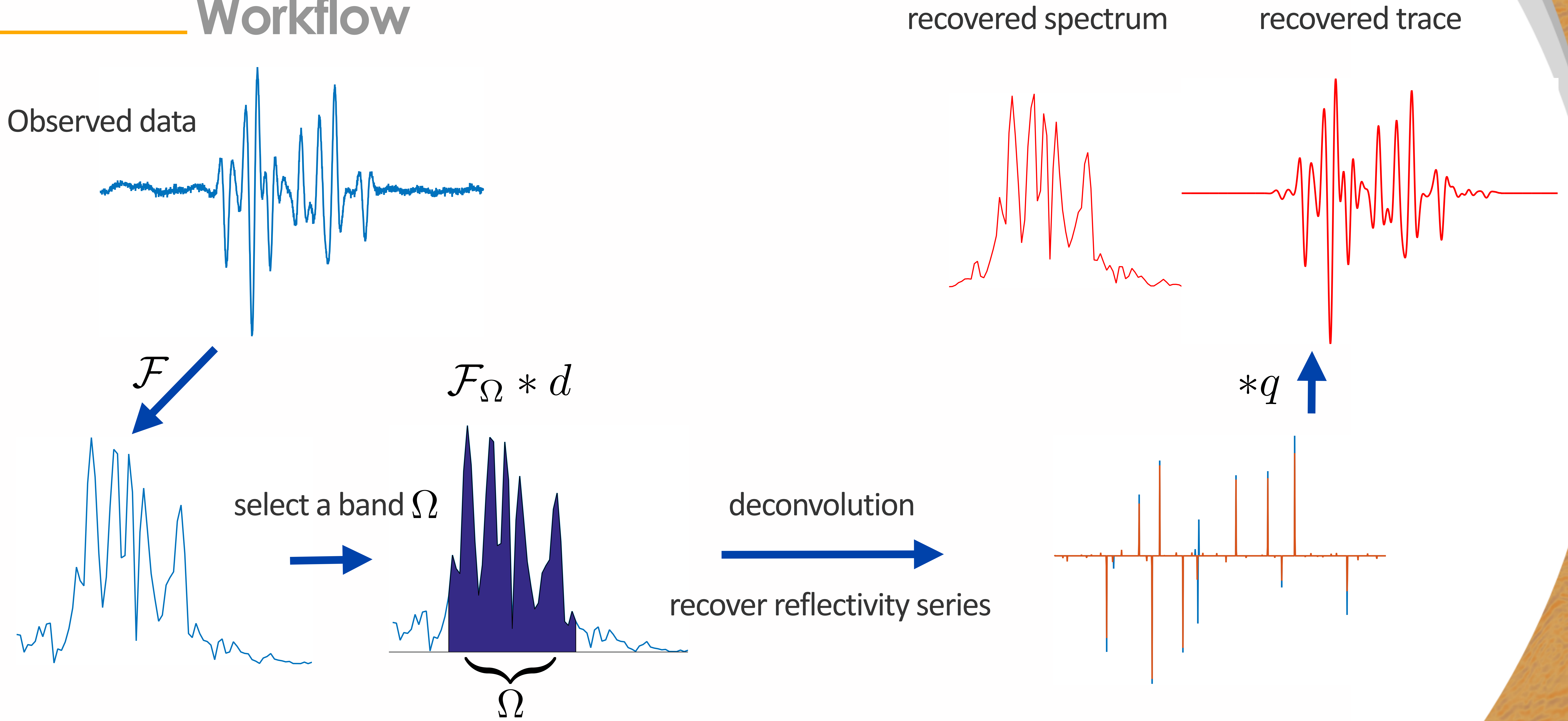
Observed data
$\mathcal{F}$
$\mathcal{F}_\Omega * d$
select a band Ω
Ω

Workflow

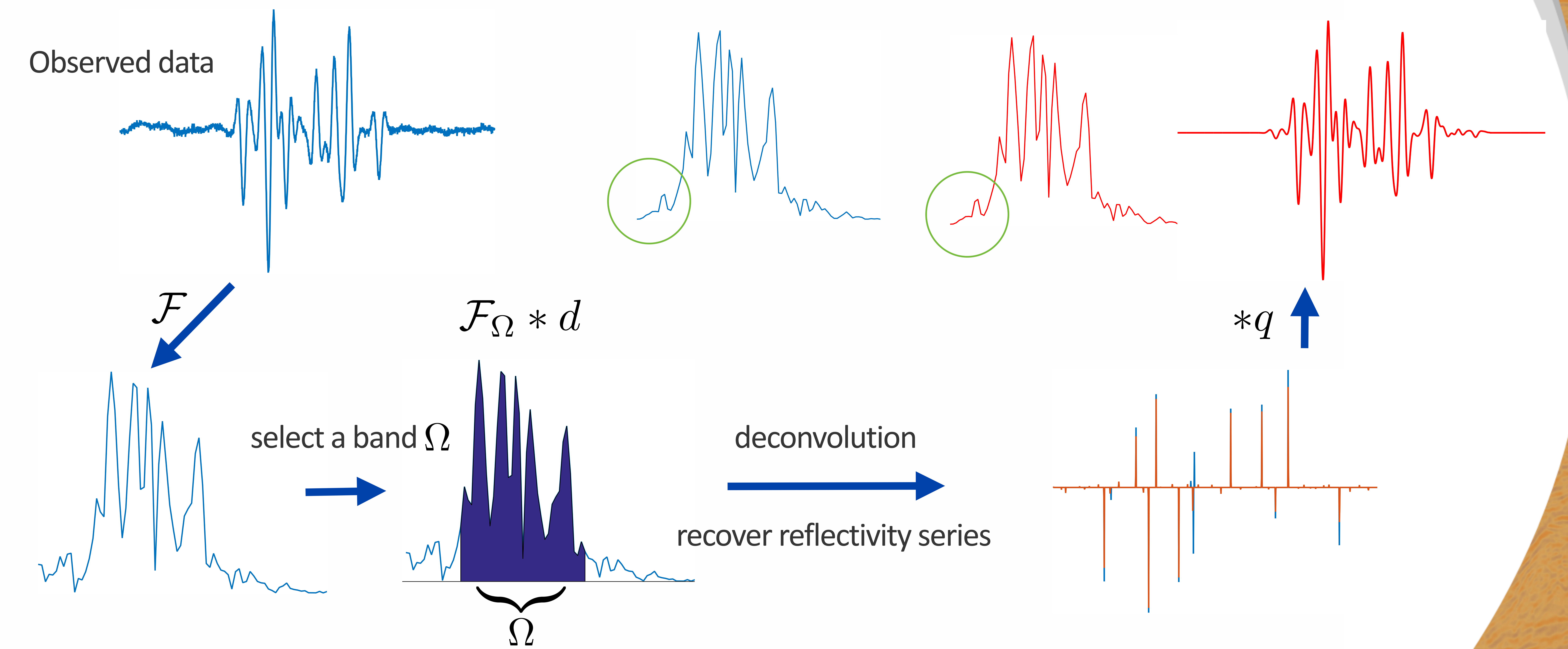


Observed data
$\mathcal{F}$
$\mathcal{F}_\Omega * d$
select a band Ω
Ω
deconvolution
recover reflectivity series

Workflow

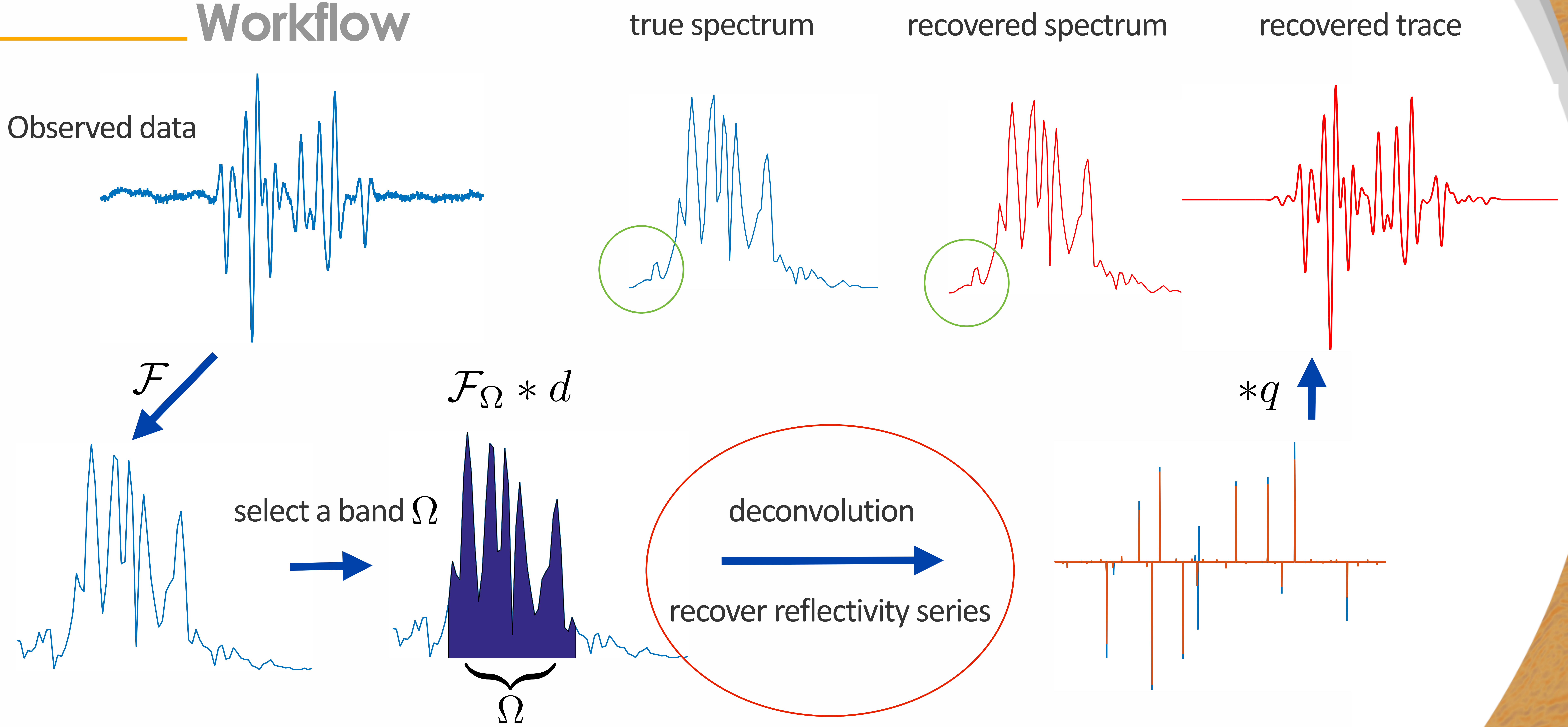

recovered spectrum
recovered trace
Observed data
$\mathcal{F}$
$\mathcal{F}_\Omega * d$
$*q$
select a band Ω
deconvolution
recover reflectivity series
Ω

Workflow



true spectrum
recovered spectrum
recovered trace
Observed data
$\mathcal{F}$
$\mathcal{F}_\Omega * d$
$*q$
select a band Ω
deconvolution
recover reflectivity series
Ω

Workflow



true spectrum
recovered spectrum
recovered trace
Observed data
$\mathcal{F}$
$\mathcal{F}_\Omega * d$
$*q$
select a band Ω
deconvolution
recover reflectivity series
Ω

[1] Schmidt, R.O, "Multiple Emitter Location and Signal Parameter Estimation," IEEE Trans. Antennas Propagation, Vol. AP-34 (March 1986), pp.276-280.
[2] H .L. Taylor, S. C. Banks, J. F. McCoy, "Deconvolution with the L1 norm." Geophysics (1979): 39-52.
[3] Candès, E. J., & Fernandez-Granda, C. (2013). Super-resolution from noisy data. J Fourier Anal Appl, *19*(6), 1229-1254.
[4] Candès, Emmanuel J., and Carlos Fernandez−Granda. "Towards a Mathematical Theory of Super−resolution." Commun Pur Appl Math, 67.6 (2014): 906-956.

Two approaches to deconvolution

- MUltiple SIgnal Classification (MUSIC)
  - needs only 2s+1 measurements
  - needs prior information on the number of events
  - has some stability w.r.t. noise
- L1 minimization (Linear Programming)
  - has greater stability
  - fits data exactly
  - needs constraint on minimal distance between opposite spikes

L1 minimization

$$\min_{\mathbf{r}} \|\mathbf{r}\|_1$$

$$\text{subject to } \mathcal{F}_\Omega(\mathbf{w} * \mathbf{r}) = \mathcal{F}_\Omega \mathbf{d}$$

$\mathbf{d}$: data

$\mathbf{w}$: wavelet

$\mathbf{r}$: reflectivity series

$\mathcal{F}_\Omega$: bandpass filter

$\Omega = [f_L, f_H]$: pass bands

user defined

[1] J.F. Clear bout and F.Muir. “Robust modelling of erratic data“, Geophysics, (1973), 826-844
[2] H .L. Taylor, S. C. Banks, J. F. McCoy, “Deconvolution with the L1 norm.” Geophysics (1979): 39-52.
[3] M. Rudelson and R. Vershynin. “On sparse reconstruction from Fourier and Gaussian measurements.” Commun Pur Appl Math 61.8 (2008), 1025-1045.
[4] Candès, Emmanuel J., and Carlos Fernandez−Granda. "Towards a Mathematical Theory of Super−resolution." Commun Pur Appl Math 67.6 (2014): 906-956.
[5] C. Dossal and S. Mallat. “Spare spike deconvolution with minimum scale”, Proc. SPASSR (2005)), 123-126

Prior art

- L1 based deconvolution first appeared in geophysics literature in 1970s [1,2]
- Compressed Sensing provided theoretical support for randomly selected Fourier coefficients [3]
- Candes et al. established a super-resolution theory assuming high frequency is missing [4]
- Dossal et al. introduced the minimal scale condition [5]

Theoretical foundation for low-frequency extrapolation

Theorem [RW,2016] r(t) can be exactly recovered by L1 minimization if the spikes are separated by 3.5 wavelengths* and the available bandwidth is greater than 60Hz. If noise exists, then the error in the estimate of r(t) is proportional to the energy of the noise.

$$*\text{wavelength} : \lambda_c = \frac{1}{f_H - f_L}$$

Theoretical foundation for low-frequency extrapolation

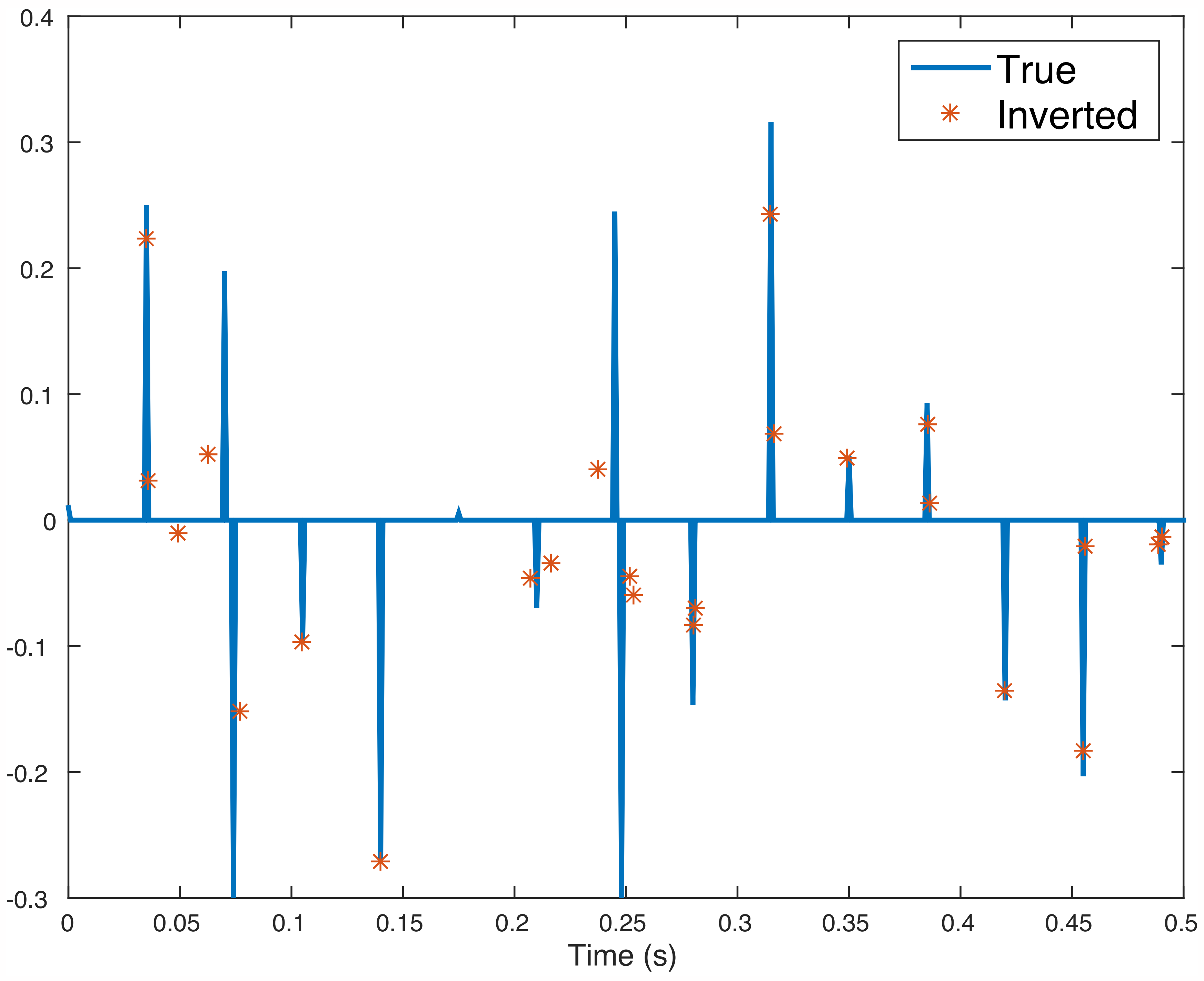
Theorem [RW,2016] r(t) can be exactly recovered by L1 minimization if the spikes are separated by 3.5 wavelengths* and the available bandwidth is greater than 60Hz. If noise exists, then the error in the estimate of r(t) is proportional to the energy of the noise.

$$^{*}\text{wavelength} : \lambda_c = \frac{1}{f_H - f_L}$$

*From numerical experiments: 1.5 wavelength is sufficient

When the minimal distance condition is not satisfied …

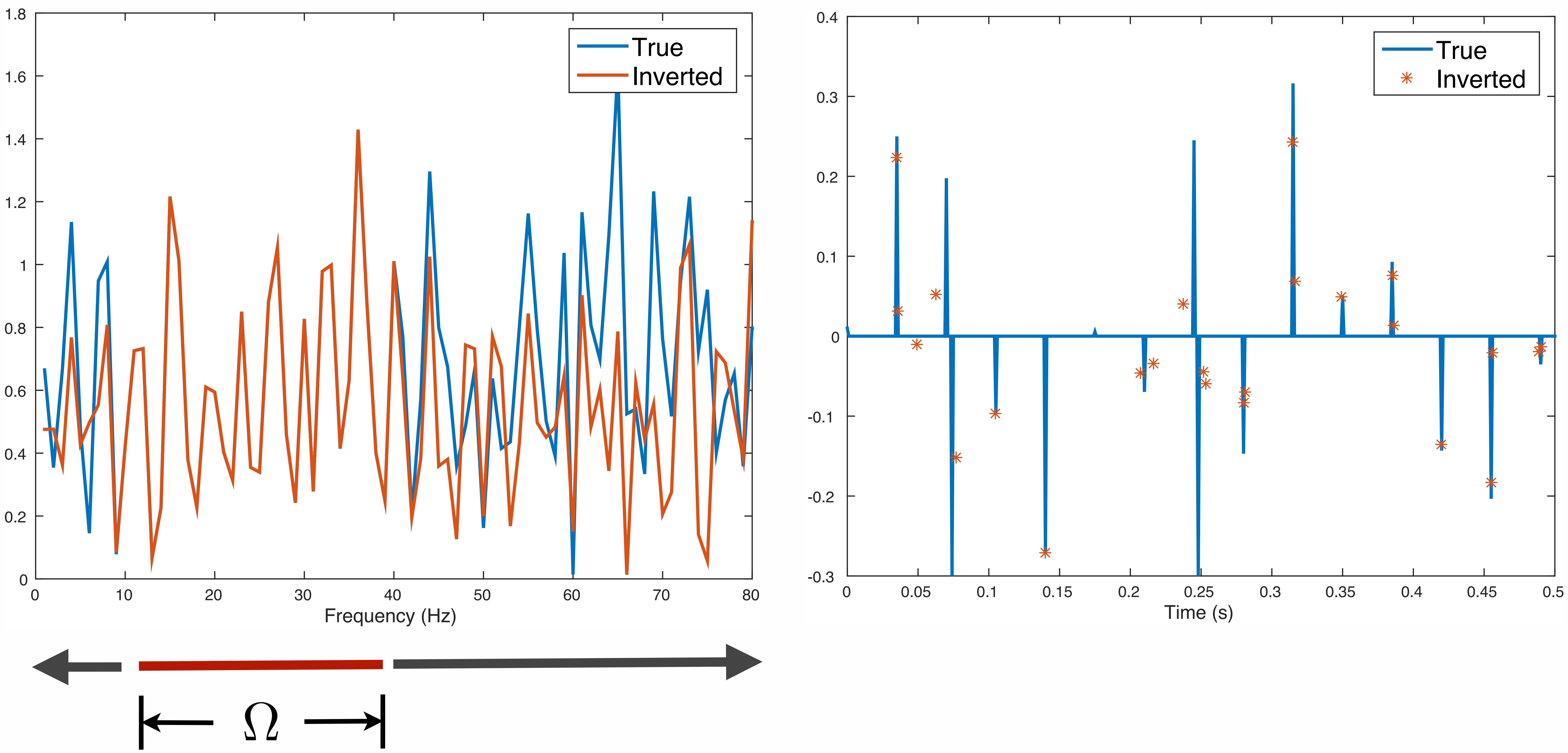
Reconstructions are affected



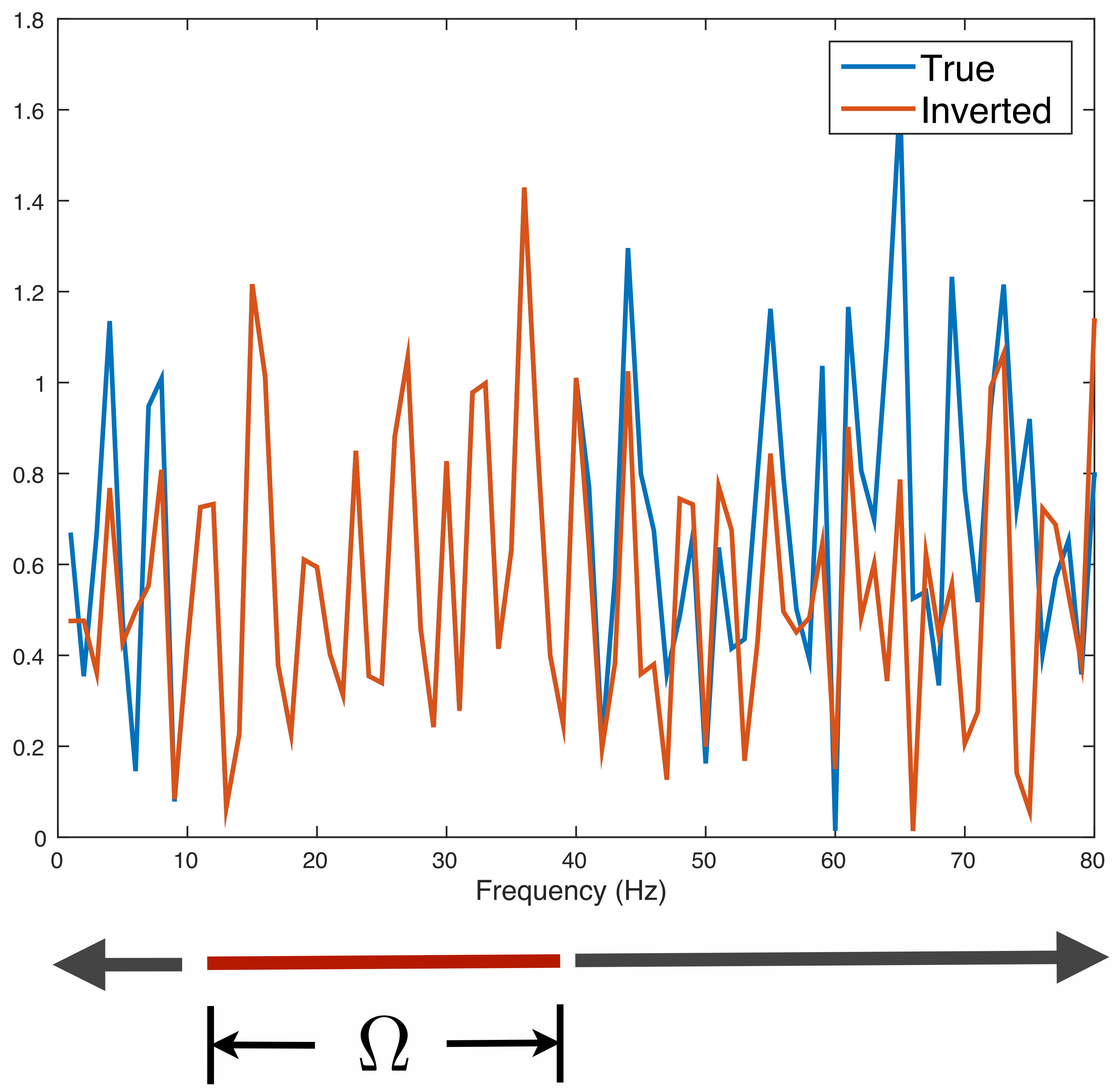
Left: L1 reconstruction of reflectivity series using frequencies $\Omega = \{10,\ldots,40\}$Hz

Deconvolution result is independent of wavelet choice

Error is proportional to distance to Ω



Error is proportional to distance to Ω



It can be shown that

- As distance from $\Omega \uparrow$, error $\uparrow$
- extrapolation towards low frequencies is more stable than towards high frequencies

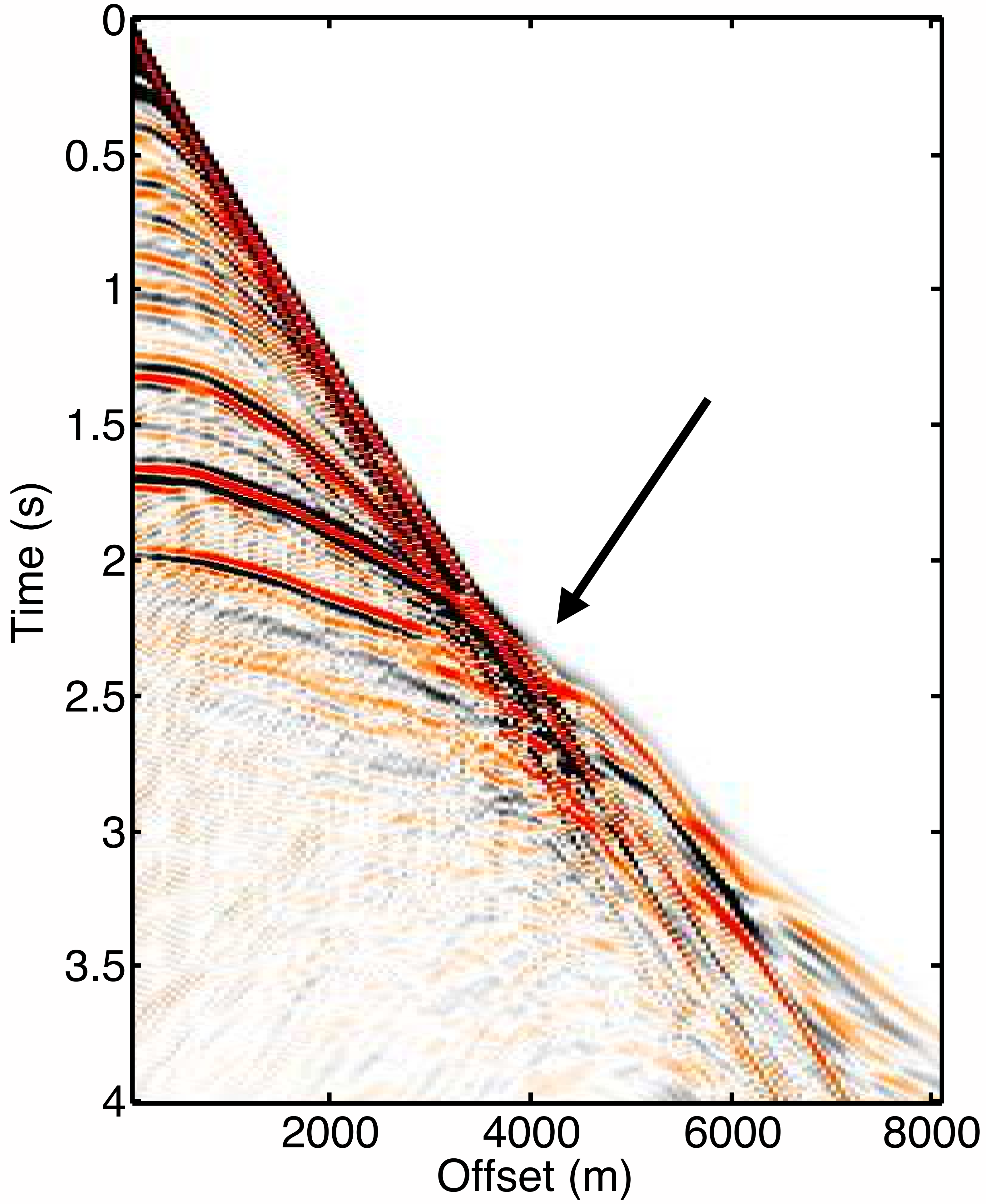
Difficulties in extrapolation

- FWI requires high accuracy in both phase and amplitude of low frequency data
- wavelet estimation is not accurate
- existence of dispersion
- 2D modeling: reflectivity series do not contain perfect spikes
- existence of very close spikes at crossing of events

noise

causes L1 to fail

The L1 minimization w/ TV-norm stabilizer

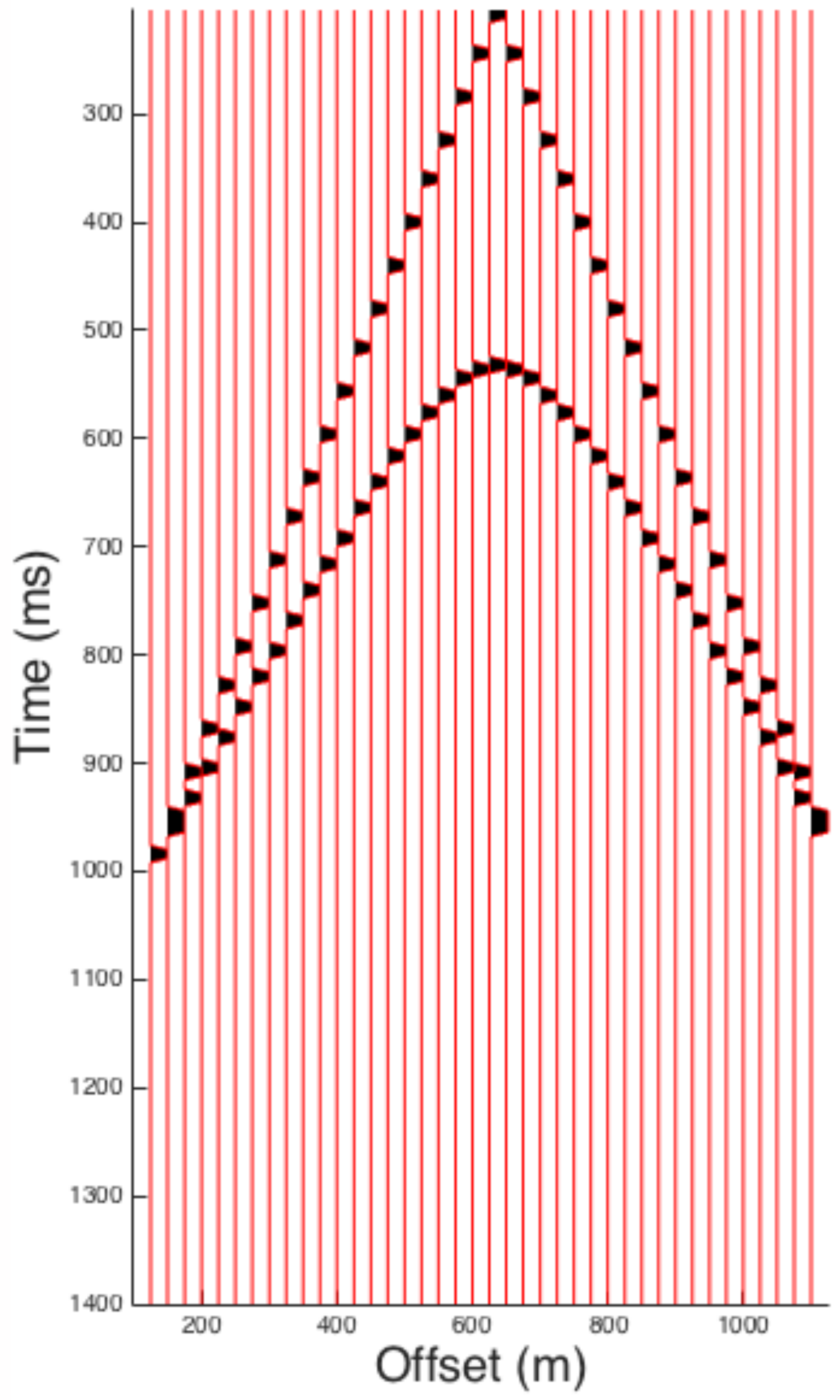


Conflicting events generate very close spikes

L1 minimization has trouble in the indicated region

Goal: utilize spatial correlations

Spatial similarity between adjacent traces



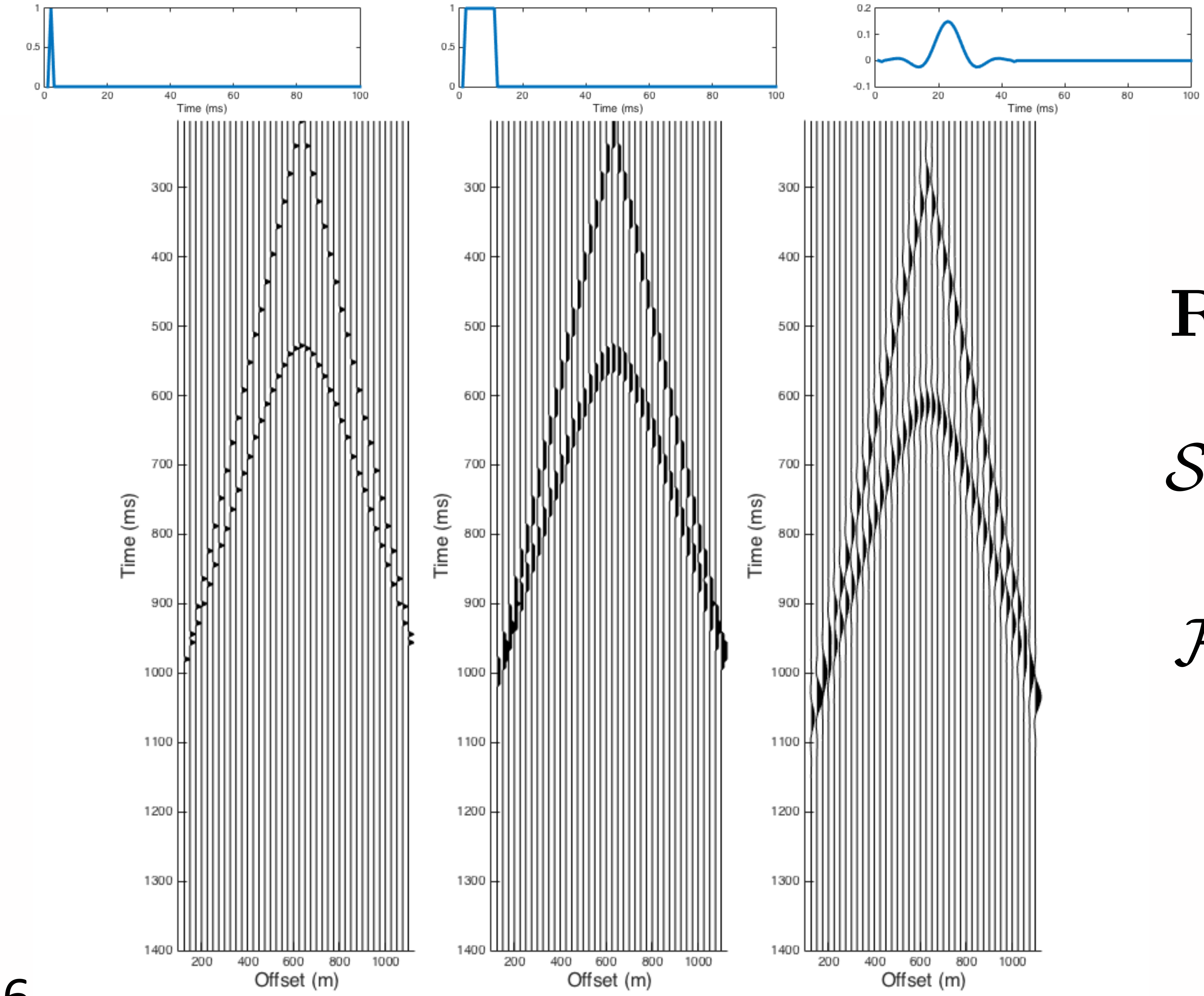
Define spatial similarity by

$$\mathcal{S}(\mathbf{R}) = \frac{\sum_{j=1}^{m-1} (\|\mathbf{R}_j\|_2^2 + \|\mathbf{R}_{j+1}\|_2^2)}{\sum_{j=1}^{m-1} \|\mathbf{R}_j - \mathbf{R}_{j+1}\|_2^2}$$

G has no spatial similarity when $\mathcal{S}(\mathbf{R}_m) \leq 1$

Left: visually continuous but has no spatial similarity

Increase spatial similarity by low-pass filtering



$$\mathbf{R}_m = \mathbf{f} * \mathbf{R}, \quad \text{where} \quad \mathbf{f} = (1, ..., 1, 0, ..., 0)$$

$$\mathcal{S}(\mathbf{R}_m) \geq \mathcal{S}(\mathbf{R})$$

$$\mathcal{F}_\Omega(\mathbf{w} * \mathbf{R}_m) = \mathcal{F}_\Omega(\mathbf{w} * \mathbf{R}) * \mathcal{F}_\Omega(\mathbf{f}) = \mathbf{d} * \mathcal{F}_\Omega(\mathbf{f})$$

TV norm minimization for Gm

NESTA is used to solve the following optimization problem

$$\mathbf{R}_{m,\text{est}} = \arg\min_{\mathbf{R}_m} \|\mathbf{R}_m\|_{TV}^{\alpha,\beta},$$

$$\text{subject to} \quad \mathbf{R}_m(\omega) = \mathbf{R}(\omega)\hat{\mathbf{f}}(\omega) = \hat{\mathbf{d}}_\Omega(\omega)\hat{\mathbf{f}}(\omega), \omega \in \Omega,$$

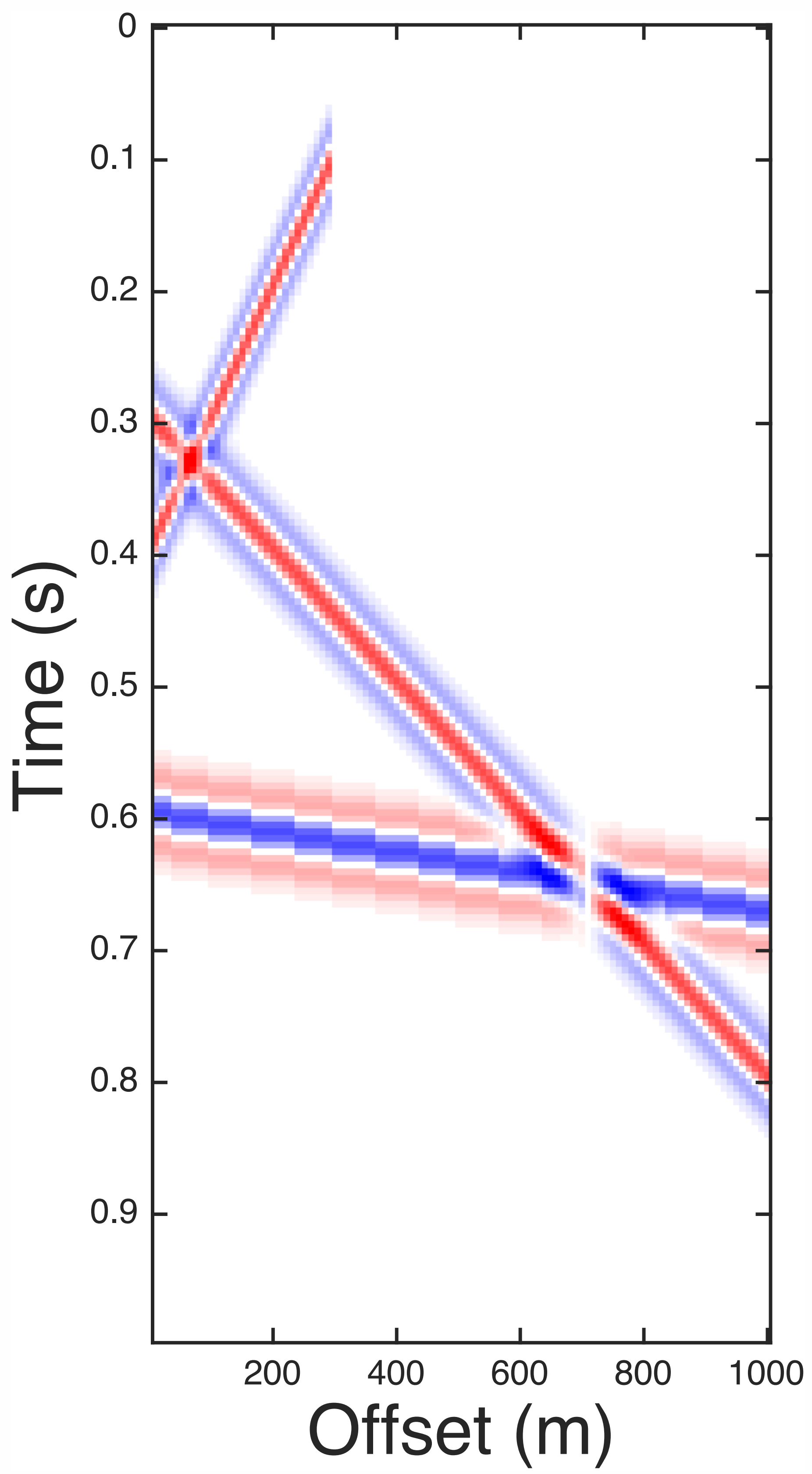
where

$$\|\mathbf{R}_m\|_{TV}^{\alpha,\beta} := \sum_{i,j} \|\nabla_{\alpha,\beta}\mathbf{R}_m(i,j))\|_{\ell_1}$$

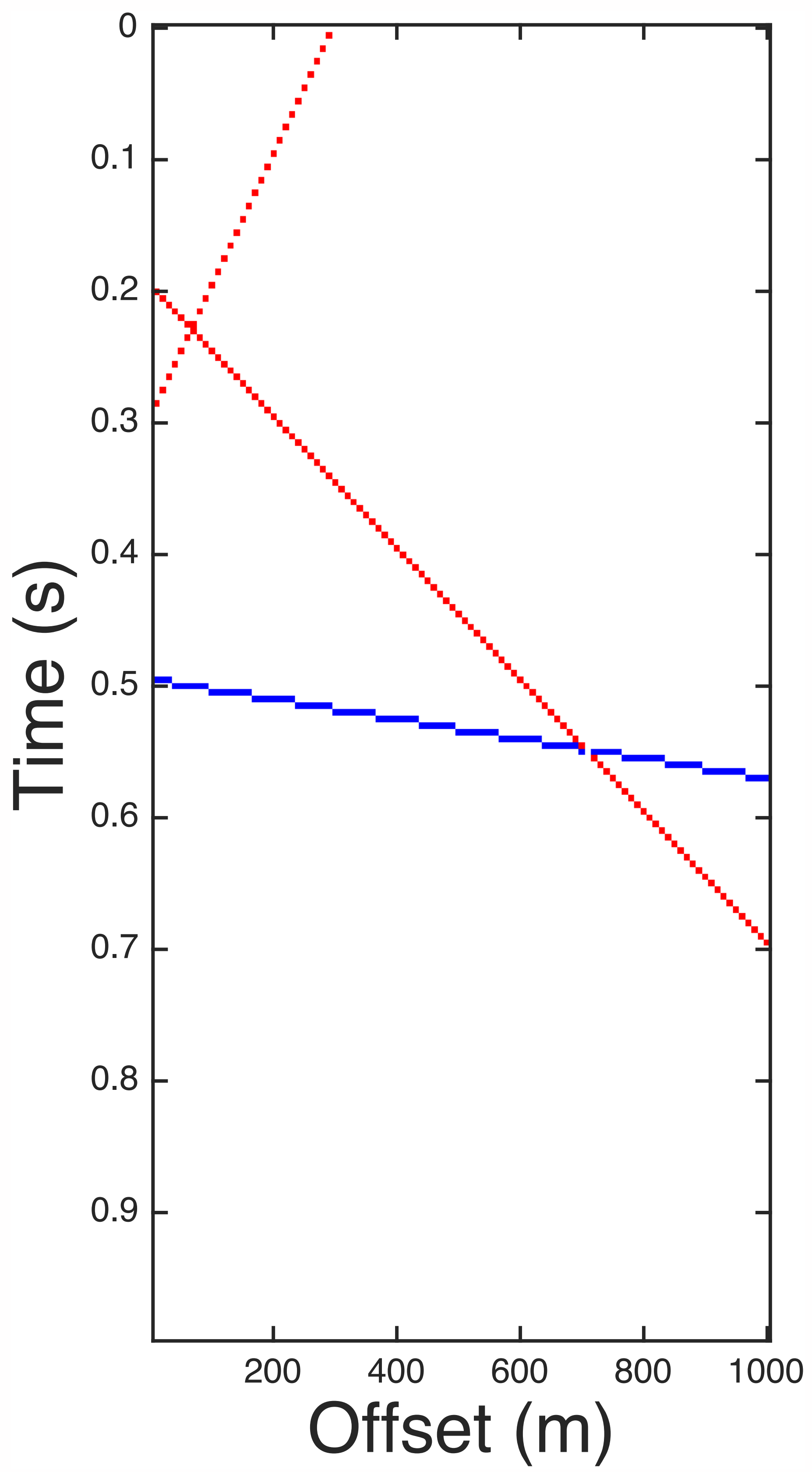
$$\nabla_{\alpha,\beta}\mathbf{R}_m(i,j) = \begin{bmatrix} \alpha(\mathbf{R}_m(i,j) - \mathbf{R}_m(i,j+1)) \\ \beta(\mathbf{R}_m(i,j) - \mathbf{R}_m(i+1,j)) \end{bmatrix}.$$

A stylized example

Data

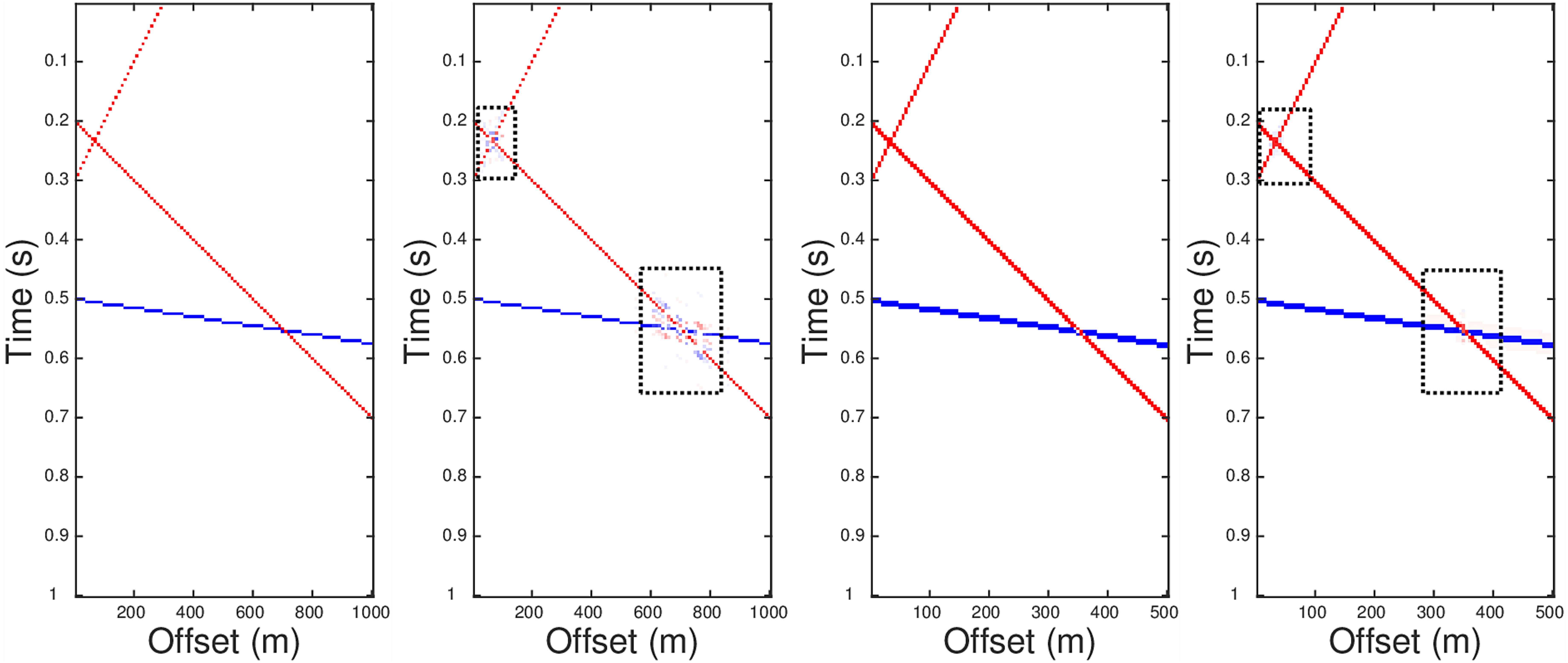


Reflectivity gather



- Three events
- Ω = 5-15Hz
- Receiver spacing : 10m
- Maximum offset : 1km

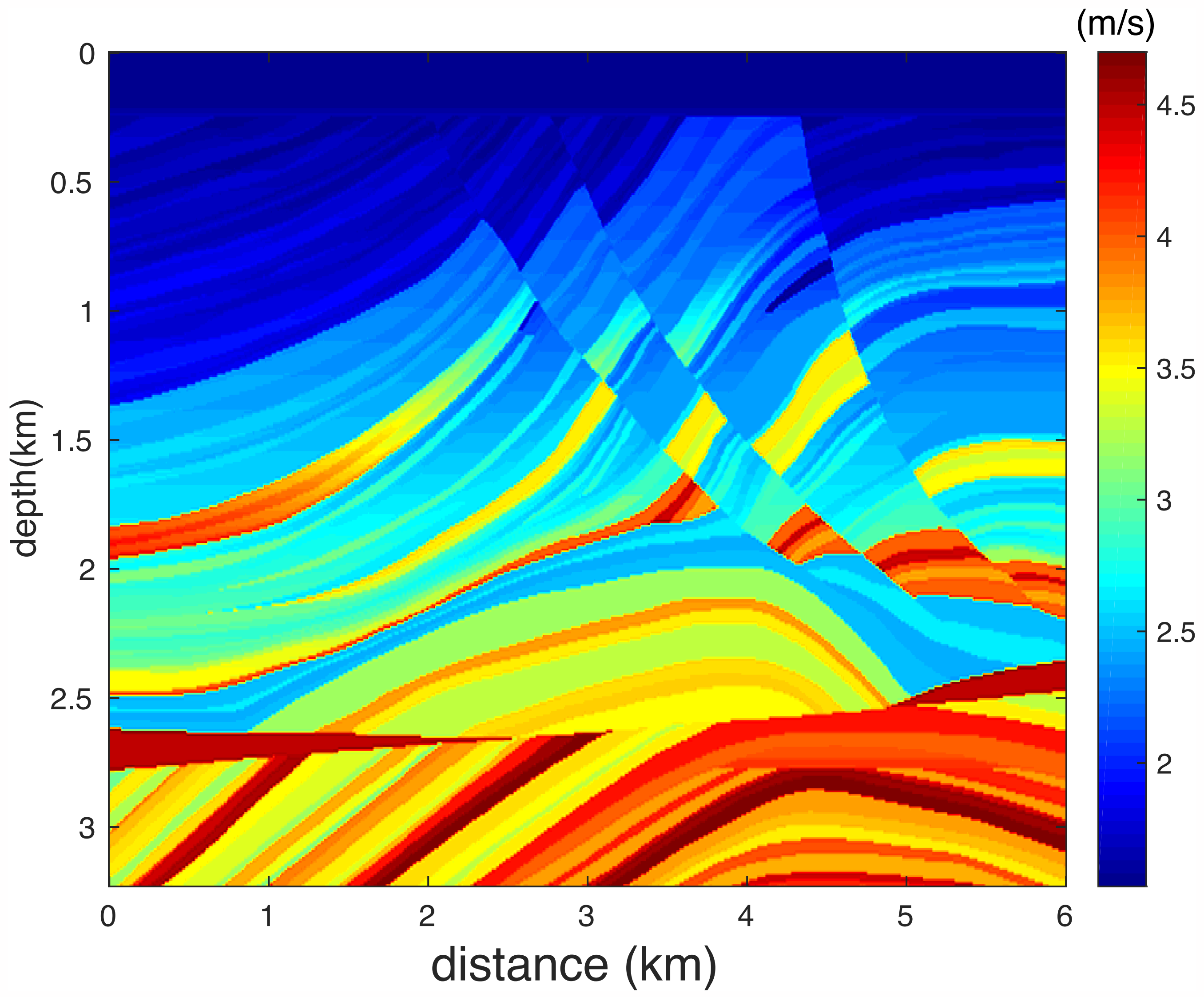
A stylized example

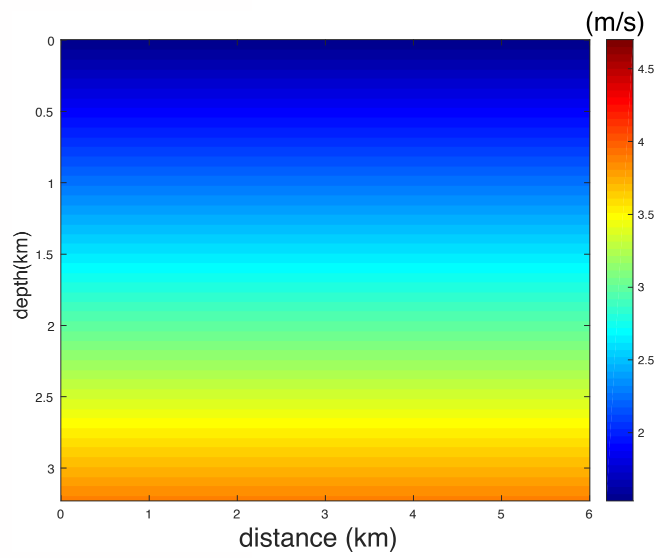


True reflectivity gather | Inverted with L1 | Filtered reflectivity gather | Inverted with TV

Synthetic data - Non-inversion crime

- IWAVE generated data
- inversion using time harmonics
- 3 frequency sweeps, 40 l-BFGS-iterations for each batch
- 20Hz Ricker wavelet
- source spacing : 0.2km
- receiver spacing : 20m
- maximum offset : 2km
- model size : 3.2×6km
- Ω = 5-15Hz





Synthetic data - Non-inversion crime

Ω = 5-15Hz

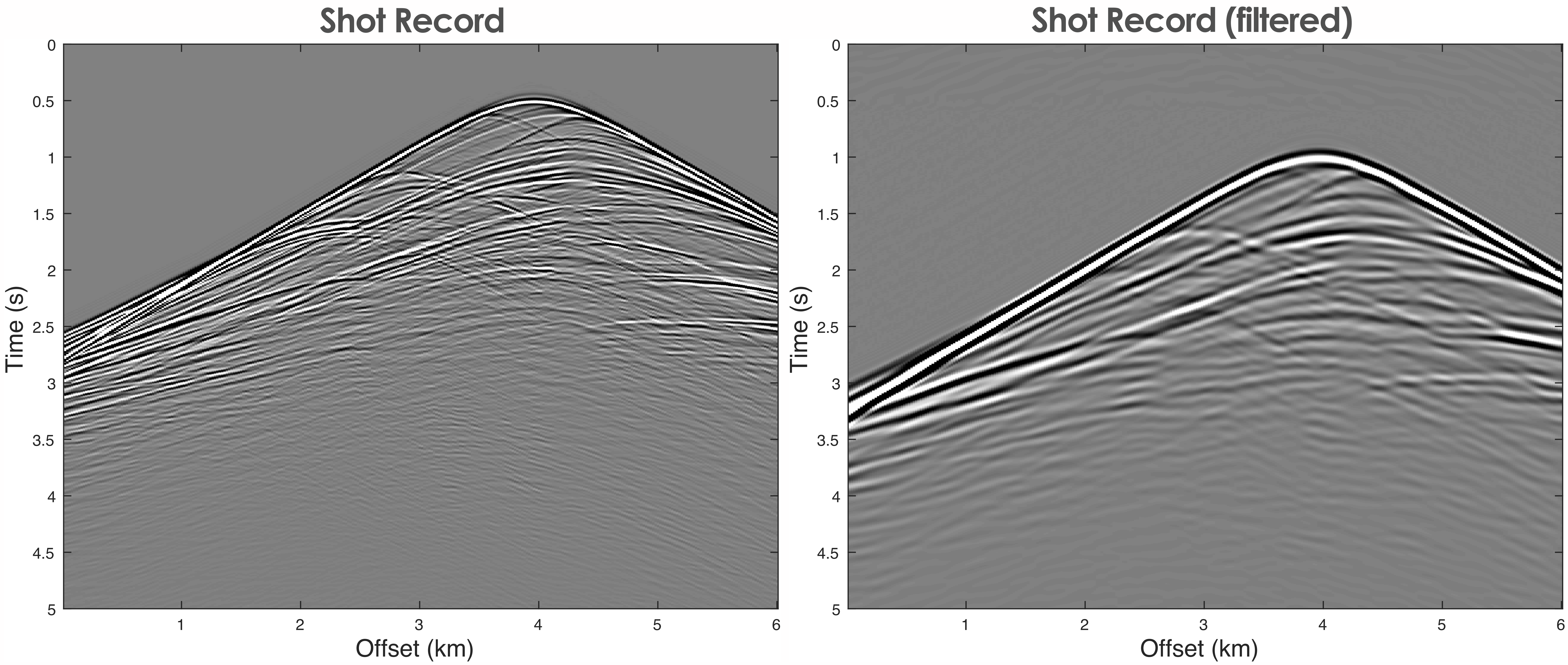
Direct inversion

- frequency continuation with batches [5,5.25]Hz,[5.5,5.75]Hz, [6,6.25]Hz....[15,15.25]Hz
- perform the previous step two more sweeps

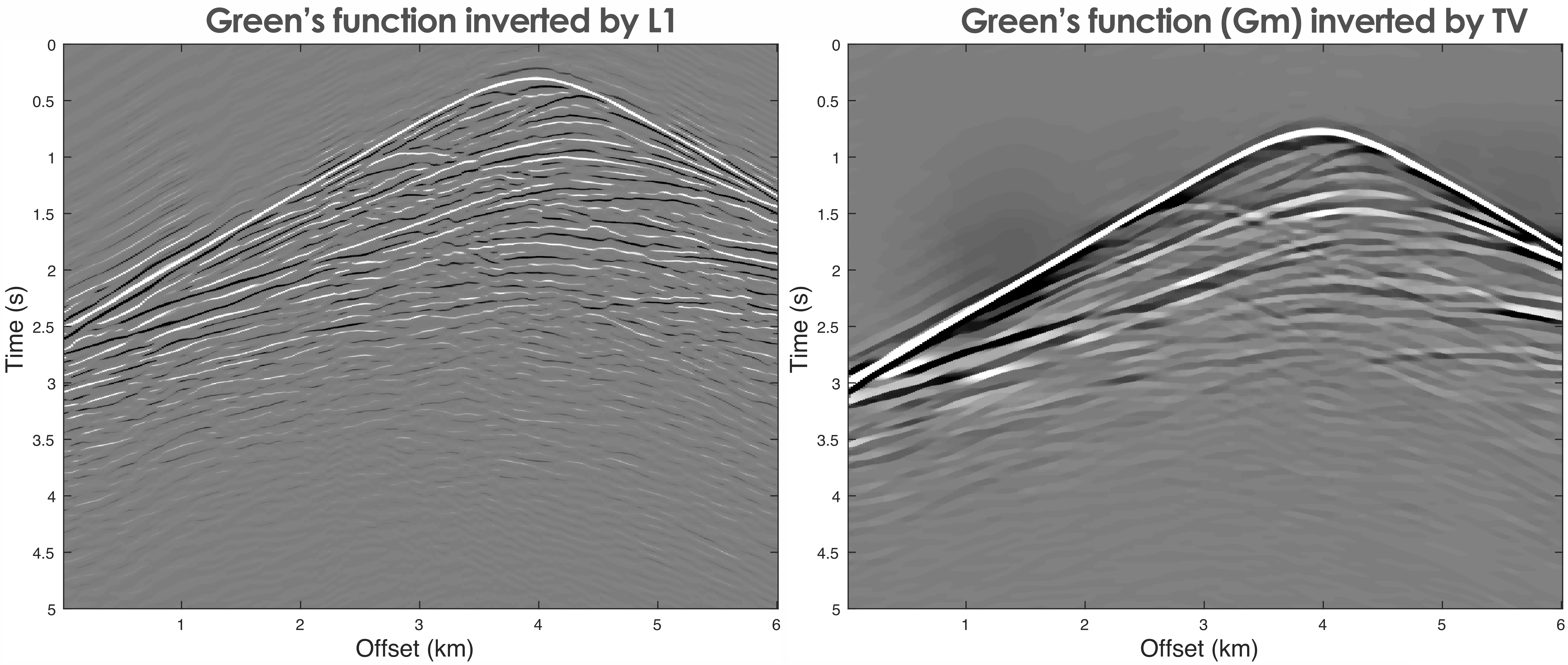
Inversion with extrapolation

- extrapolation from {5,...,15}Hz to {1,...5}Hz
- frequency continuation with batches [1,1.25]Hz,[1.5,1.75]Hz,....[4.5,4.75]Hz
- frequency continuation with batches [5,5.25]Hz,[5.5,5.75]Hz, [6,6.25]Hz.... [15,15.25]Hz
- perform the previous step two more sweeps

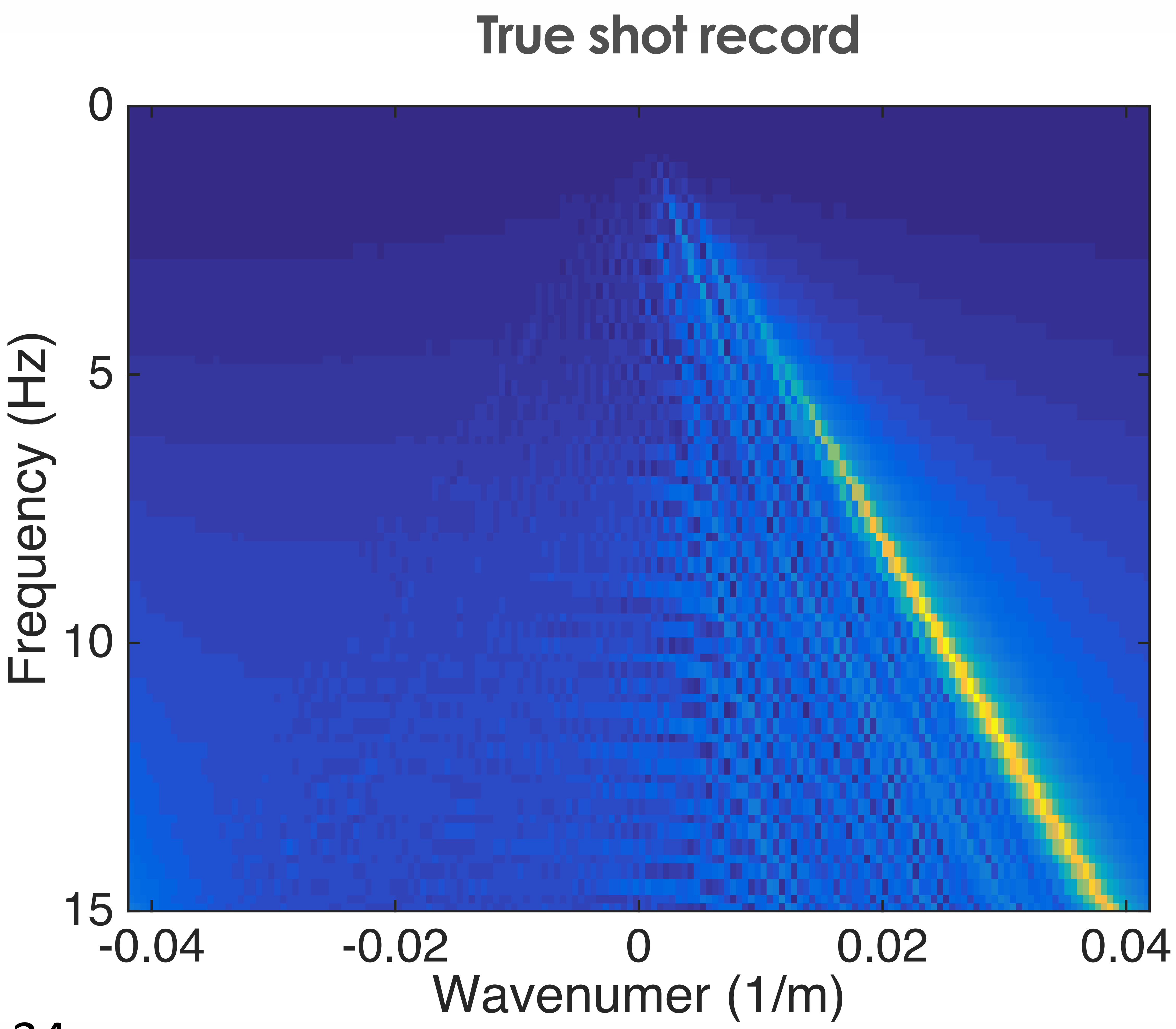
The effect of filtering

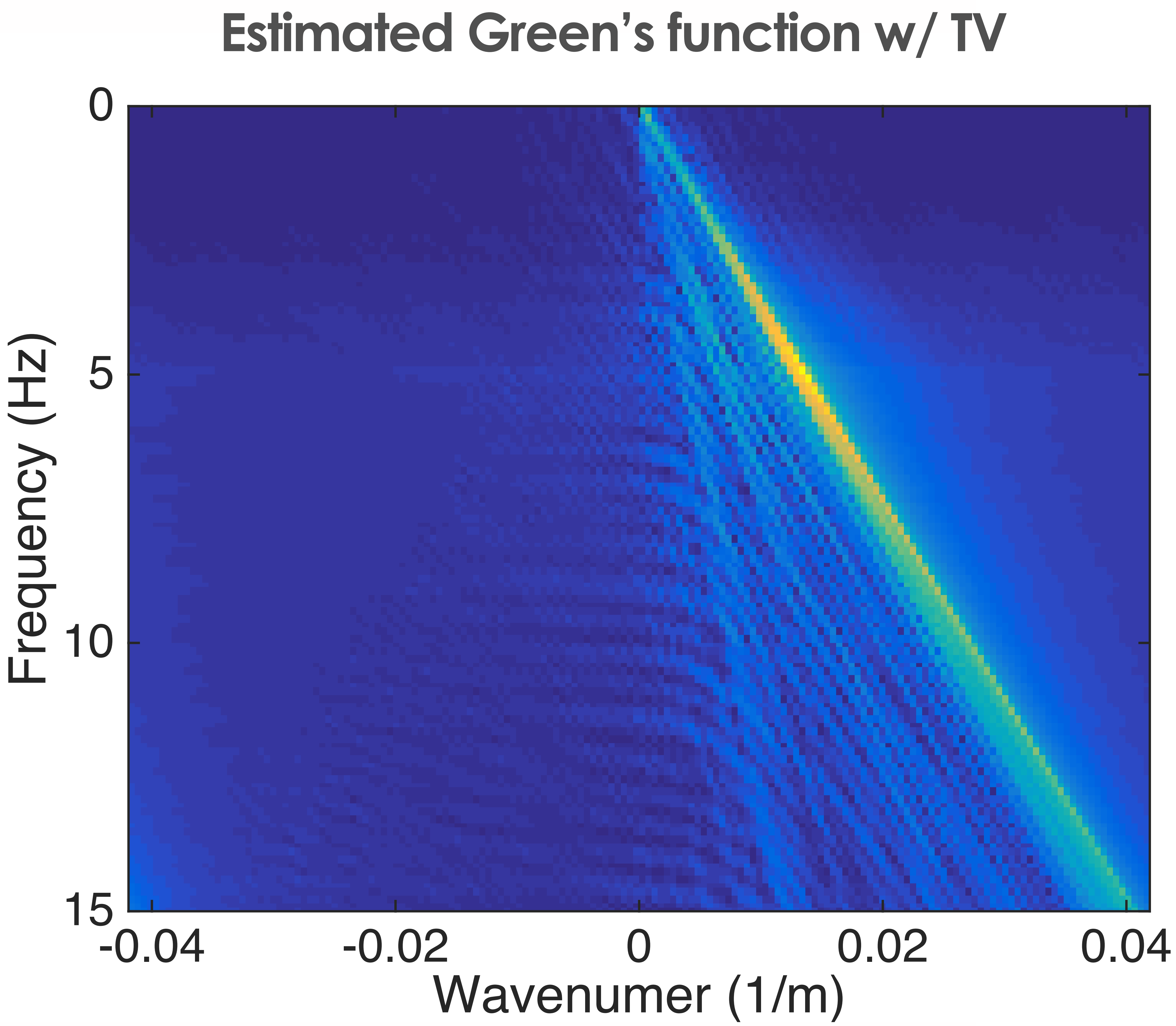


Deconvolution result using 5-15Hz data

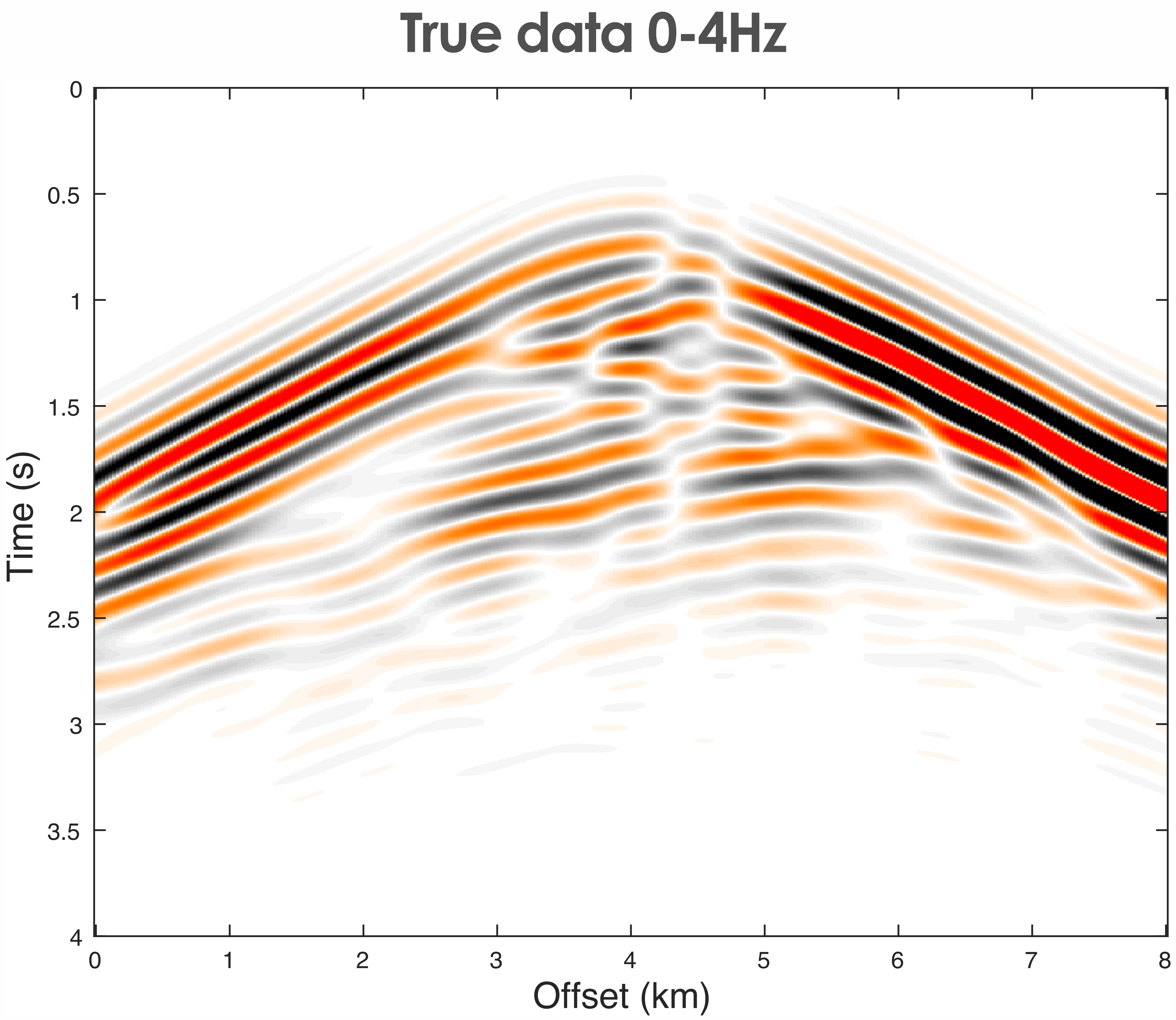


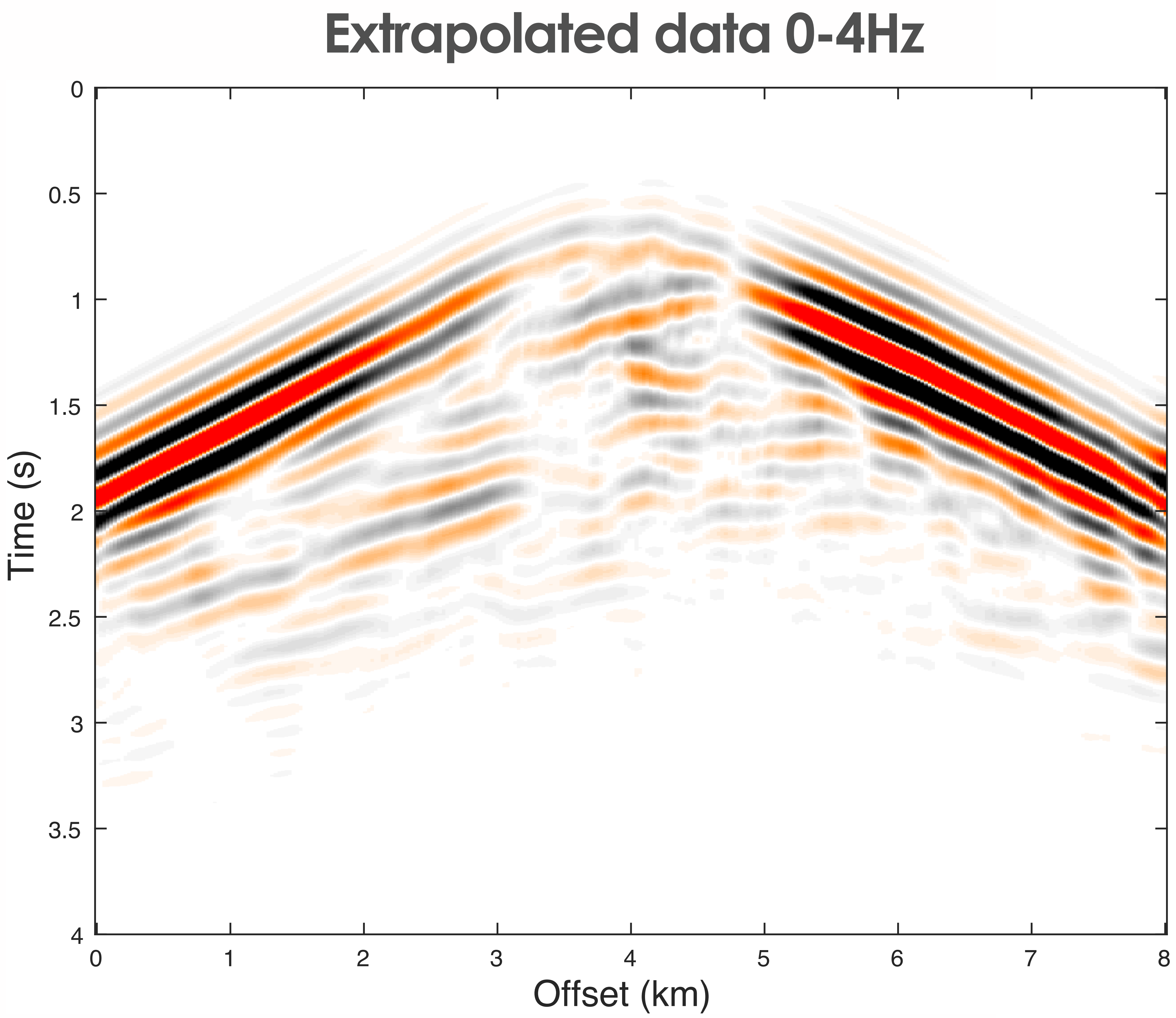
Deconvolution result from 5-15Hz data



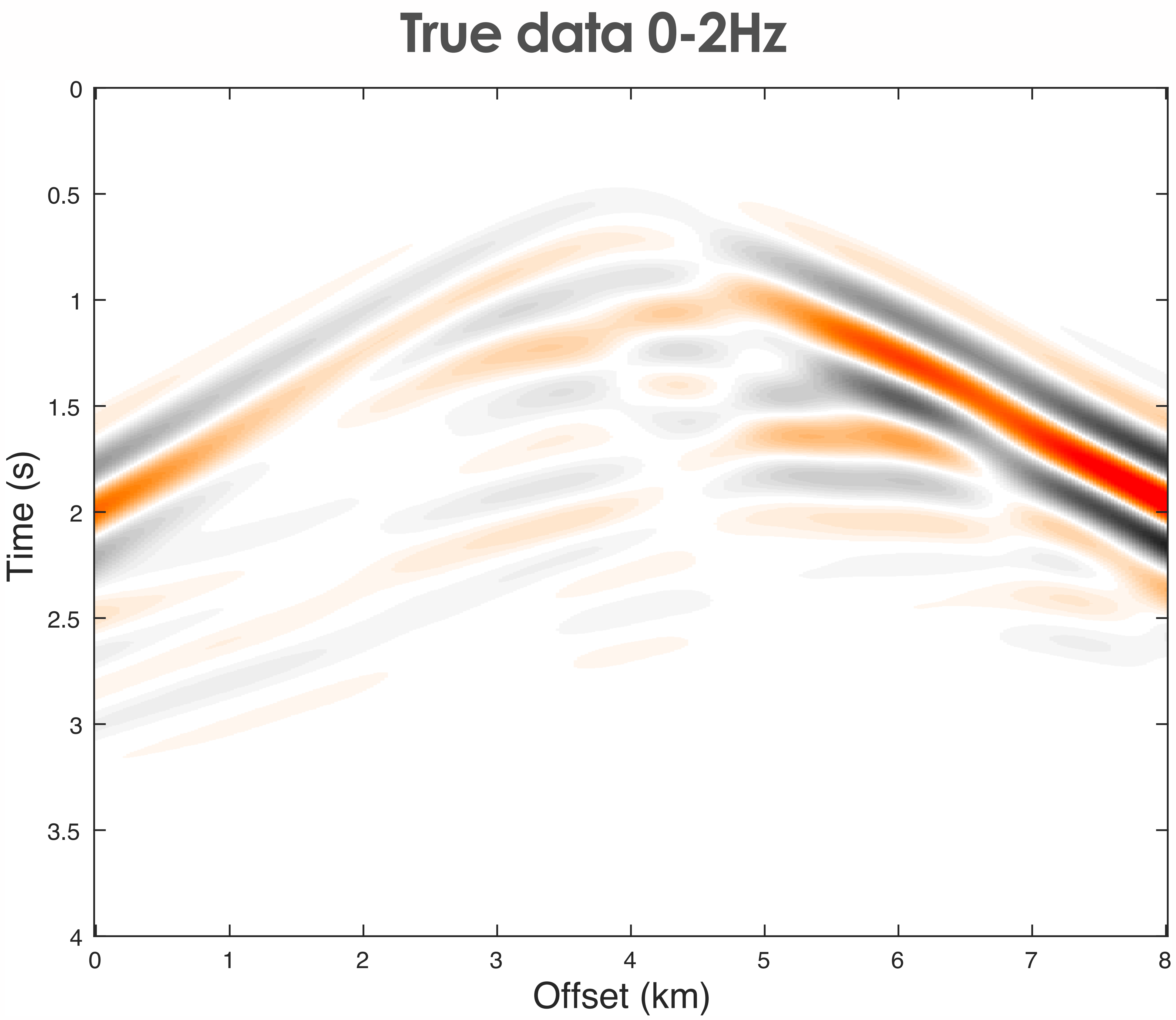


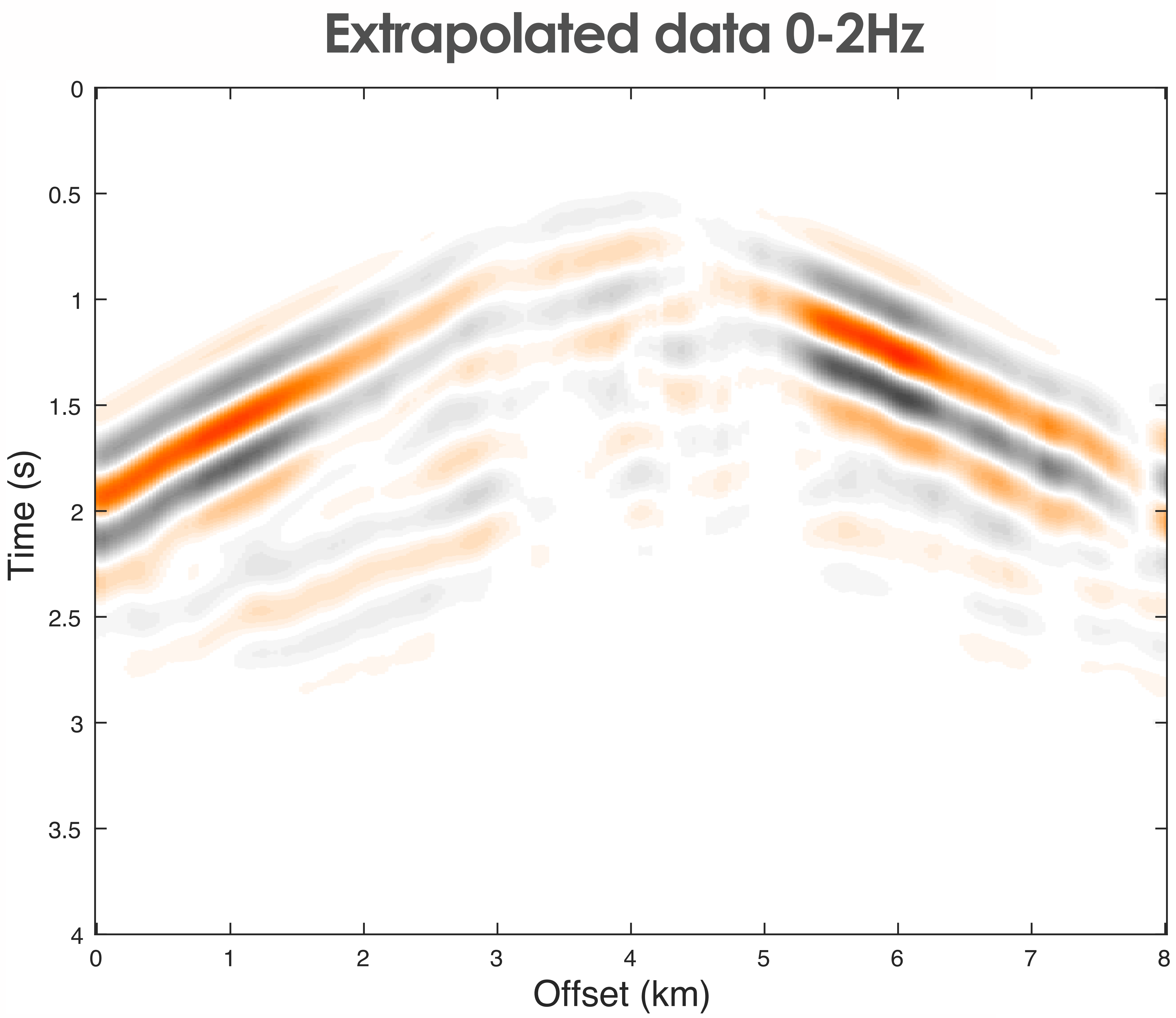
Deconvolution result from 5-15Hz data





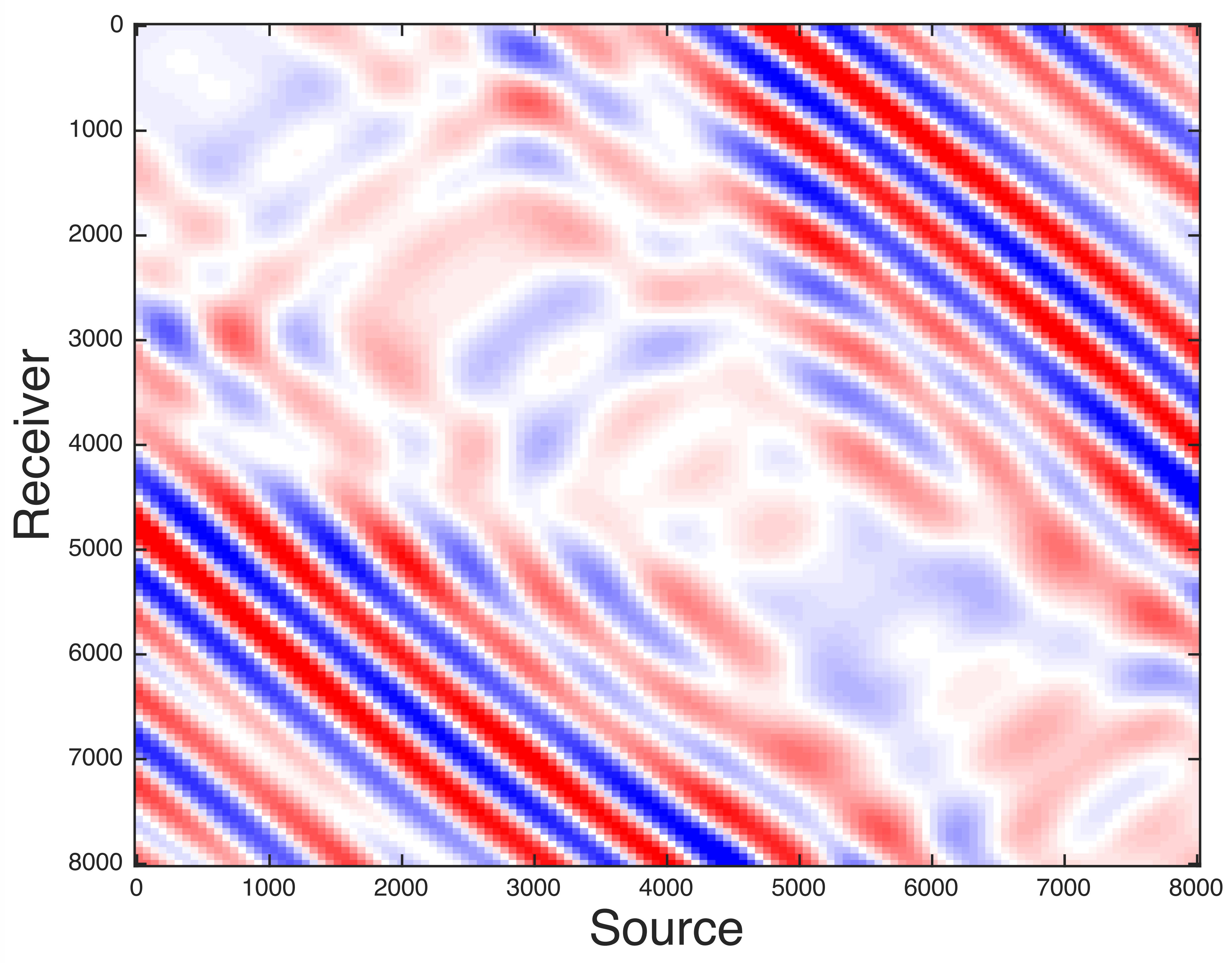
Deconvolution result from 5-15Hz data



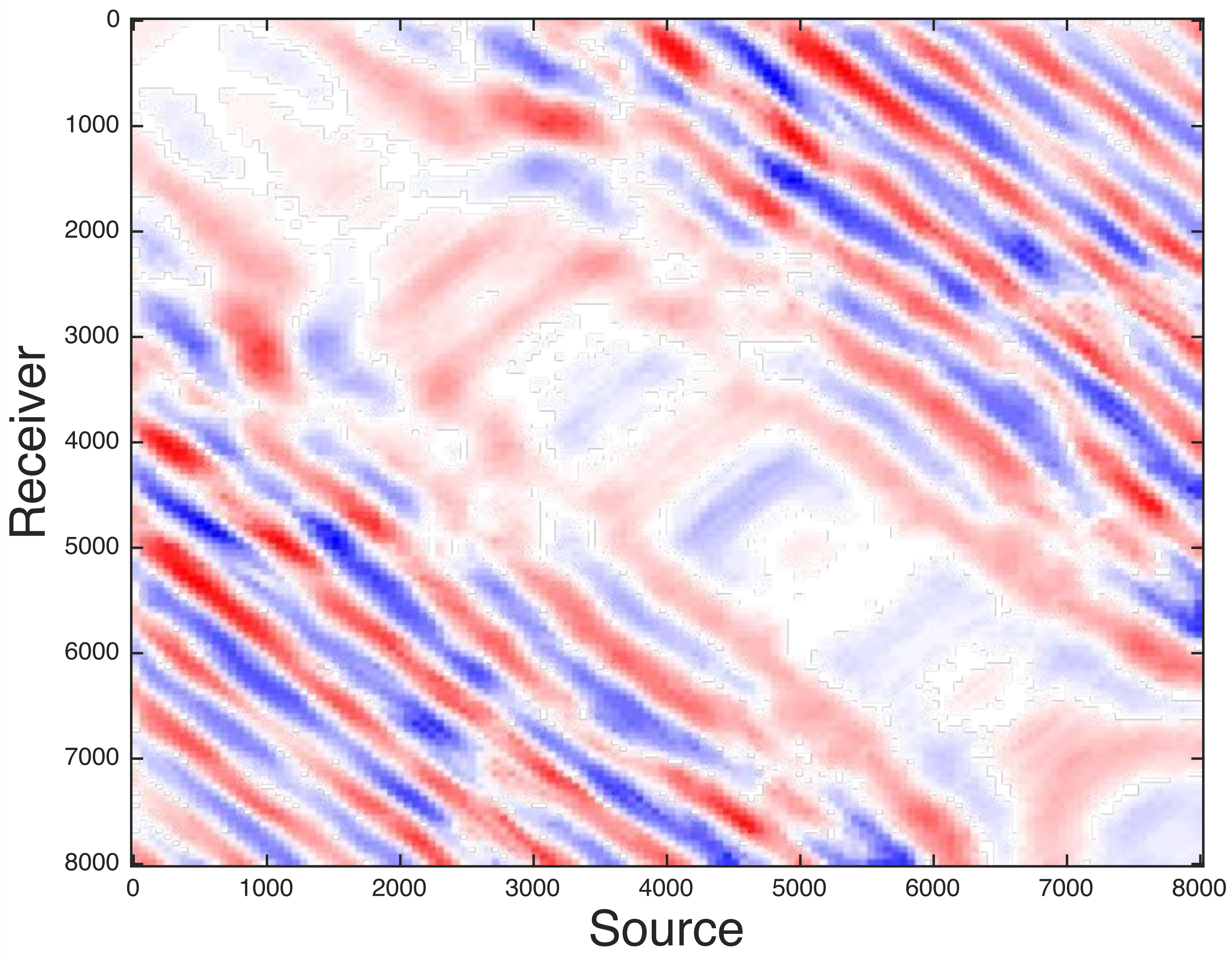


Deconvolution result from 5-15Hz data

True data 3Hz

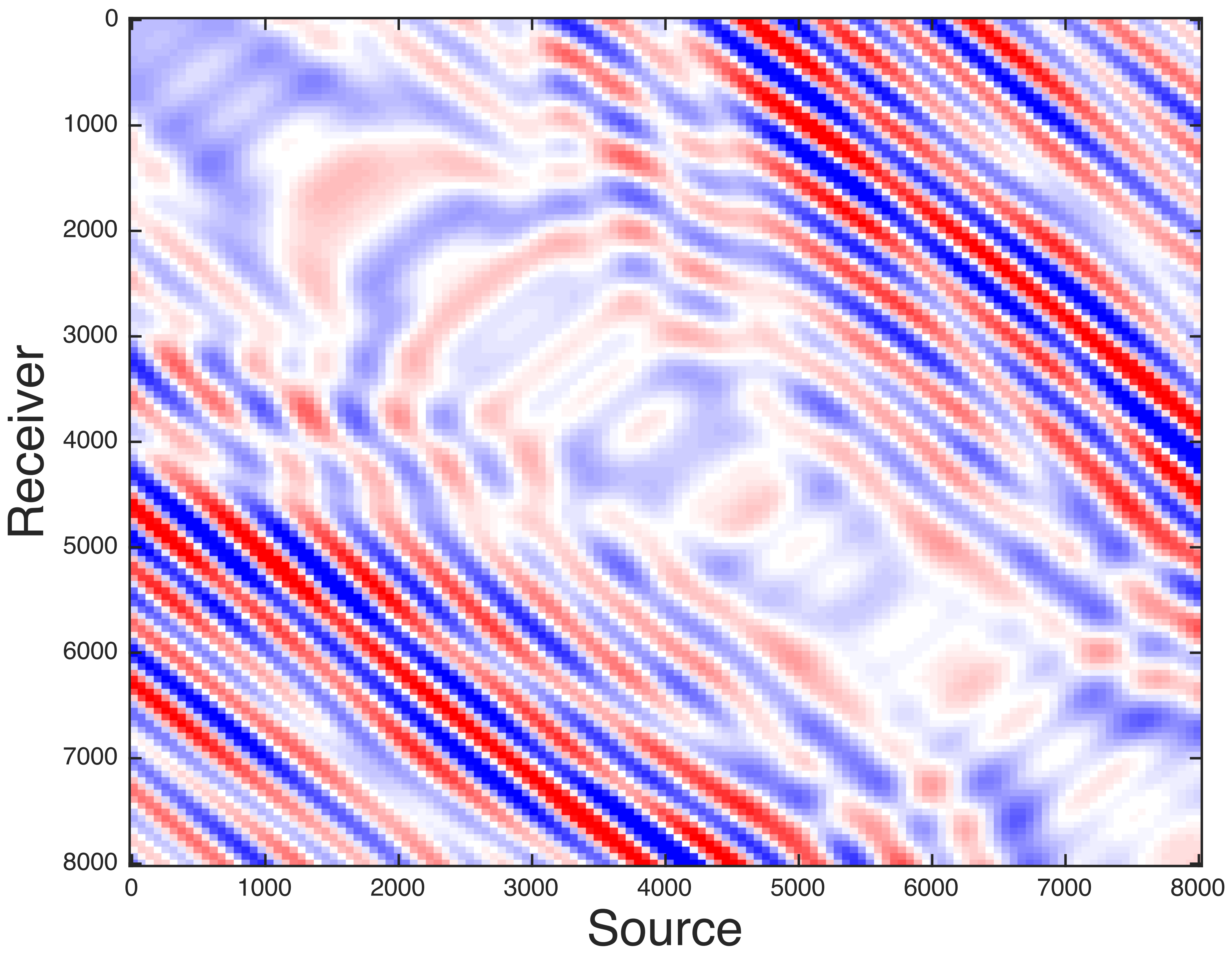


Extrapolated data 3Hz

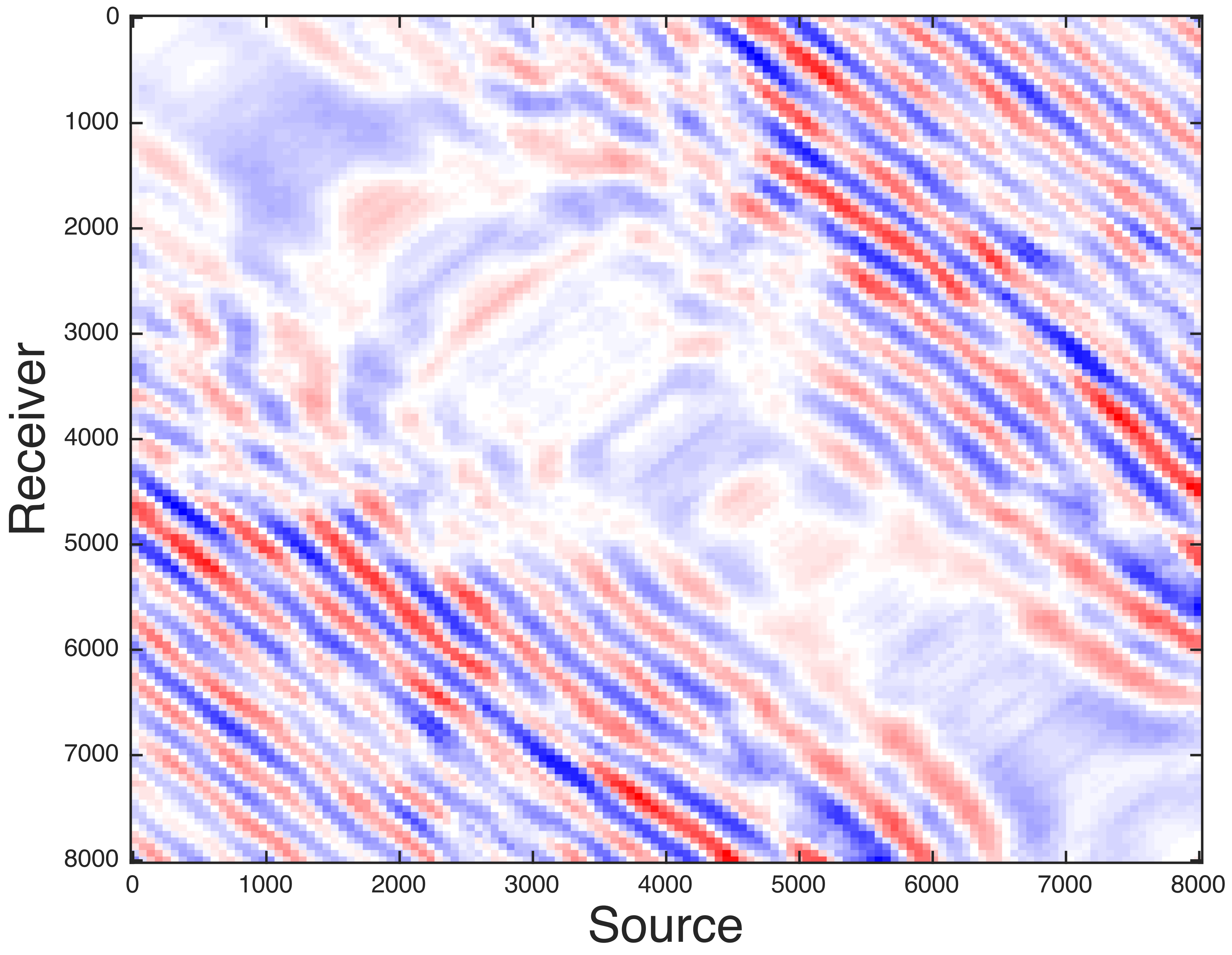


Deconvolution result from 5-15Hz data

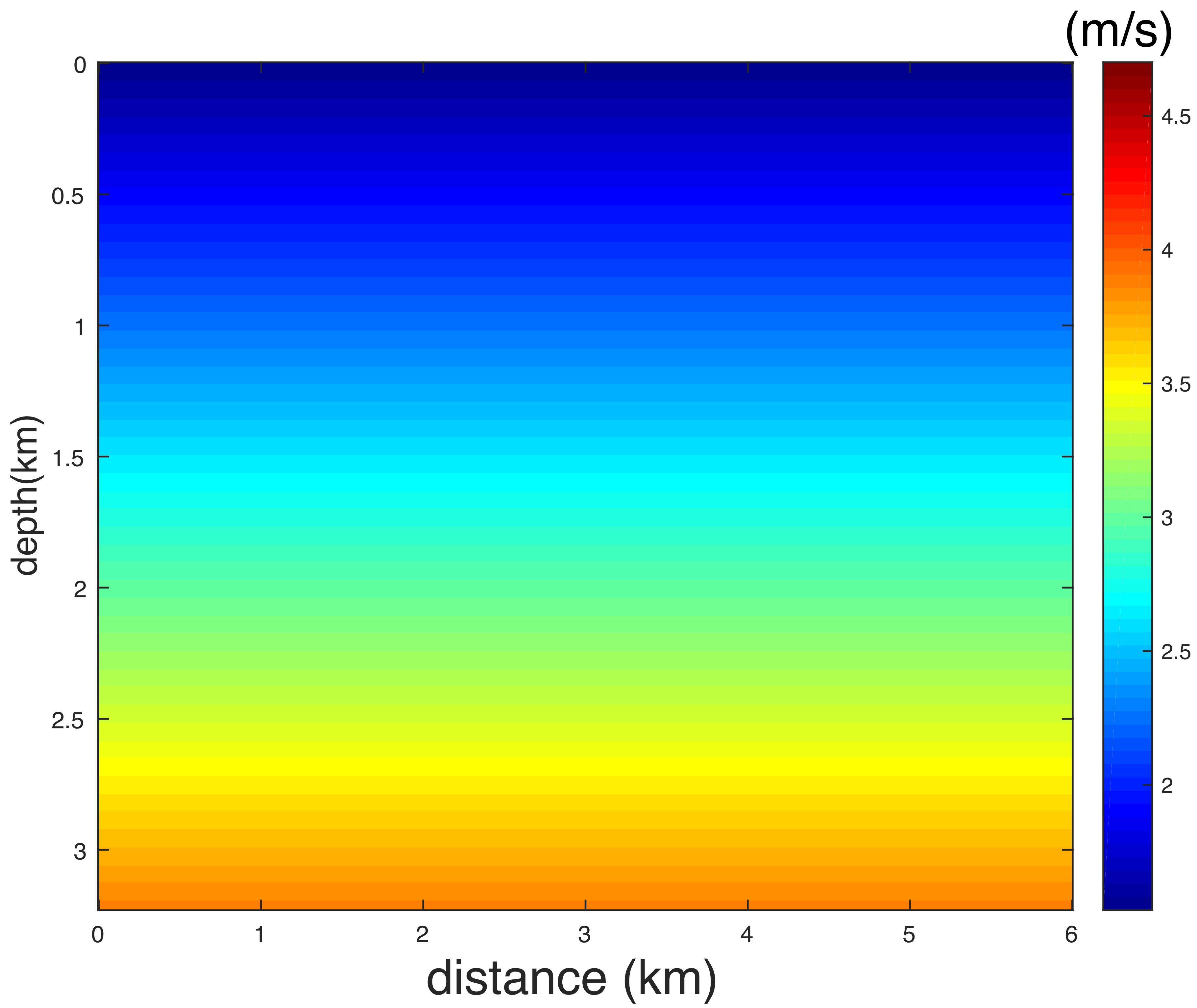
True data 4Hz

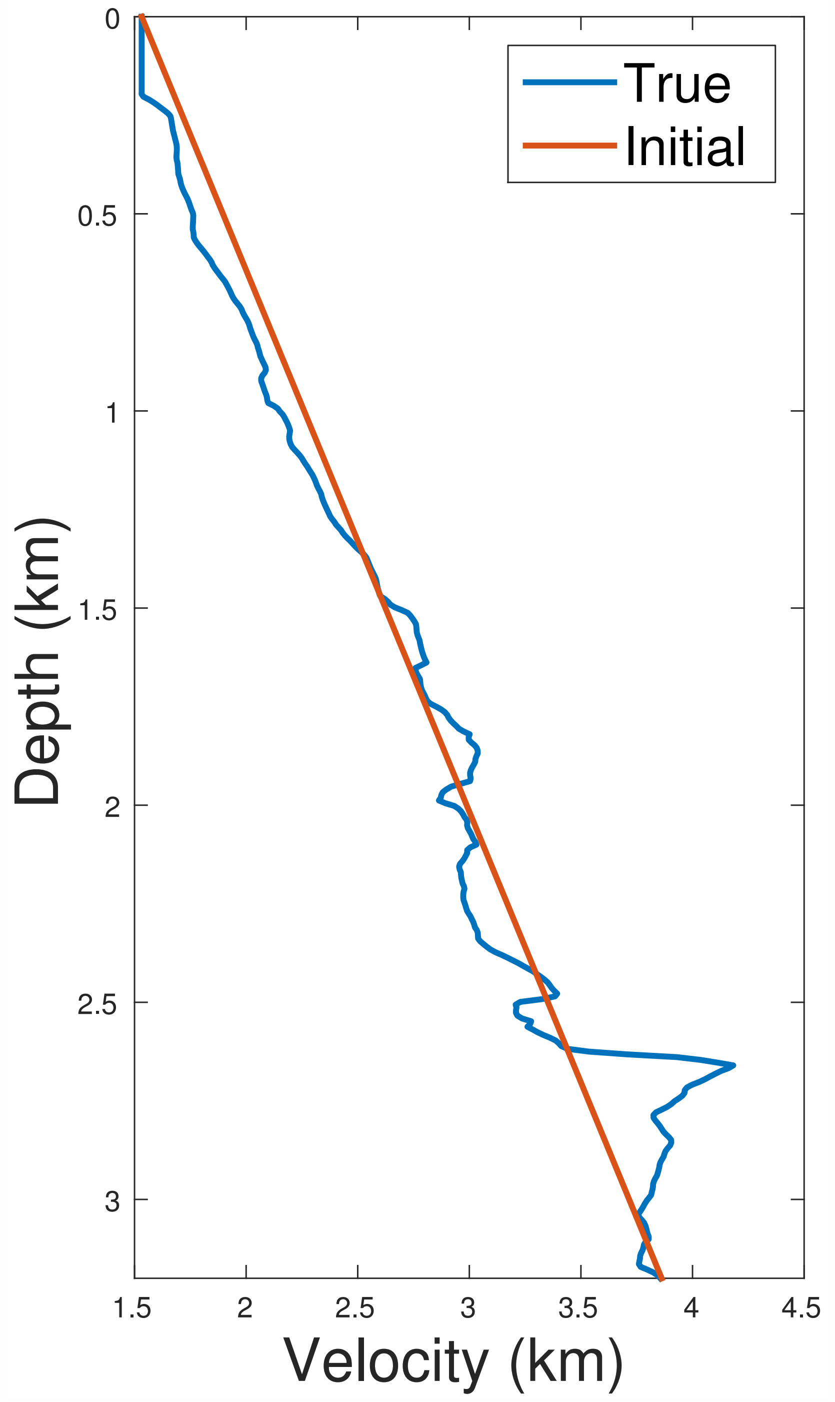


Extrapolated data 4Hz



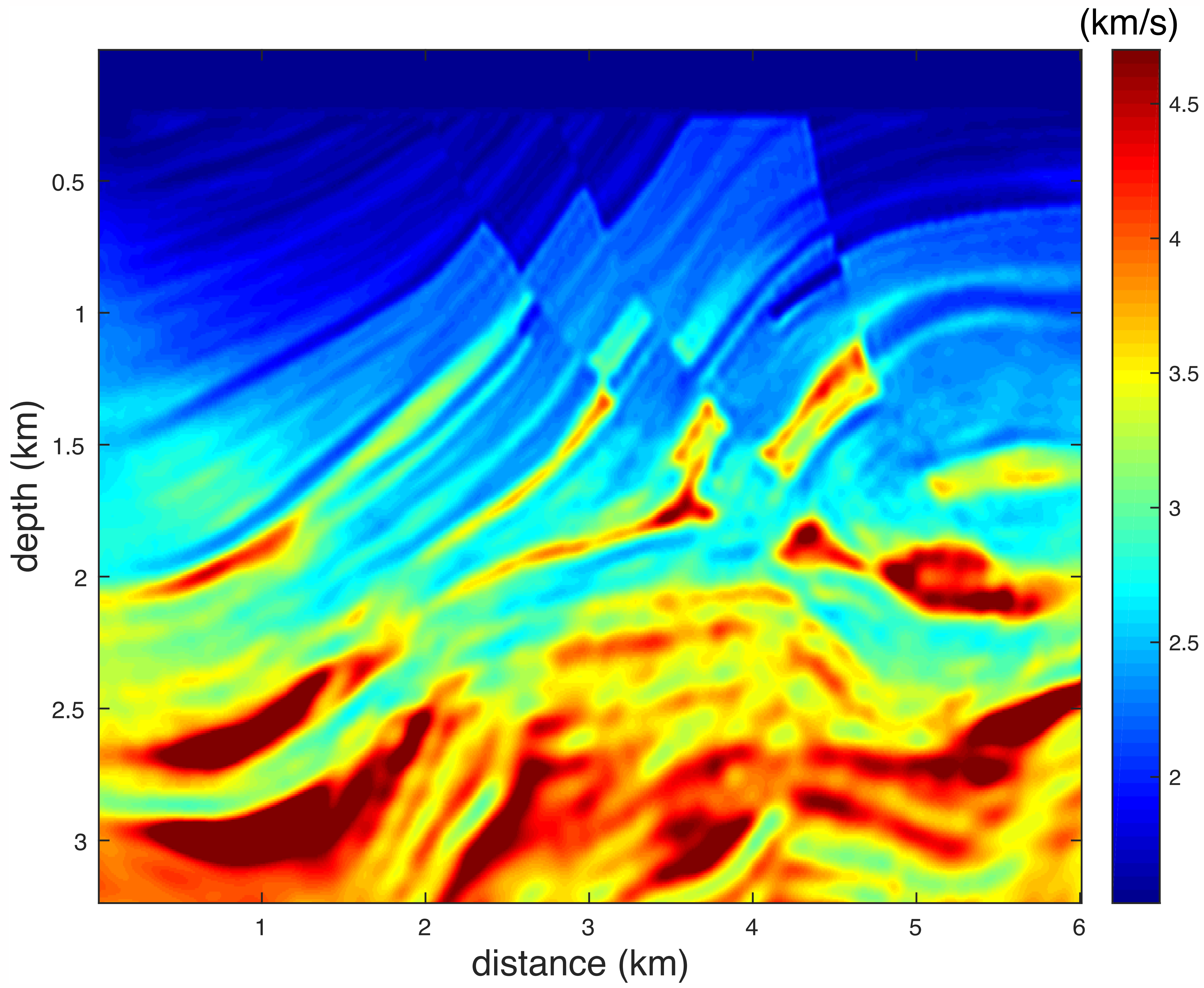
Initial guess



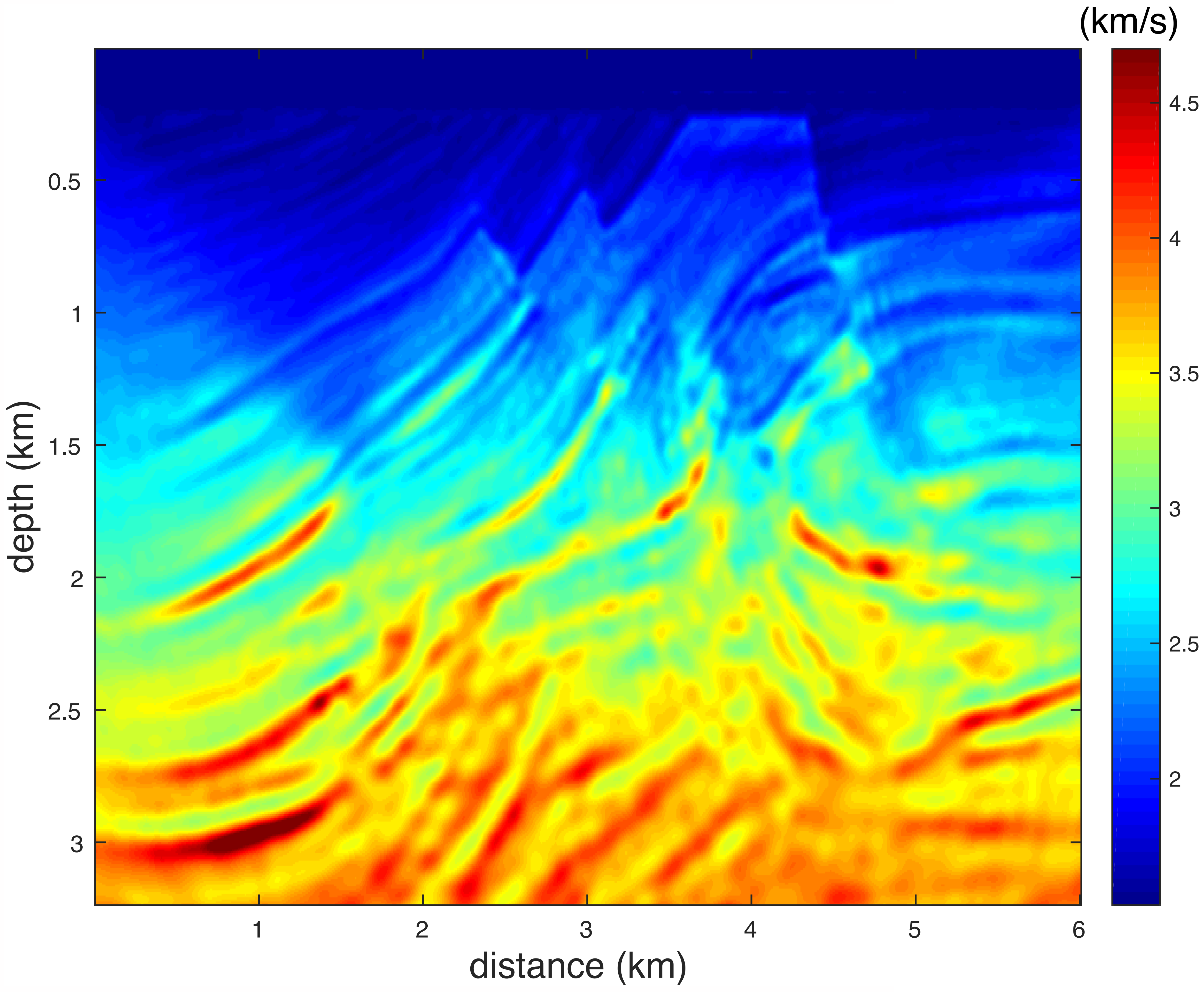


Recovery of FWI

1-15Hz true data

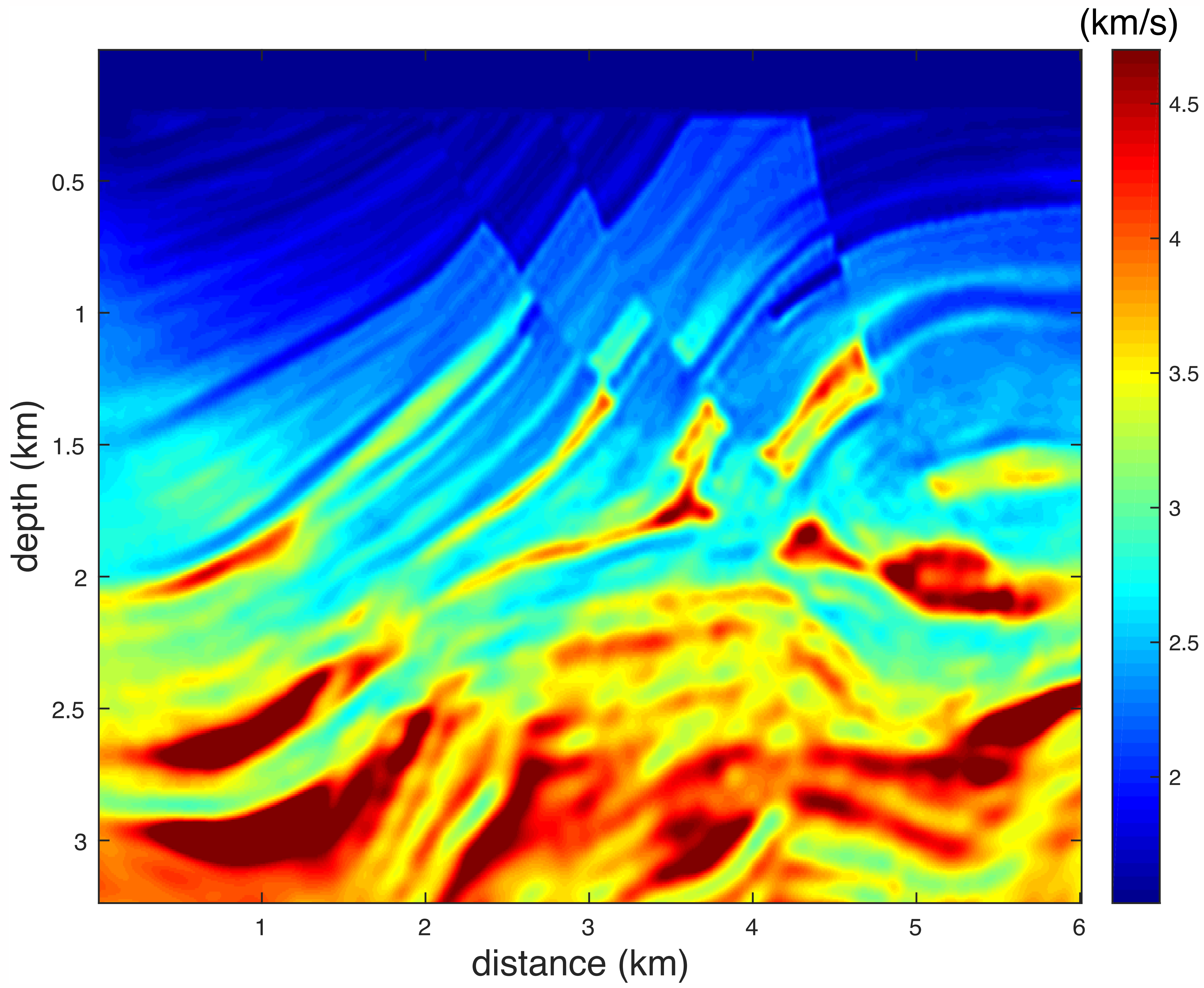


direct inversion with 5-15Hz data

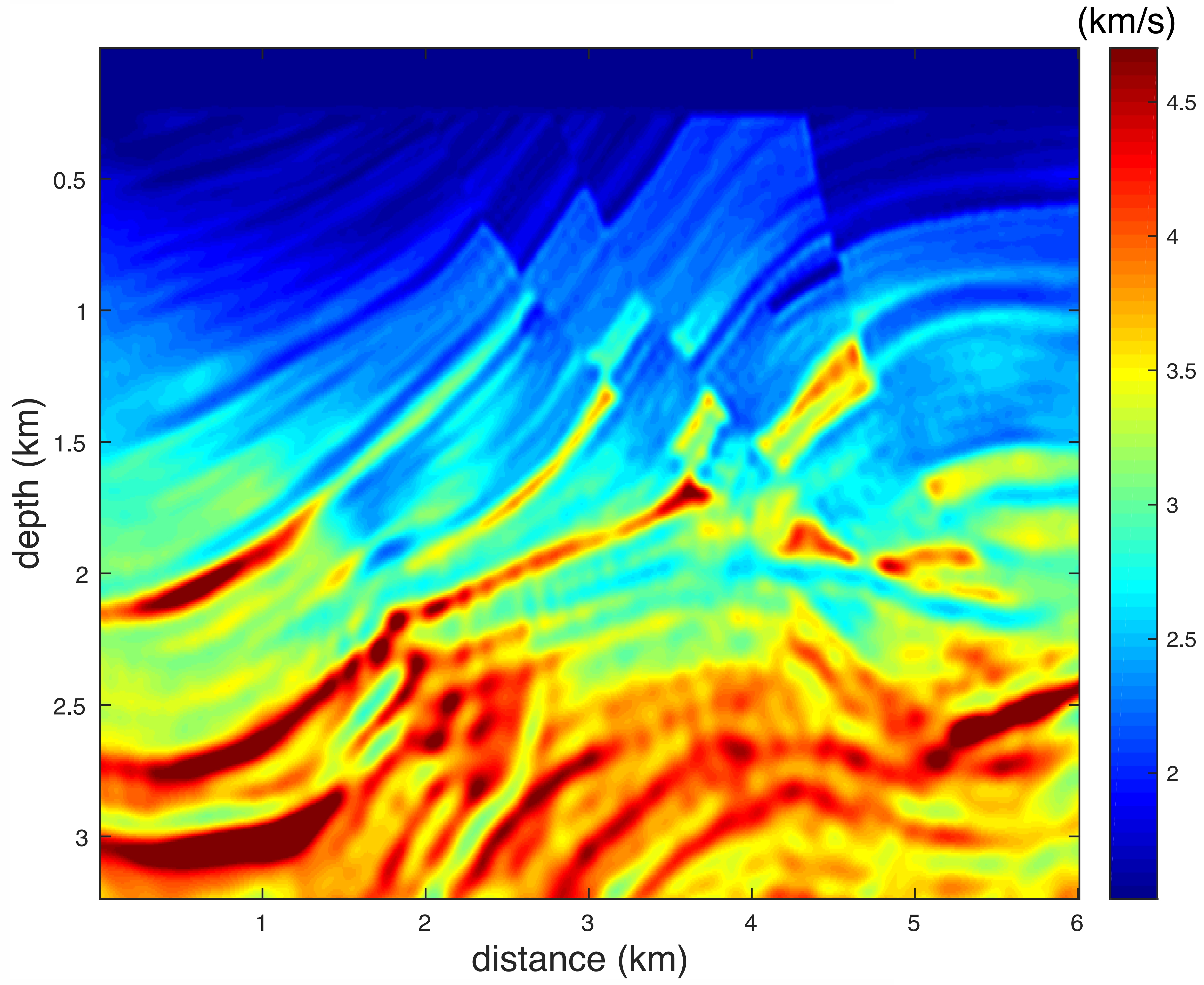


Inversion result via FWI

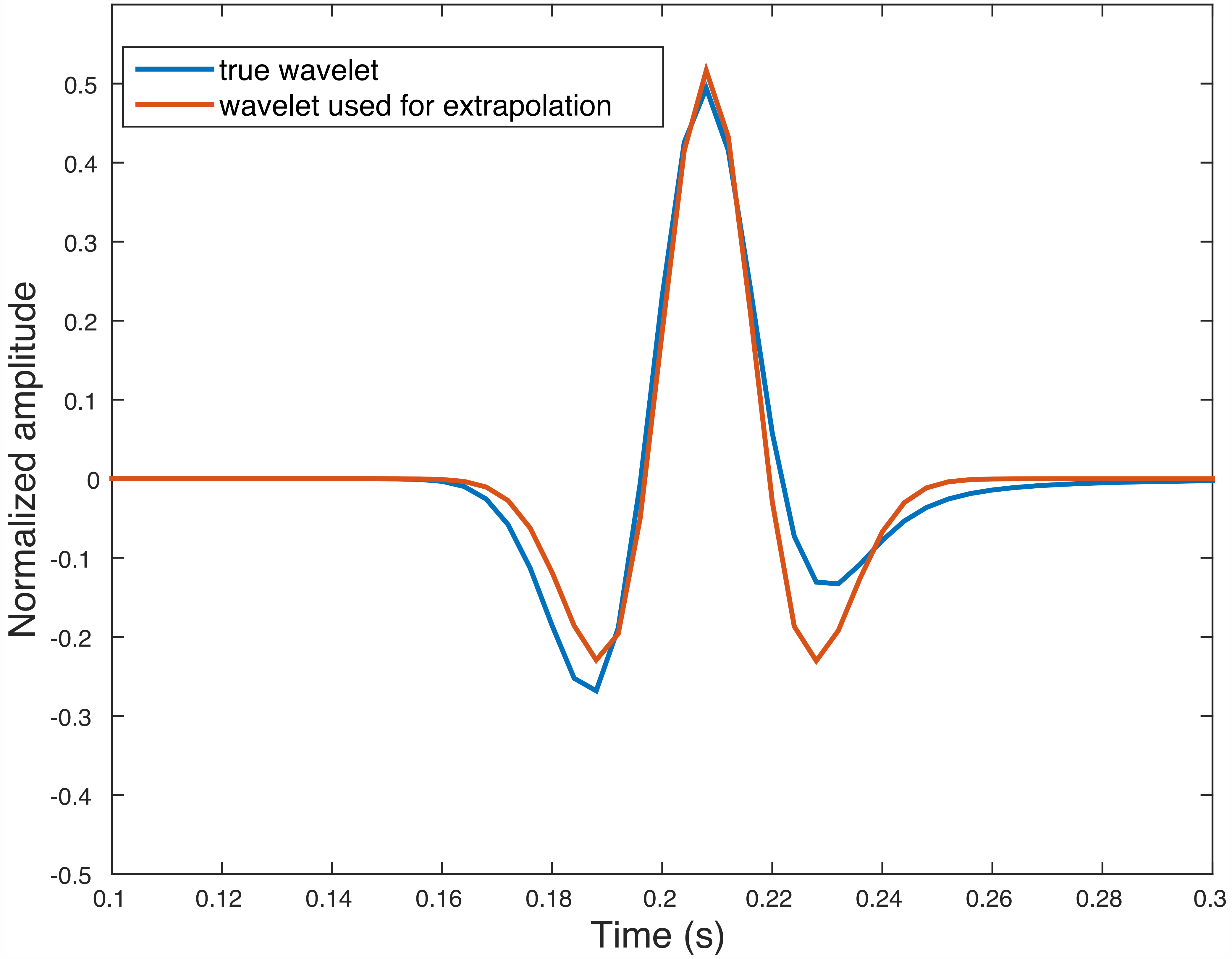
1-15Hz true data

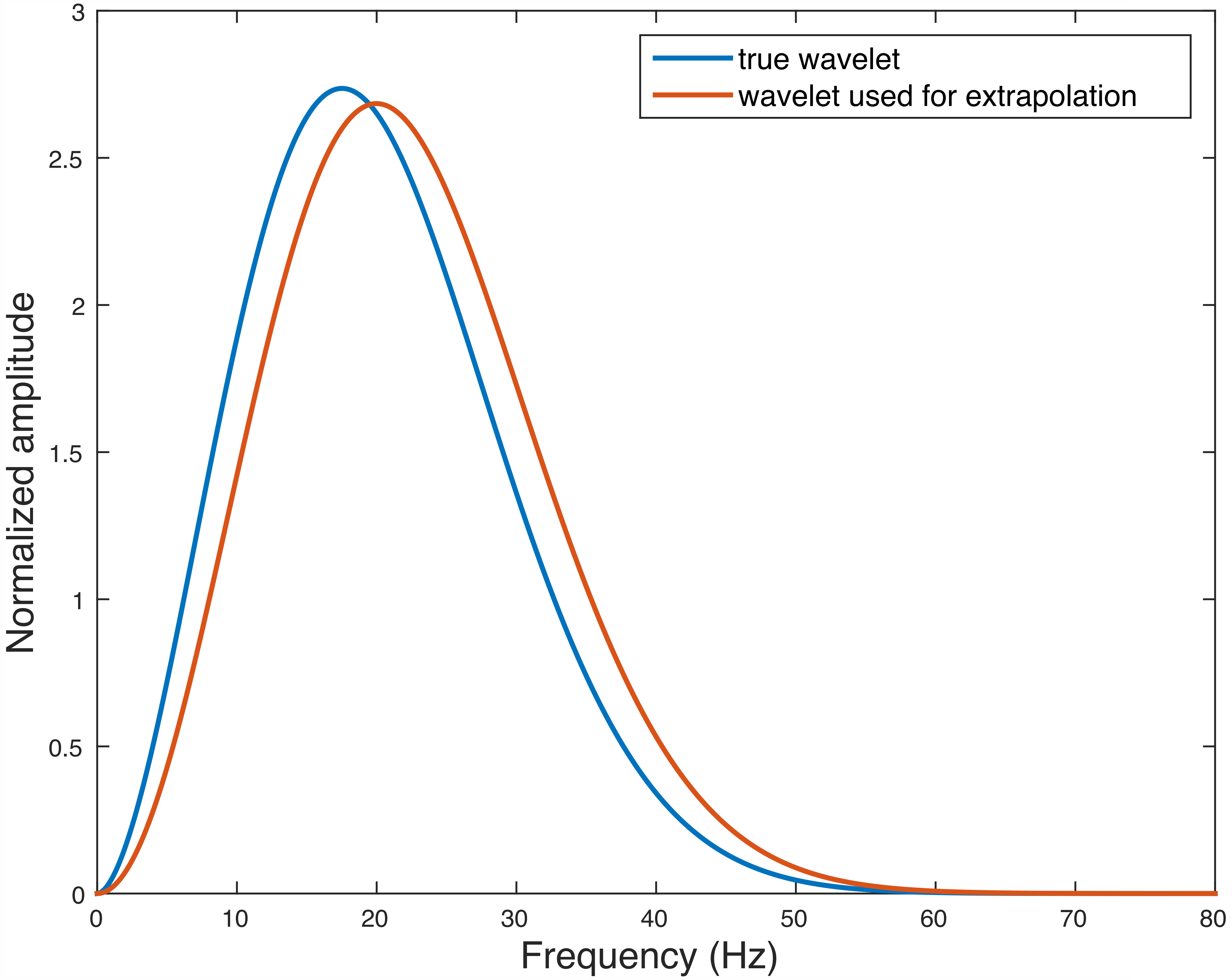


5-15Hz data with extrapolation



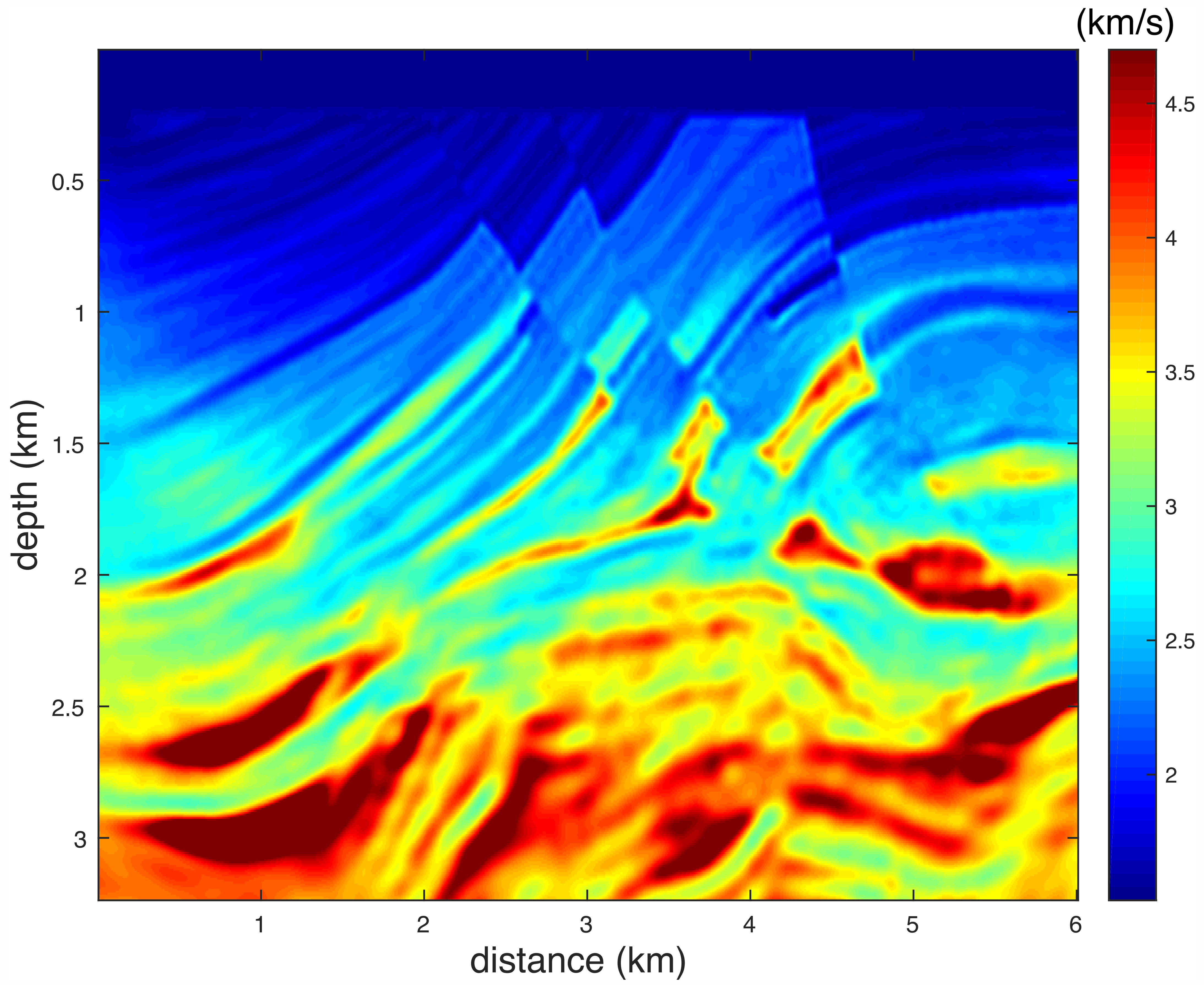
Stability test



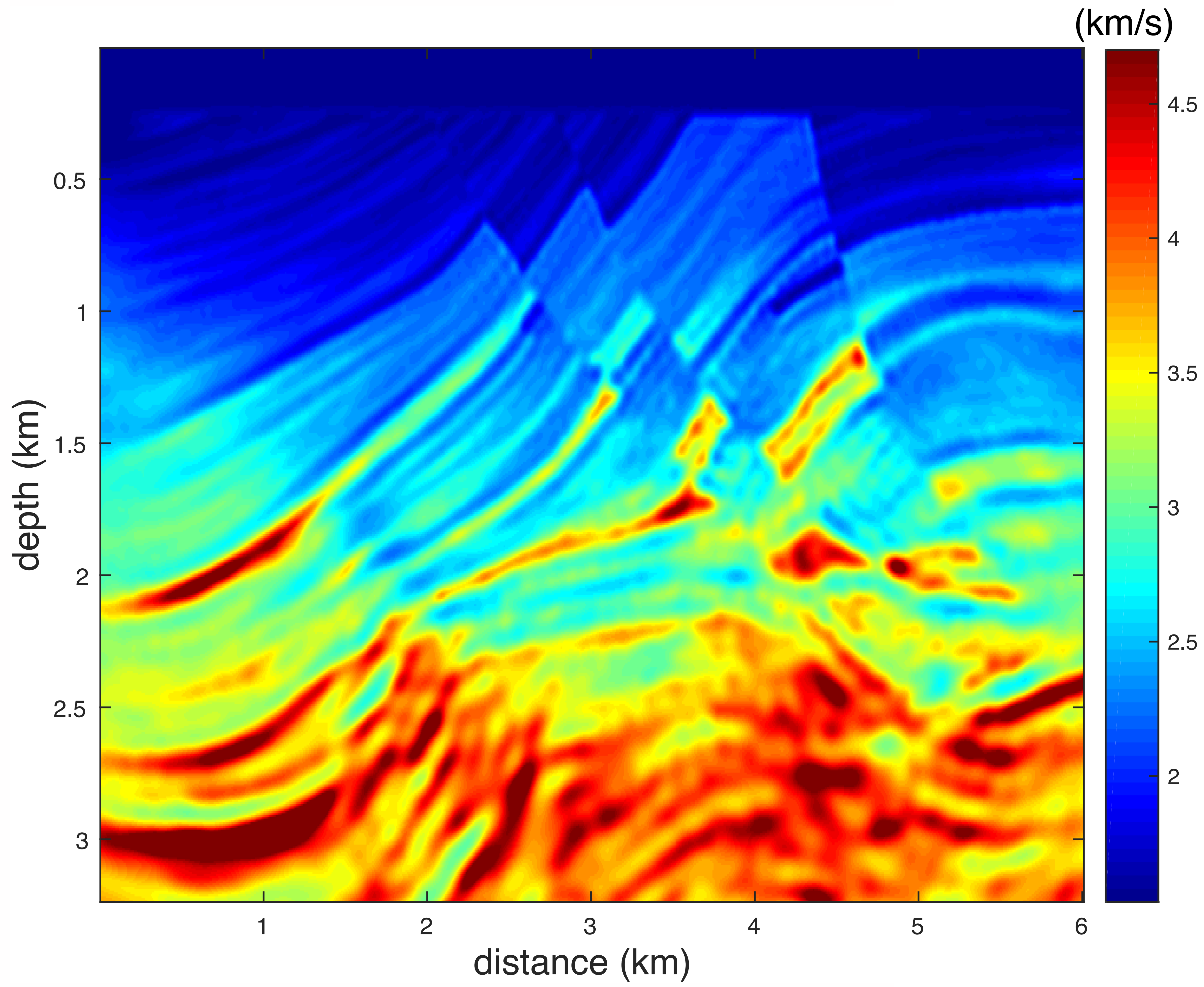


Inversion result via FWI

1-15Hz data

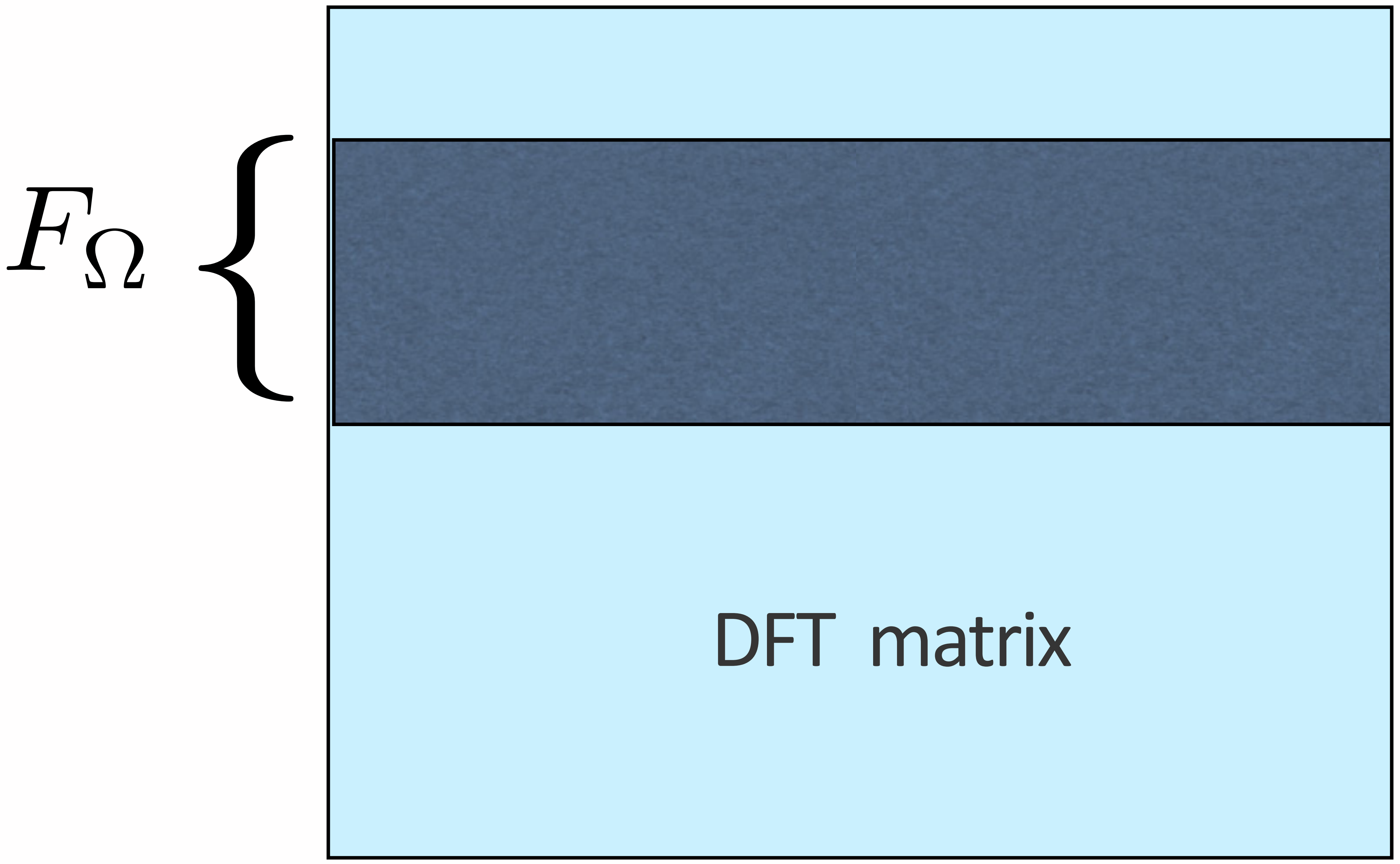


5-15Hz data with extrapolation



Lq norm minimization: to overcome the minimal distance barrier

Discrete setting



Signal: $x \in \mathbb{R}^N$

Wavelet: $w \in \mathbb{R}^N$

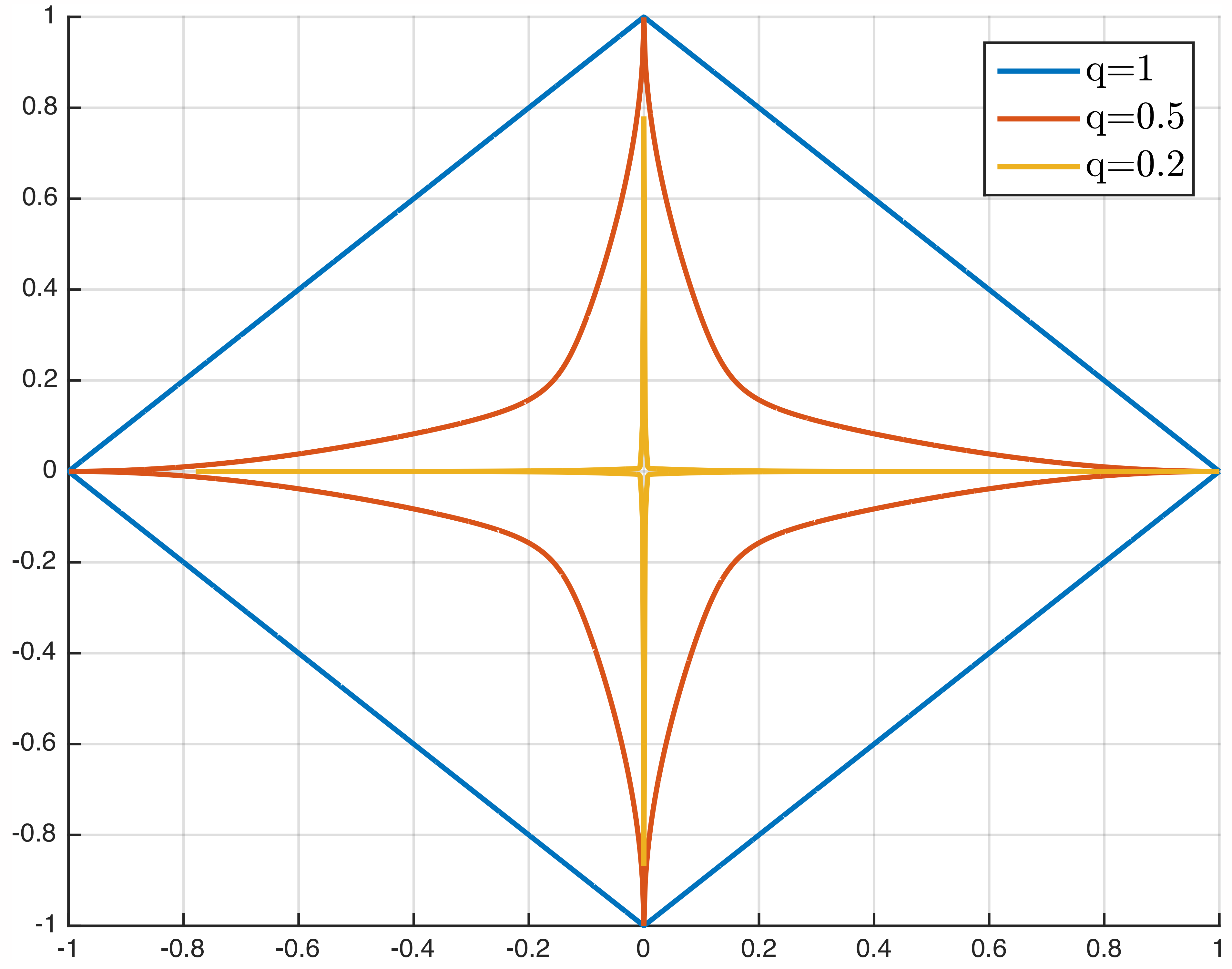
Bands in use: $\Omega = \{m_1, \ldots, m_2\}$

Data: $y = \underbrace{F_\Omega^* F_\Omega}_{\text{discrete filter}}(w * x) \in \mathbb{C}^N$

The Lq norm

$$\|x\|_q = (x_1^q + x_2^q + \cdots x_n^q)^{1/q} \qquad q \leq 1$$

Lq unit ball



$q \rightarrow 0$

more spiky, less convex

Lq objective

Lq norm minimization

$$\min_{\mathbf{r}} \|\mathbf{r}\|_q \qquad 0 < q < 1$$

$$\text{subject to } F_\Omega(\mathbf{w} * \mathbf{r}) = F_\Omega \mathbf{d}$$

$\mathbf{d}$: data
$\mathbf{w}$: wavelet
$\mathbf{r}$: reflectivity series
F_Ω: DFT matrix restricted to $\Omega = [m_1, m_2]$

Lq objective

Lq norm minimization

$$\min_{\mathbf{r}} \|\mathbf{r}\|_q \qquad 0 < q < 1$$
$$\text{subject to } F_\Omega(\mathbf{w} * \mathbf{r}) = F_\Omega \mathbf{d}$$

Exact recovery is guaranteed if

$$q \leq \frac{C}{s \log(N/(m_2 - m_1))}$$

The algorithm is less stable as q gets smaller

SLIM

L1+Lq

Weighted Lq

$$\min_{\mathbf{r}} \|\mathbf{r}\|_1 + \lambda \|\mathbf{r}\|_q^q$$

$$\text{subject to } F_\Omega(\mathbf{w} * \mathbf{r}) = F_\Omega \mathbf{d} \qquad 0 < q < 1$$

Theoretical analysis suggested choice

$$\lambda \in \left[c_q \frac{\|x\|_1}{\|x\|_q^q}, C_q \frac{\|x\|_1}{\|x\|_q^q} \right]$$

Approximate recovery guaranteed by

$$q \leq \frac{C}{\log N} \text{ and } s \leq c(m_2 - m_1)$$

Solver: Reweighted Least Squares

To solve

$$\min_{\mathbf{x}} \|\mathbf{x}\|_q^q$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

Write it as

$$\min_{\mathbf{x}} \|\mathbf{x}^{q-1} \odot \mathbf{x}\|_1$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

RLS iterates as

$$\mathbf{x}_{k+1} = \min_{\mathbf{x}} \|\mathbf{x_k}^{q-1} \odot \mathbf{x}\|_1$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

SLIM

Solver: Reweighted Least Squares

$$\mathbf{x}_{k+1} = \min_{\mathbf{x}} \|{\mathbf{x_k}}^{q-1} \odot \mathbf{x}\|_1$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

can be written more formally as

$$\mathbf{x}_{k+1} = \min_{\mathbf{x}} \|\mathbf{a}_k \odot \mathbf{x}\|_1 \qquad \text{(RLS)}$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{a}_{k+1} = (|\mathbf{x}_k| + \epsilon)^{q-1}$$

RLS for L1+Lq

Initialize weights: $\mathbf{a}_0 = [1, ..., 1]$

For $k = 1, ..., K$**:**

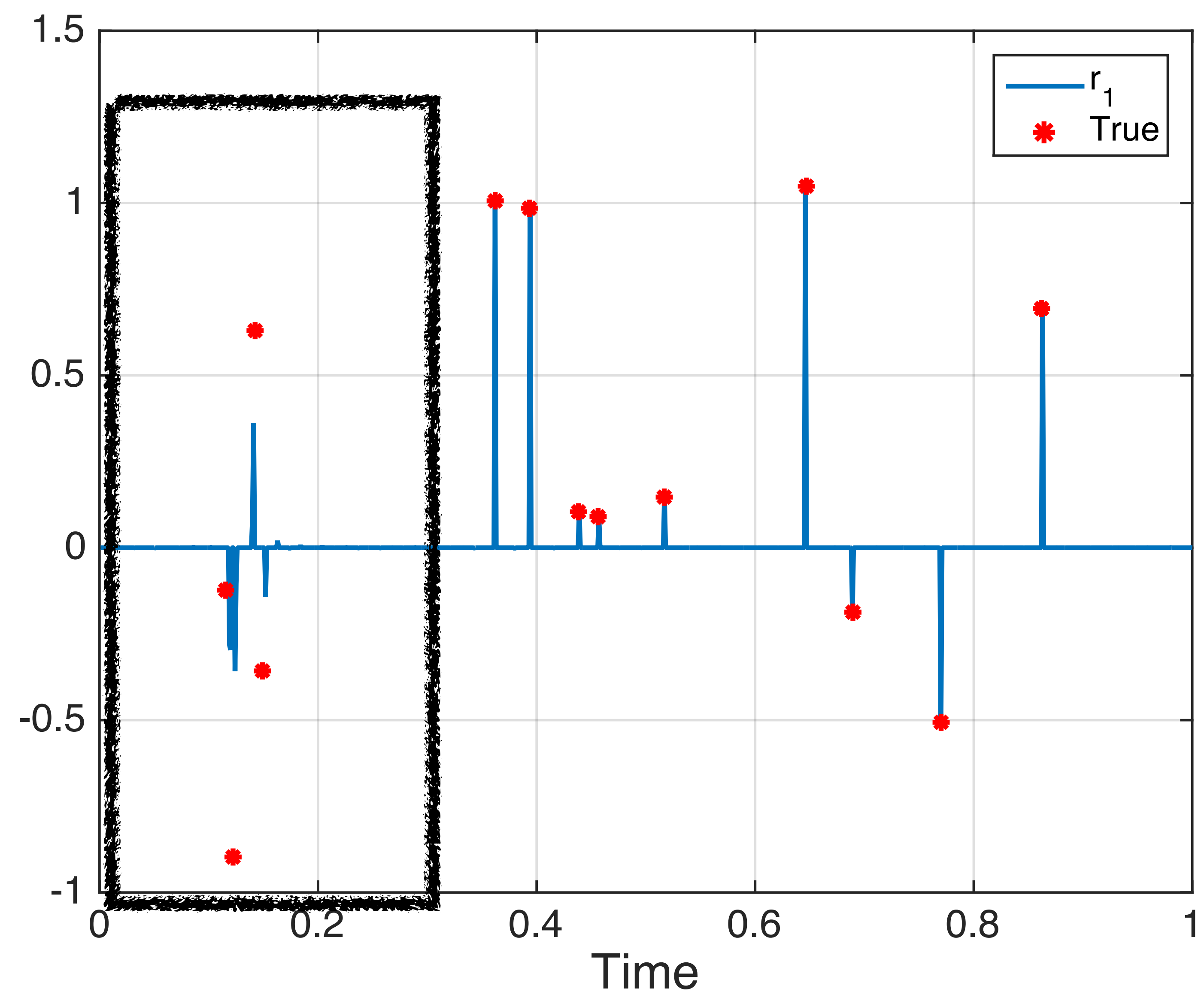
Define weight: $\mathbf{a}_k = [\mathbf{a}_k(1), ..., \mathbf{a}_k(n)], \quad \mathbf{a}_k(i) = \dfrac{1}{\epsilon + |\mathbf{r}_{k-1}(i)|^{1-p}}$

Update $\mathbf{r}_k$: $\quad \mathbf{r}_{k+1} = \min_{\mathbf{r}} \|\mathbf{r}\|_1 + \lambda \|\mathbf{a}_k \otimes \mathbf{r}\|_1$

$\quad$ subject to $F_\Omega(\mathbf{w} * \mathbf{r}) = F_\Omega \mathbf{d}$

End

Why RLS does not work for our case?

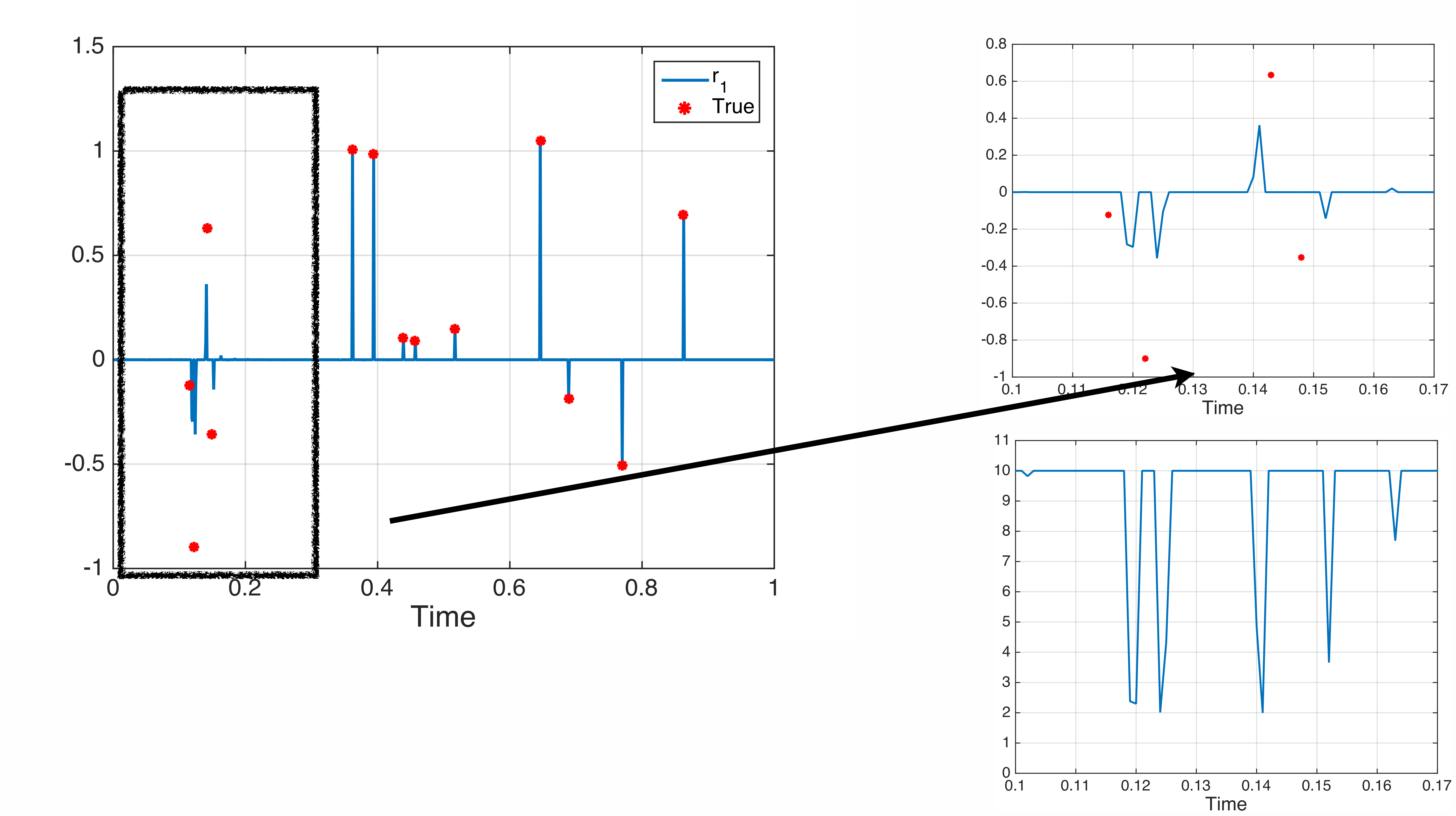


$$N = 1000$$

$$\Omega = [m_1, m_2] = [10, 50]$$

r1: estimated reflectivity series after the first step of RLS

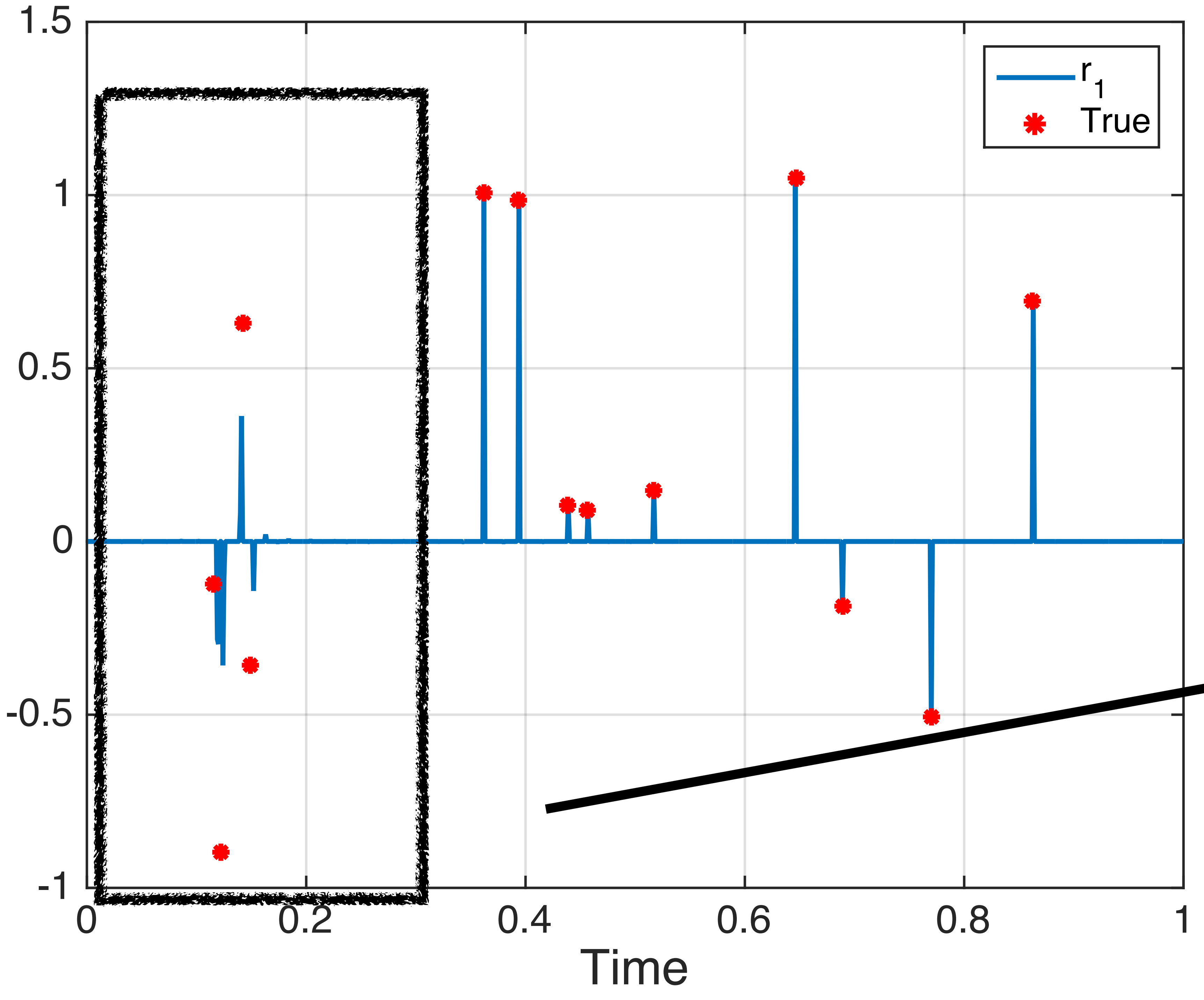
Why RLS does not work for our case?

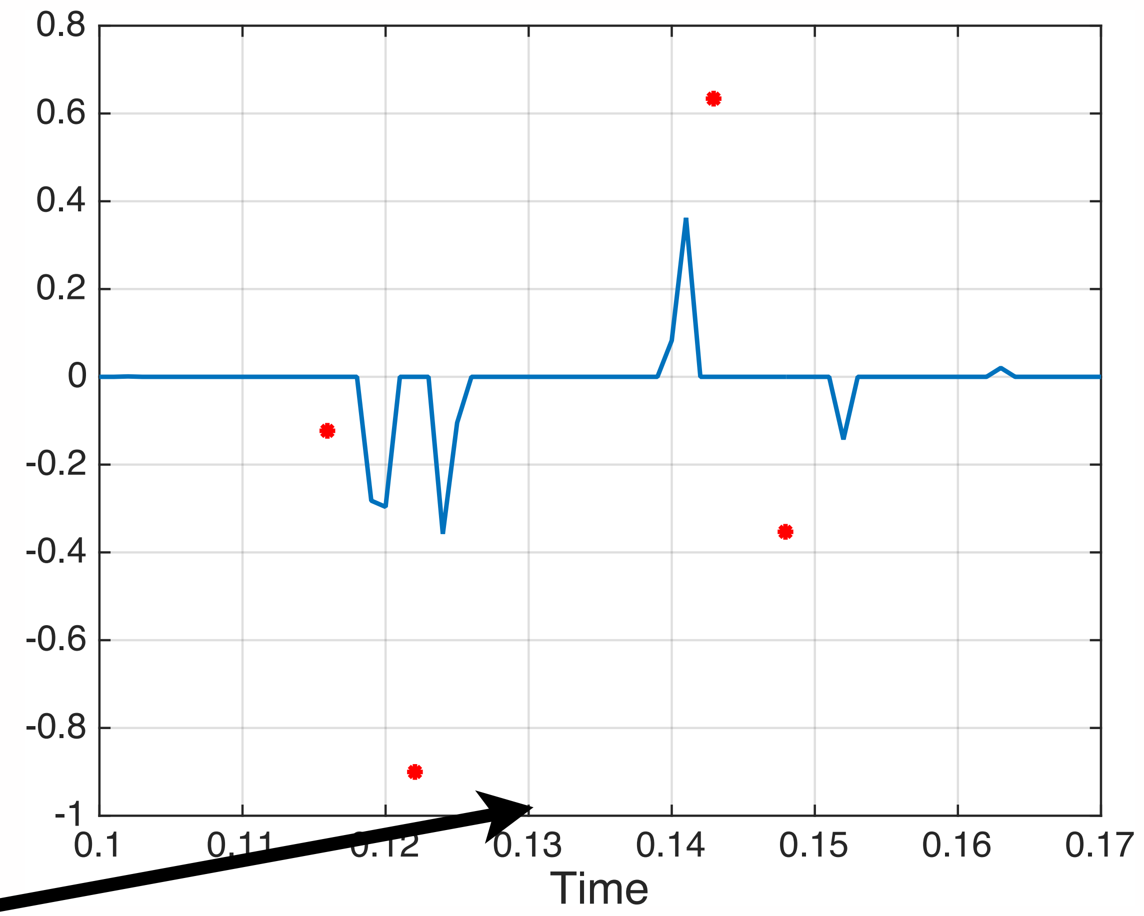


$\mathbf{r}_1$

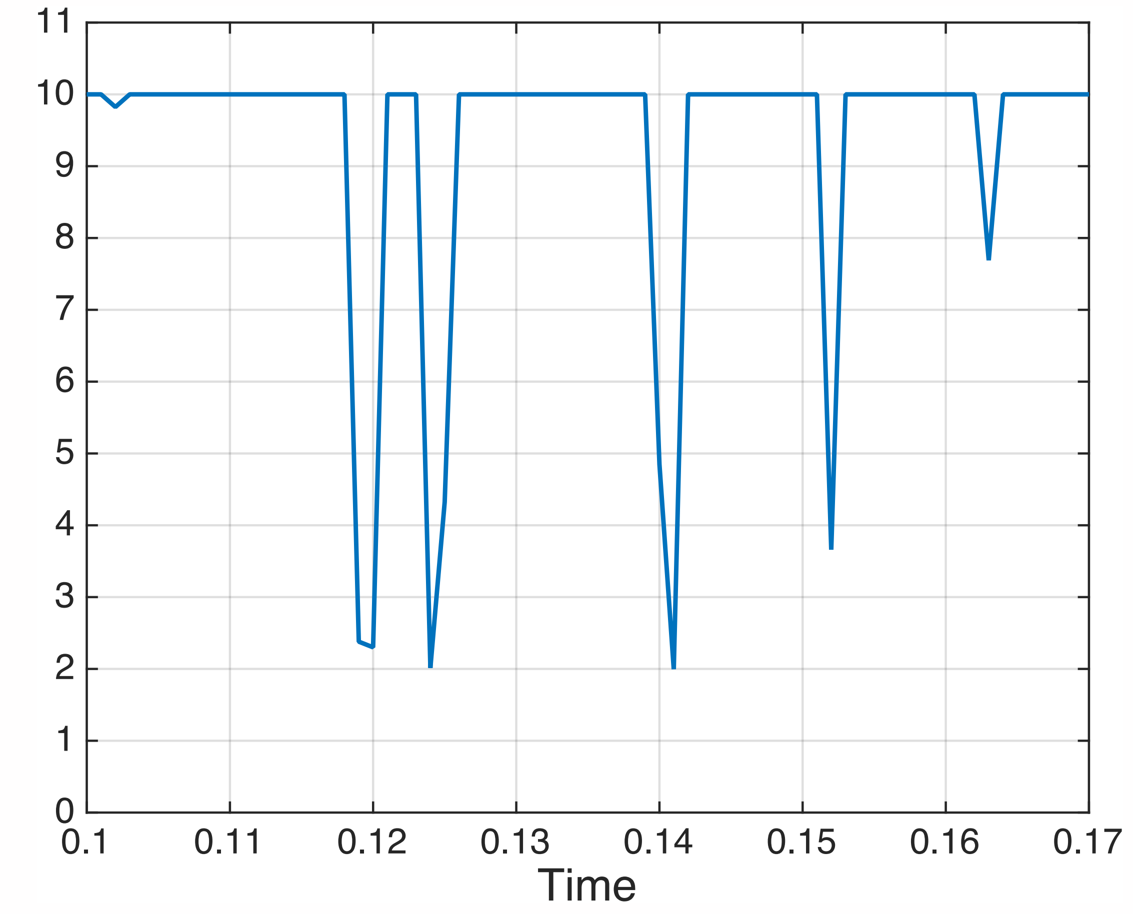
$\mathbf{a}_2$

Why RLS does not work for our case?





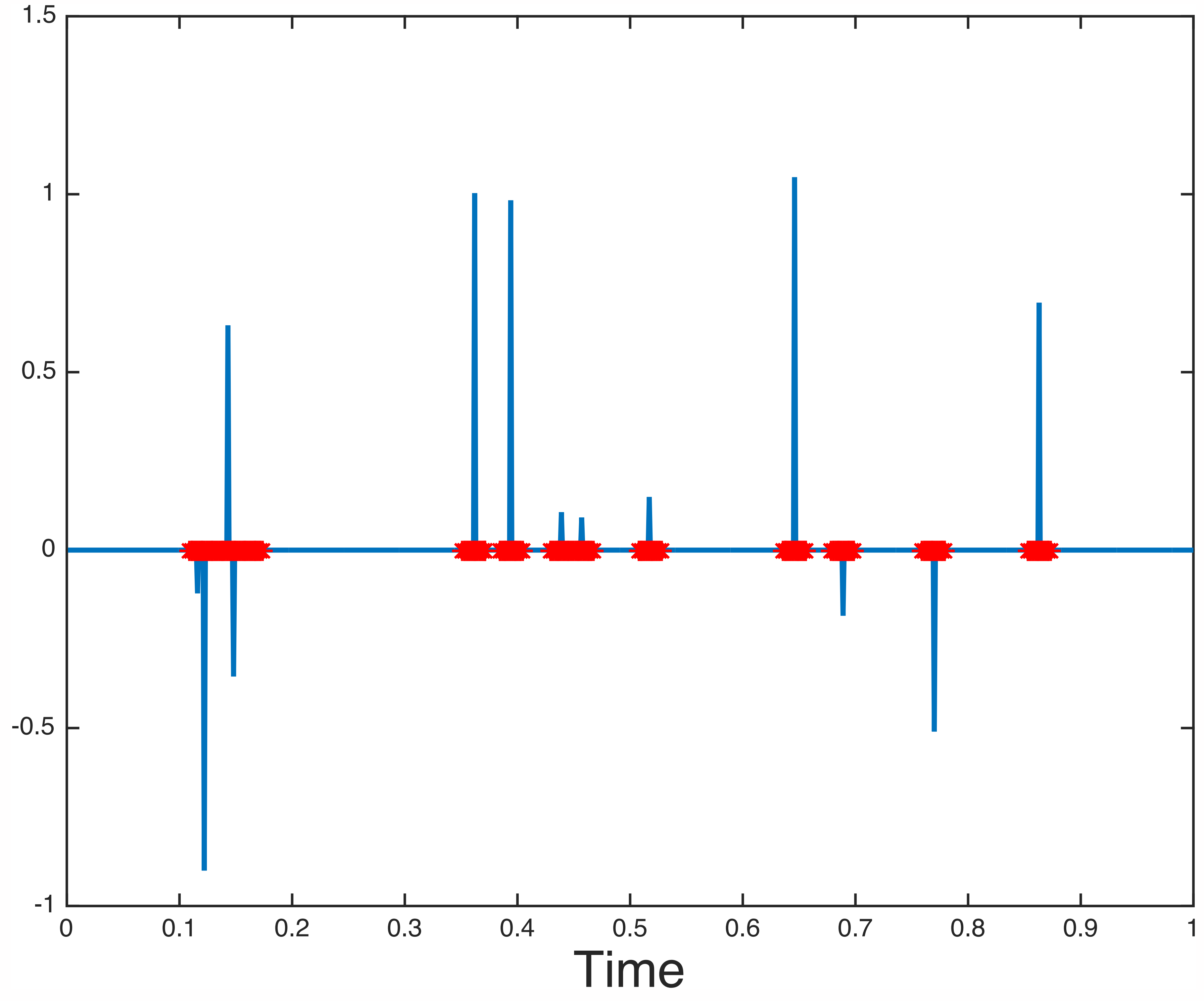
$\mathbf{r}_1$



Weights promotes
the previously detected
wrong supports

Solution

The true support is within 1.5 wavelength of the nonzeros of $\mathbf{r}_1$



Modified Reweighted Least Squares

For $l = 1, ..., L$**:**

$$\text{size}(\mathcal{N}_i) = \frac{1.5N}{m_1 - m_2}\frac{1}{l}, \quad \text{for } i = 1, ..., N$$

For $k = 1, ..., K$**:**

Define weight: $\mathbf{a}_k = [\mathbf{a}_k(1), ..., \mathbf{a}_k(n)], \quad \mathbf{a}_k(i) = \dfrac{1}{\epsilon + \min_{j \in \mathcal{N}_i} |\mathbf{r}_k(j)|^{1-p}}$

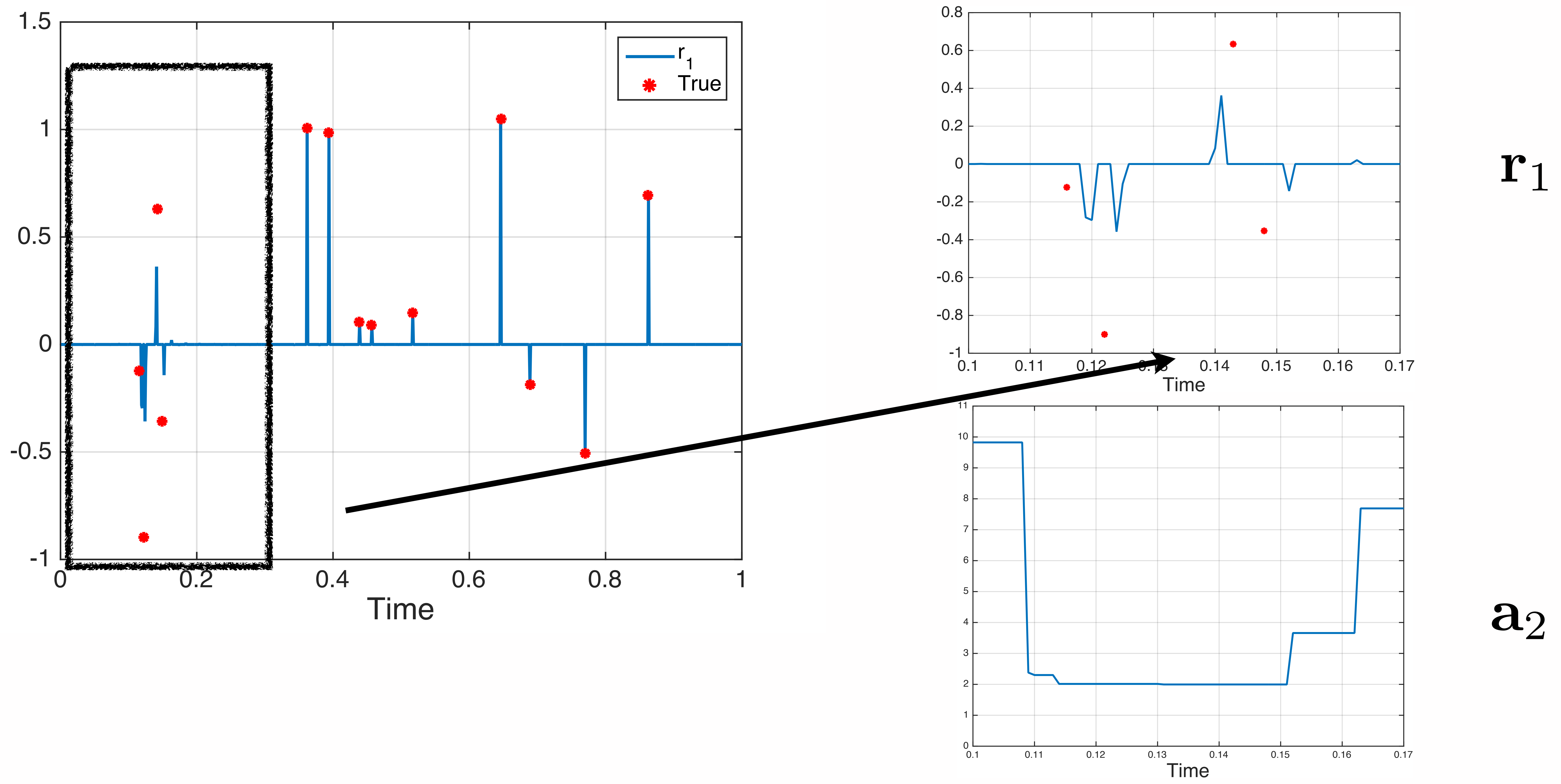
Update $\mathbf{r}_k$: $\quad \mathbf{r}_{k+1} = \min_{\mathbf{r}} \|\mathbf{r}\|_1 + \lambda \|\mathbf{a}_k \odot \mathbf{r}\|_1$

subject to $F_\Omega(\mathbf{w} * \mathbf{r}) = F_\Omega \mathbf{d}$

End

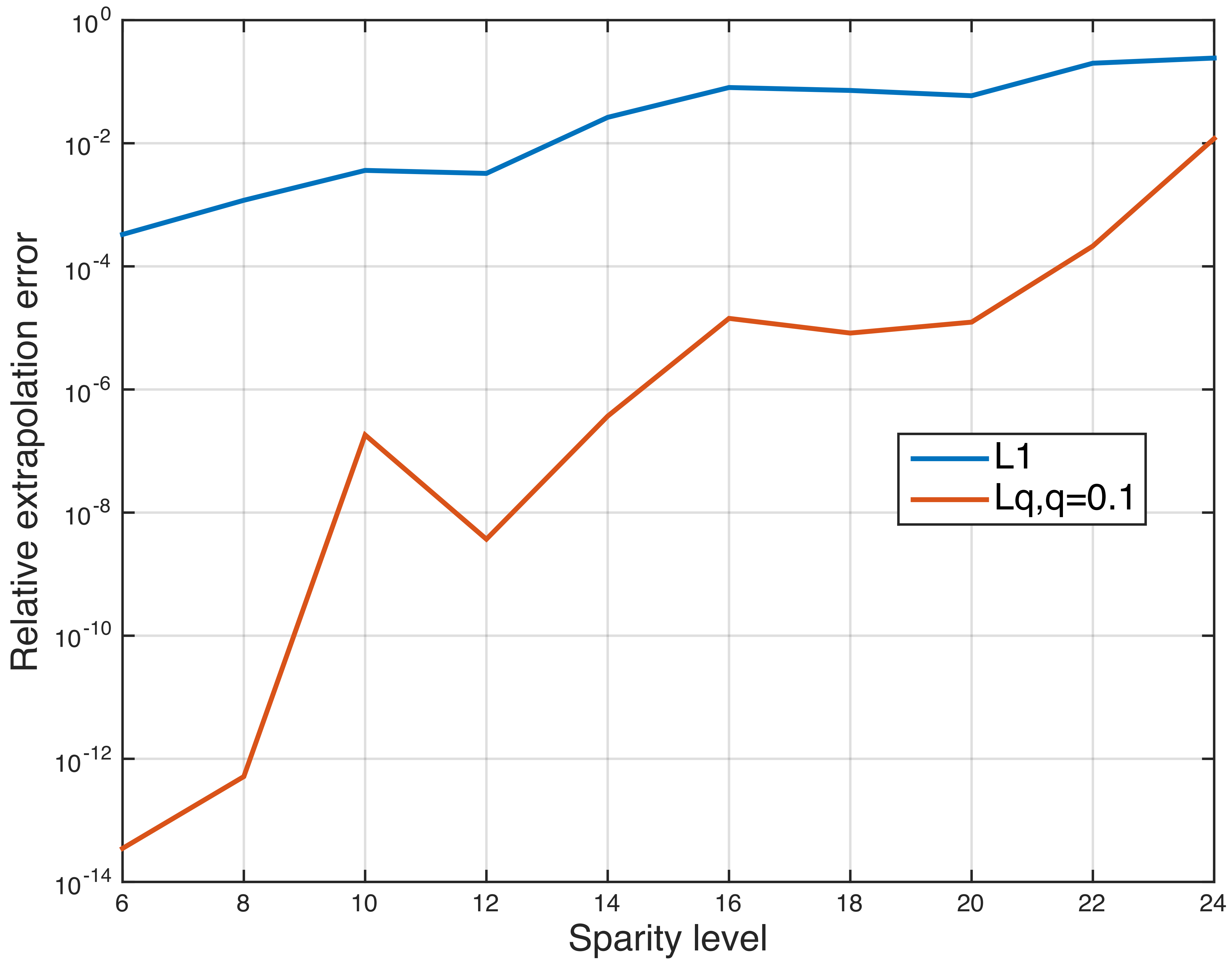
End

Back to previous example



Numerical test

Clean data



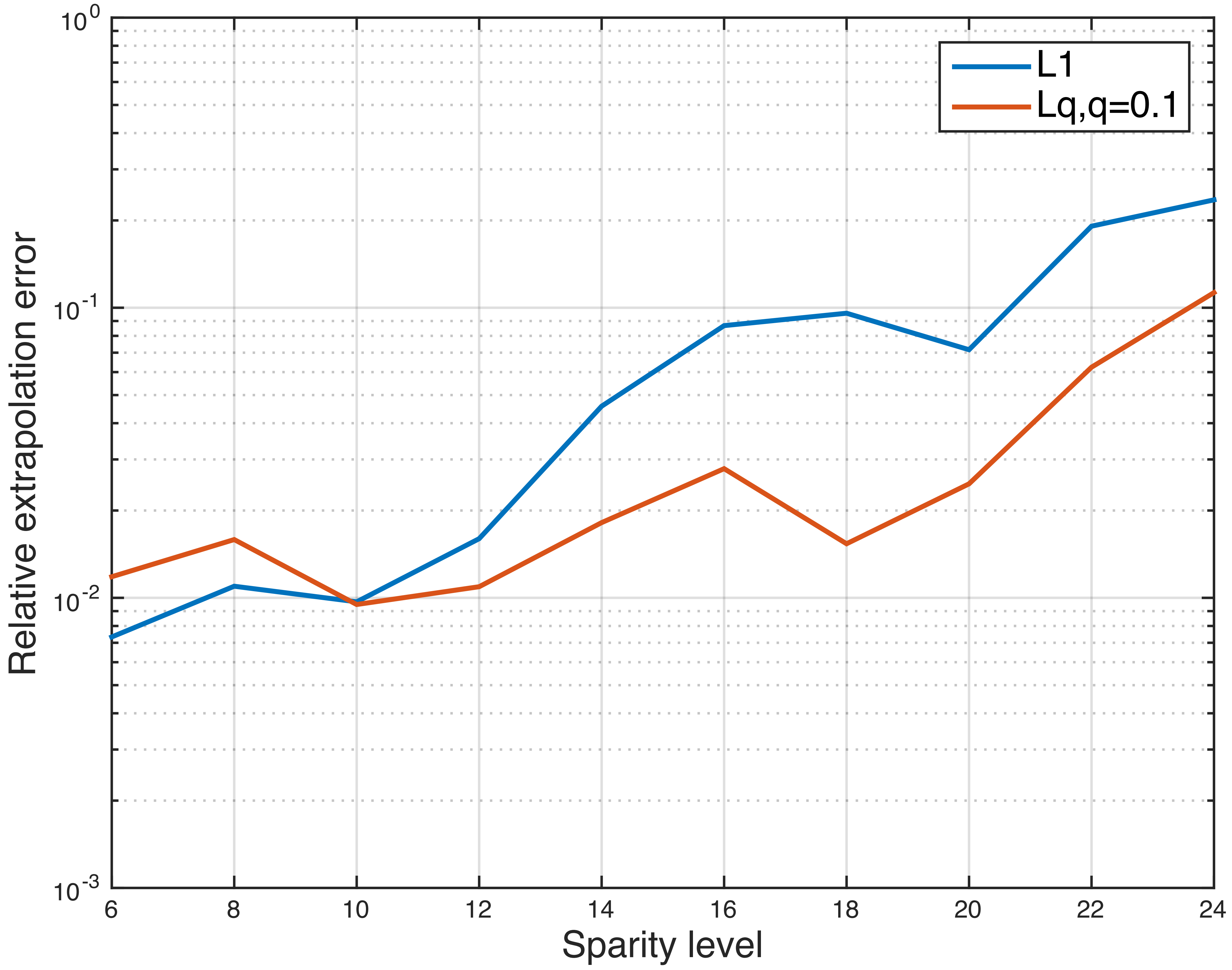
Extrapolation from {10,..,50}Hz towards {1,...,10}Hz

Each point is the mean error on 50 random draws of sparse signals

N=1000, m1=10, m2=50

Numerical test

SNR=20



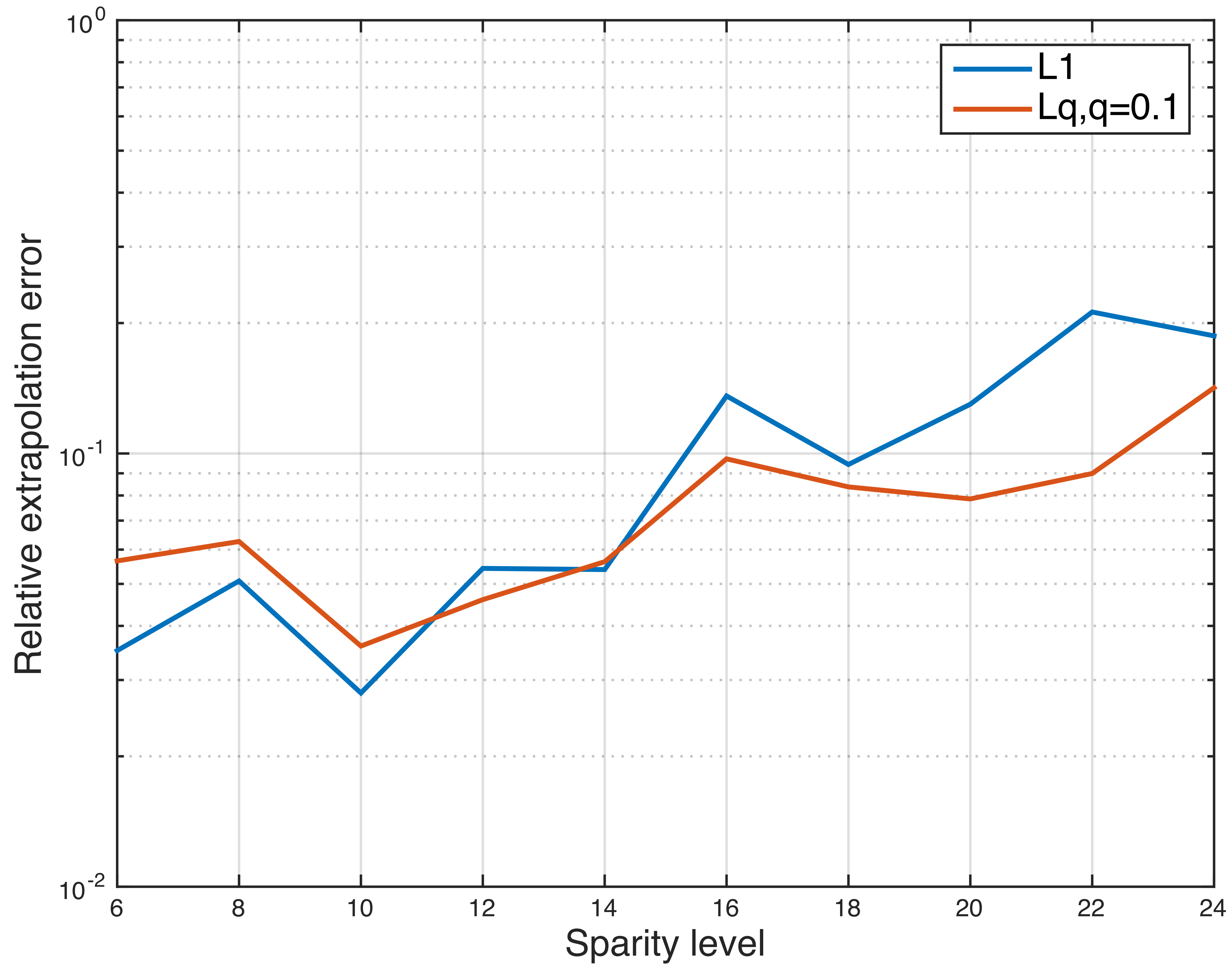
Extrapolation from {10,..,50}Hz towards {1,...,10}Hz

Each point is the mean error on 50 random draws of sparse signals

N=1000, m1=10, m2=50

Numerical test

SNR=13



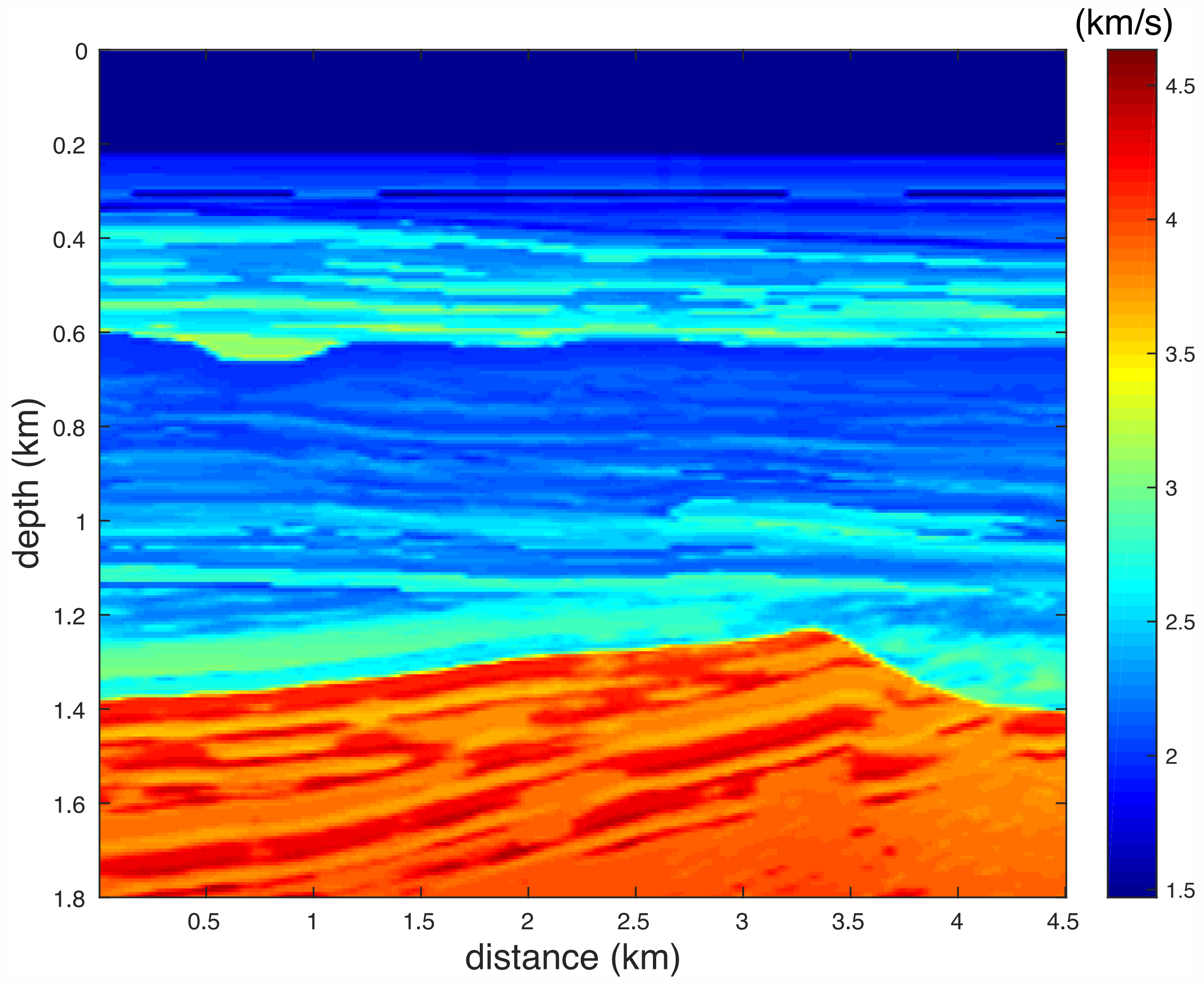
Extrapolation from {10,..,50}Hz towards {1,...,10}Hz

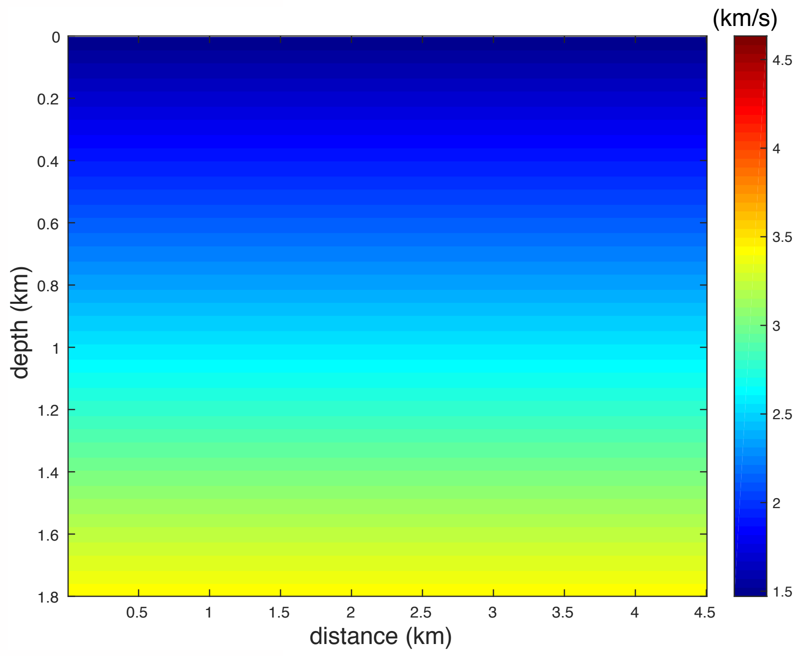
Each point is the mean error on 50 random draws of sparse signals

N=1000, m1=10, m2=50

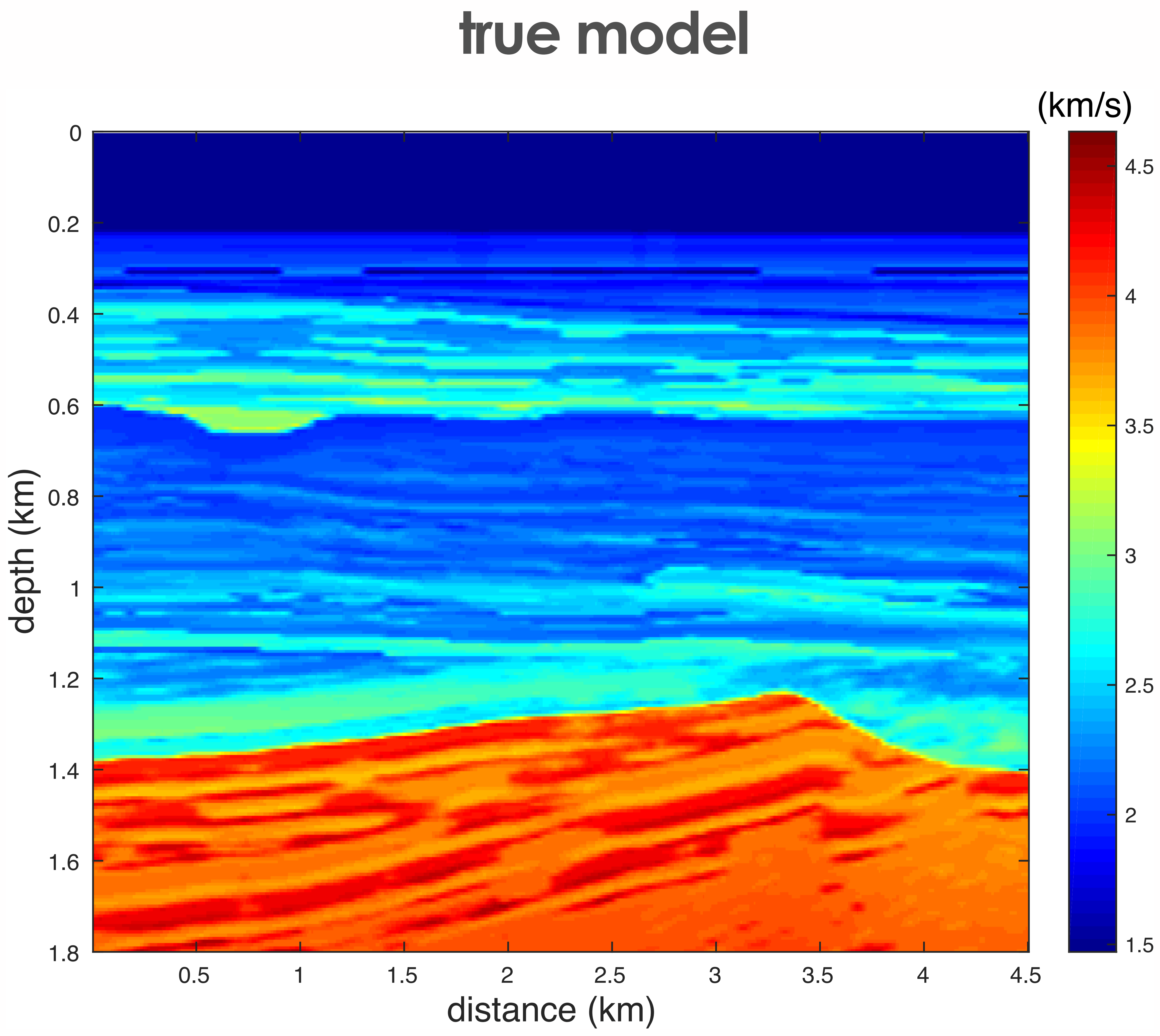
Synthetic data - Non-inversion crime

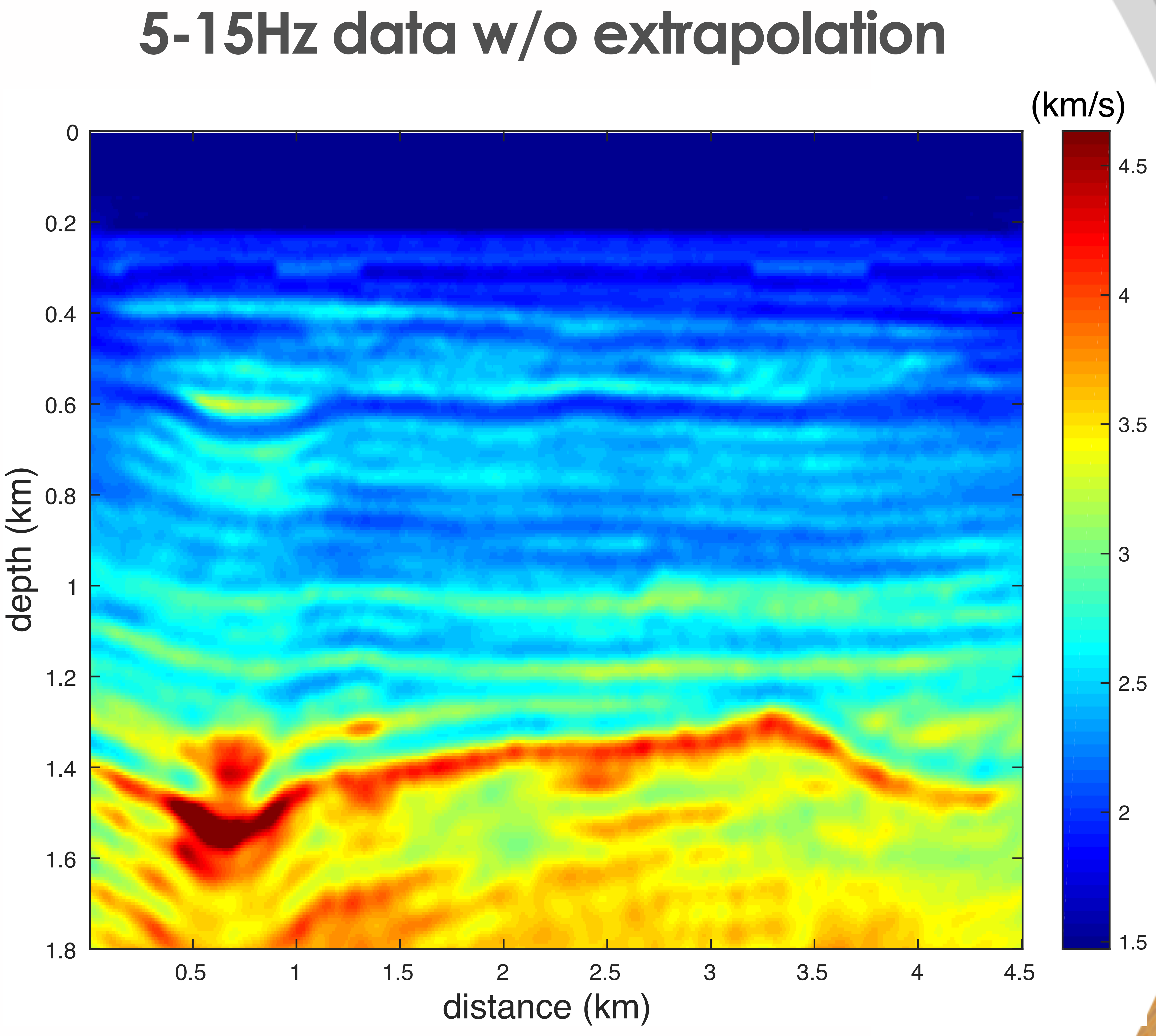
- IWAVE generated data
- inversion using time harmonics
- 3 frequency sweeps, 20 lbfgs-iters for each batch
- 20Hz Ricker wavelet
- source spacing : 0.2km
- receiver spacing : 20m
- maximum offset : 1km
- model size : 1.8km × 4.5km



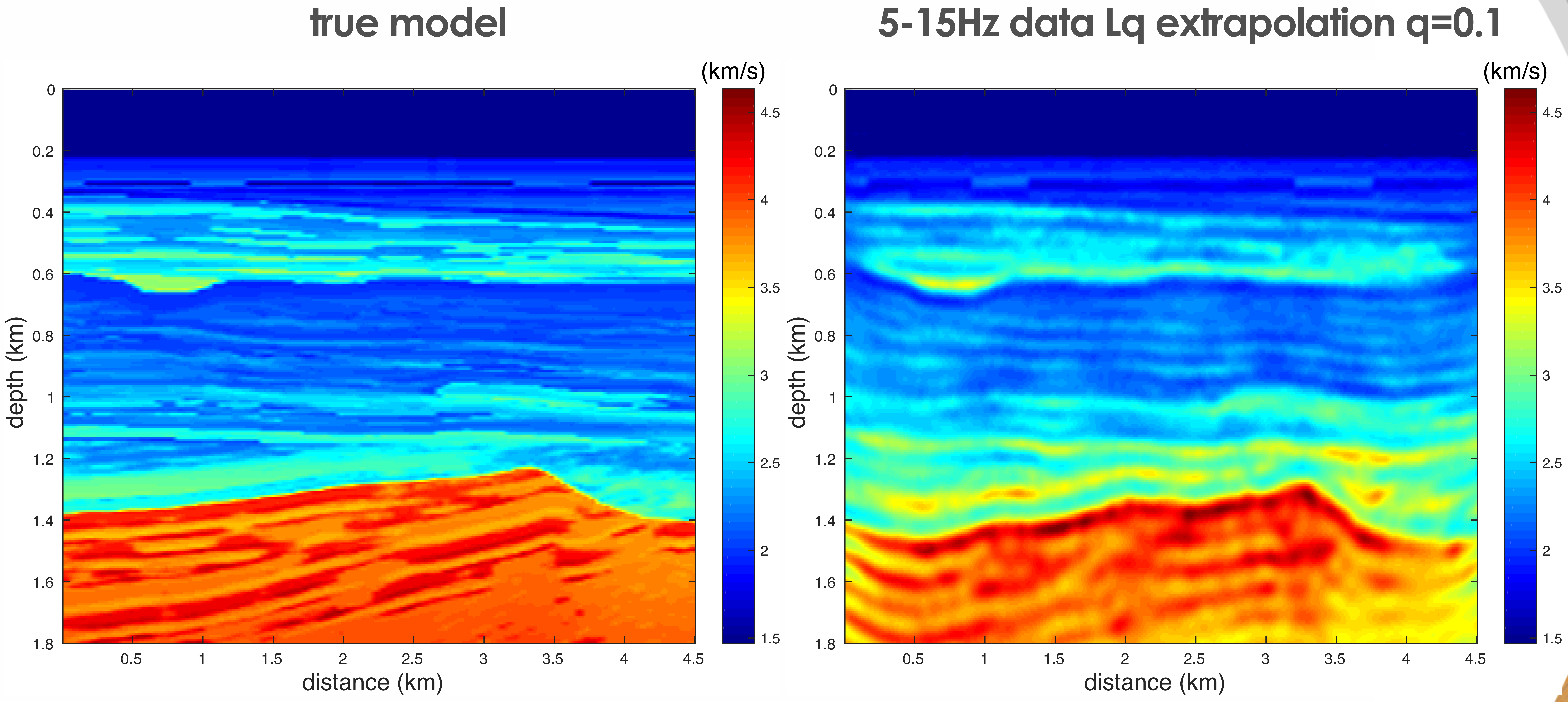


Recovery of FWI

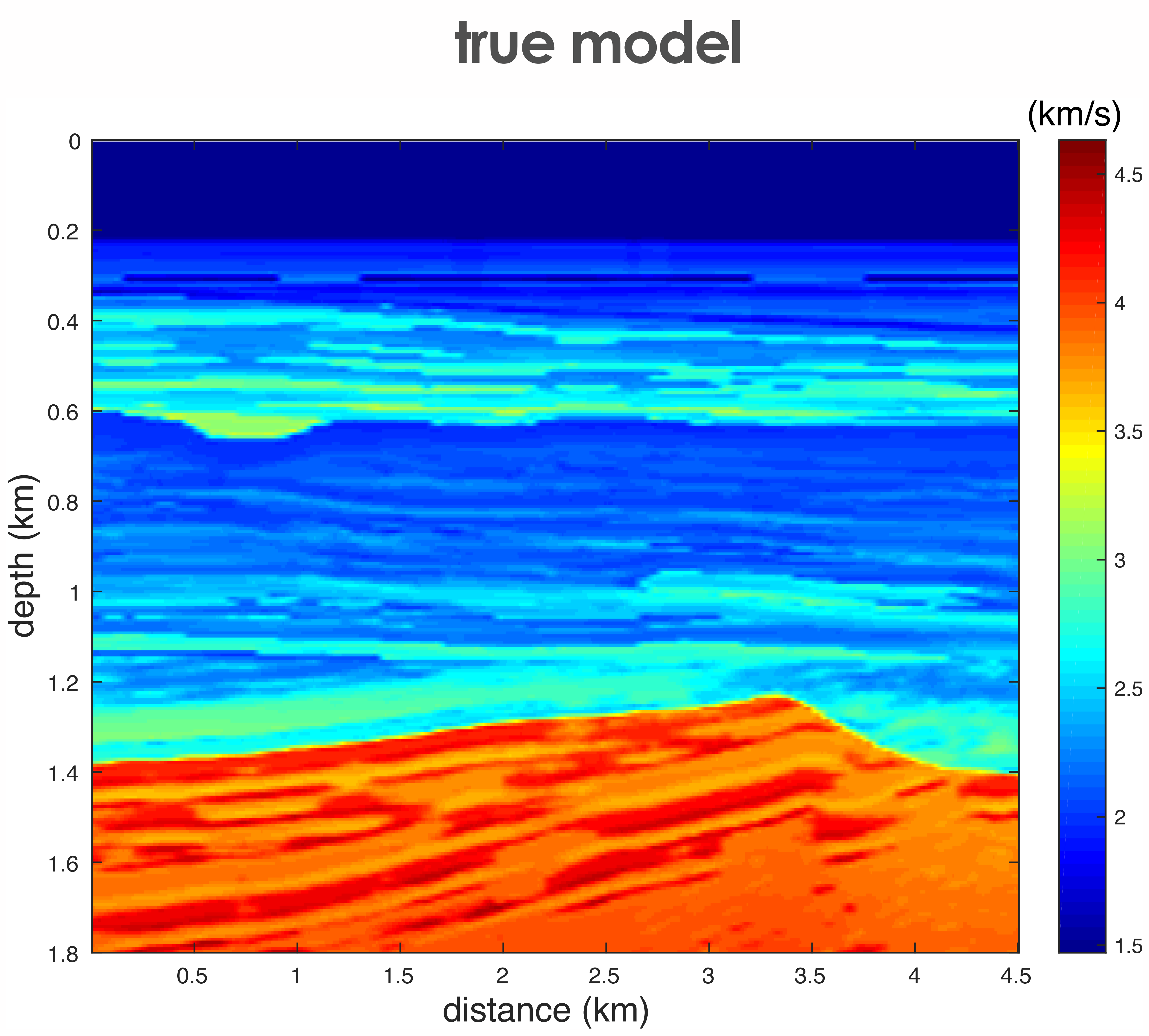


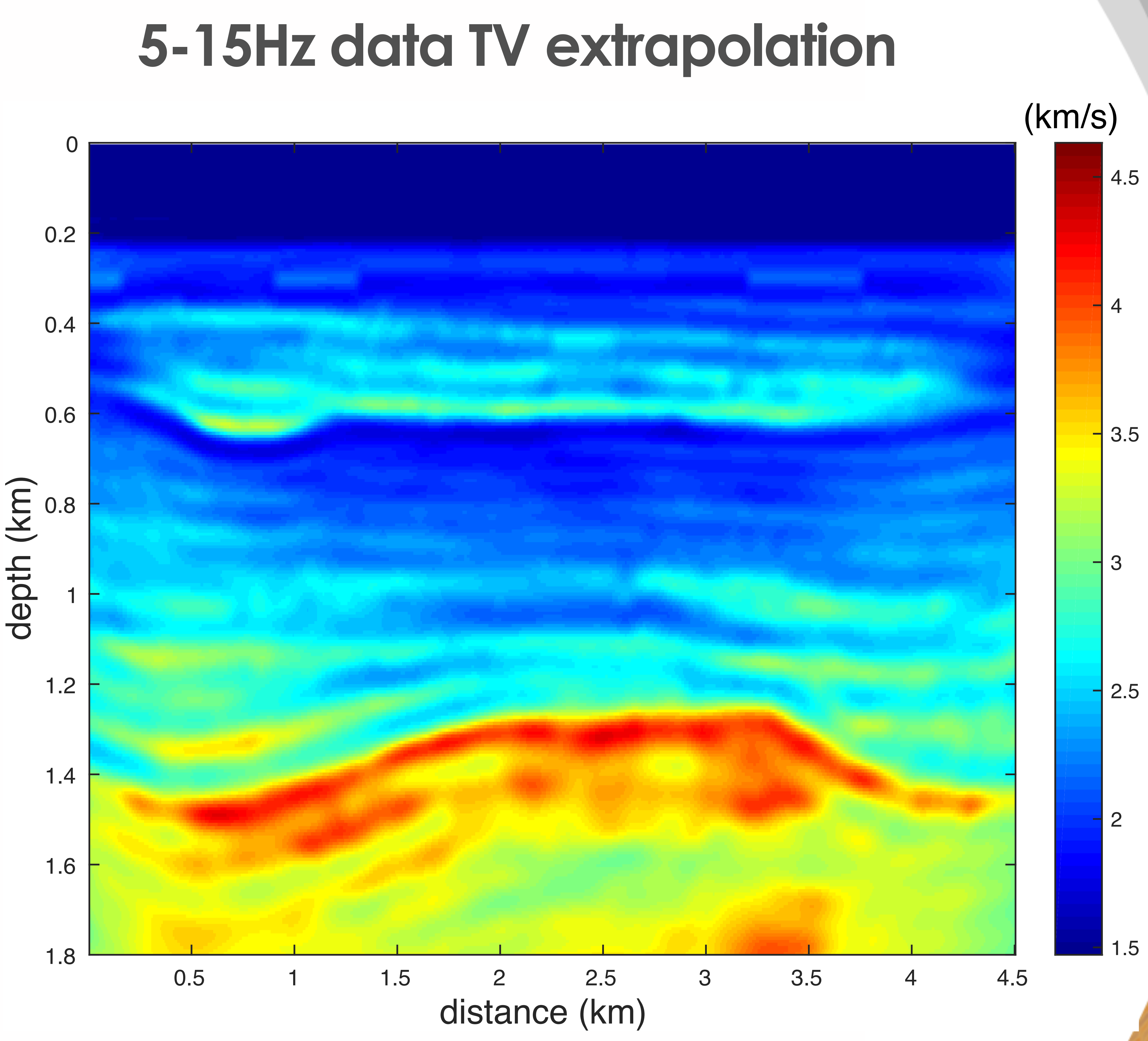


Recovery of FWI

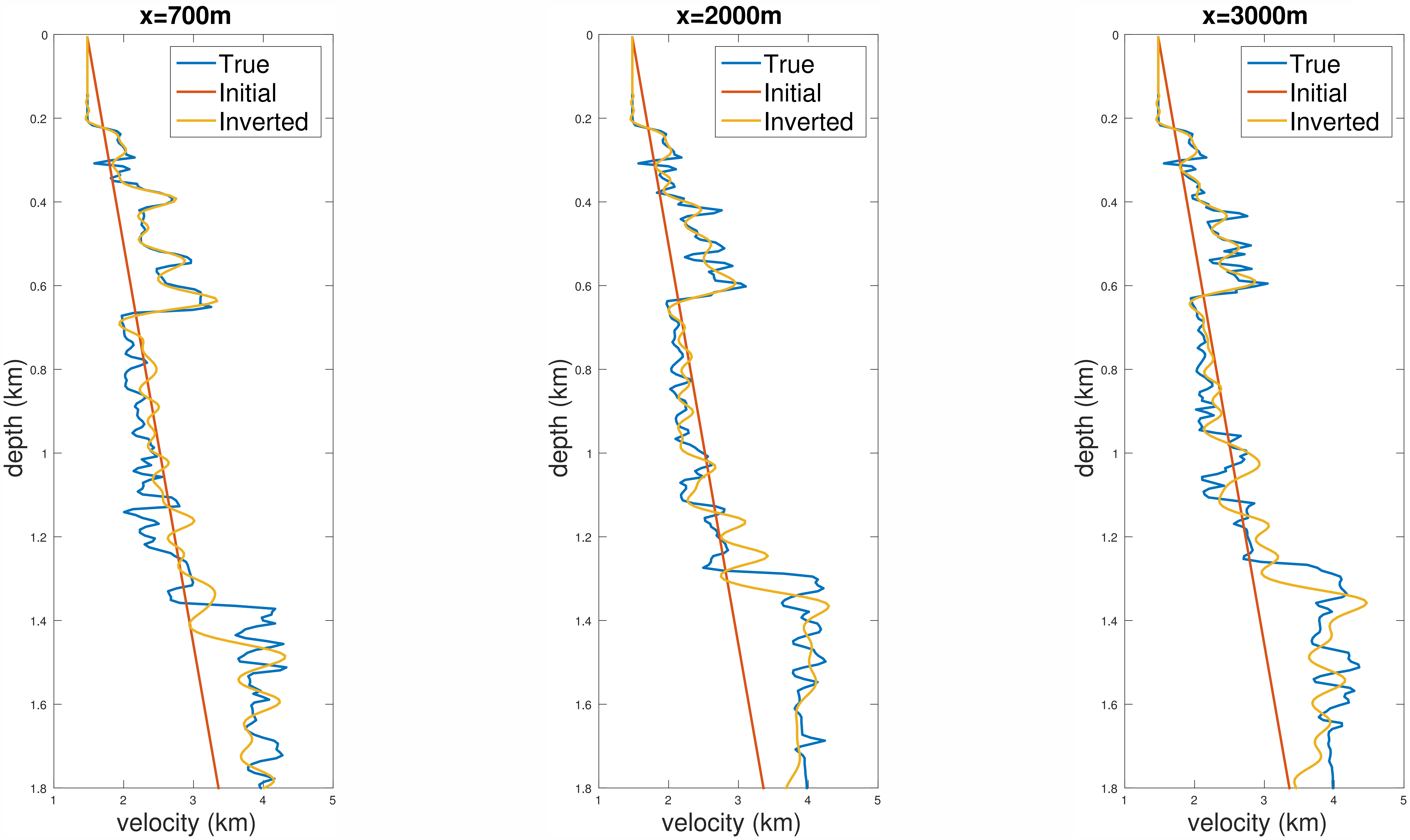


Recovery of FWI





Cross-sections for the inverted model by Lq



Conclusions

- We provide theoretical understanding of the classical L1 minimization.
- We use the L1 based approach for low frequency extrapolation.
- We used TV norm regularization to enhance stability of the classical L1 minimization.
- The efficacy of the method is still limited by the minimal distance of opposite events.
- The method is stable with respect to additive noise and dispersion.

Acknowledgements

This research was carried out as part of the SINBAD project with the support of the member organizations of the SINBAD Consortium.