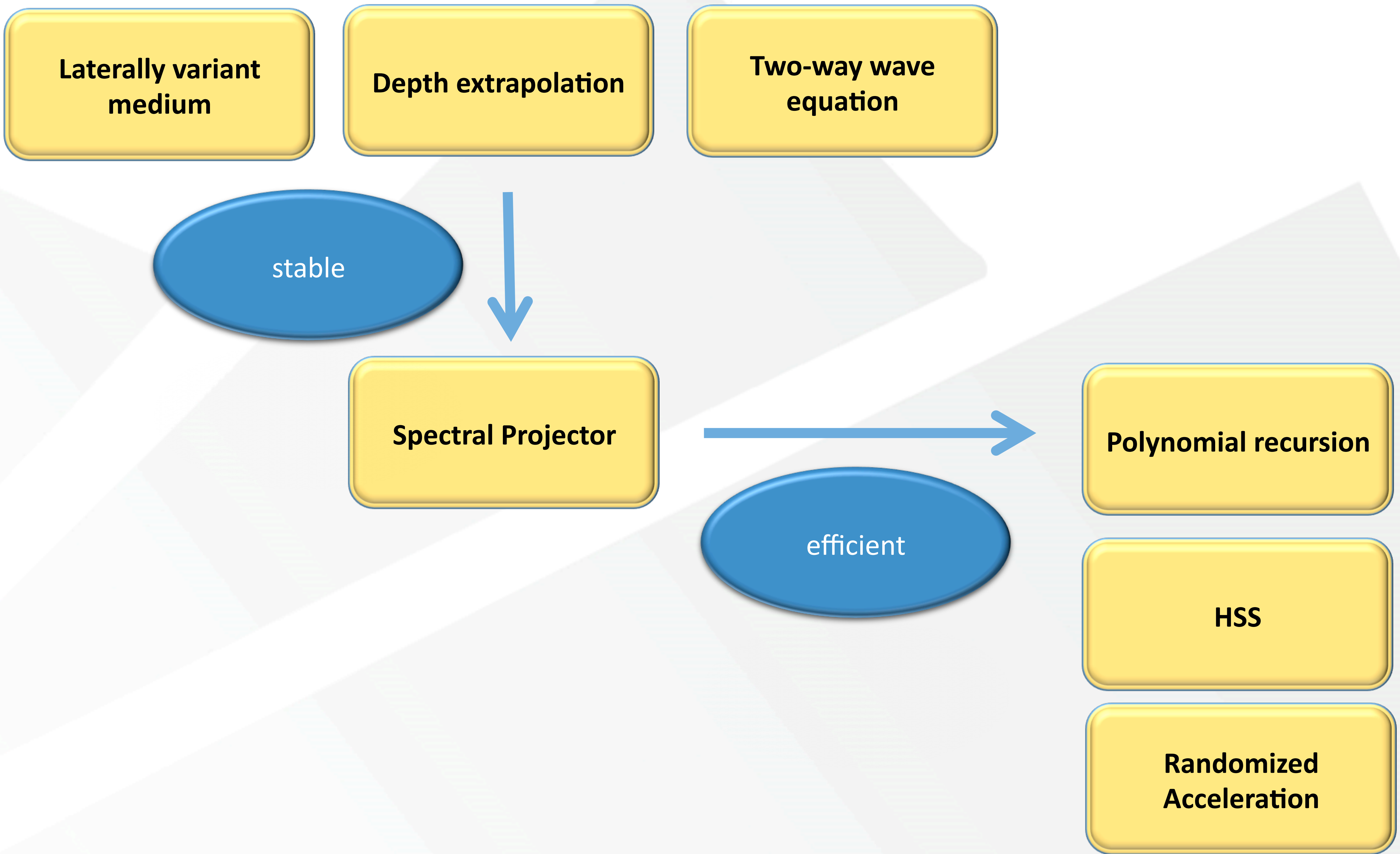
RANDOMIZED HSS ACCELERATION FOR FULL-WAVE-EQUATION DEPTH MIGRATION

Lina Miao, Polina Zheglova, and Felix Herrmann,
University of British Columbia

K. Sandberg and G. Beylkin. Full-wave-equation depth extrapolation for migration. Geophysics, 74(6):WCA121–WCA128, 2009.



K. Sandberg and G. Beylkin. Full-wave-equation depth extrapolation for migration. Geophysics, 74(6):WCA121–WCA128, 2009.

PROBLEM FORMULATION

$$p_{tt} = v(z,x)^2 (p_{xx} + p_{zz})$$

$$p(x,0,t) = 0, \ \ t \le 0$$

$$\begin{cases} p(x,z,0) = f(x) \\ p_t(x,z,0) = g(x) \end{cases}$$

$$\xrightarrow{\text{Fourier Transform}}$$

$$\hat{p}_{zz} = \left[-\left(\frac{\omega}{v(x,z)} \right)^2 - D_{xx} \right] \hat{p} \equiv L\hat{p}$$

$$\begin{cases} \hat{p}(x,z,\omega) = q(x,z,\omega) \\ \hat{p}_z(x,z,\omega) = q_z(x,z,\omega) \end{cases}$$

Two way wave-equation
as an initial value problem

Unstable with evanescent
wave components

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Spectral Projector $\mathcal{P}$

$$\hat{p}_{zz} = \mathcal{P}L\mathcal{P}\hat{p}$$

$$\hat{p}(x, z_n, \omega) = q(x, z_n, \omega)$$

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Two way wave-equation as an initial value problem

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Stabilized initial value problem with spectral projector

L. Auslander and A. Tsao. On parallelizable eigensolvers. Citeseer, 1991.
N. J. Higham. Functions of matrices: theory and computation. Siam, 2008.

SPECTRAL PROJECTOR

Geophysically → Unstable extrapolation operator with propagating and **evanescent** waves

$$L = V\Lambda V^*$$

Mathematically → Indefinite matrix with negative and **positive** eigenvalues

Get rid of all **evanescent** waves

$$k_z^2 = \left(\frac{\omega}{v(x,z)}\right)^2 - k^2 \geq 0$$

→

$$\lambda = -\left(\frac{\omega}{v(x,z)}\right)^2 + k^2 \leq 0$$

Set all **positive** eigenvalue to be 0

Stable extrapolation operator with only propagating waves

$$\mathcal{P}L\mathcal{P} = V\tilde{\Lambda}V^*$$

Definite matrix with only negative eigenvalues

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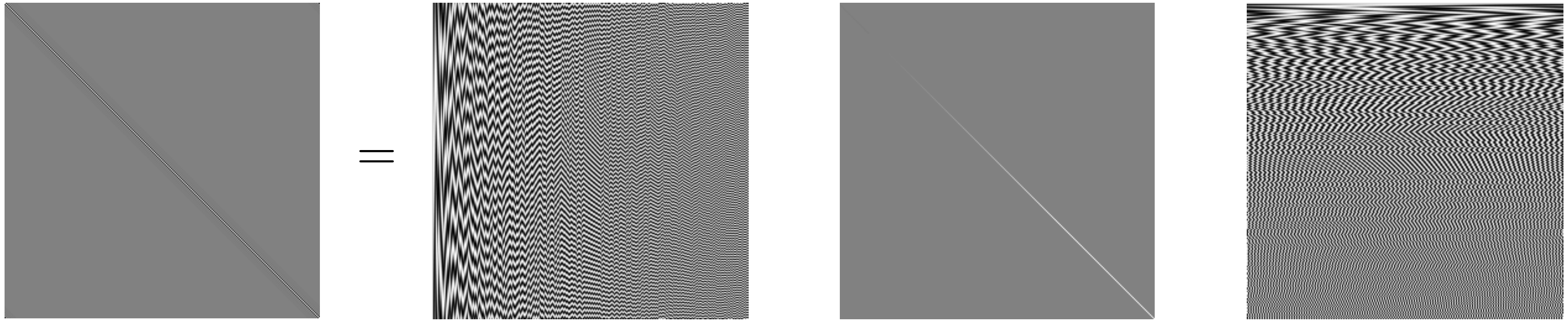
Definite matrix with only negative eigenvalues

$$\mathcal{P} = (I - \text{sign}(L))/2$$

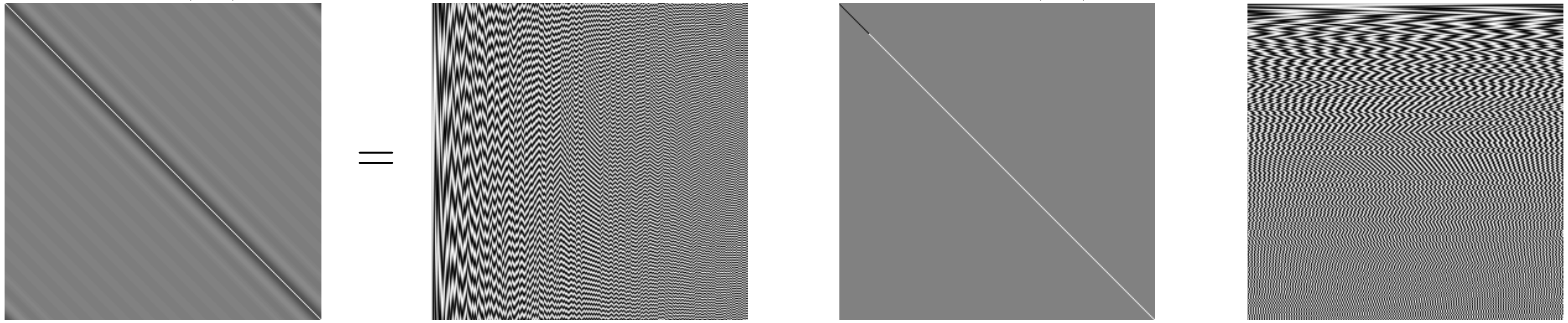
$$\text{sign}(L) = V \text{sign}(D) V^*$$

MATRIX SIGN FUNCTION

$$L = V * \Lambda * V^*$$



$$\mathrm{sign}(L) = V * \mathrm{sign}(\Lambda) * V^*$$



C. S. Kenney and A. J. Laub. The matrix sign function. Automatic Control, IEEE Transactions on, 40(8):1330–1348, 1995.

NEWTON-SCHULZ ITERATION FOR MATRIX SIGN FUNCTION

Algorithm Newton-Schulz Iteration for the Matrix Sign Function (Auslander and Tsao (1991))

Input: Self adjoint matrix L
Output: $S := \text{sign}(L)$

1. Initialize $S_0 = L/||L||_2$, where $||L||_2$ stands for the 2 norm of matrix L
2. For $k = 1......N$, $S_{k+1} = \frac{3}{2}S_k - \frac{1}{2}S_k^3$

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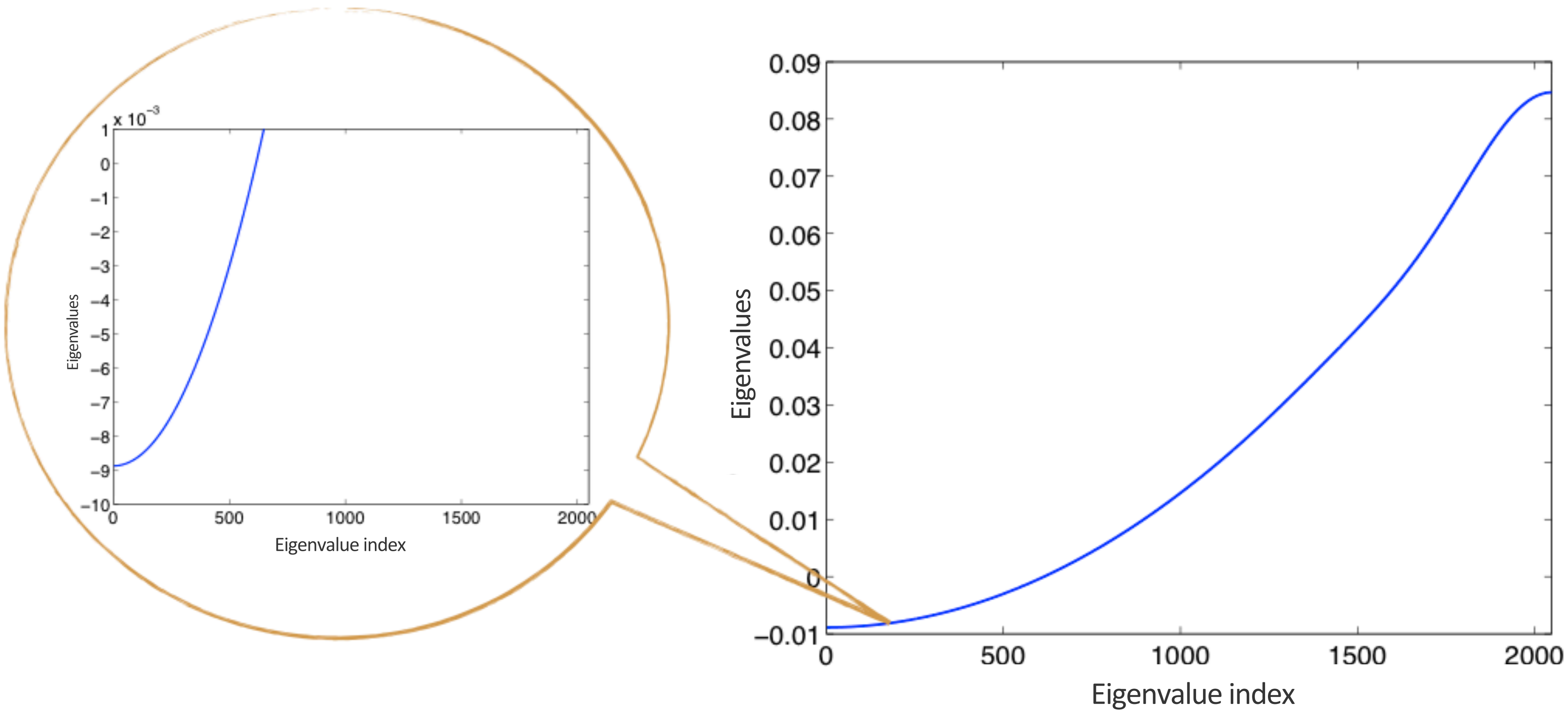
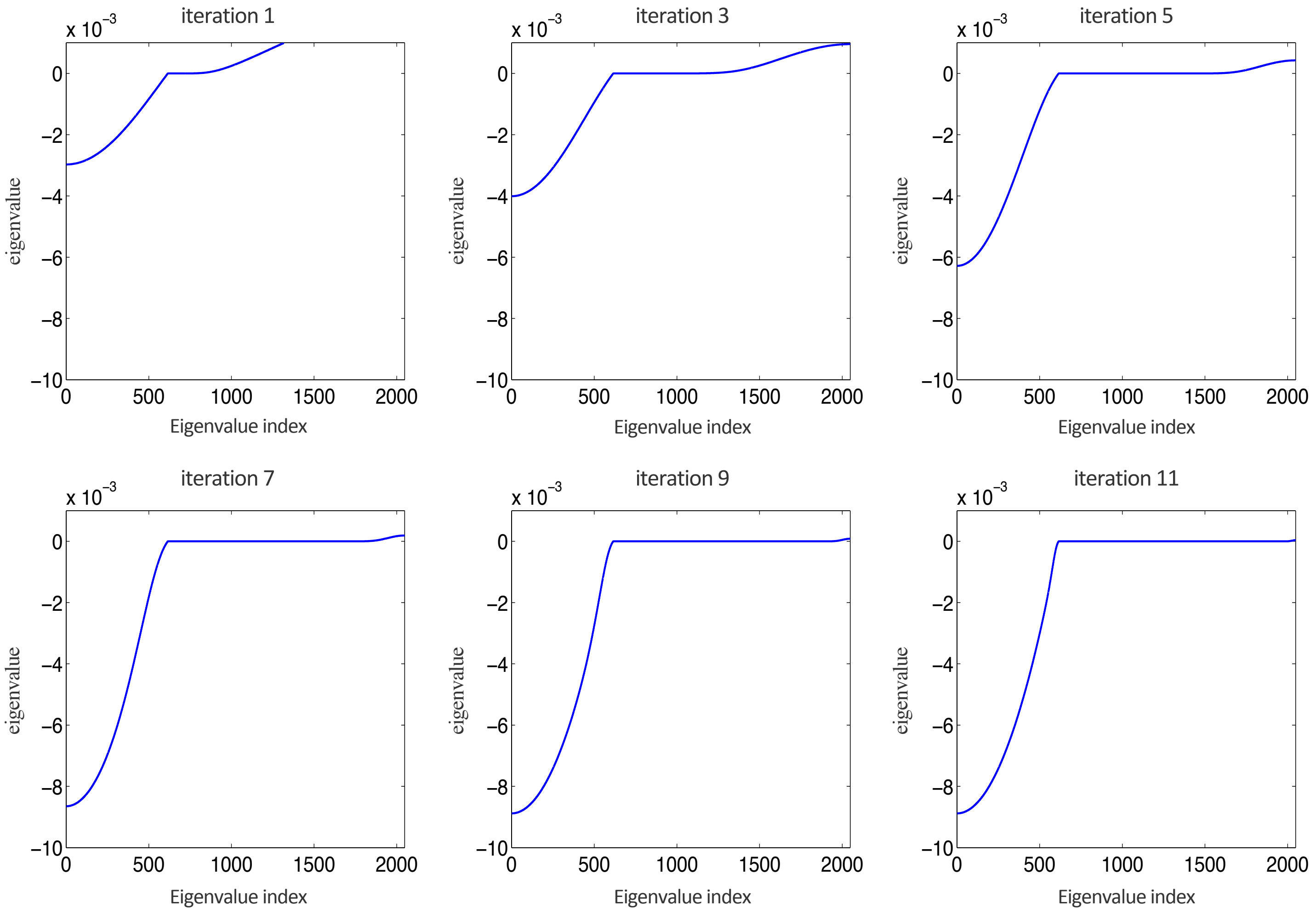
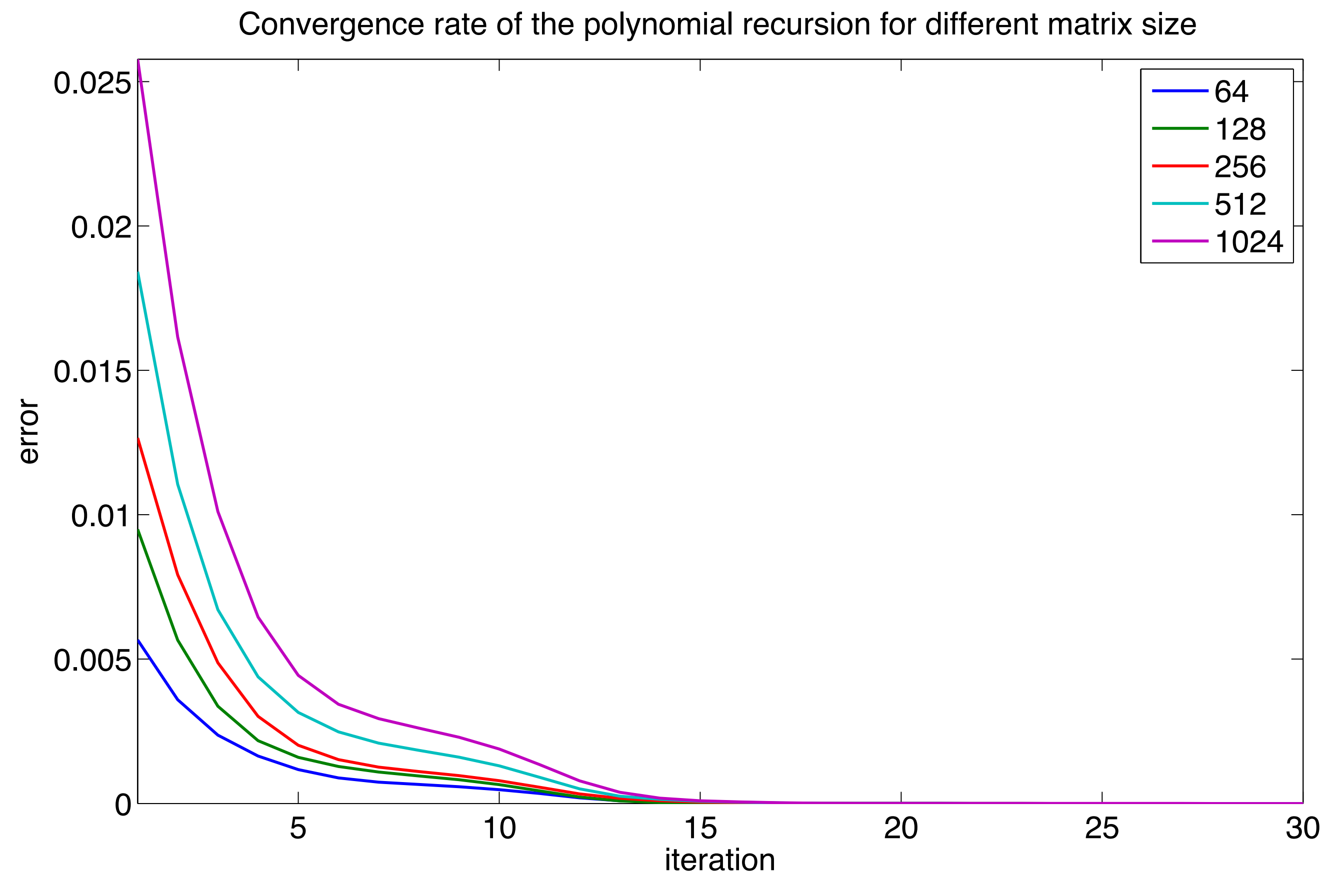
SPECTRAL PROJECTOR (MATRIX SIGN FUNCTION) VIA POLYNOMIAL RECURSION



SPECTRAL PROJECTOR (MATRIX SIGN FUNCTION) VIA POLYNOMIAL RECURSION



SIGN FUNCTION COMPUTATION VIA POLYNOMIAL RECURSION



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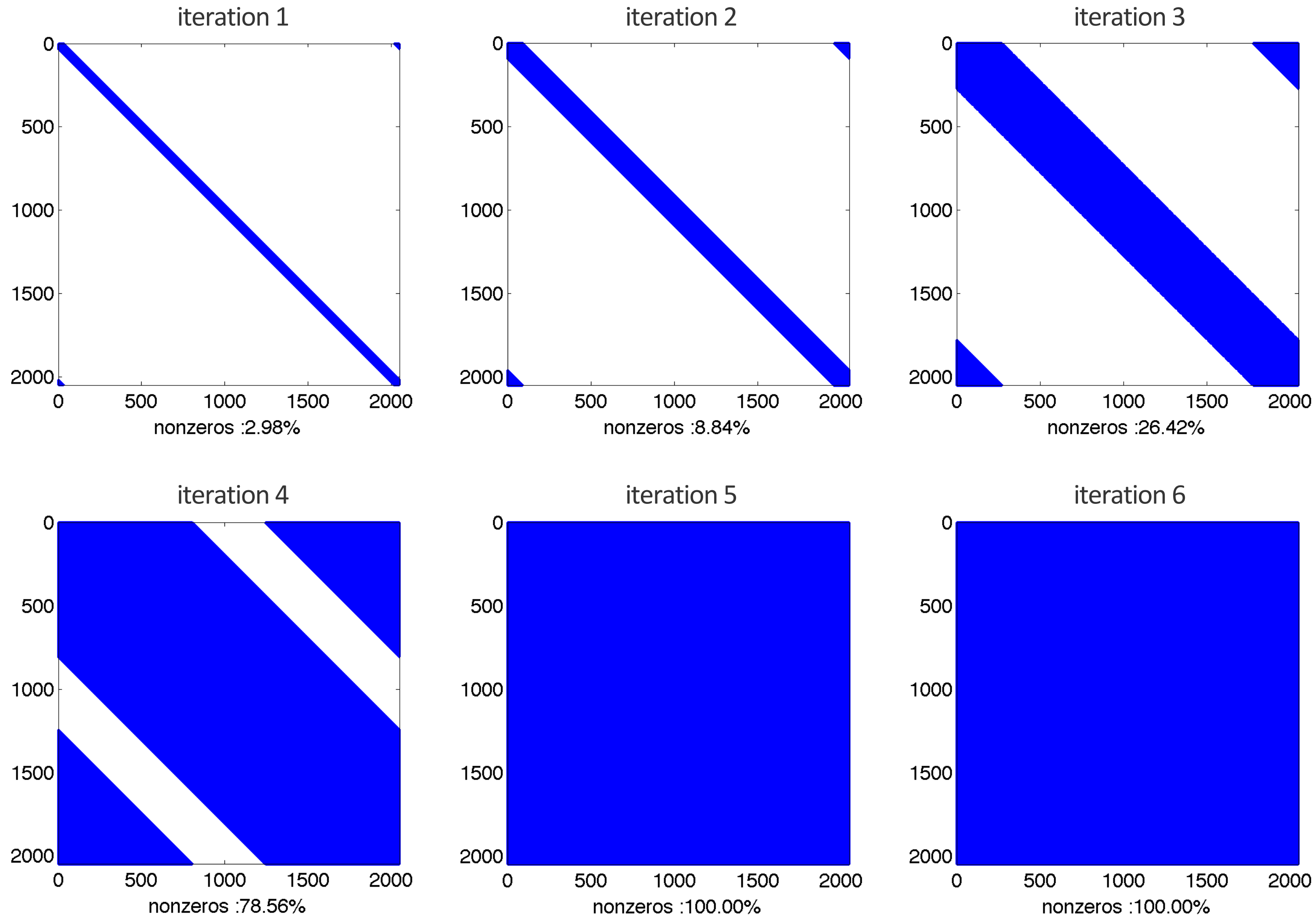
MATRIX SPARSITY IN THE POLYNOMIAL RECURSION



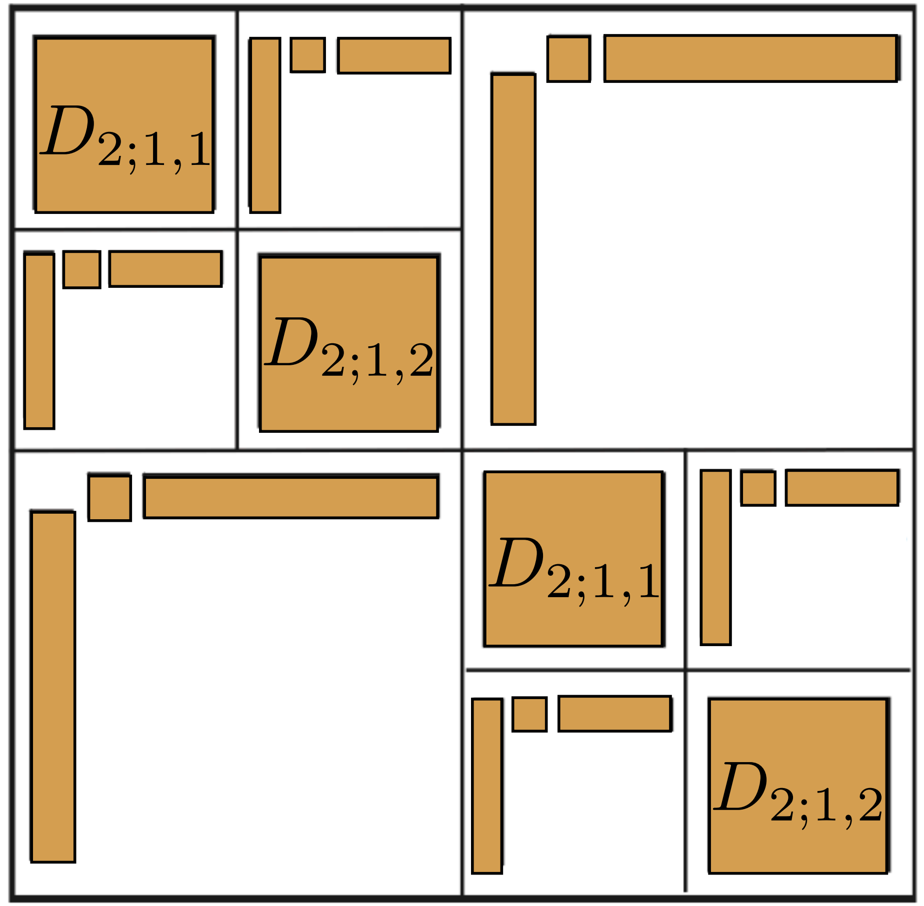
J. Xia. On the complexity of some hierarchical structured matrix algorithms. SIAM Journal on Matrix Analysis and Applications

HIERARCHICALLY SEMI-SEPARABLE MATRIX REPRESENTATION

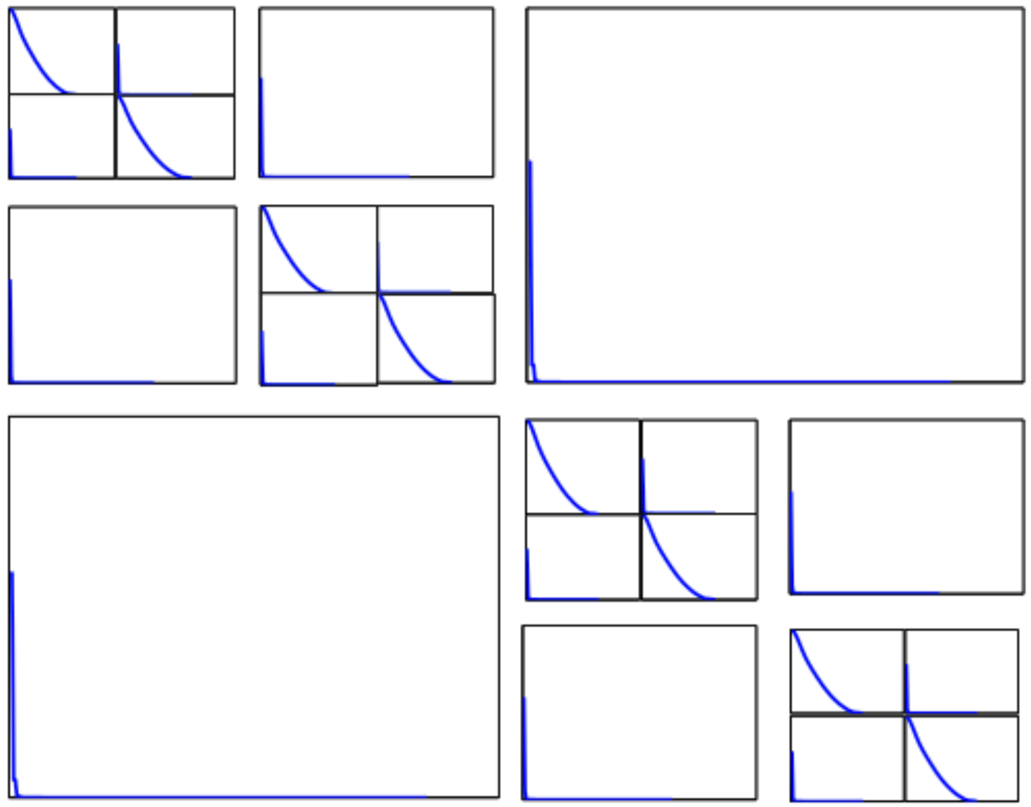
$$A = \begin{pmatrix} A_{1;1,1} & A_{1;1,2} \\ A_{1;2,1} & A_{1;2,2} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} D_{1;1,1} & (USV^*)_{1;1,2} \\ (USV^*)_{1;2,1} & D_{1;2,2} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \begin{pmatrix} D_{2;1,1} & (USV^*)_{2;1,2} \\ (USV^*)_{2;2,1} & D_{2;2,2} \end{pmatrix} & (USV^*)_{1;1,2} \\ (USV^*)_{1;2,1} & \begin{pmatrix} D_{2;1,1} & (USV^*)_{2;1,2} \\ (USV^*)_{2;2,1} & D_{2;2,2} \end{pmatrix} \end{pmatrix}$$

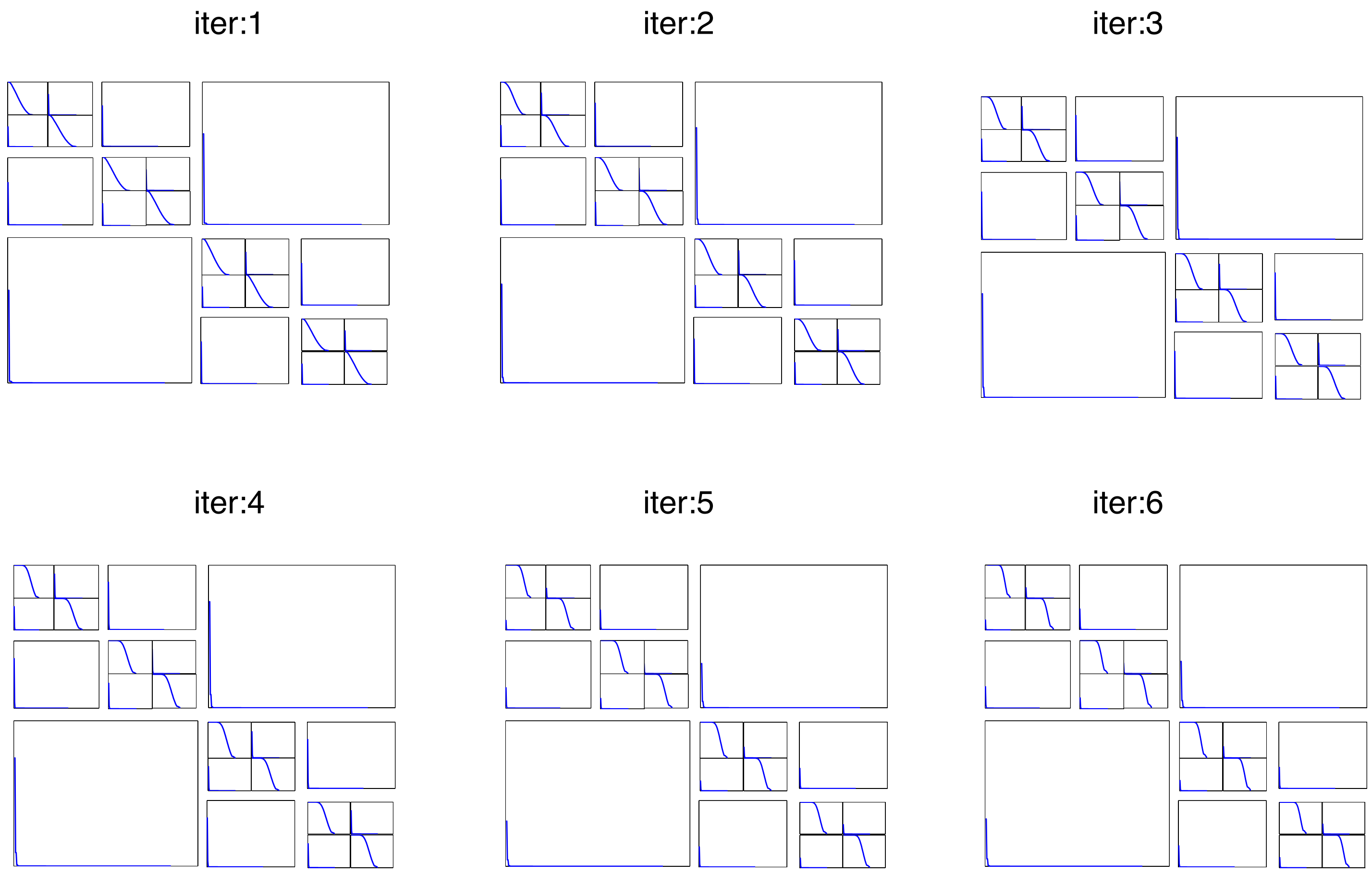


MATRIX SPECTRUM WITH HSS PARTITION



J. Xia, Y. Xi, and M. Gu. A superfast structured solver for toeplitz linear systems via randomized sampling
W. Lyons. Fast algorithms with applications to pdes. PhD thesis, UNIVER- SITY of CALIFORNIA, 2005.

MATRIX SPECTRUM WITH HSS PARTITION IN NEWTON-SCHULZ ITERATION



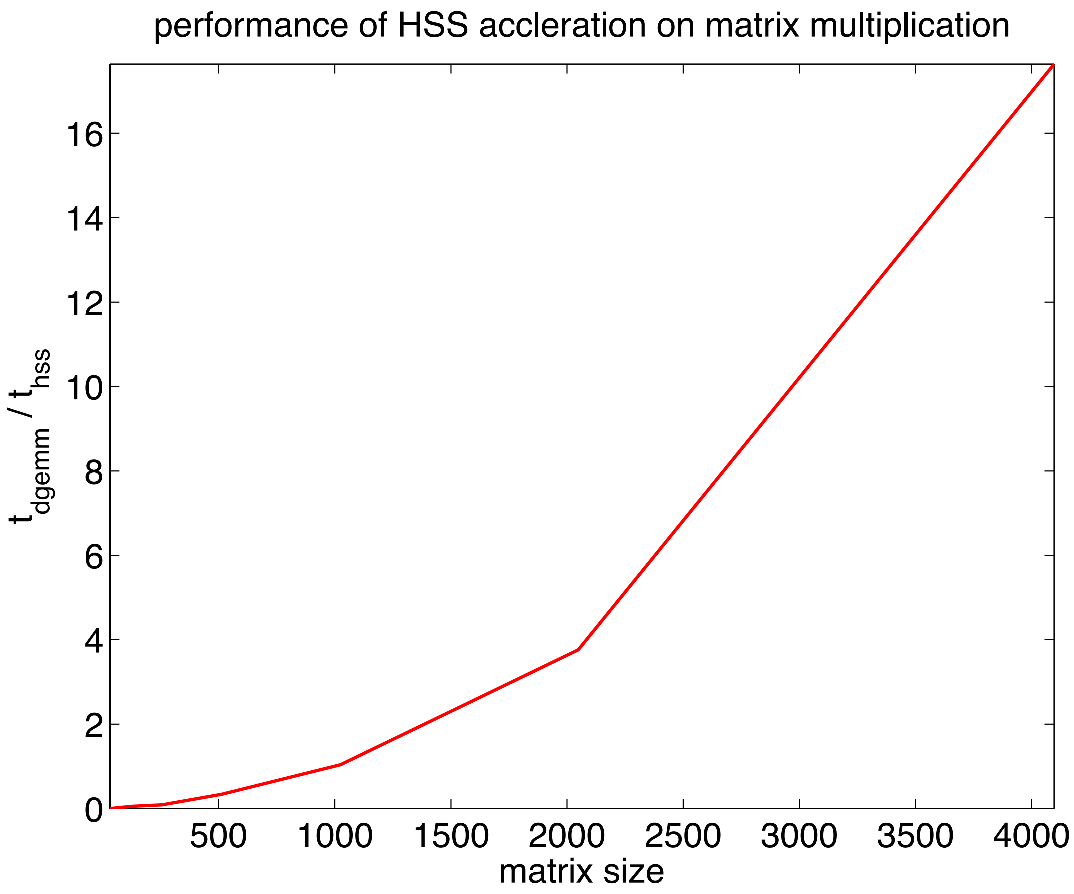
Z. Sheng, P. Dewilde, and S. Chandrasekaran. Algorithms to solve hierar- chically semi-separable systems. pages 255–294, 2007.

COMPLEXITY ANALYSIS OF HSS OPERATIONS

Operation	Complexity with HSS	Complexity without HSS
Matrix-Vector Multiplication	$\mathcal{O}(nk^2)$	$\mathcal{O}(n^2)$
Matrix-Matrix Multiplication	$\mathcal{O}(nk^3)$	$\mathcal{O}(n^3)$
Matrix addition	$\mathcal{O}(nk^2)$	$\mathcal{O}(n^2)$
Compression	$\mathcal{O}(nk^3)$	Not Applicable
LU Decomposition	$\mathcal{O}(nk^3)$	$\mathcal{O}(n^3)$
Inverse	$\mathcal{O}(nk^3)$	$\mathcal{O}(n^3)$
Transpose	$\mathcal{O}(nk)$	$\mathcal{O}(n^2)$
Construct HSS (Xia (2012))	$\mathcal{O}(n^2r)$	Not Applicable

W. Lyons. Fast algorithms with applications to pdes. PhD thesis, UNIVER- SITY of CALIFORNIA, 2005.

COMPUTATION OF MATRIX-MATRIX MULTIPLICATION WITH HSS



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$$\downarrow$$

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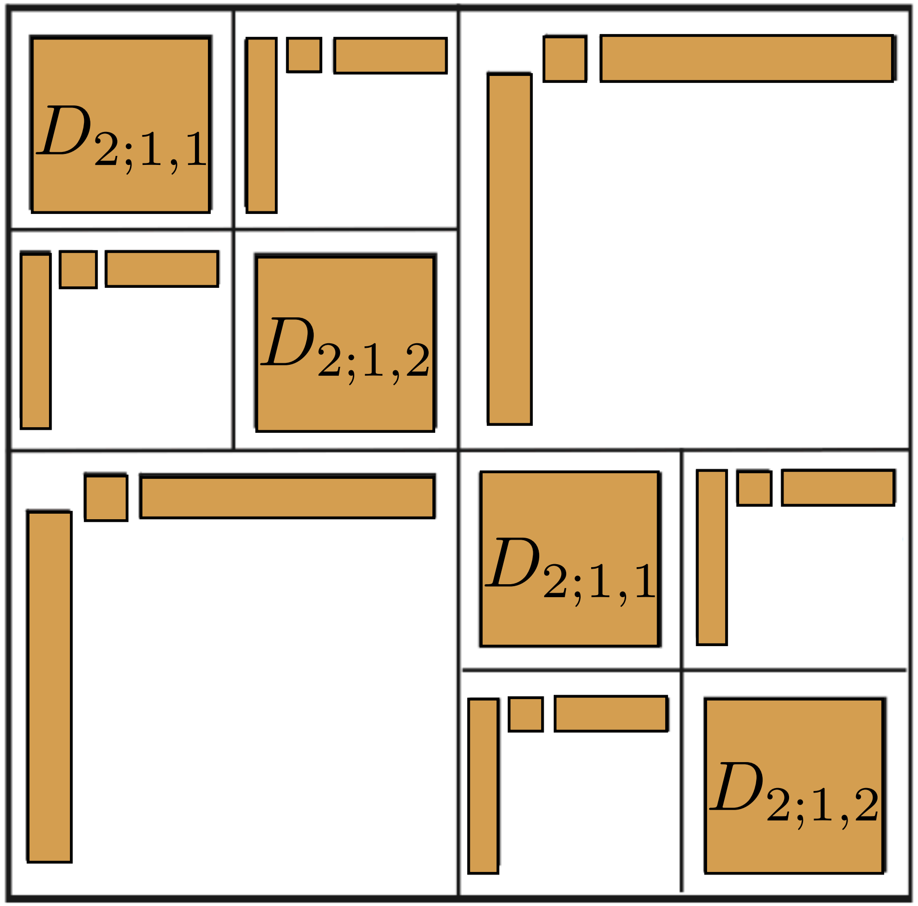


M. Tygert and Rokhlin. A randomized algorithm for approximating the svd of a matrix. 2007

RANDOMIZED HSS ALGORITHM

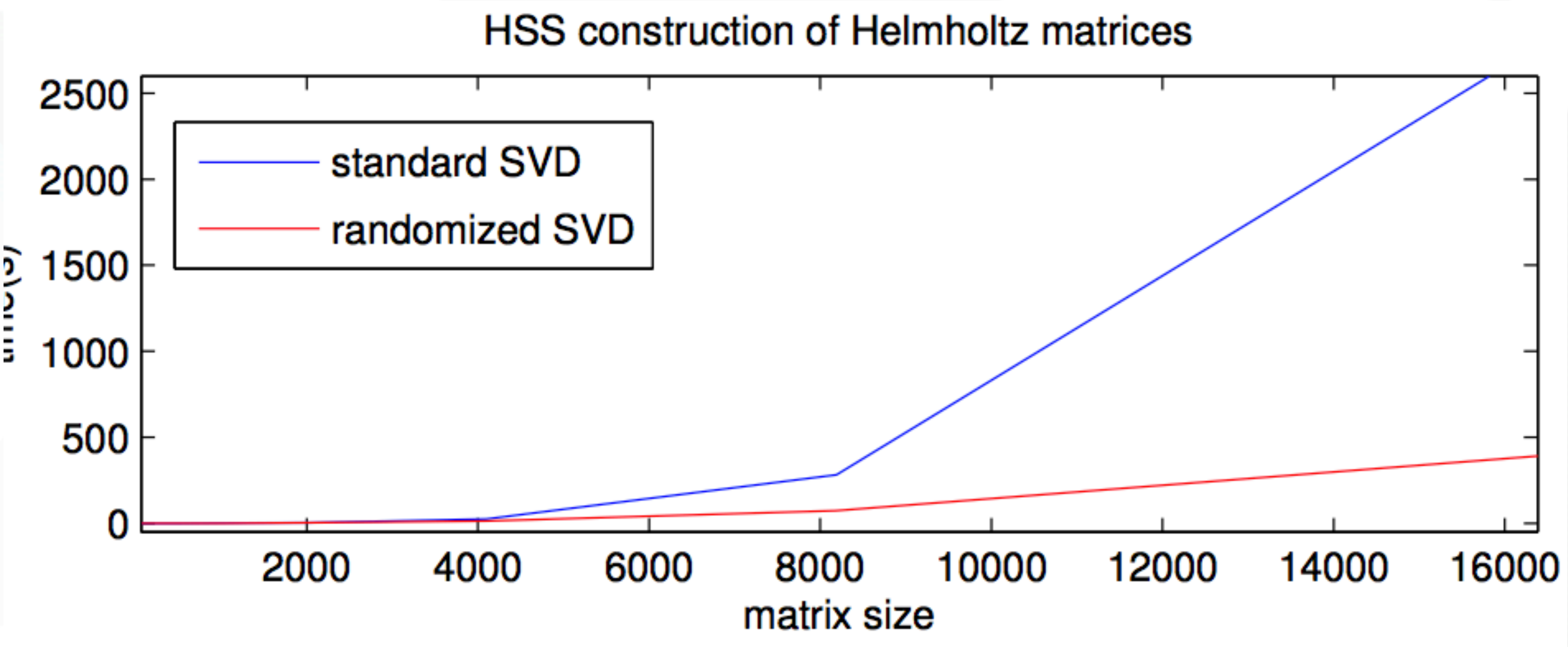
Algorithm The basic rSVD algorithm (Tygert and Rokhlin (2007))

Input: An m x n matrix A with rank k, $k \leq n \leq m$
output: The randomized SVD of A, U, Σ, V

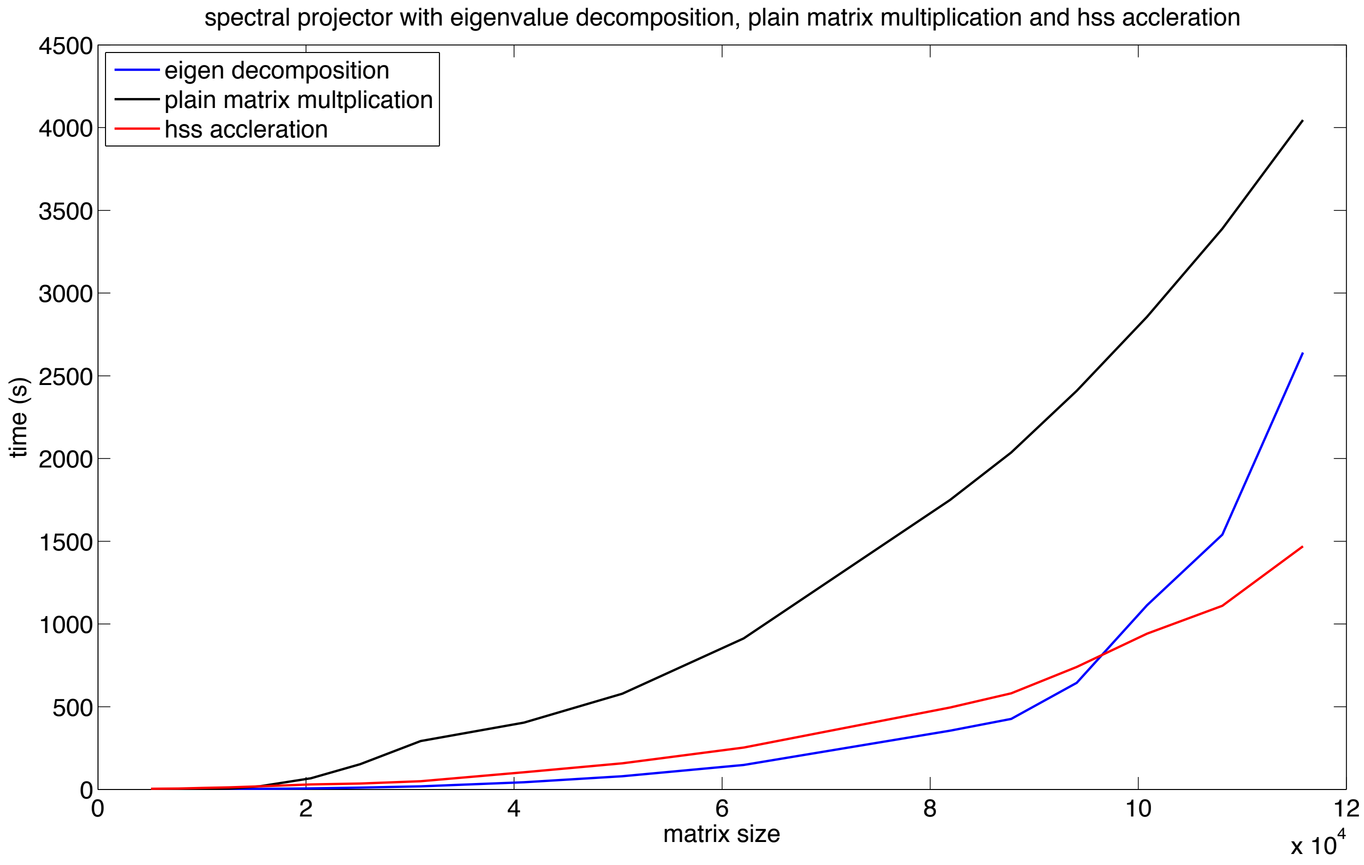
1. Form $P_{m\times(k+20)} = A_{m\times n}R_{n\times(k+20)}$; R is random
2. Form $Q_{m\times k}$, the matrix whose columns consist of the left singular vectors corresponding to the k greatest singular values of $P_{m\times(k+20)}$
3. Form $S_{n\times k} = A^T_{n\times m}Q_{m\times k}$
4. Compute the SVD : $T_{k\times k}\Sigma_{k\times k}V^T_{k\times n}$ of $S^T_{k\times n}$
5. Form $U_{m\times k} = Q_{m\times k}T_{k\times k}$

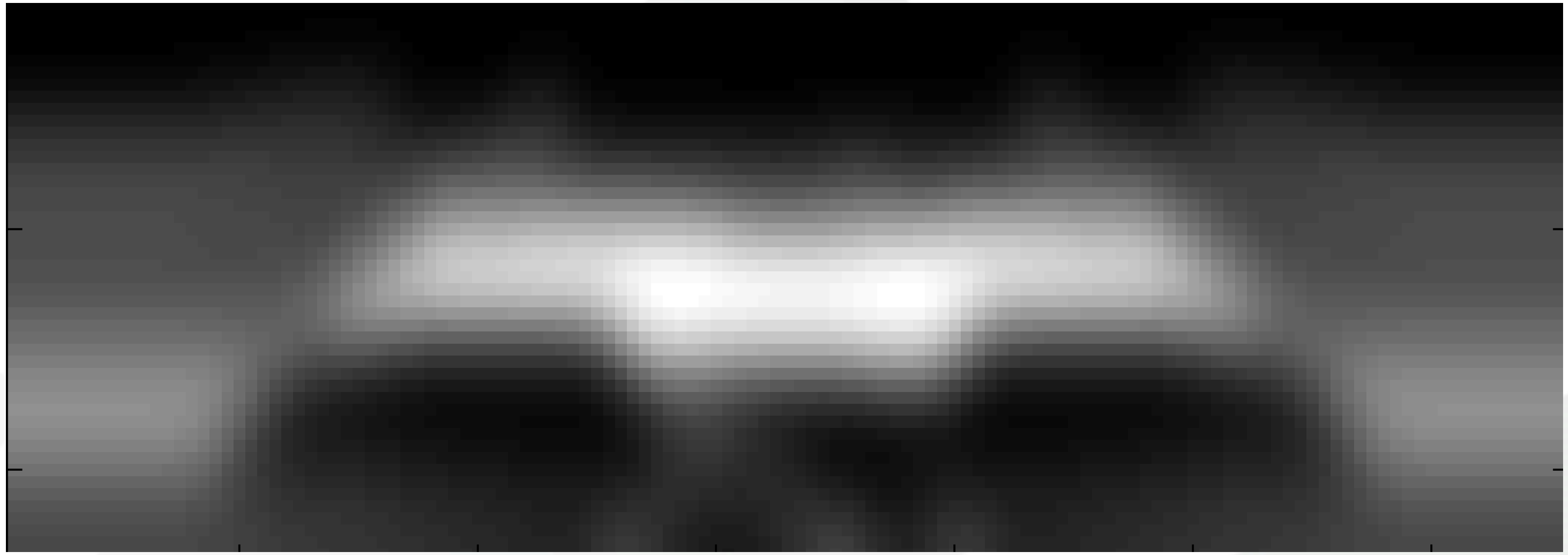
Halko, Nathan, Per-Gunnar Martinsson, and Joel A. Tropp. "Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions.”

RANDOMIZED HSS

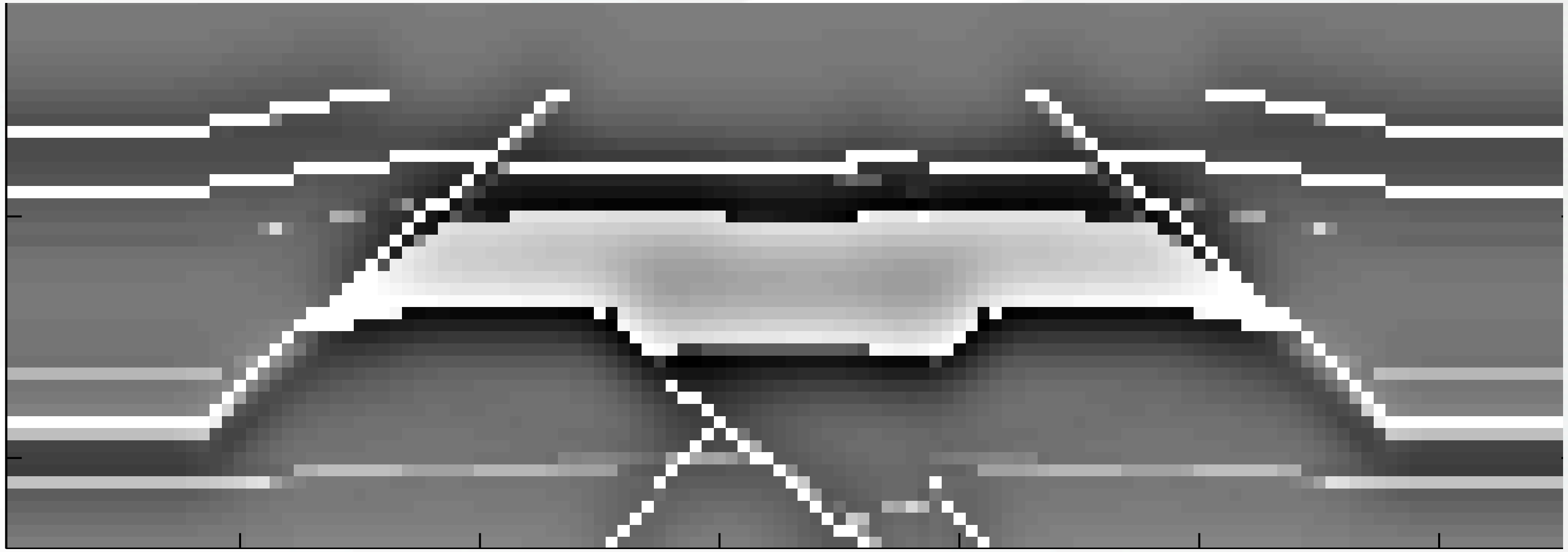


COMPUTATION OF SPECTRAL PROJECTOR WITH HSS

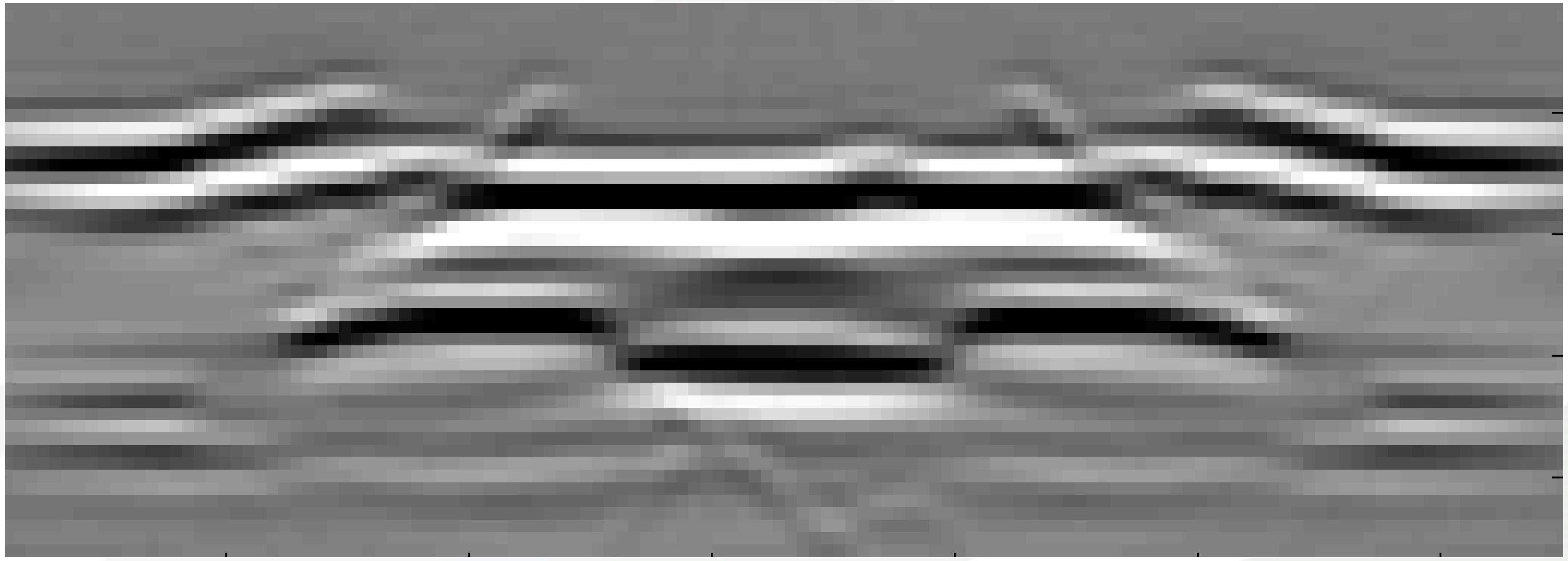




m_0



dm



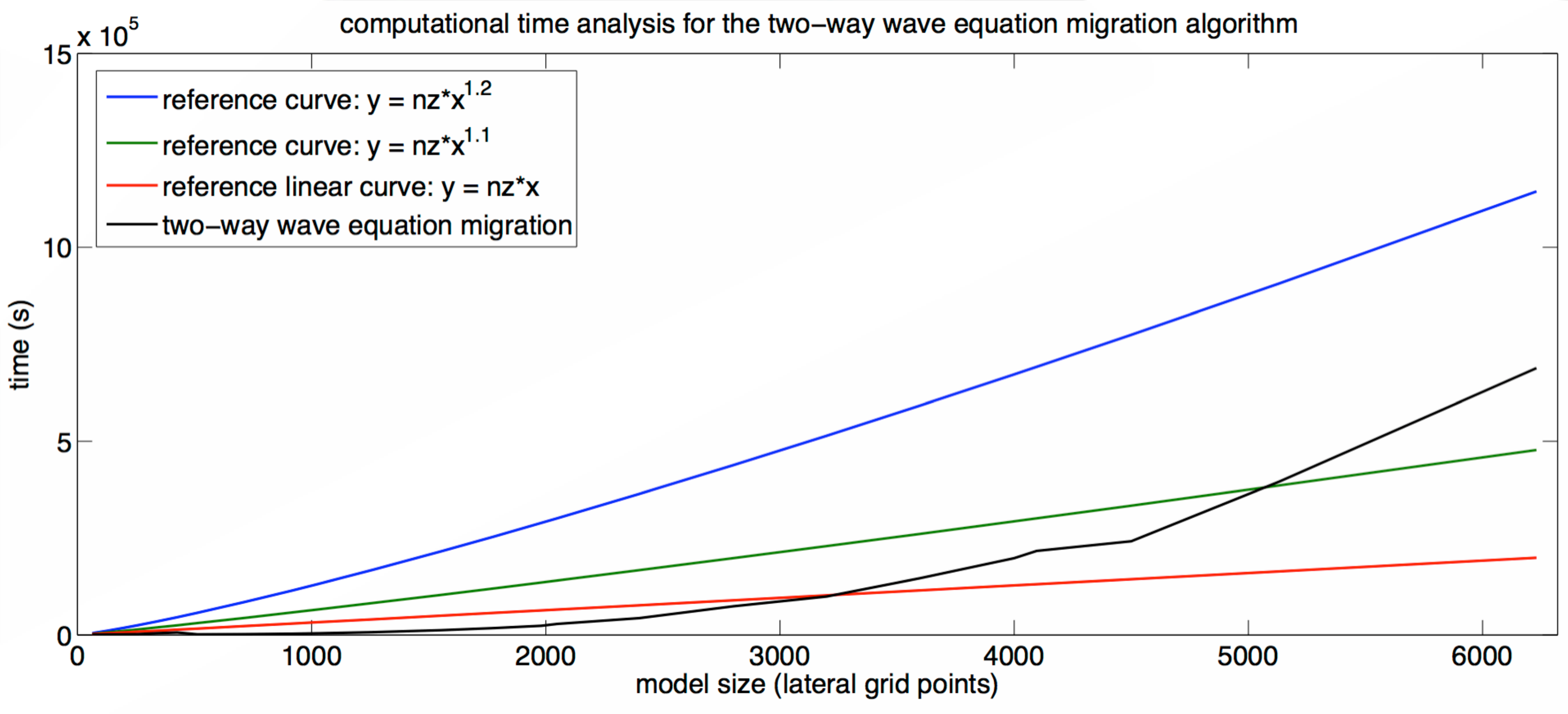
two-way wave-equation depth migration

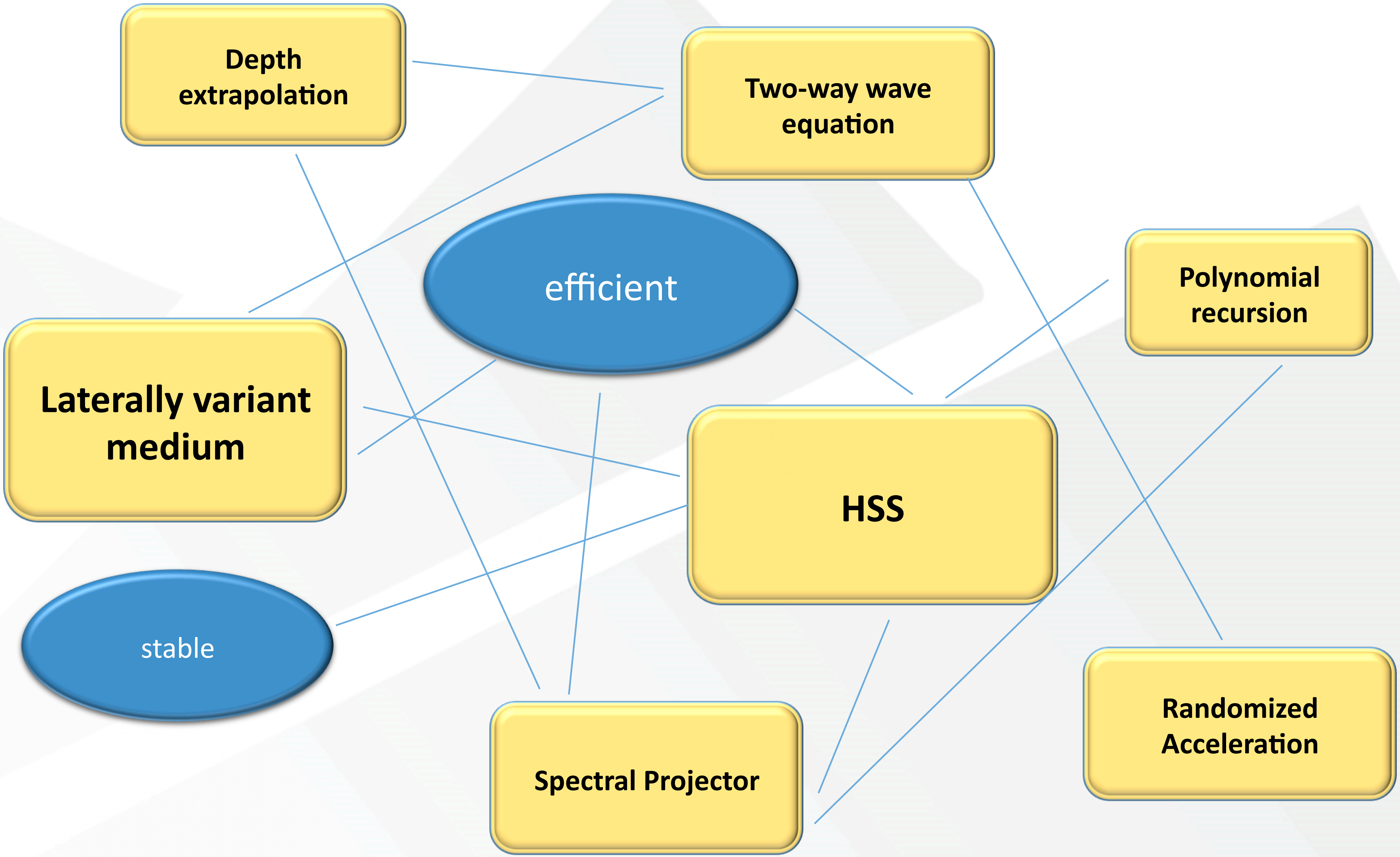
Reverse time migration

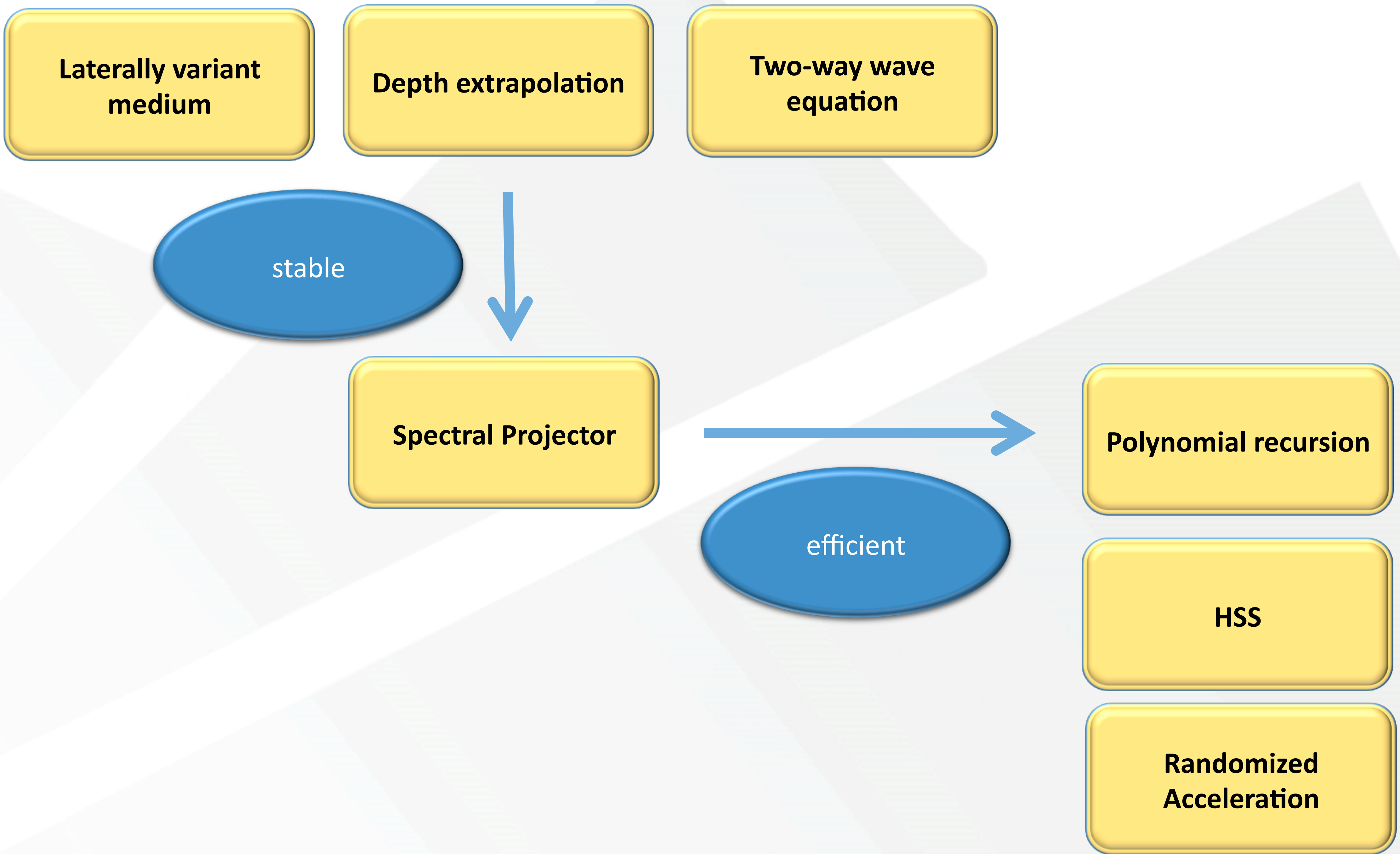
MIGRATION TIME COMPLEXITY

Operation	Complexity
rSVD	$\mathcal{O}(n)$
HSS construction	$\mathcal{O}(n)$
HSS matrix matrix addition	$\mathcal{O}(n)$
HSS matrix matrix multiplication	$\mathcal{O}(n)$
HSS matrix scalar multiplication	$\mathcal{O}(n)$
Depth extrapolation (Sandberg and Beylkin (2009))	$\mathcal{O}(n)$
The two-way wave equation depth extrapolation migration	$\mathcal{O}(n_\omega n_z n)$

MIGRATION TIME COMPLEXITY




Depth extrapolation
Two-way wave equation
efficient
Polynomial recursion
Laterally variant medium
HSS
stable
Spectral Projector
Randomized Acceleration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CONCLUSION

- **Spectral projector** computed with **polynomial recursion** using **randomized HSS techniques** provides an **efficient two-way** wave equation based **depth extrapolation** migration for **laterally variant medium**.
- Better image quality with the steep events (90 degrees) compared to reverse time migration (RTM).

ACKNOWLEDGEMENT

- This work was in part financially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant (22R81254) and the Collaborative Research and Development Grant DNOISE II (375142-08). This research was carried out as part of the SINBAD II project with support from the following organizations: BG Group, BGP, BP, CGG, Chevron, ConocoPhillips, ION, Petrobras, PGS, Statoil, Total SA, WesternGeco, and Woodside.

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