Wavefield Reconstruction Inversion (WRI) – a new take on wave-equation based inversion

Felix J. Herrmann

SLIM
University of British Columbia

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Ernie Esser, Tristan van Leeuwen*, and Bas Peters

* now @ Mathematical Institute at Utrecht University

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Motivation

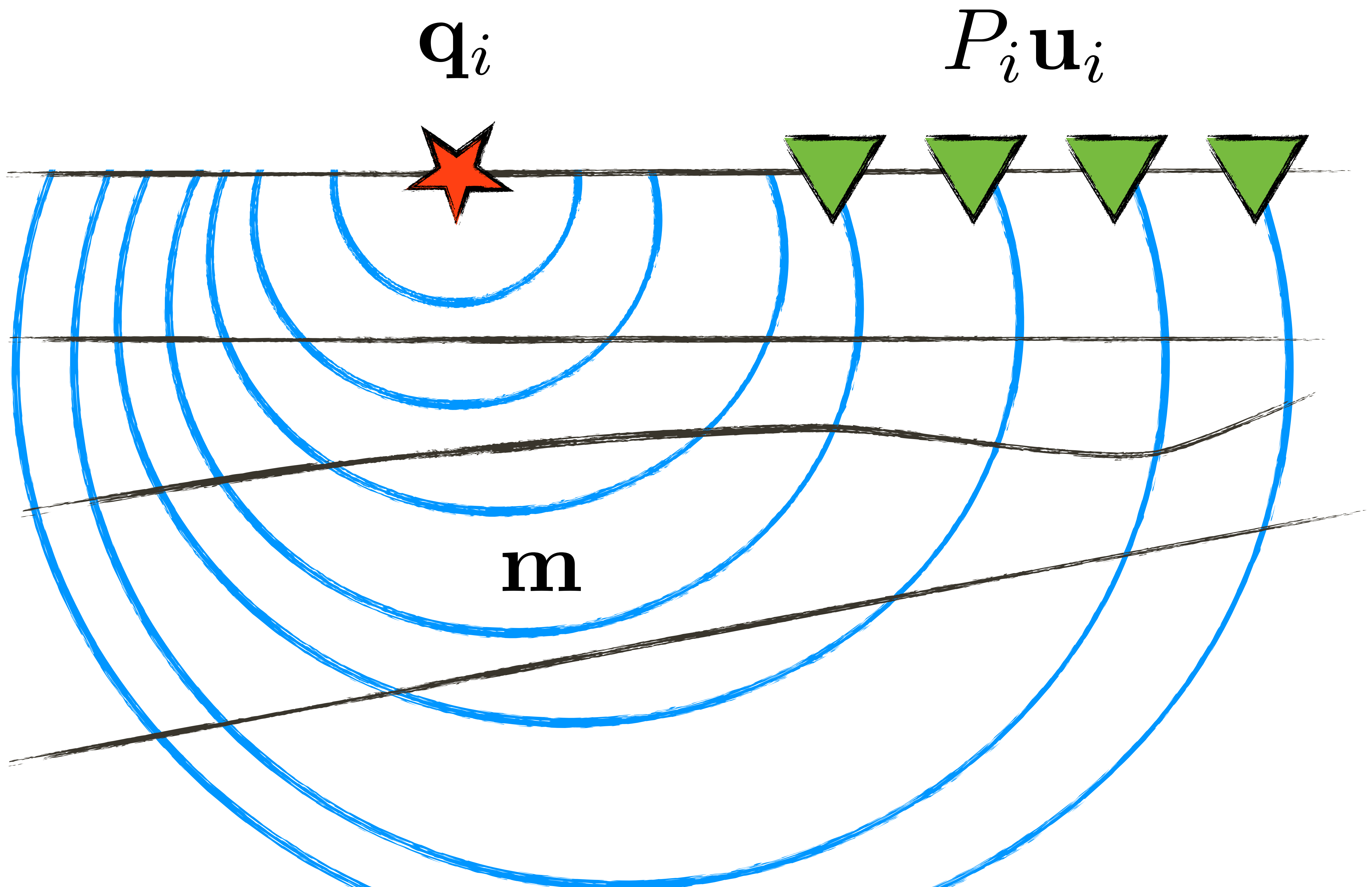
Full-waveform inversion is plagued with local minima

Derive an alternative extended formulation

- less prone to local minima
- computationally feasible
- relaxes the physics while staying solidly grounded

Waveform inversion

Retrieve the medium parameters from partial measurements of the solution of the wave-equation: $A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$



[Tarantola, ’84; Pratt, ’98; Haber, ’00; Plessix, ’06]

Waveform inversion

Adjoint-state/reduced-space methods:

- Optimize over earth models to minimize the misfit between observed and simulated data while solving the wave equation exactly for each earth model.

Full-space or all-at-once methods:

- Optimize over earth models & wavefields jointly to minimize the misfit between observed and simulated data subject to wavefields that satisfy the wave equation.

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Waveform inversion

Both approaches assume *flawless* wave physics–i.e.,

"known" physics → $A(\mathbf{m})$; "known" source → $\mathbf{q}_i$; "unknown" wavefield → $\mathbf{u}_i$

$$A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

- holds exactly for each source *i*
- differ on insisting wave equations to hold for each iteration
- different unknowns: **m** ⟷ **m** & **u**

[Richter, '81]

Equation error approach

If we "know" the wavefields everywhere, we solve for $\mathbf{m}$ from

$$A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

via

$$\min_{\mathbf{m}} \|A(\mathbf{m})P_i^{-1}\mathbf{d}_i - \mathbf{q}_i\|_2^2 \qquad \left(\text{cf. } \min_{\mathbf{m}} \|P_i A(\mathbf{m})^{-1}\mathbf{q}_i - \mathbf{d}_i\|_2^2\right)$$

The challenge is to reconstruct wavefields from partial measurements...

[van Leeuwen & FJH, 2013]

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WRI – Wavefield Reconstruction Inversion

For $\mathbf{m}$ fixed, reconstruct wavefields by jointly fitting observed shots

$$P\mathbf{u}_i \approx \mathbf{d}_i$$

and wave-equations

$$A(\mathbf{m})\mathbf{u}_i \approx \mathbf{q}_i$$

via least-squares solutions of the data-augmented wave-equation

$$\min_{\mathbf{u}_i} \left\| \begin{pmatrix} P_i \\ A(\mathbf{m}) \end{pmatrix} \mathbf{u}_i - \begin{pmatrix} \mathbf{d}_i \\ \mathbf{q}_i \end{pmatrix} \right\|_2^2$$

followed by fixing $\mathbf{u}_i$ and solving

$$\min_{\mathbf{m}} \|A(\mathbf{m})\mathbf{u}_i - \mathbf{q}_i\|_2^2$$

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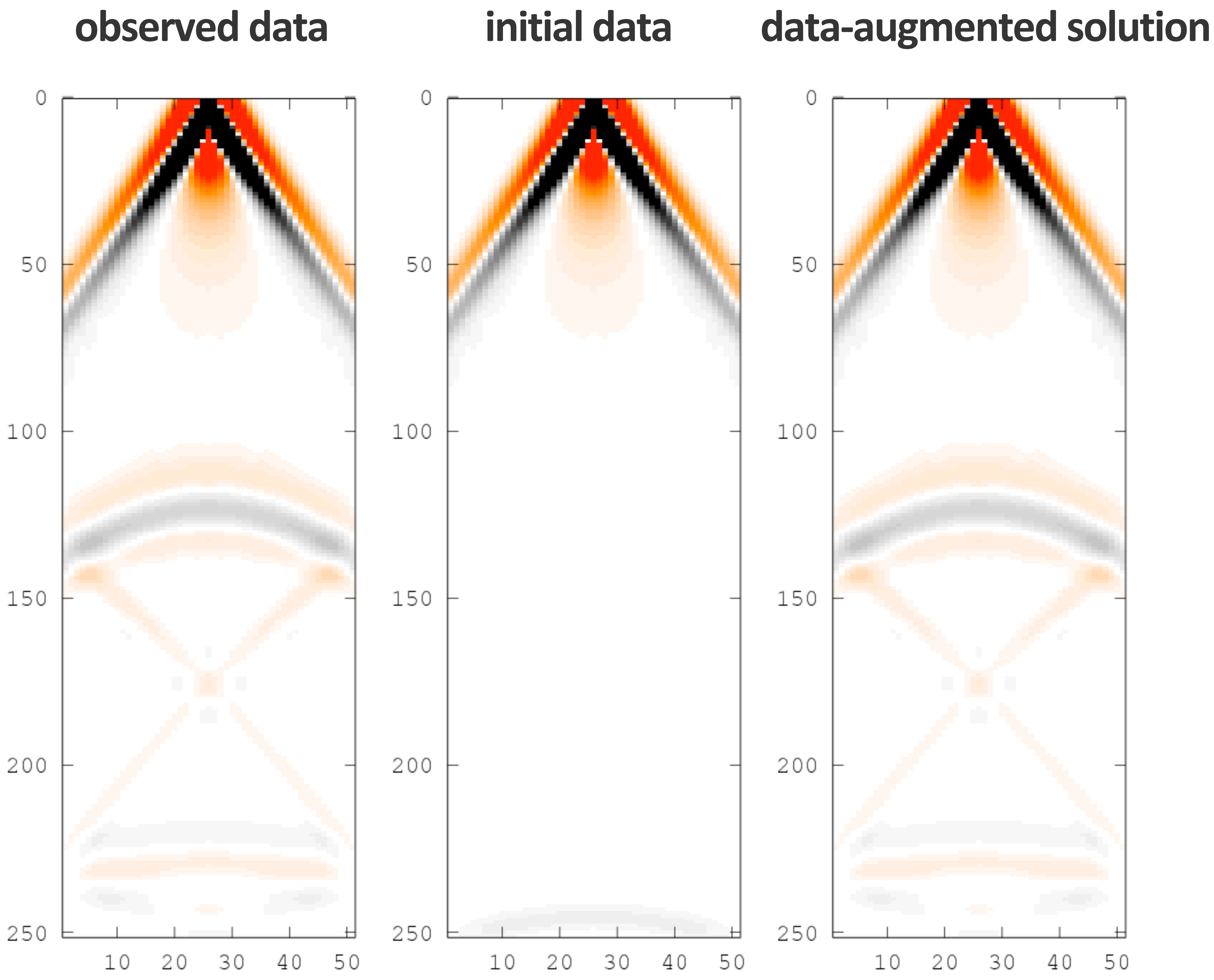
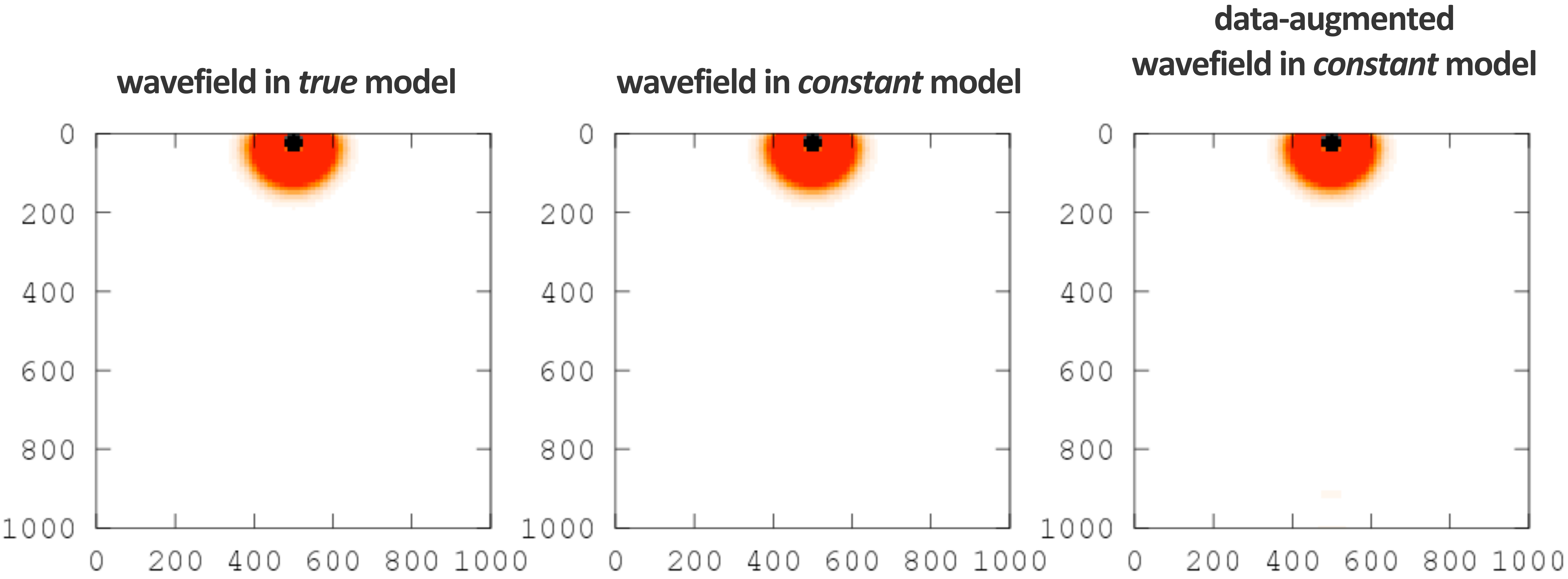
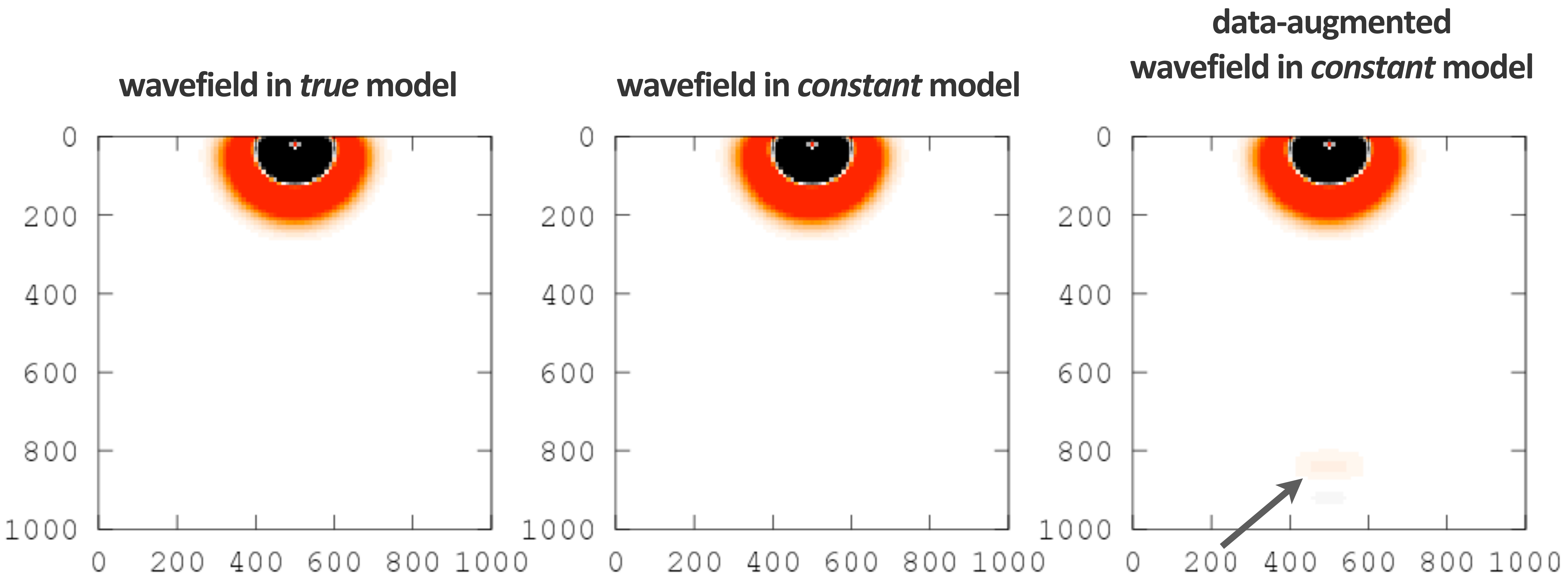
wave-equation X wavefield = source

versus

(wave-equation / sampling operator) X wavefield = (source / data)


observed data
initial data
data-augmented solution
0
50
100
150
200
250
10
20
30
40
50


wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
0
200
400
600
800
1000
0 200 400 600 800 1000


wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
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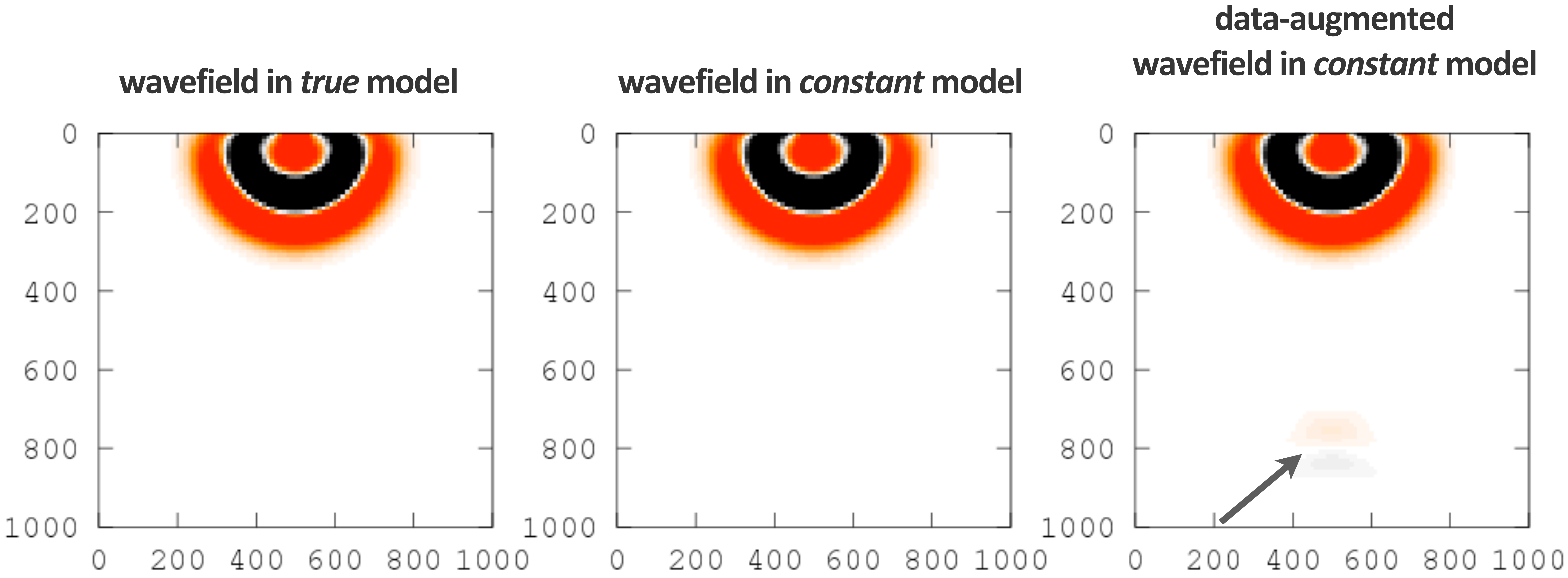
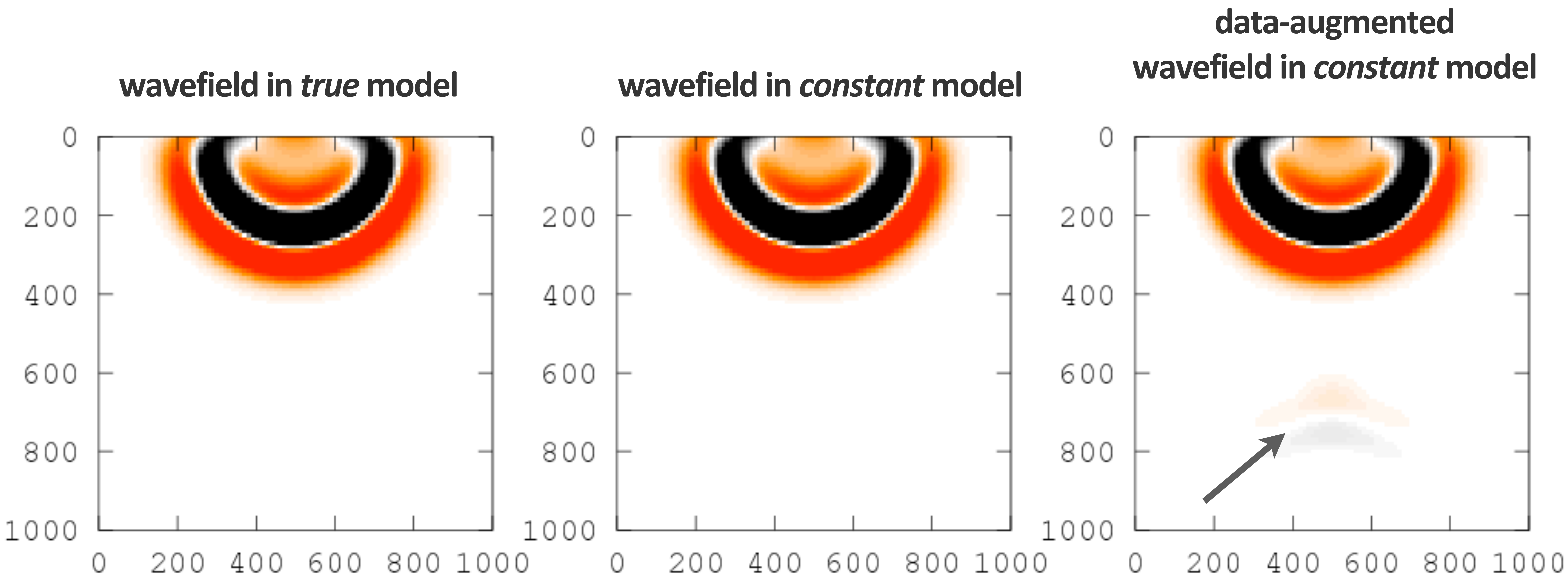
800
1000


wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model



wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
0
200
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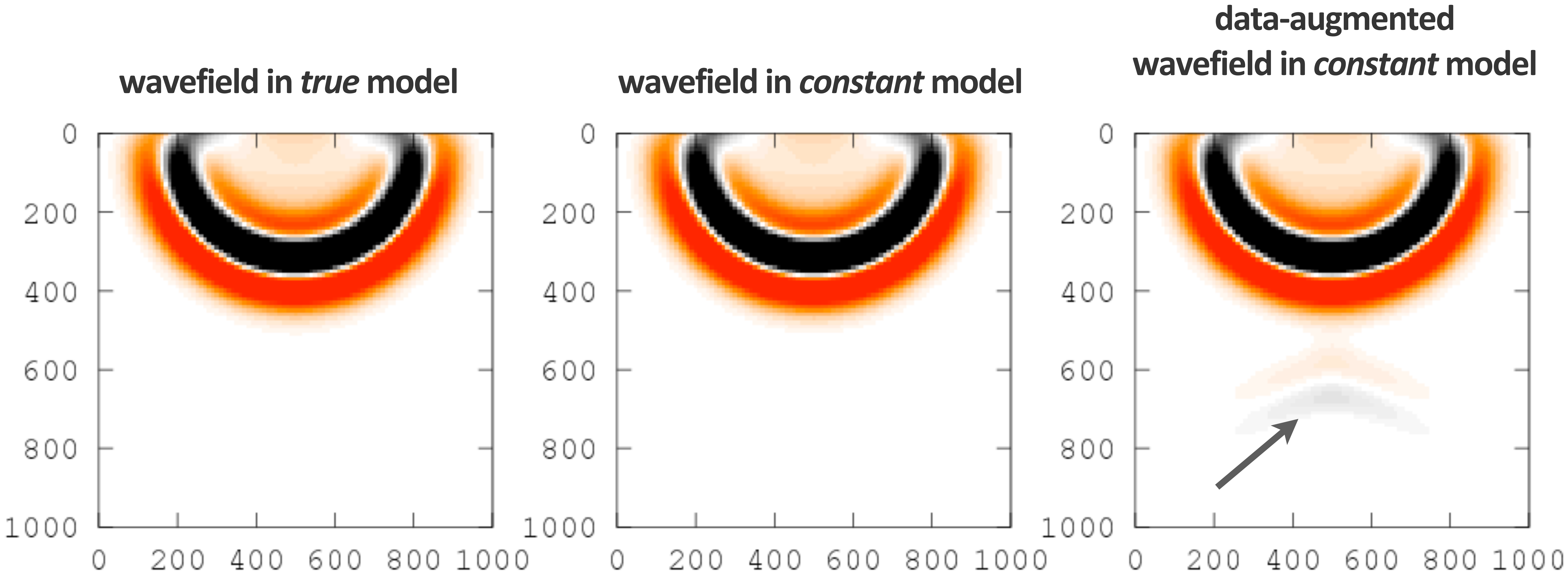
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data-augmented
wavefield in *constant* model
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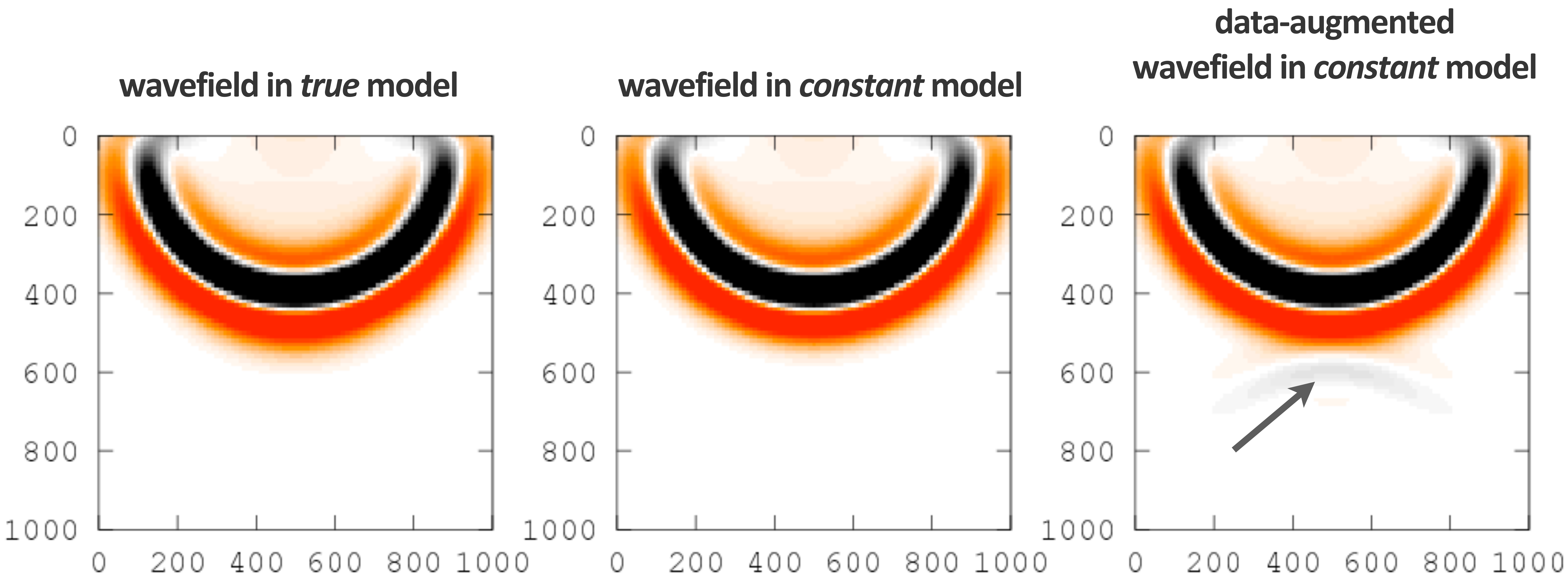
wavefield in *true* model
wavefield in *constant* model
data-augmented wavefield in *constant* model
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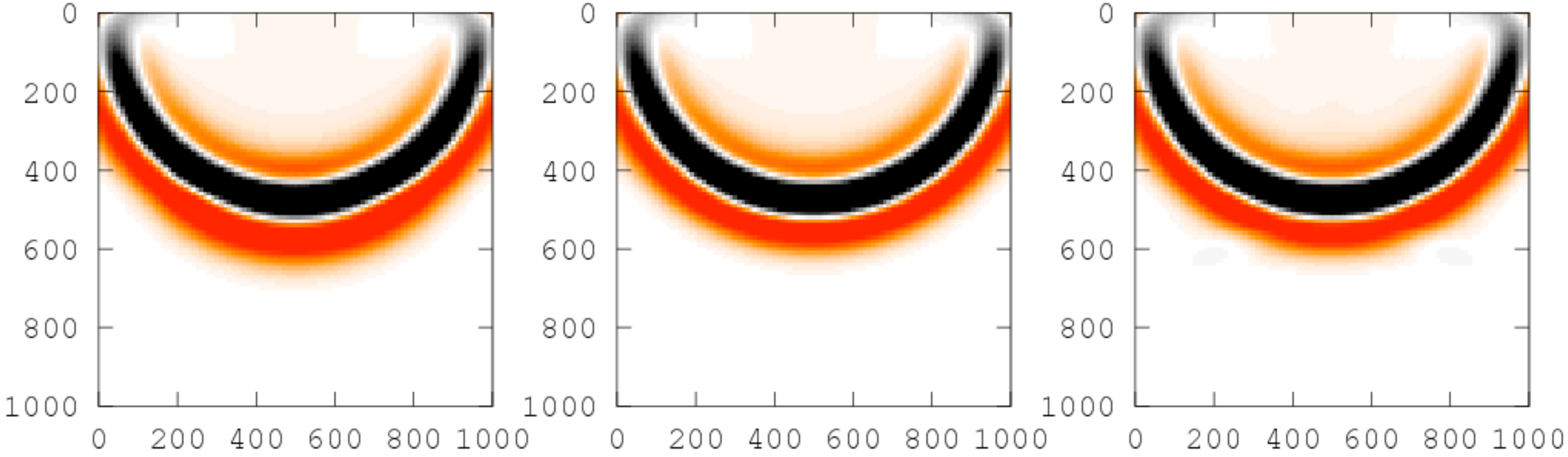
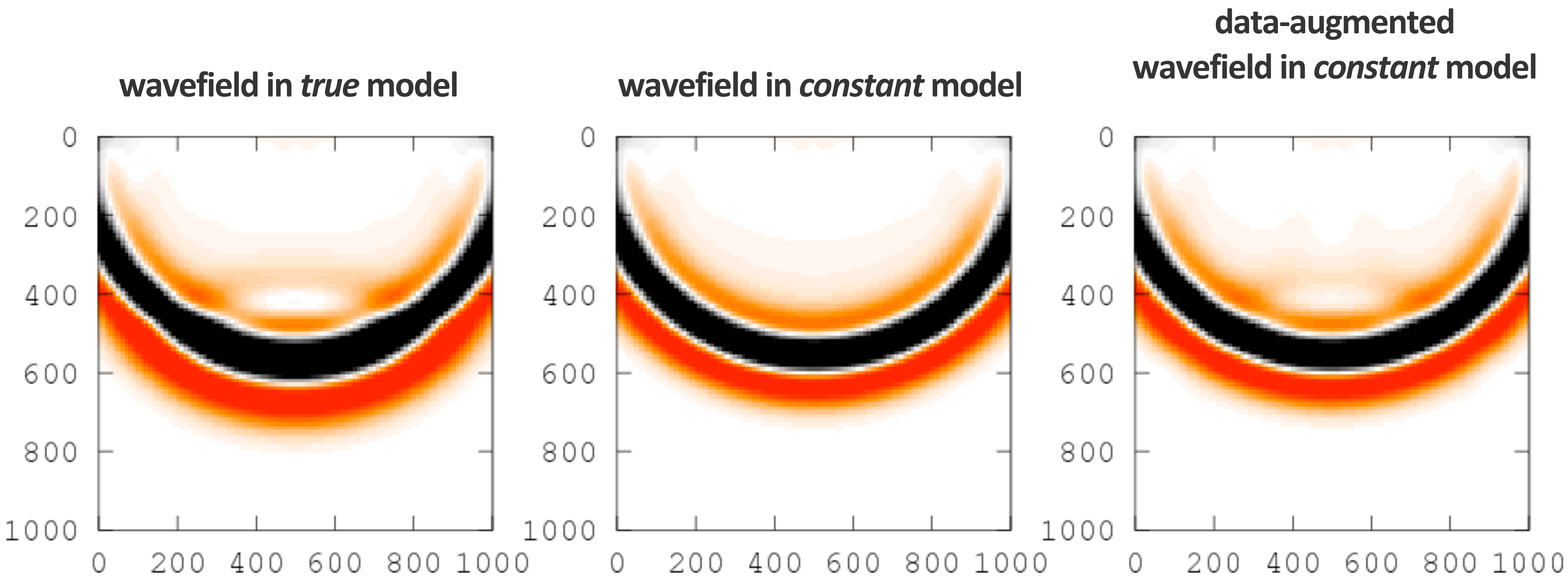
1000

wavefield in *true* model

wavefield in *constant* model

data-augmented wavefield in *constant* model





wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
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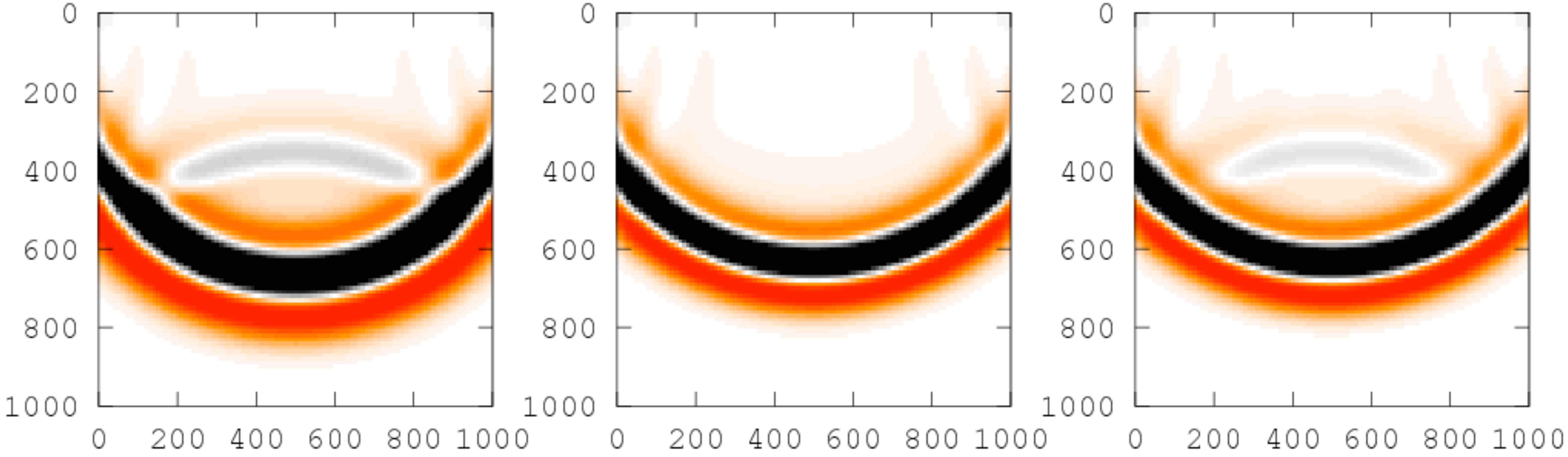
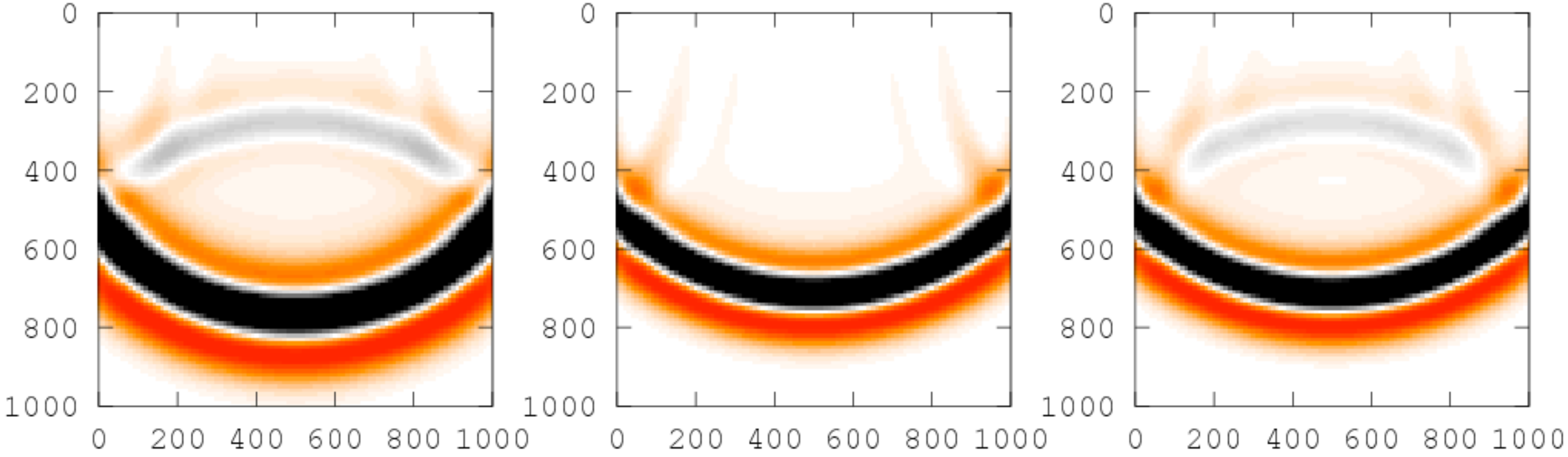
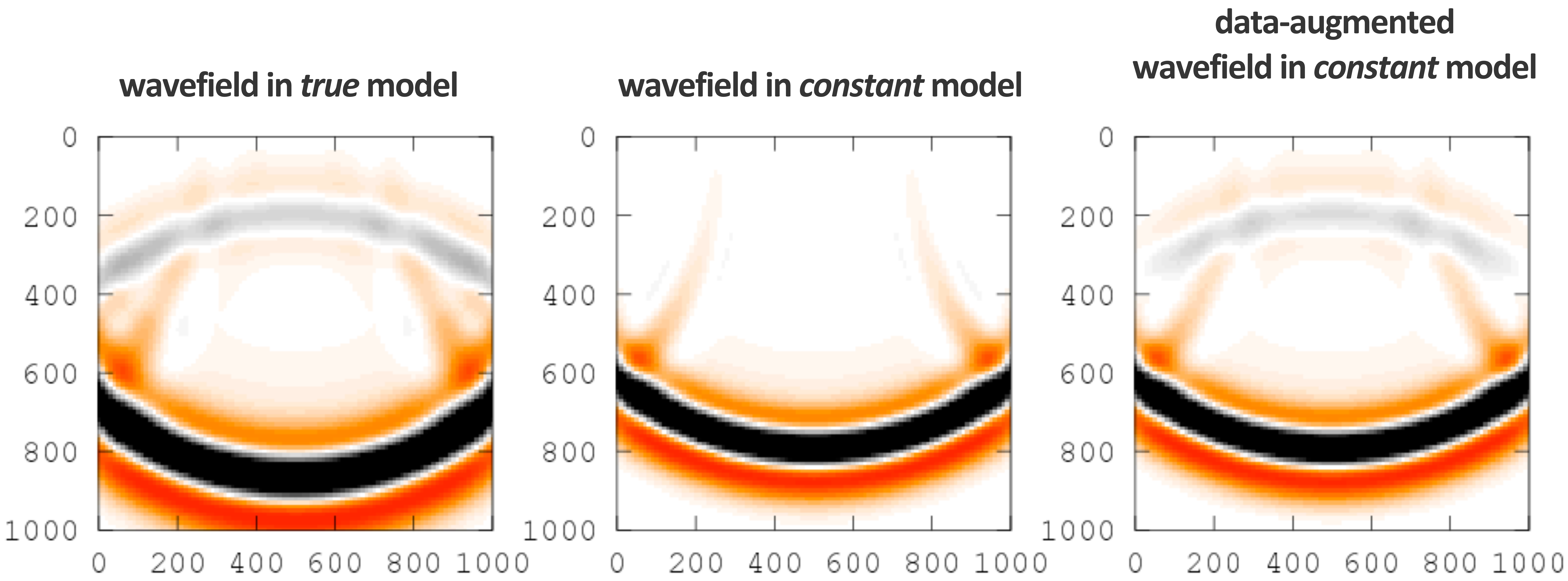
0 200 400 600 800 1000

wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**



wavefield in *true* model | **wavefield in *constant* model** | **data-augmented wavefield in *constant* model**





wavefield in *true* model
wavefield in *constant* model
data-augmented
wavefield in *constant* model
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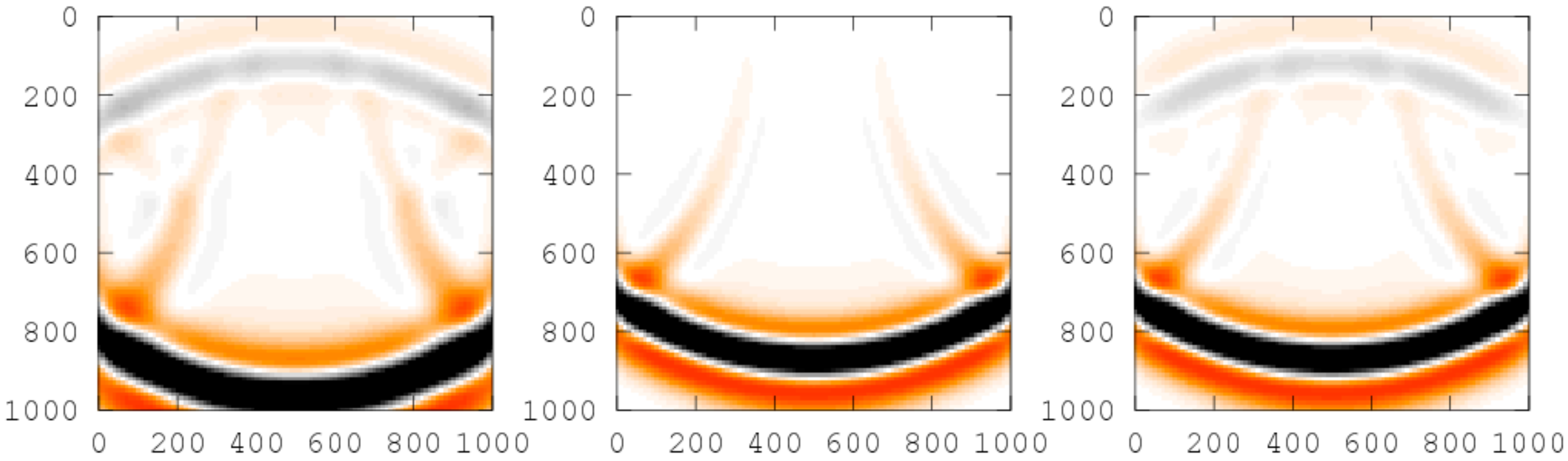
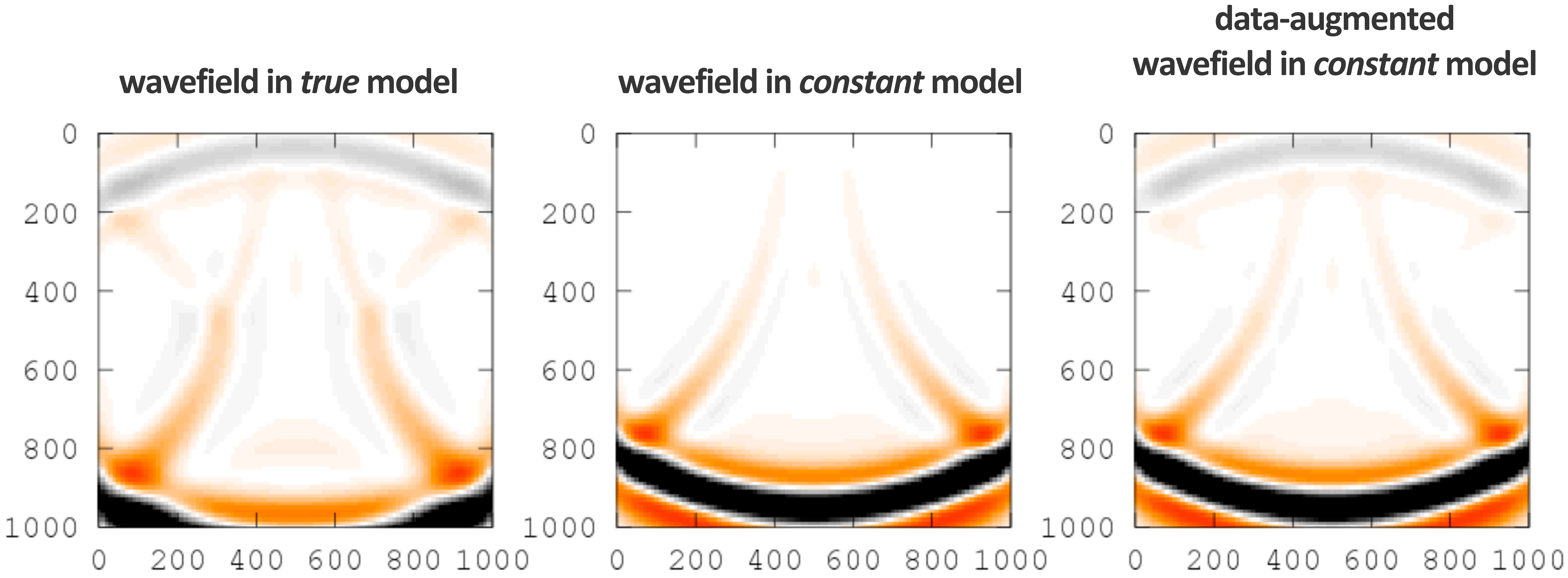
wavefield in *true* model | wavefield in *constant* model | data-augmented wavefield in *constant* model

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data-augmented wavefield in *constant* model
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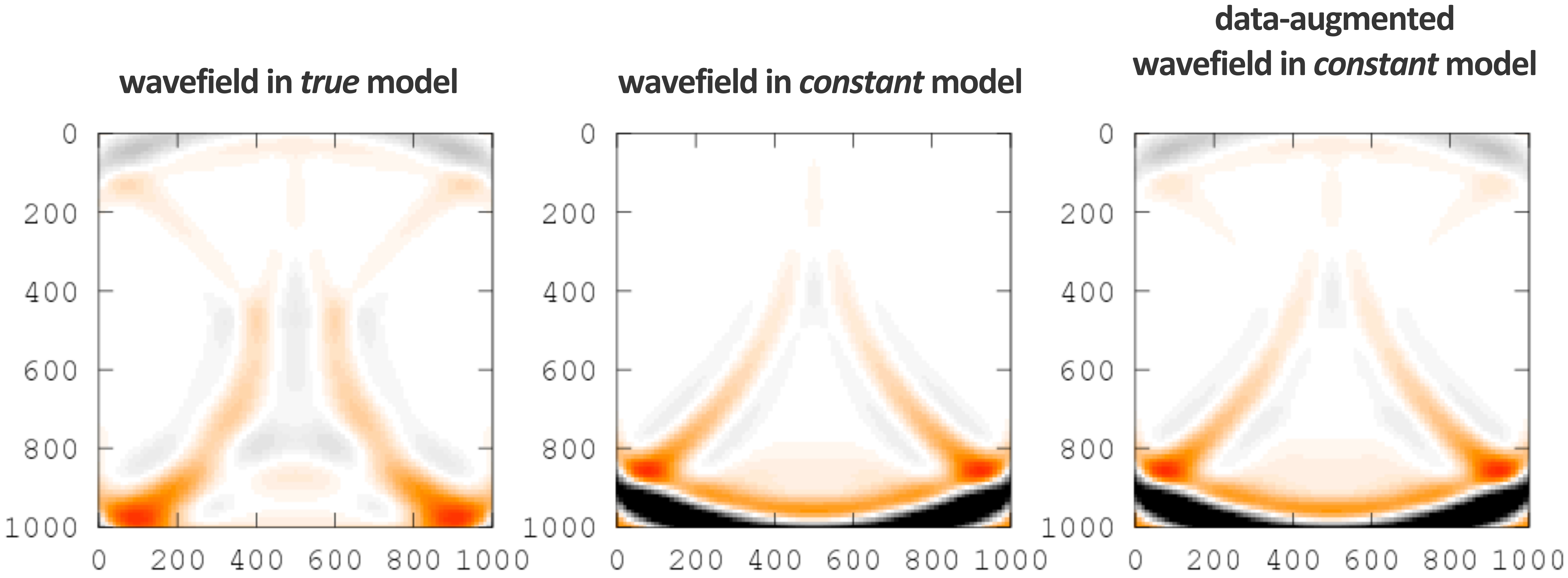

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wavefield in *constant* model
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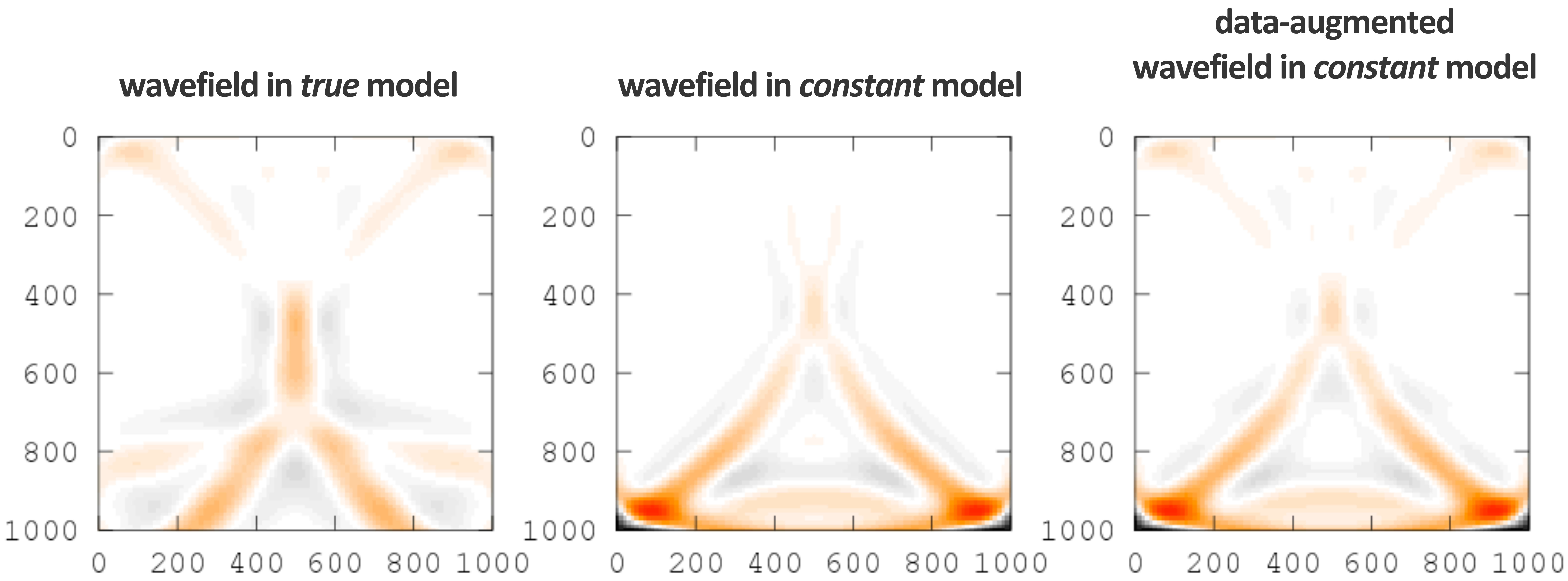
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wavefield in *constant* model
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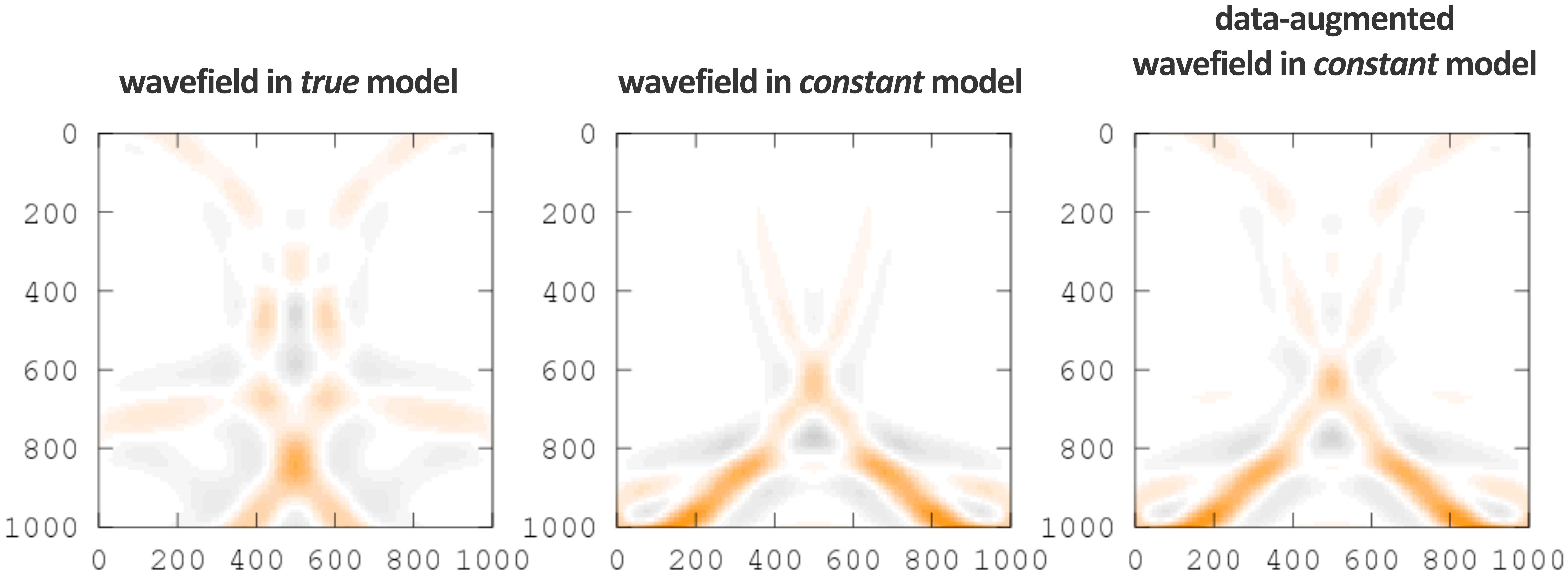
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wavefield in *constant* model
data-augmented
wavefield in *constant* model
0
200
400
600
800
1000

[Heinkenschloss, ’98 , Haber, ’00]

PDE-constrained optimization
all-at-once full-space approach

simulated data (↓ $P_i\mathbf{u}_i$), simulated wavefield (↓ $\mathbf{u}_i$)

$$\min_{\mathbf{m},\mathbf{u}} \sum_{i=1}^{M} \|P_i\mathbf{u}_i - \mathbf{d}_i\|_2^2 \quad \text{s.t.} \quad A_i(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

observed data (↑ $\mathbf{d}_i$), Helmholtz equation (↑ A_i), source (↑ $\mathbf{q}_i$)

- avoids having to solve the PDE explicitly
- sparse (GN) Hessian
- requires storing all variables **(m,u)**
- does **not** scale to industry-scale seismic problems

[Tarantola ’84; Pratt, ’98; Plessix, ’06]

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Adjoint-state/reduced-space formulation

Elimination of the constraint leads for all sources to

$$\min_{\mathbf{m}} \phi_{\text{red}}(\mathbf{m}) = \sum_{i=1}^{M} \|P_i A_i(\mathbf{m})^{-1}\mathbf{q}_i - \mathbf{d}_i\|_2^2$$

- no need to store all wavefields (block-elimination)
- suitable for black-box optimization (e.g., l-BFGS)
- need to solve forward & adjoint PDEs
- very non-linear in earth model (m)
- dense (GN) Hessian, involves additional PDE solves
- paints you in a corner by insisting on the physics...

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WRI – penalty formulation

Instead of eliminating, we add constraints as penalties–i.e.,

$$\min_{\mathbf{m},\mathbf{u}} \phi_\lambda(\mathbf{m},\mathbf{u}) = \sum_{i=1}^{M} \|P\mathbf{u}_i - \mathbf{d}_i\|_2^2 + \lambda^2 || A_i(\mathbf{m})\mathbf{u}_i - \mathbf{q}_i\|_2^2$$

coincides with original problem when $\lambda \uparrow \infty$

[Aravkin & van Leeuwen, ’12; van Leeuwen & FJH, ’13]

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Variable projection

Solve data-augmented wave equation for each source

$$\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u}_{i,\lambda} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$$

Define reduced objective with proxy wavefields

$$\phi_\lambda(\mathbf{m}) = \phi_\lambda(\mathbf{m}, \bar{\mathbf{u}}_\lambda) = \|P\bar{\mathbf{u}}_\lambda - \mathbf{d}\|_2^2 + \lambda^2 \|A(\mathbf{m})\bar{\mathbf{u}}_\lambda - \mathbf{q}\|_2^2$$

[van Leeuwen & FJH, ’13]

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Wavefield Reconstruction Inversion

WRI method

for each source i

solve $\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u}_{\lambda,i} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$

$\mathbf{g} = \mathbf{g} + \lambda^2 \omega^2 \mathsf{diag}(\bar{\mathbf{u}}_{i,\lambda})^* (A(\mathbf{m}) \bar{\mathbf{u}}_{i,\lambda} - \mathbf{q}_i)$

$\mathbf{m} = \mathbf{m} - \alpha \mathbf{g}$

end

correlation proxy wavefield & PDE residual

Conventional method

for each source i

solve $A(\mathbf{m}) \mathbf{u}_i = \mathbf{q}_i$

solve $A(\mathbf{m})^* \mathbf{v}_i = P_i^* (P_i \mathbf{u}_i - \mathbf{d}_i)$

$\mathbf{g} = \mathbf{g} + \omega^2 \mathsf{diag}(\mathbf{u}_i)^* \mathbf{v}_i$

$\mathbf{m} = \mathbf{m} - \alpha \mathbf{g}$

end

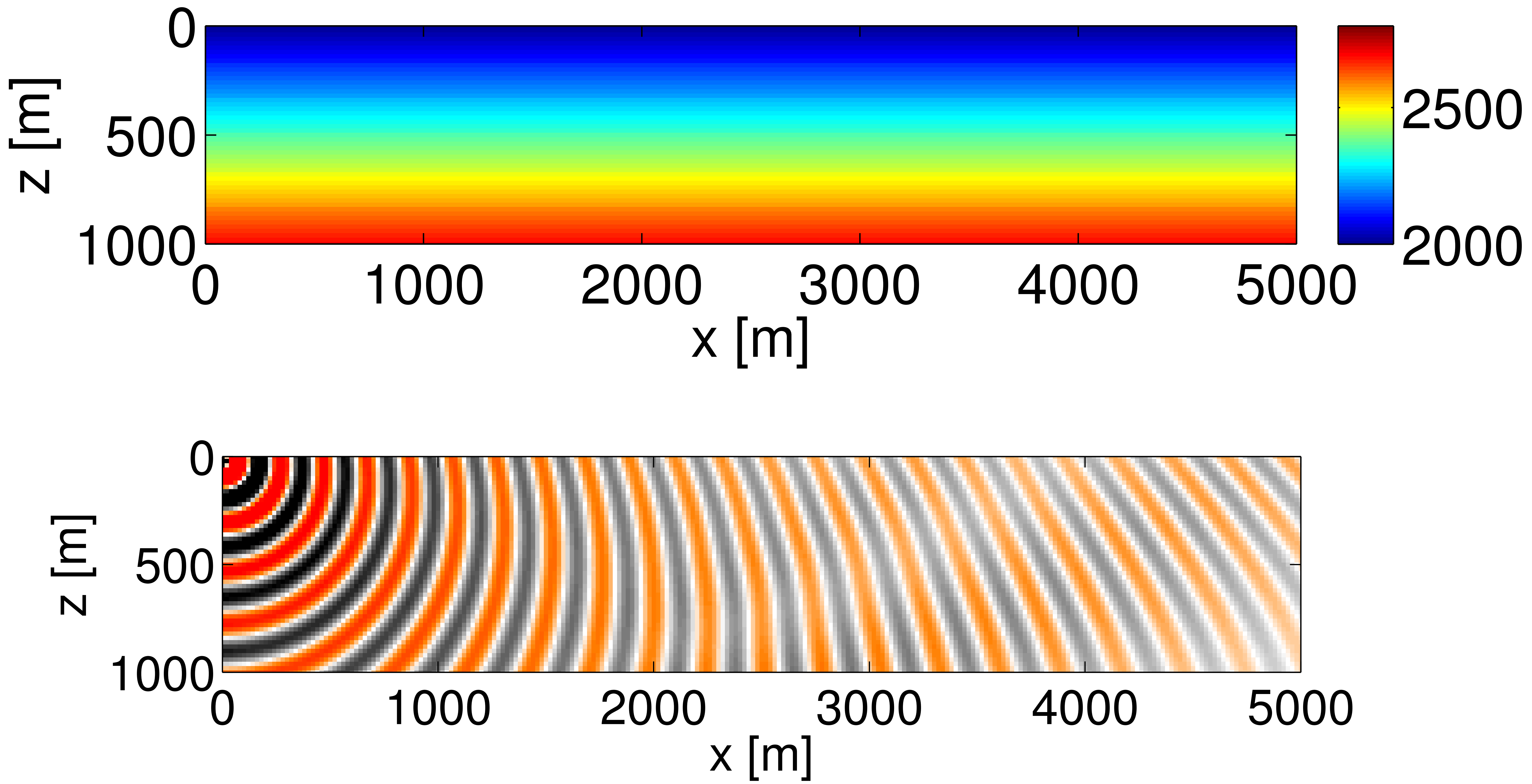
correlation wavefield & data residual

Wavefield Reconstruction Inversion

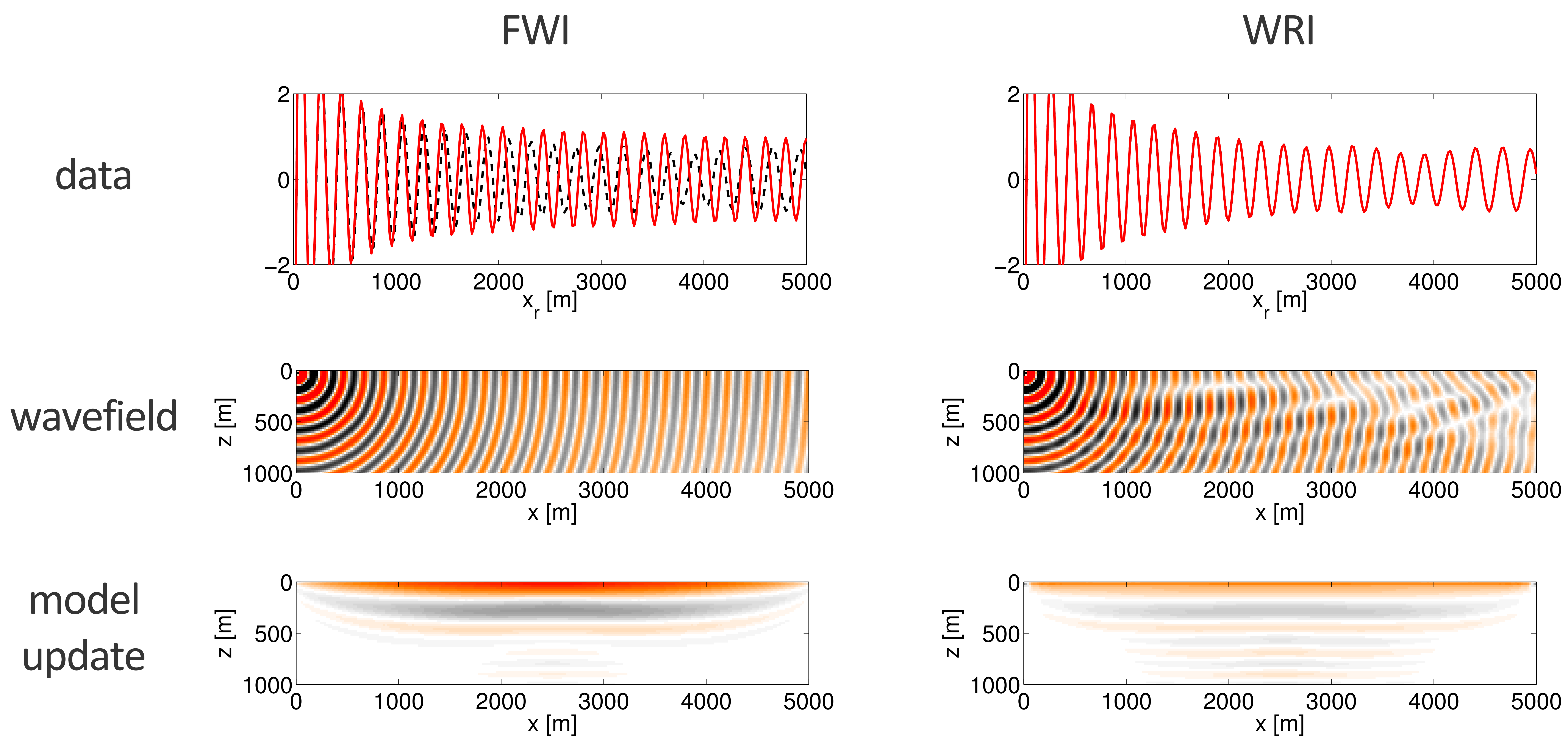
- **no** need to store all the fields (**u**)
- **no** adjoint solves
- sparse approximation of GN Hessian for small λ
- less non-linear in **m**
- need to solve overdetermined PDEs
- ...

Diving wave example

true model and wavefield

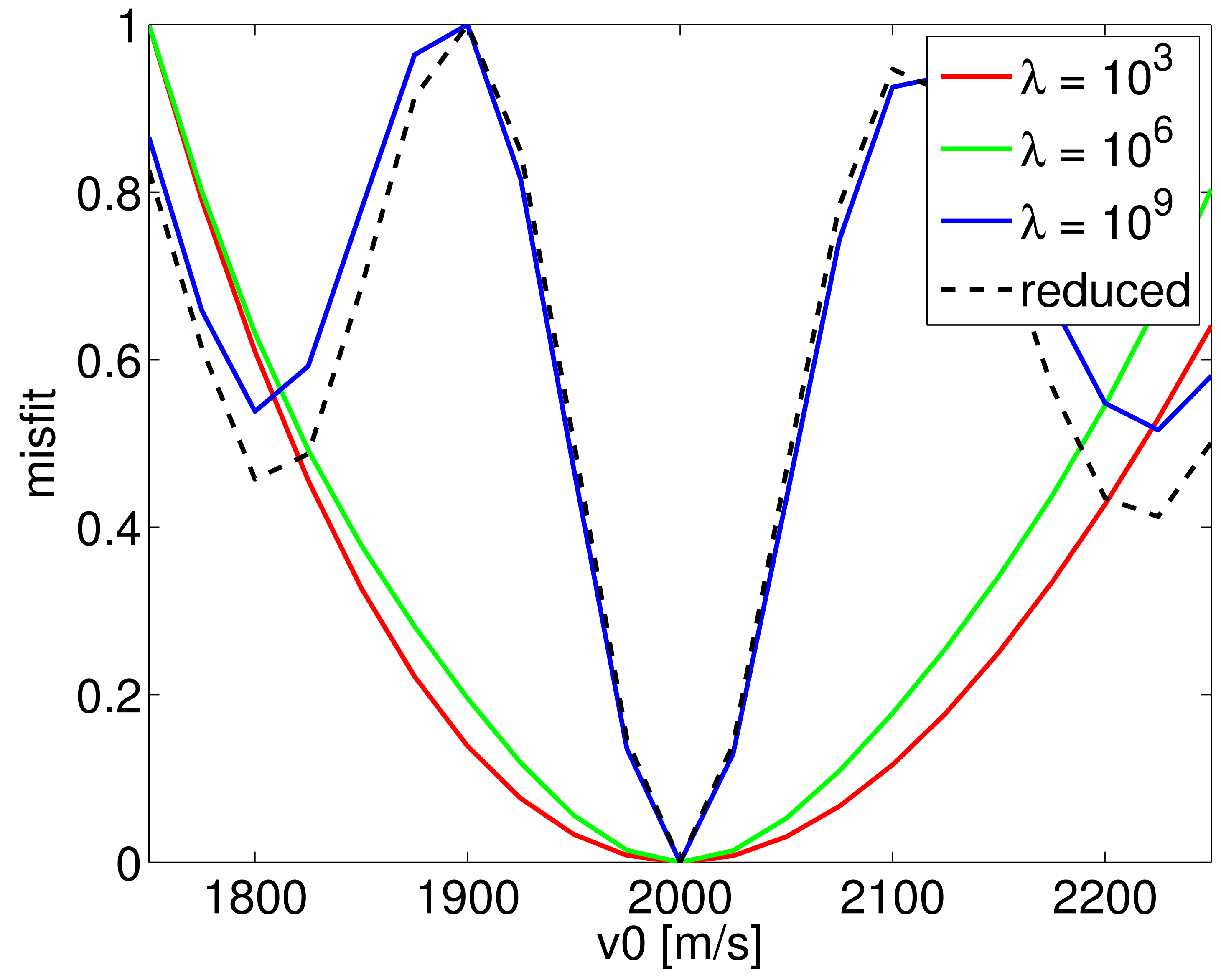


Wavefields in *homogeneous* background

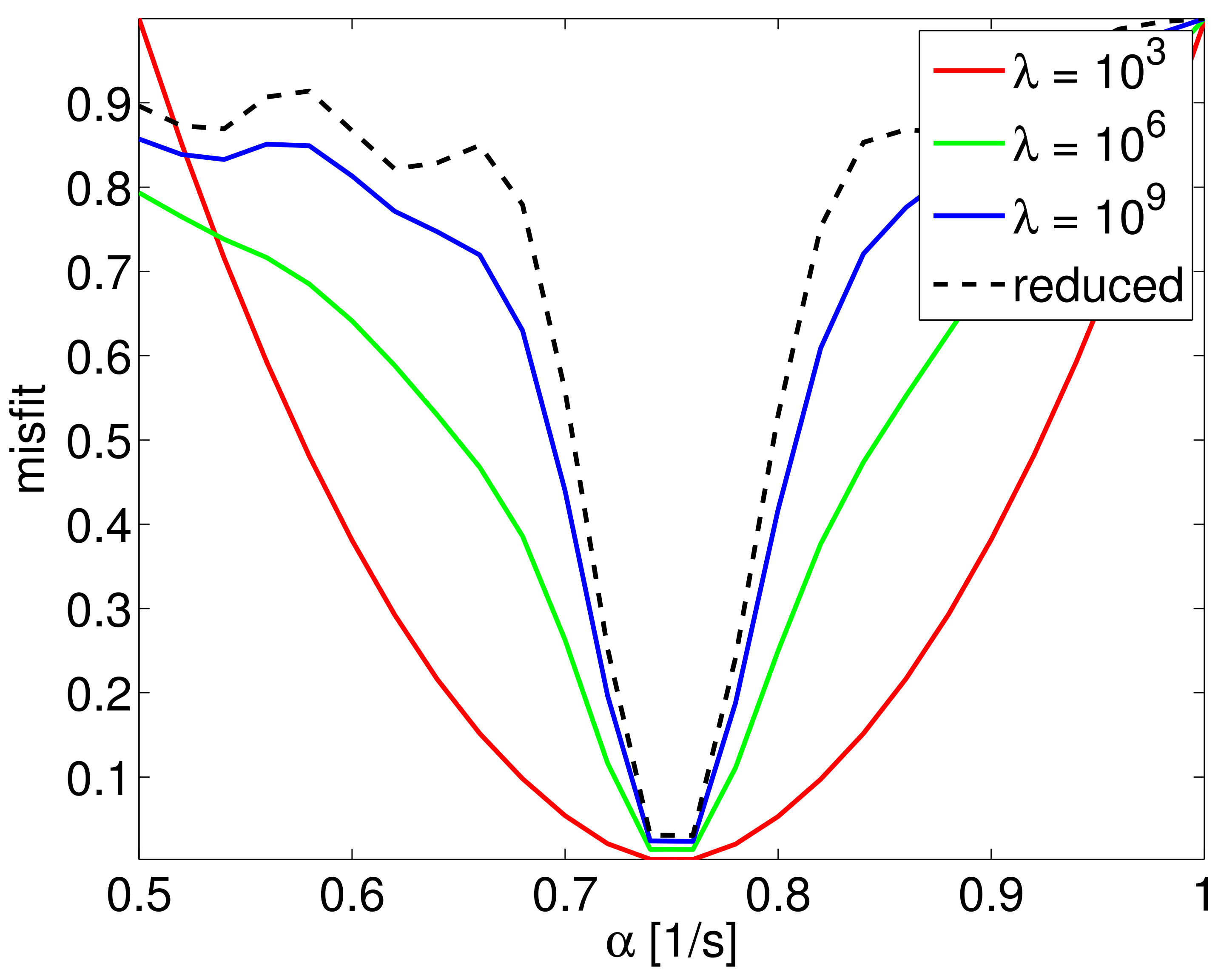


Local minima

single shot, single frequency data for *linear* velocity profile $v(z) = v_0 + \alpha z$,
misfit as function of (v_0, α)



constant



slope

Connections

[Aria Abubakar et. al. ’09]

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Related work

Contrast-source formulation

- combined objective is similar
- but does not eliminate wavefields via variable projection
- **requires storage of wavefields for all sources**

Extended modelling

The penalty formulation

$$\min_{\mathbf{m},\mathbf{u}} ||P\mathbf{u} - \mathbf{d}||_2^2 + \lambda^2 ||A(\mathbf{m})\mathbf{u} - \mathbf{q}||_2^2$$

can be interpreted as

$$\min_{\tilde{\mathbf{m}}} \text{misfit}(\tilde{\mathbf{m}}) + \text{annihilator}(\tilde{\mathbf{m}})$$

with

$$\tilde{\mathbf{m}} = (\mathbf{m}, \mathbf{u})$$

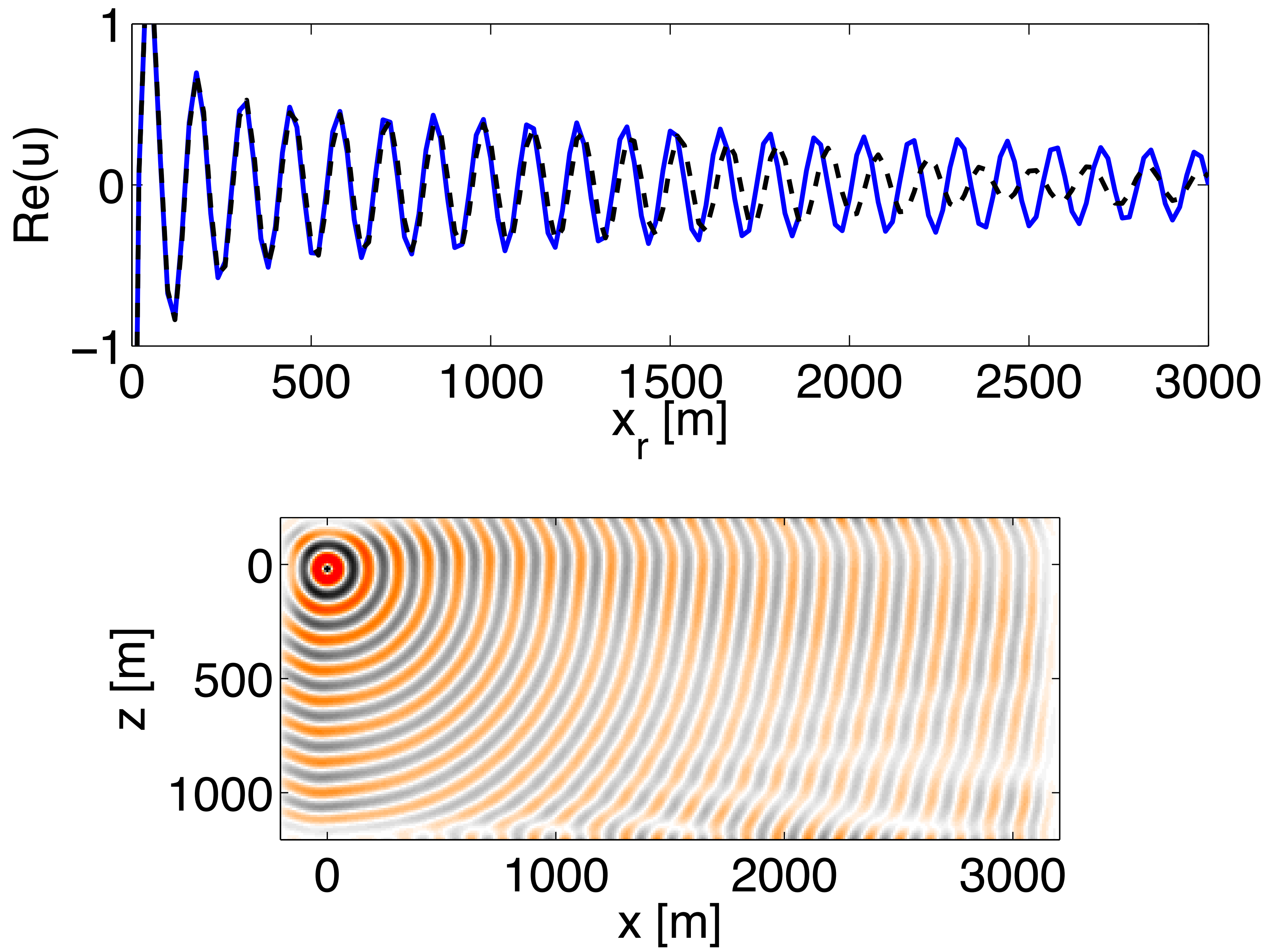
For a physically plausible model we have

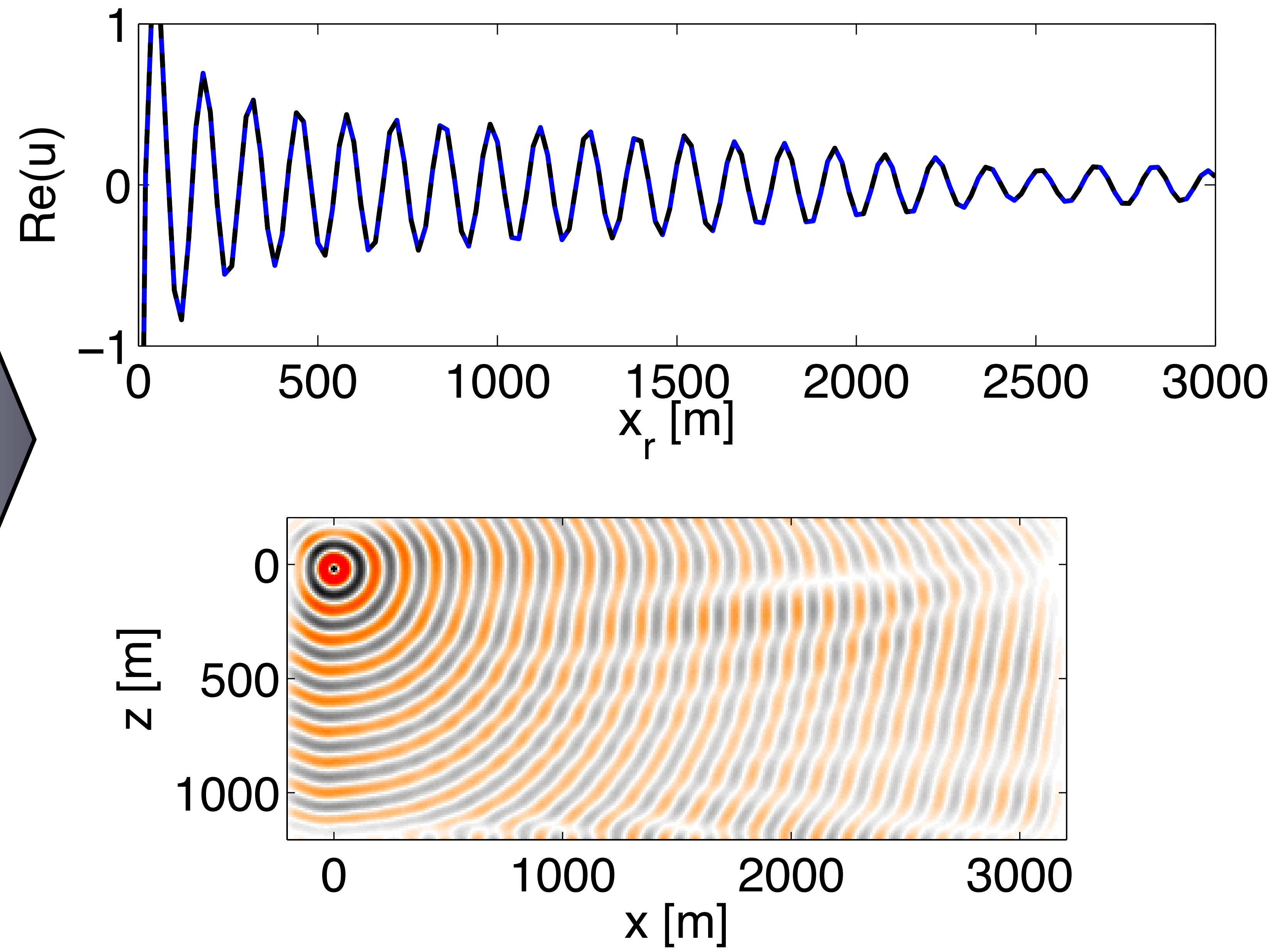
$$\text{annihilator}(\tilde{\mathbf{m}}) = 0$$

[Symes, personal communication]

Warping

The overdetermined WE is a way of warping



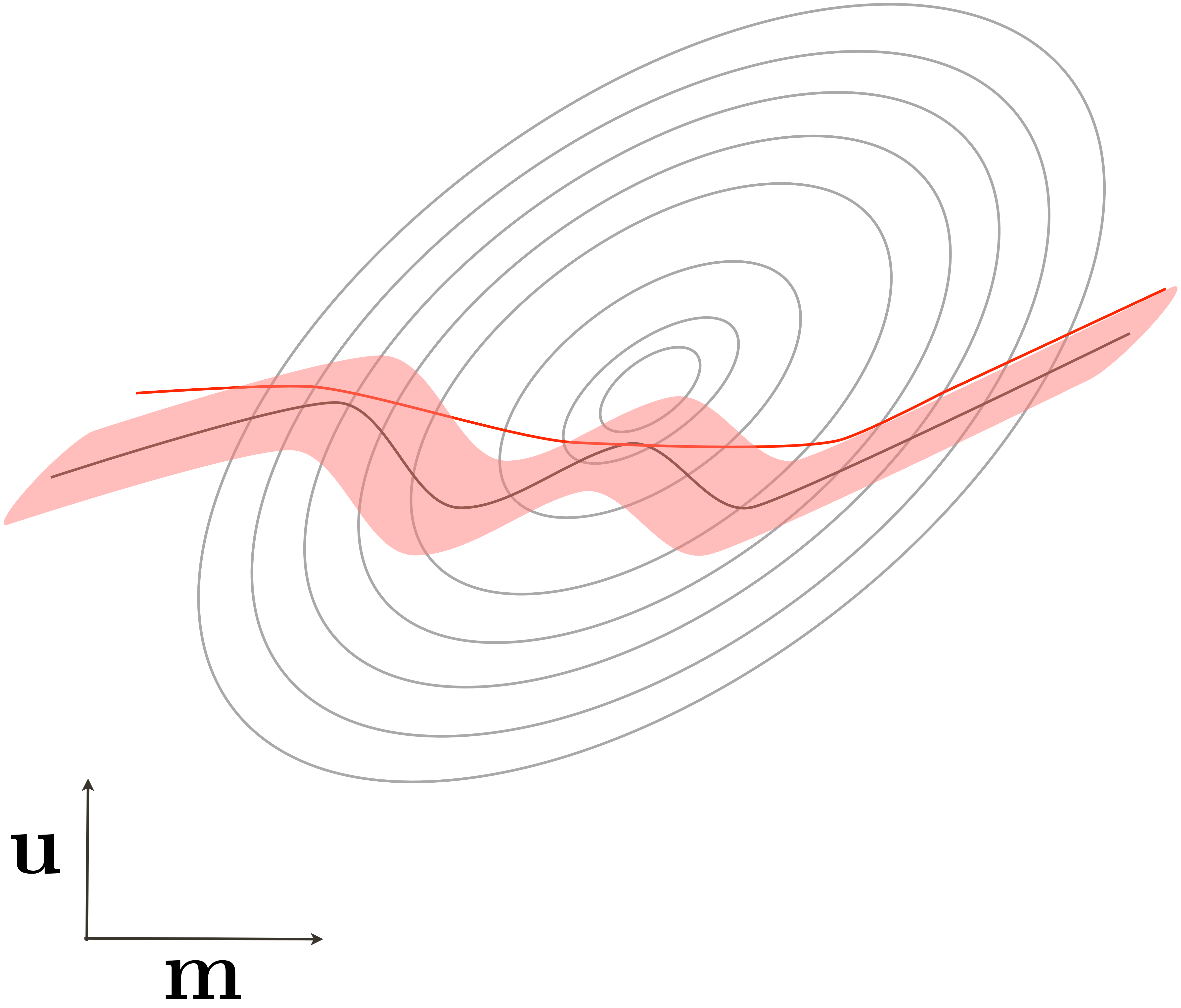


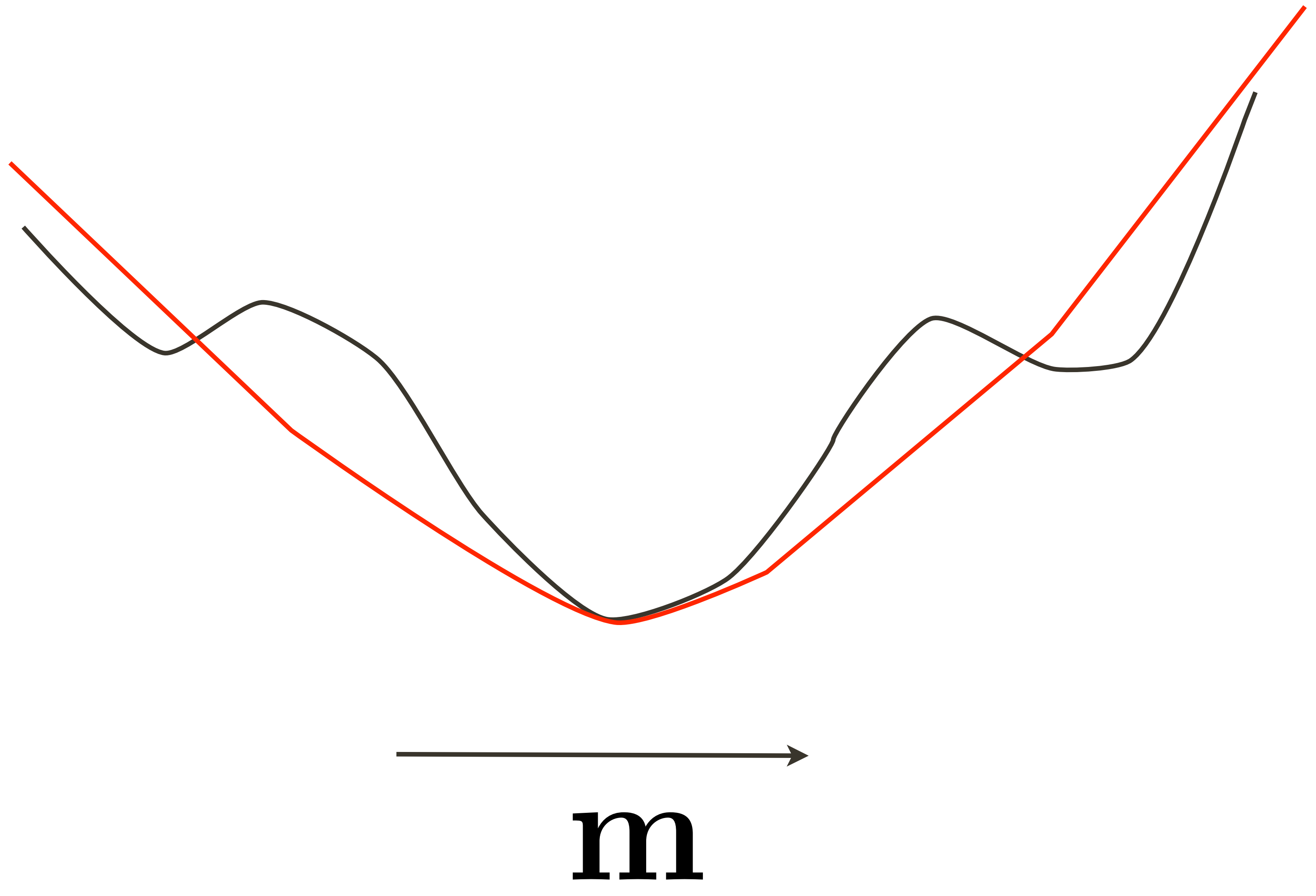
[Baek ’13, Ma ’13]

WRI vs. FWI

Larger # of degrees of freedom

“more convex”





Wavefield Reconstruction Inversion

WRI method

for each source i

solve $\begin{pmatrix} P_i \\ \lambda A_i(\mathbf{m}) \end{pmatrix} \mathbf{u}_{\lambda,i} \approx \begin{pmatrix} \mathbf{d}_i \\ \lambda \mathbf{q}_i \end{pmatrix}$

$\mathbf{g} = \mathbf{g} + \lambda^2\omega^2 \text{diag}(\bar{\mathbf{u}}_{i,\lambda})^*(A(\mathbf{m})\bar{\mathbf{u}}_{i,\lambda} - \mathbf{q}_i)$

$H_{GN} = H_{GN} + \lambda^2\omega^4 \text{diag}(\mathbf{u}_i)^* \text{diag}(\mathbf{u}_i)$

$\mathbf{m} = \mathbf{m} - \alpha H_{GN}^{-1}\mathbf{g}$

end

diagonal
=
pseudo Hessian

Conventional method

for each source i

solve $A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$

solve $A(\mathbf{m})^*\mathbf{v}_i = P_i^*(P_i\mathbf{u}_i - \mathbf{d}_i)$

$\mathbf{g} = \mathbf{g} + \omega^2 \text{diag}(\mathbf{u}_i)^*\mathbf{v}_i$

$\mathbf{m} = \mathbf{m} - \alpha\mathbf{g}$

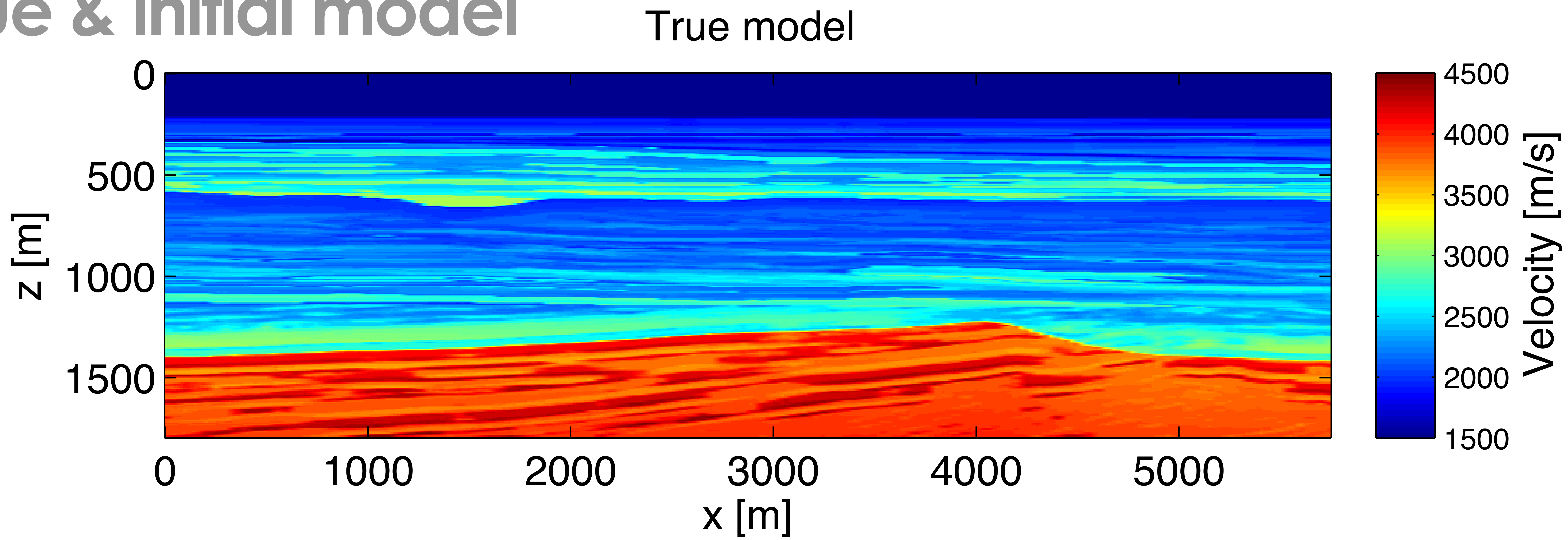
end

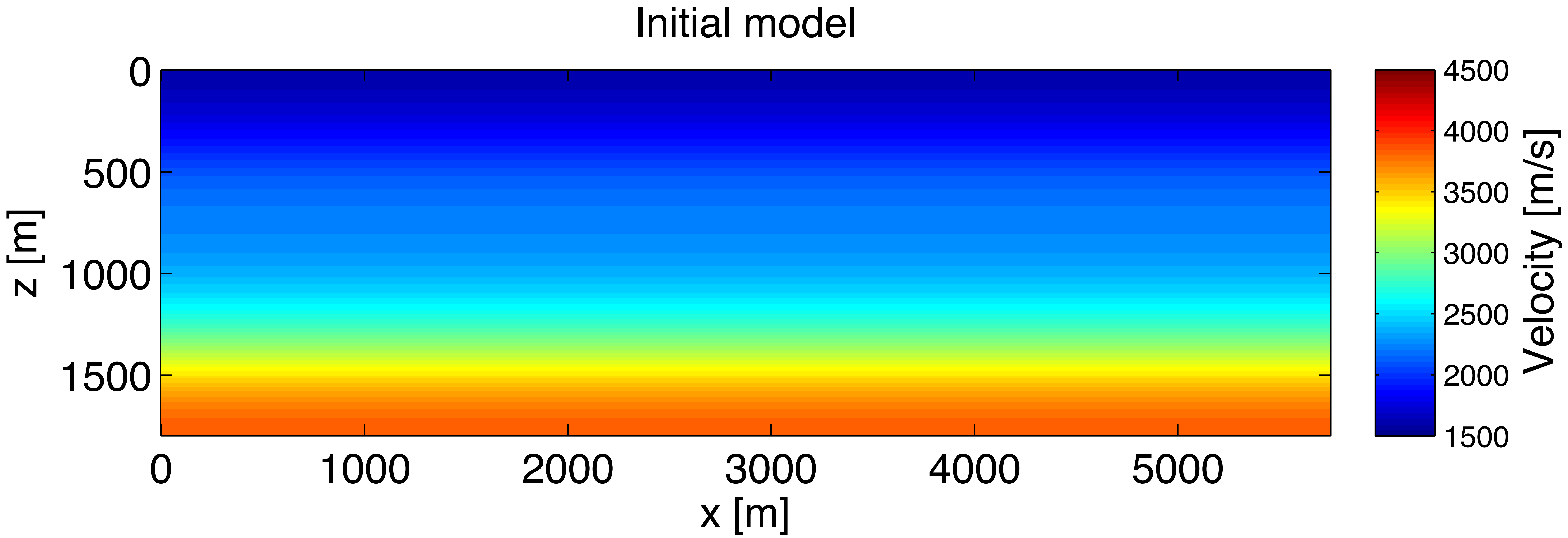
dense
&
too expsensive

Example – BG Compass model

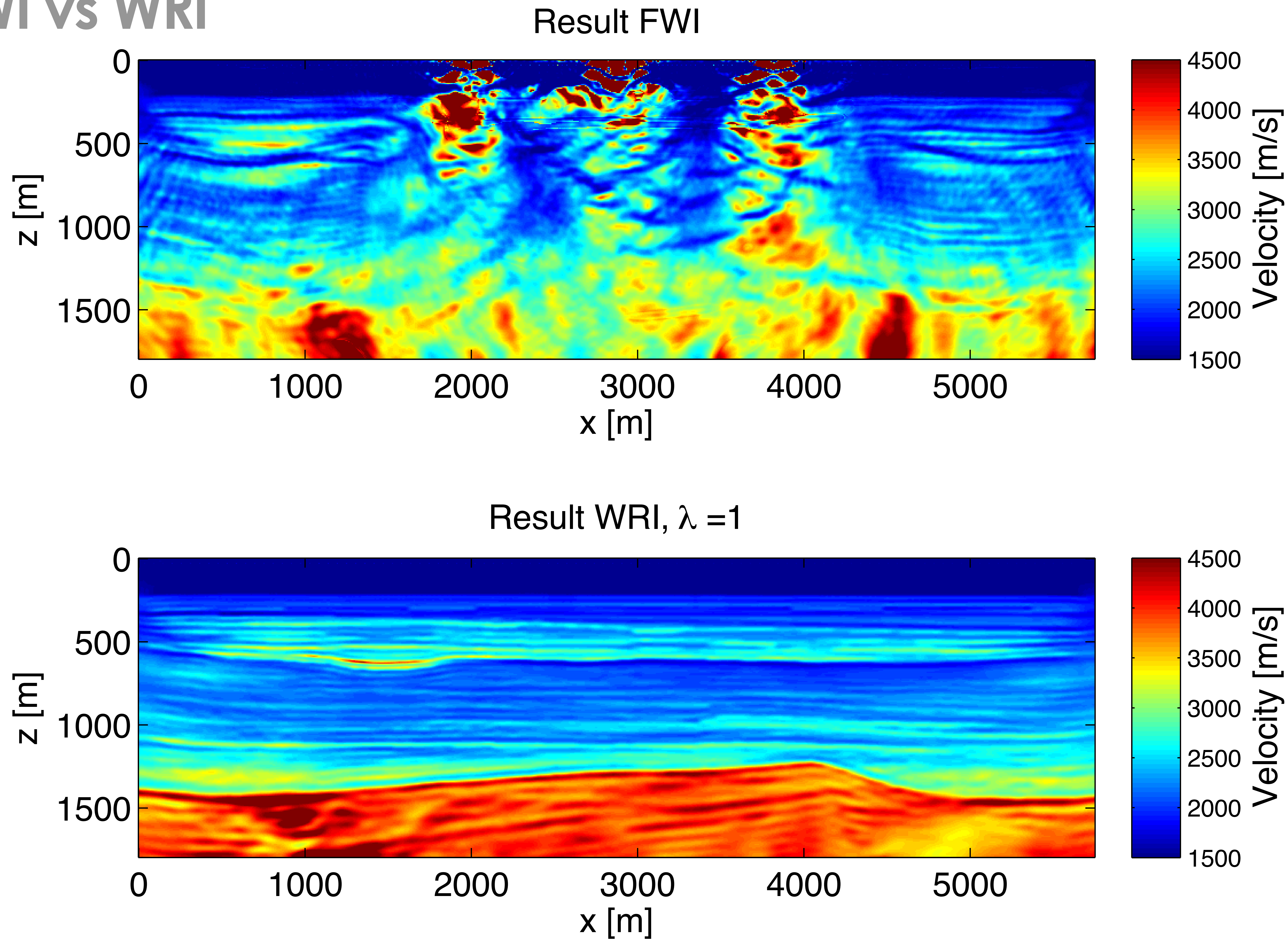
- Low frequencies missing, 24 frequency batches (15 iterations each) {5 6} ,{6 7},... ,{28 29} Hertz. Each interval contains 5 frequencies.
- 103 sources/receivers w/ 55m sample interval
- Inaccurate initial model

True & initial model



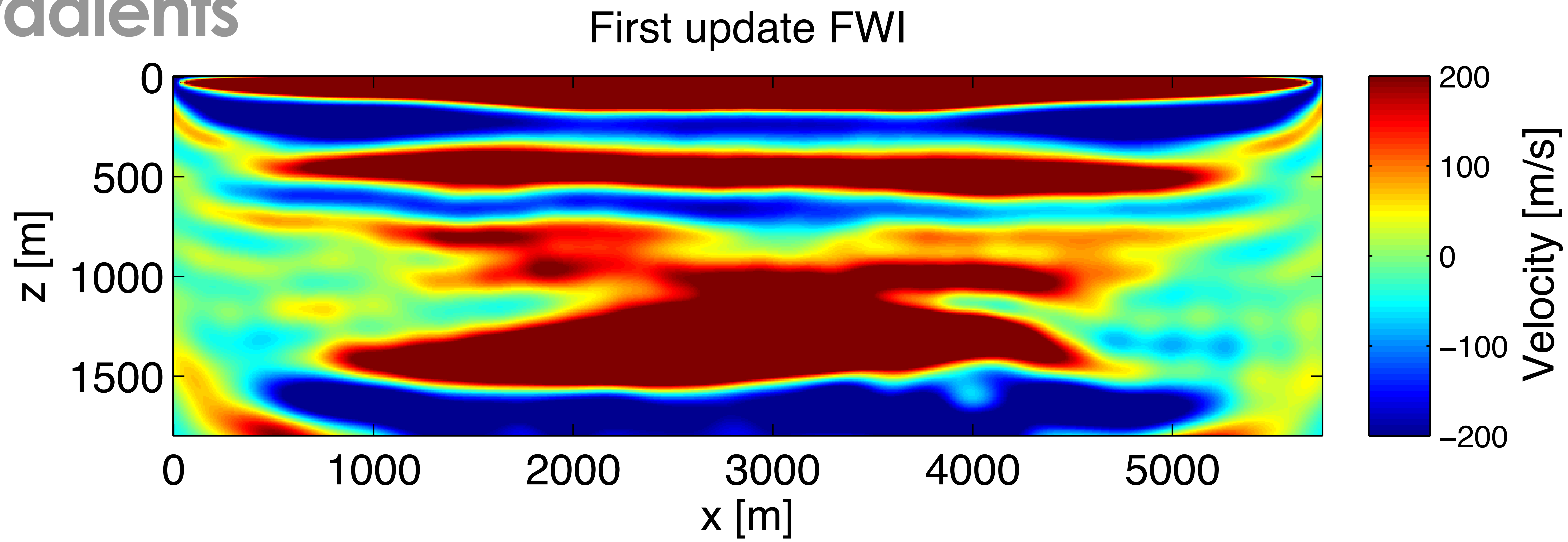


FWI vs WRI

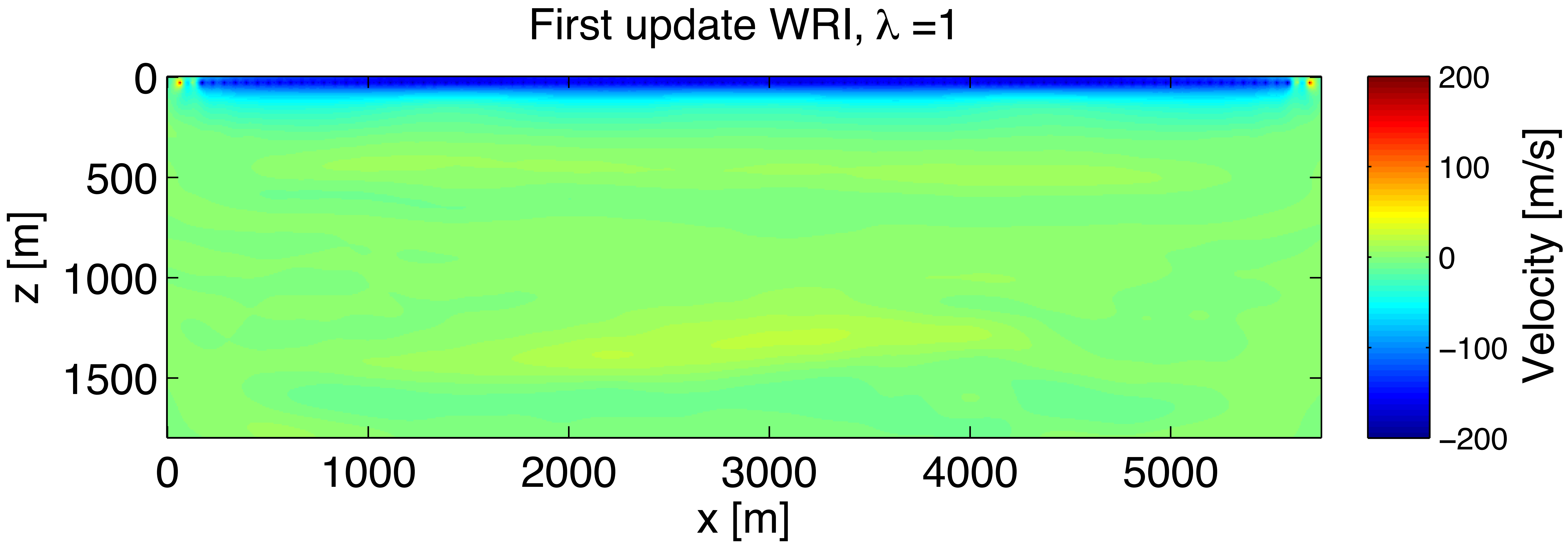


Result FWI
0
500
1000
1500
z [m]
0
1000
2000
3000
4000
5000
x [m]
4500
4000
3500
3000
2500
2000
1500
Velocity [m/s]
Result WRI, λ =1
0
500
1000
1500
z [m]
0
1000
2000
3000
4000
5000
x [m]
4500
4000
3500
3000
2500
2000
1500
Velocity [m/s]

Gradients

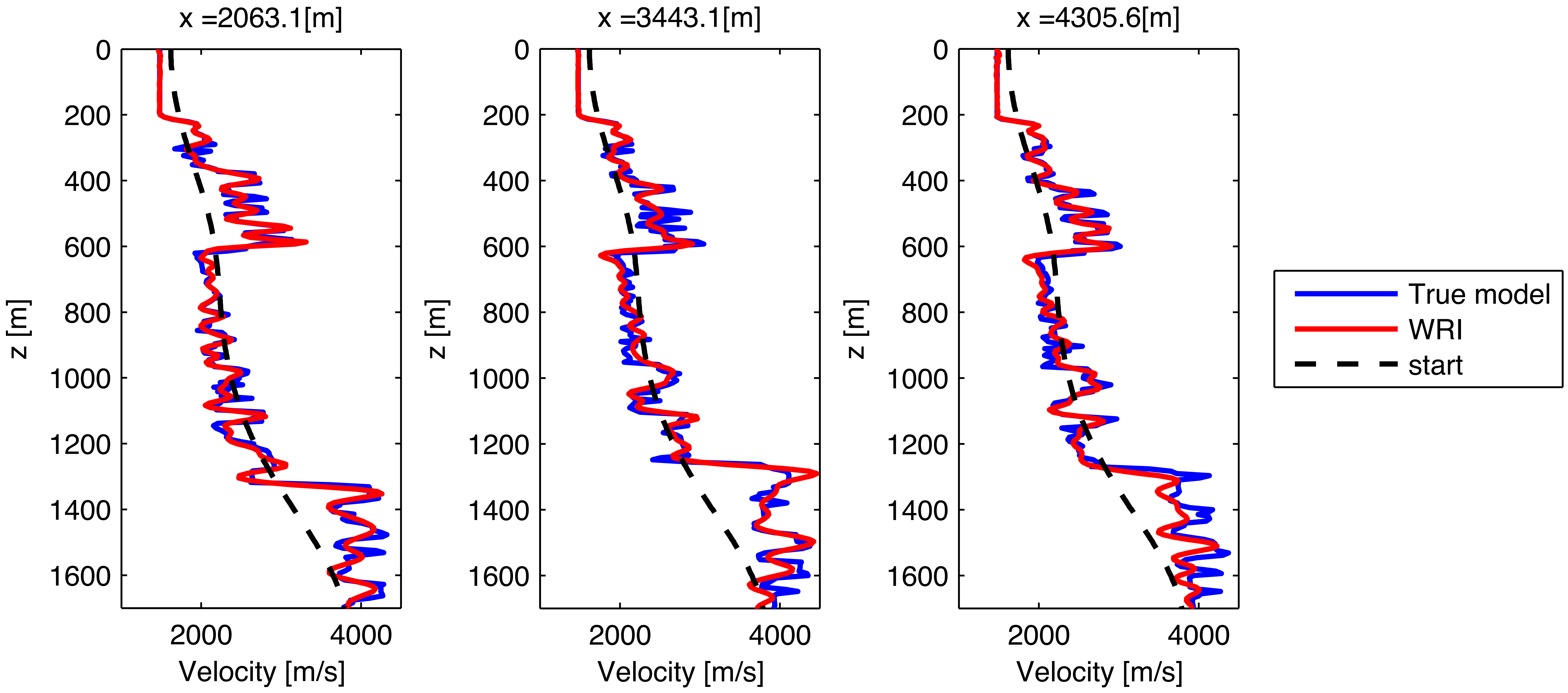


First update FWI
z [m]
0
500
1000
1500
0
1000
2000
3000
4000
5000
x [m]
200
100
0
−100
−200
Velocity [m/s]

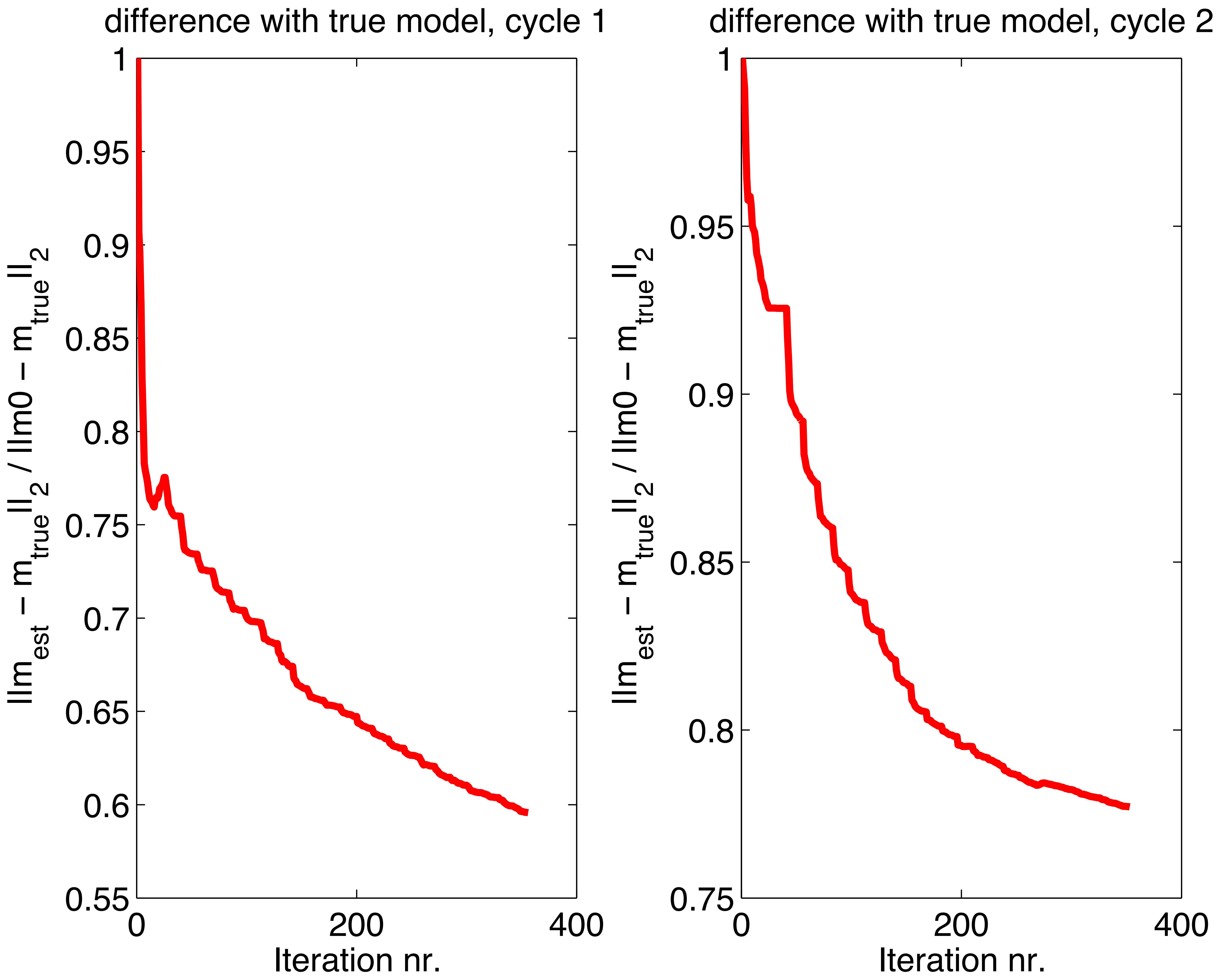


First update WRI, λ =1
z [m]
0
500
1000
1500
0
1000
2000
3000
4000
5000
x [m]
200
100
0
−100
−200
Velocity [m/s]

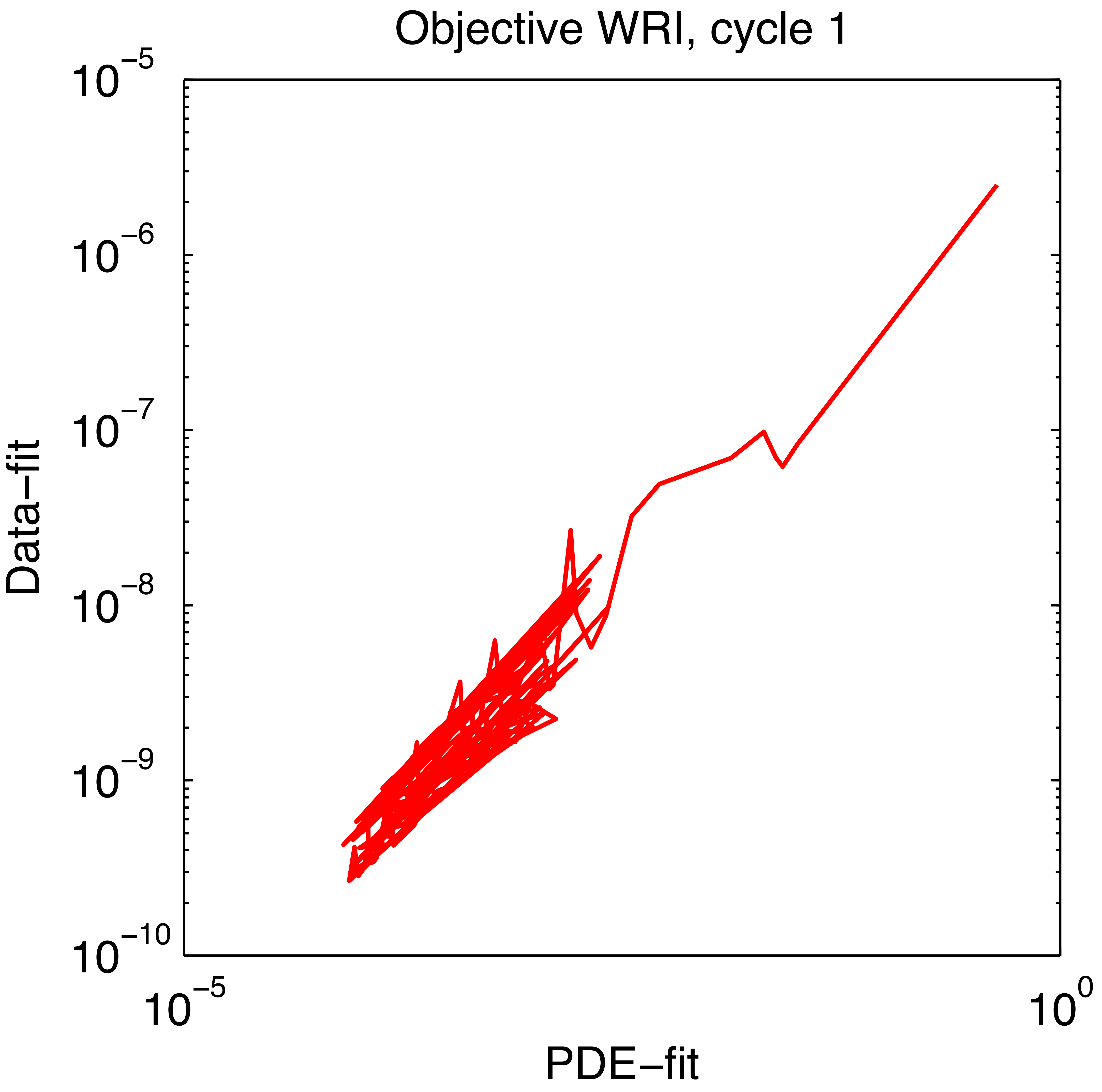
Cross sections

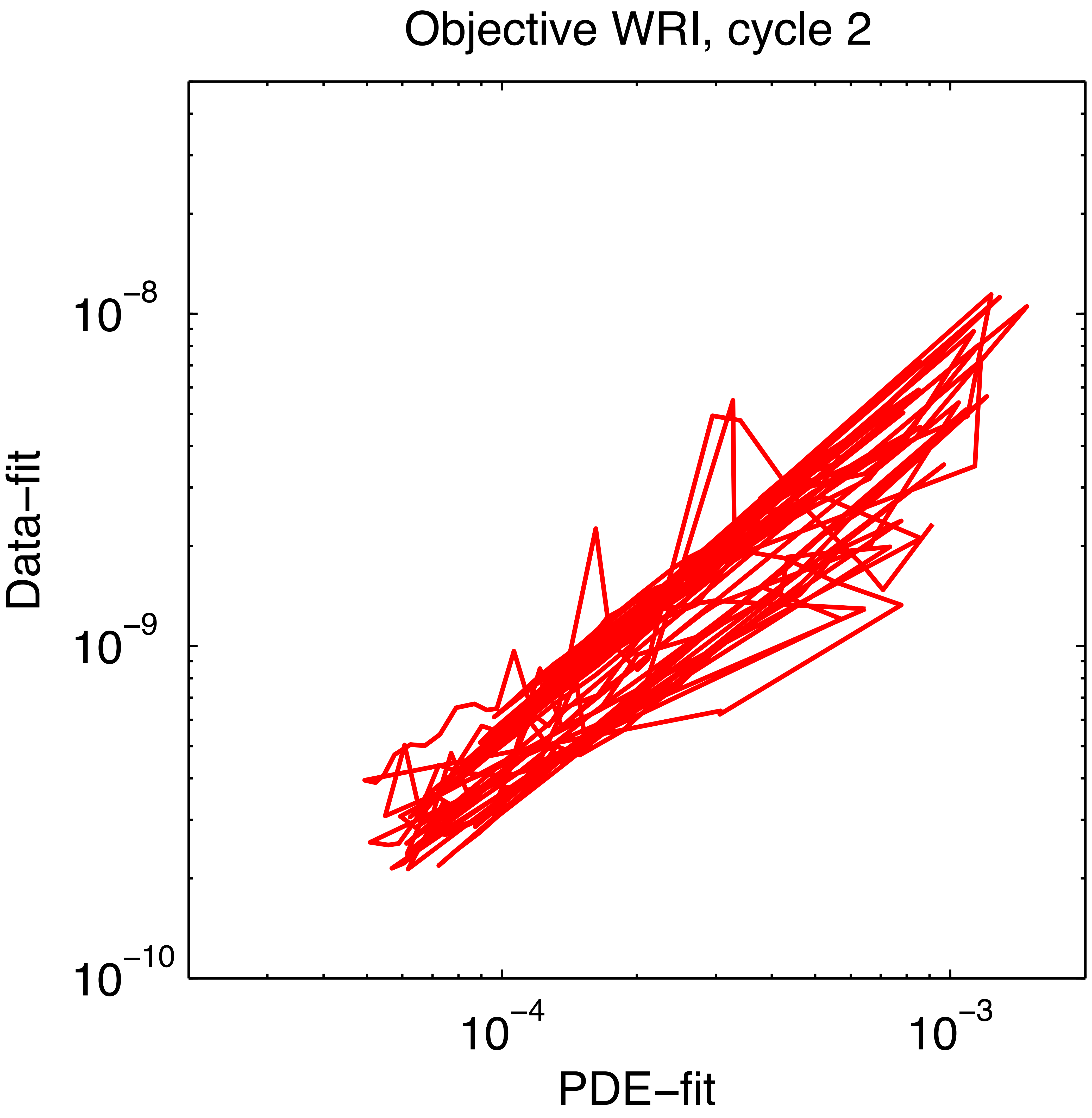


Relative model errors



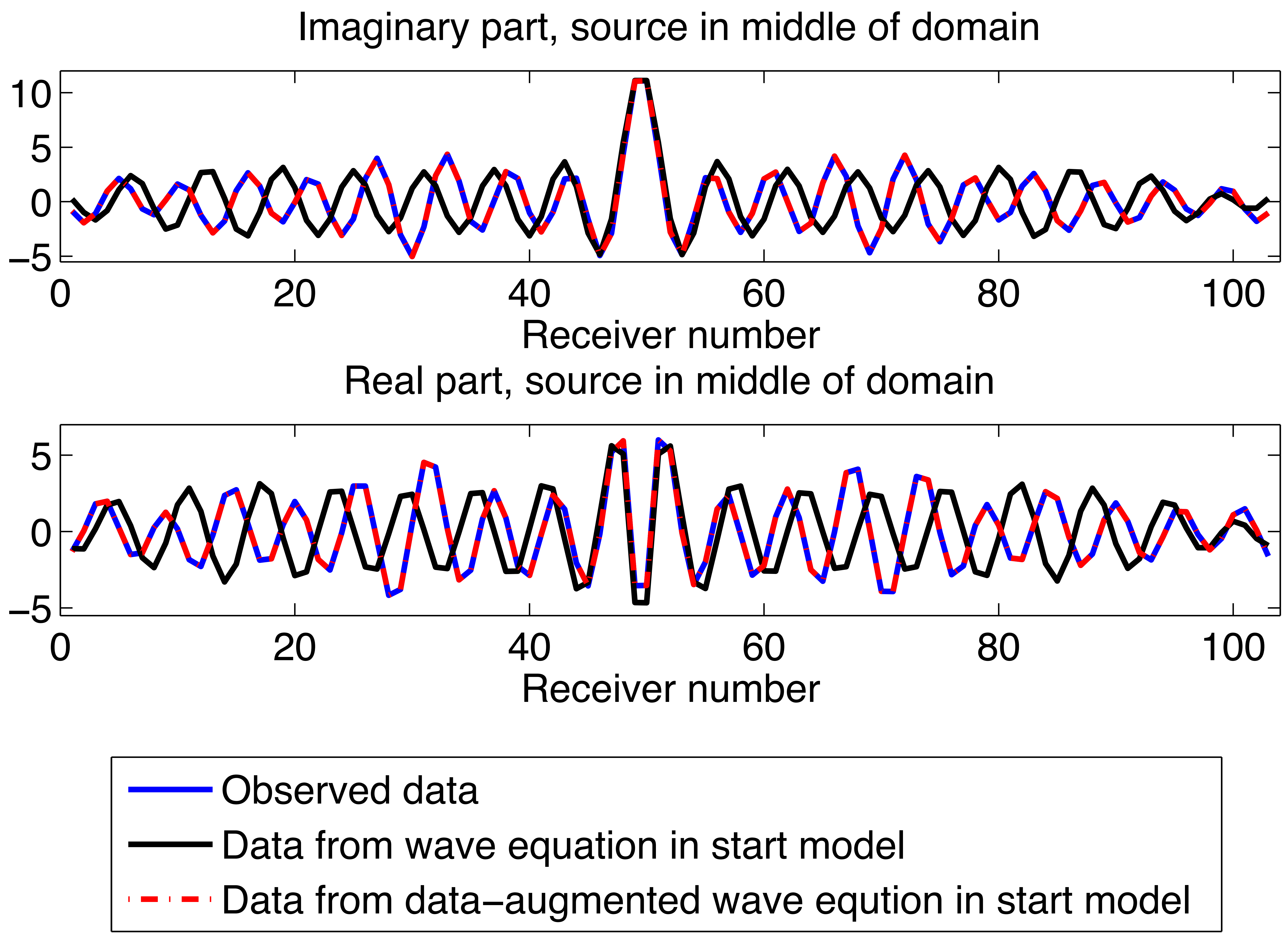
Objective function value





Data fit increases at some iterates

Data fit



Imaginary part, source in middle of domain
10
5
0
−5
0
20
40
60
80
100
Receiver number
Real part, source in middle of domain
5
0
−5
0
20
40
60
80
100
Receiver number
Observed data
Data from wave equation in start model
Data from data−augmented wave eqution in start model

Ernie Esser, Tristan van Leeuwen, Aleksandr Y. Aravkin, and Felix J. Herrmann, "**A scaled gradient projection method for total variation regularized full waveform inversion**". 2014.
Bas Peters and Felix J. Herrmann, "**A sparse reduced Hessian approximation for multi-parameter Wavefield Reconstruction Inversion**". 2014.

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Extensions

Total-variation regularization

via scaled gradient projections & bound constraints

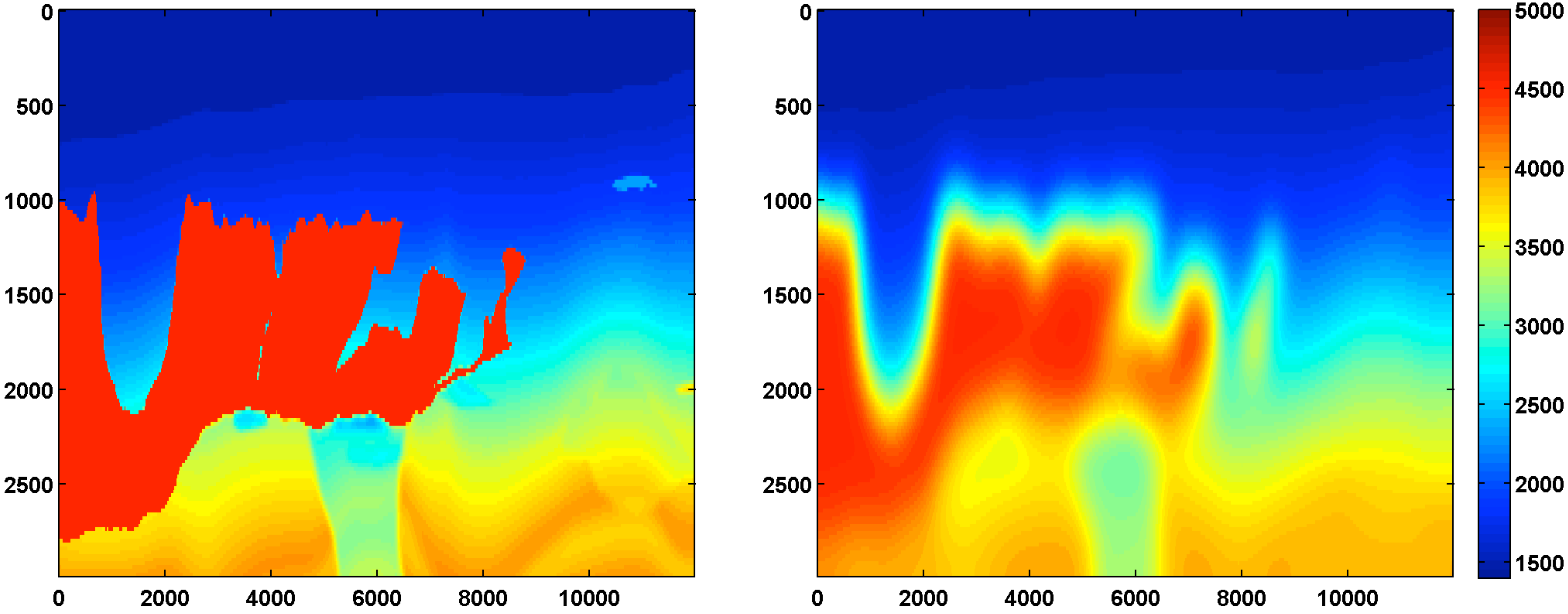
Multi-parameter case

via sparse approximate Gauss-Newton scaling

BP model

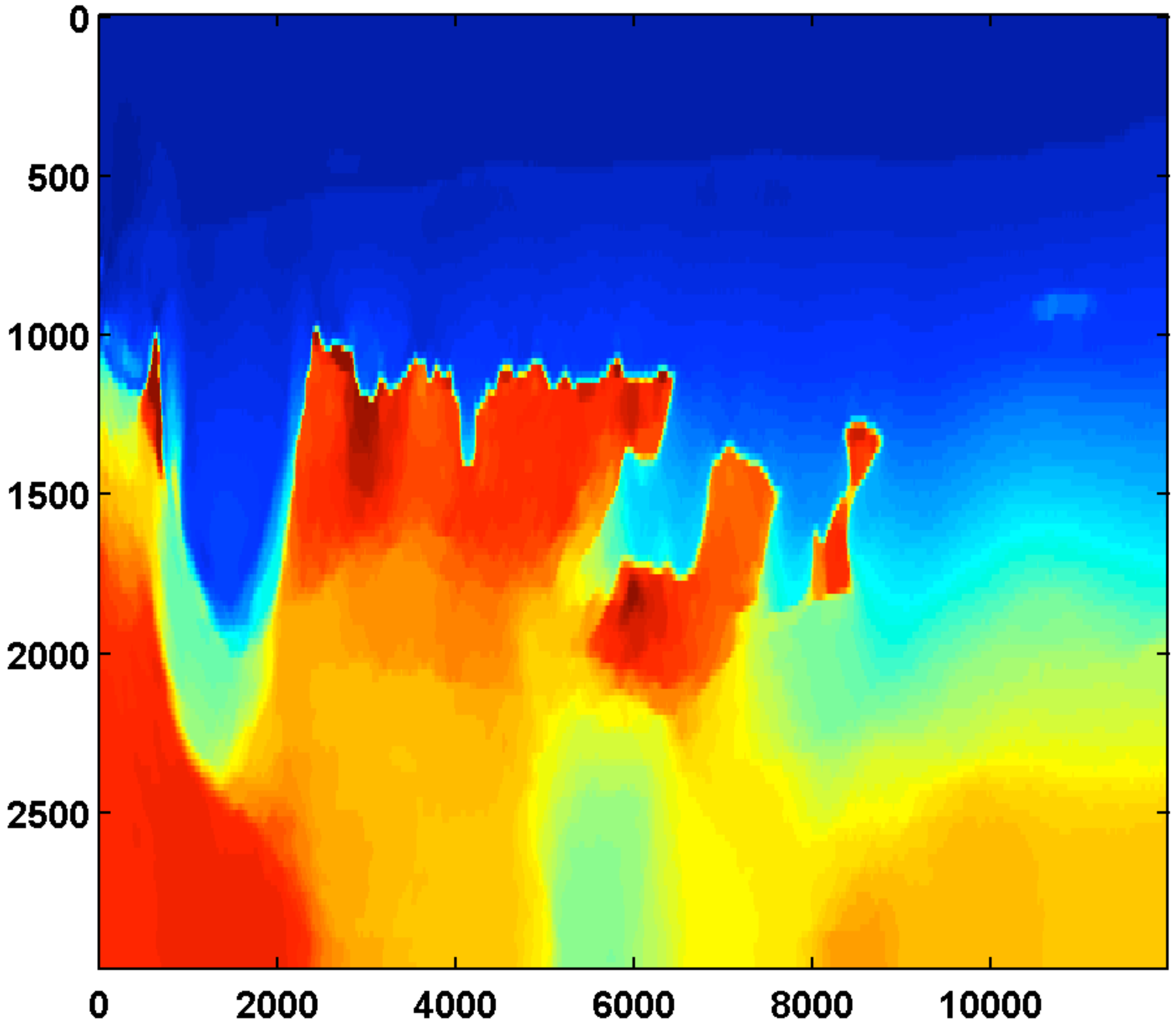
- number of sources: 126 (starting 1000m in from boundary)
- number of receivers: 299
- frequency range: 3-20Hz in overlapping batches of 2
- maximum number of outer iterations per frequency batch: 25
- maximum number of inner iterations for convex subproblems: 2000
- known Ricker wavelet sources with 15Hz peak frequency
- two simultaneous shots with Gaussian weights w/ redraws
- no added noise

True and initial velocity

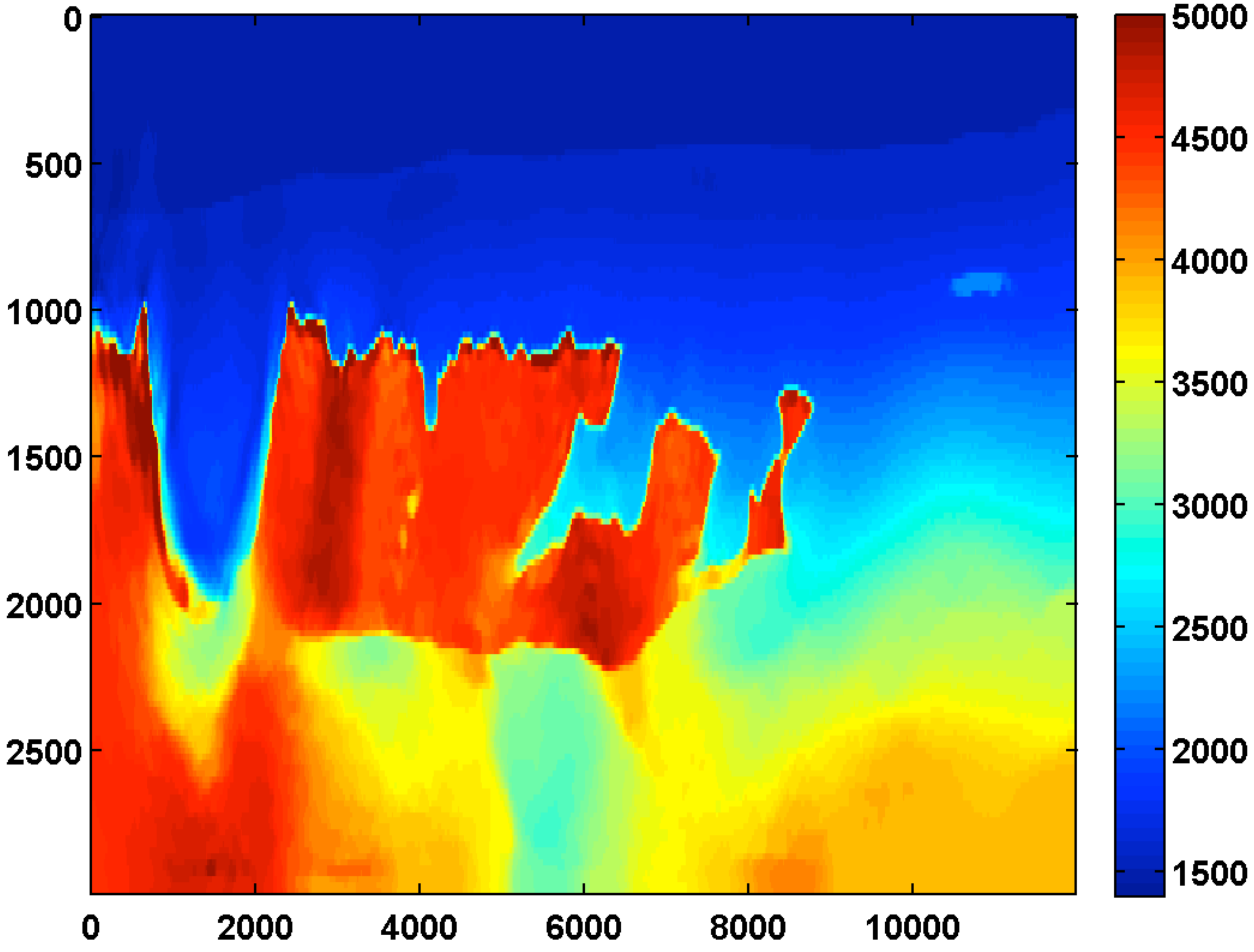


Results w/ TV

After one cycle through the frequencies



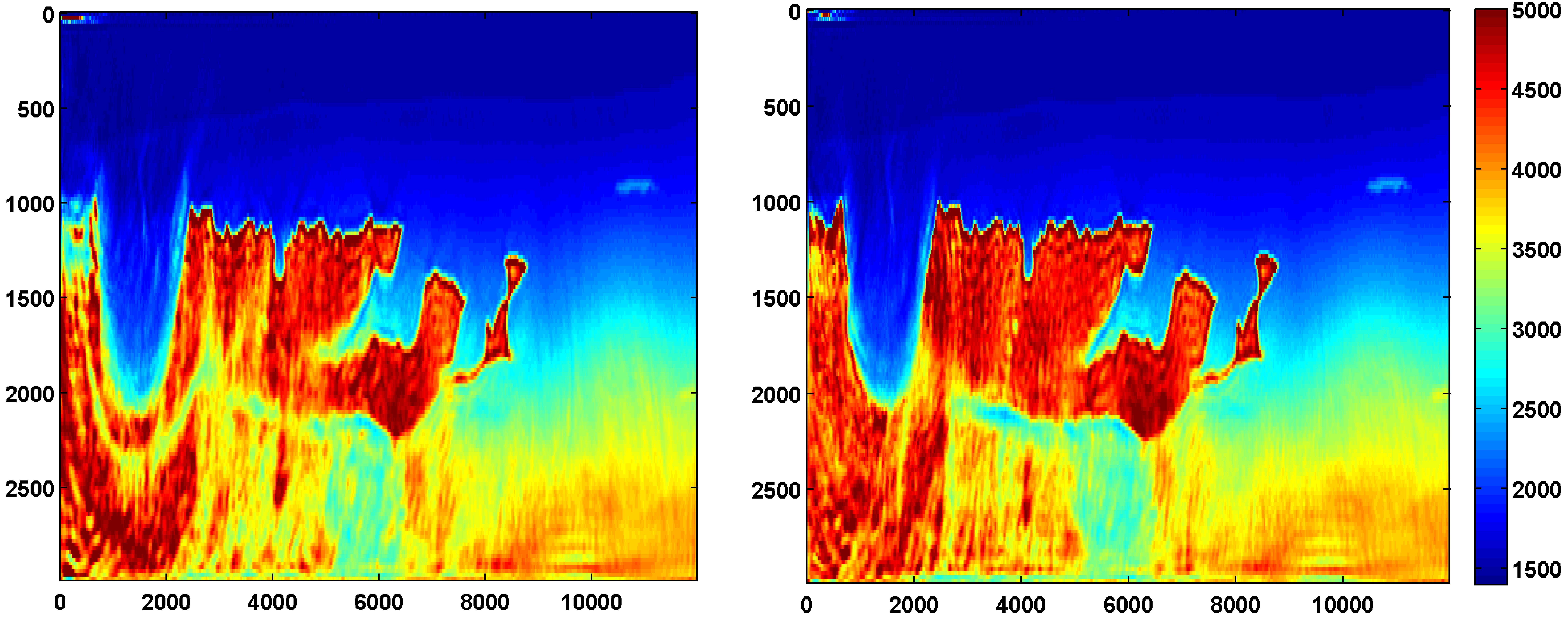
After two cycles through the frequencies



Results w/o TV

After one cycle through the frequencies

After two cycles through the frequencies



Conclusions

New alternating method for wave-equation based inversion:

- same extended search space as in all-at-once but with memory & CPU requirements as in adjoint-state approach
- no adjoints & sparse GN-Hessian approximation
- less susceptible to local minima due to data fit
- sparse GN Hessians
- bilinear

Challenge: Stationary points are not necessary global minima

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Acknowledgements

Thank you for your attention !

https://www.slim.eos.ubc.ca/

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