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An iterative multilevel method for computing wavefields in frequency-domain seismic inversion

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Outline of the Talk

Objective: iterative method for frequency-domain wavefield computation

- acoustic (Helmholtz) equation
- Helmholtz solver
  - direct vs. iterative methods
  - problems with iterative methods
- First ingredient: Preconditioning for Krylov methods
  - Shifted Laplacian
- Second ingredient: Multilevel Krylov (MK) methods
  - MK on preconditioned Helmholtz
- Numerical examples
- Conclusions

Frequency-domain acoustic wave equation

$$\mathcal{H}u = -\rho(x)\nabla \cdot \left(\frac{1}{\rho(x)}\nabla u(x)\right) - \omega^2(x)u(x) = b$$

$$\downarrow$$

$$H[\omega]u = b$$

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Illustration

Frequency domain seismic imaging

For one shot, one frequency:

- forward modeling: $Hu = b \longrightarrow u = H^{-1}b$
- back-propagation: $H^*v = \delta u \longrightarrow v = (H^*)^{-1}\delta u$
- imaging: $\delta m = \mathrm{Re}(u \odot v)$

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How to compute u and v?

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How to compute u and v?

Factorization of H into upper and lower triangular matrix

Also known as direct methods or Gaussian-elimination based methods

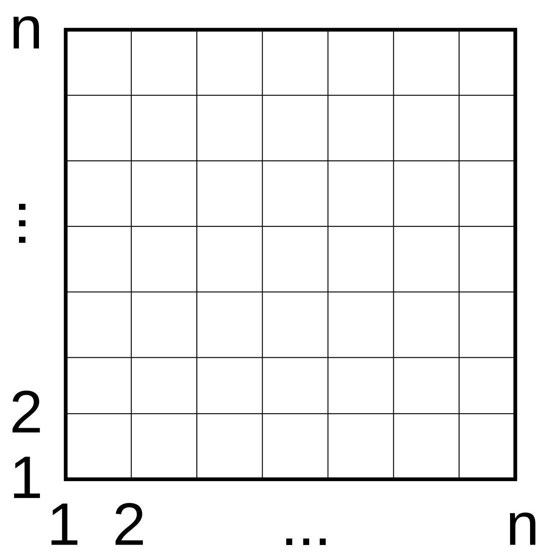
Direct Methods

Cost of computing u (and v):

$\mathcal{O}(n^{d^d})$, $d = 2, 3$

Memory:

$\mathcal{O}(n^{d-1}n^d)$, $d = 2, 3$



Typical problems, $n = 10^3$.

In 2D:

- Computational cost: $\sim 10^{12}$ flops
- Memory: $\sim 10^9$ (1 Giga) units

In 3D:

- Computational cost: $\sim 10^{27}$ flops
- Memory: $\sim 10^{15}$ (1 Peta) units

In 2D, still possible, but not in 3D!

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Illustration

Is it possible to replace direct methods?

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Yes! use iterative methods

Example:

- basic iterative methods: Jacobi, Gauß-Seidel
- Krylov methods: CG, GMRES, Bi-CGSTAB, etc

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Illustration

Is it possible to replace direct methods?

Yes! use iterative methods

Example:

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Facts:

- Computational cost: $\mathcal{O}(n^d)$ flops per iteration, in total, $n_{iter}\mathcal{O}(n^d)$
- Memory: $\mathcal{O}(n^d)$ units

- not robust methods

  Worst case, $n_{iter} = \mathcal{O}(n^d) \quad \longrightarrow \quad$ computational cost: $\mathcal{O}(n^{d+2})$ flops

  Ideally, $n_{iter} = \mathcal{O}(1) \longrightarrow$ computational cost: $\mathcal{O}(n^d)$

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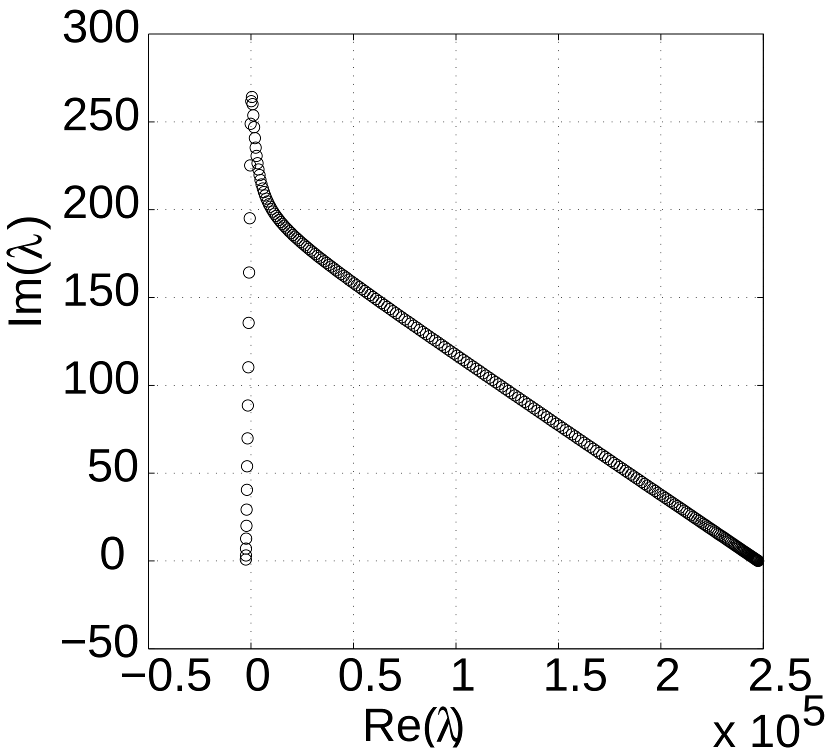
From theory:

Relation between convergence of iterative methods and eigenvalues of H

1D Helmholtz equation, $k := 2\pi fL/c = 50$.



- Small eigenvalues close to zero
- Large eigenvalues unbounded

  Ill-conditioned

- Real parts of eigenvalues:

  change signs → Indefinite

Slow convergence or divergence

Tackling ill-conditioning:

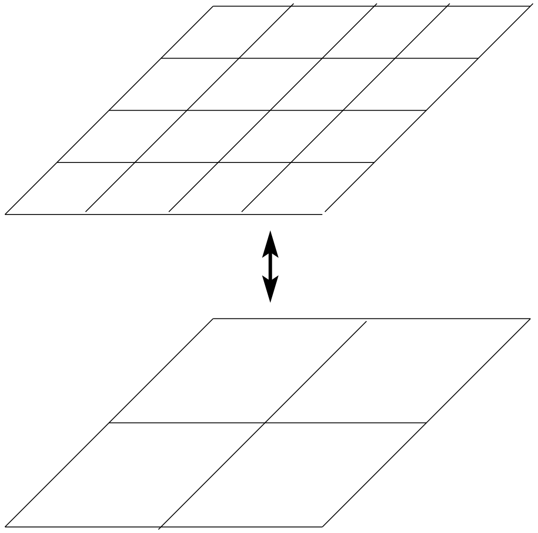
- use preconditioning matrix M
  eigenvalues of HM^{-1} are more clustered

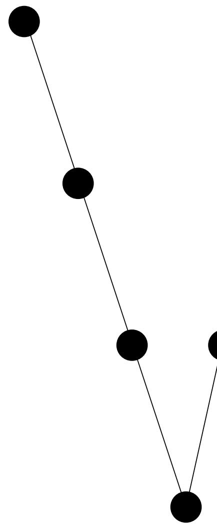
  If $M^{-1} \approx H^{-1}$, eigenvalues of HM^{-1} are close to 1 $\longrightarrow$ fast convergence

- use multilevel technique
  error components associated with small eigenvalues are corrected on the coarser grid

Indefiniteness is the most difficult part to handle!

Solve $Hu = b$ on a hierarchy of grids of different mesh size.





Multigrid (MG):

- reduce nonsmooth errors by basic iterative methods: Jacobi/Gauß-Seidel
- reduce smooth errors on the coarse grid (smaller system)

Efficient methods ($\mathcal{O}(n^d)$ methods) for non-indefinite systems

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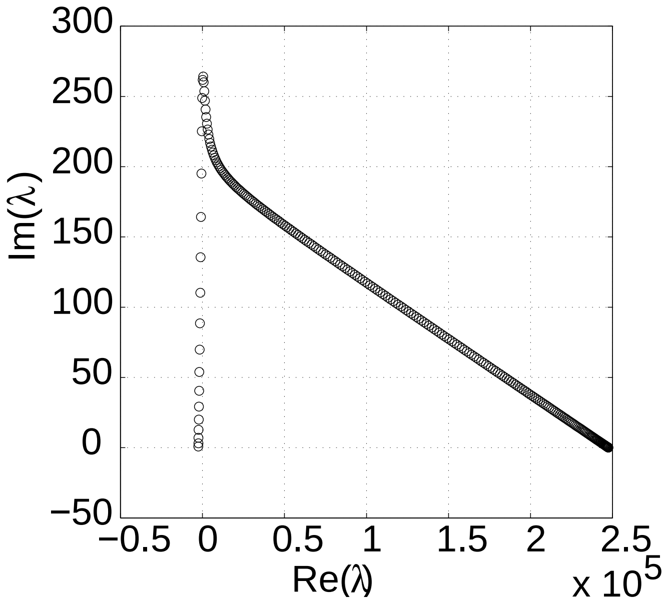
Preconditioner: tackling "indefiniteness"

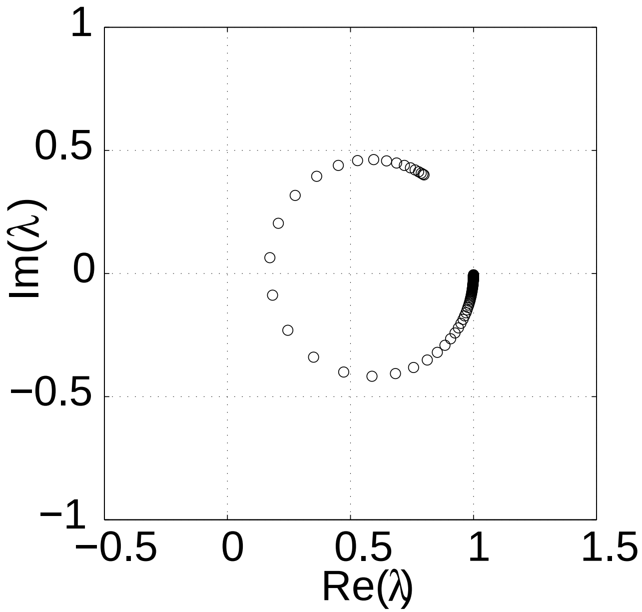
First step: preconditioner which makes indefinite system definite

In this case, use shifted-Laplace (damped Helmholtz) operator:

$$M \triangleq -\rho(x)\nabla\cdot\left(\frac{1}{\rho(x)}\nabla\right) - (1-\frac{1}{2}\hat{j})\omega^2(x), \quad \hat{j}=\sqrt{-1}.$$

Example: 1D Helmholtz, $k := \underline{2\pi fL/c} = 50.$





[E., Oosterlee, Vuik, 2006]

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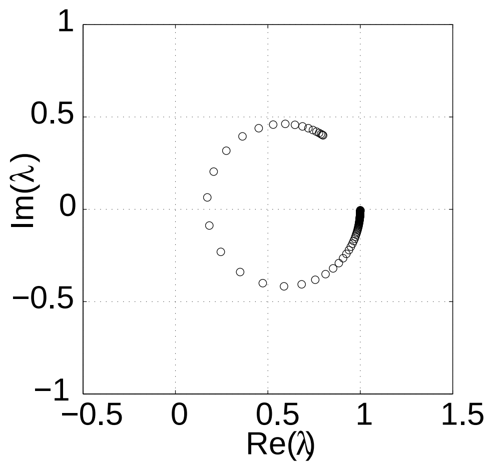
Preconditioner: tackling "indefiniteness"

Important aspects:

- the system HM^{-1} becomes definite
  Krylov methods should converge easier, meaning $n_{iter} < n^d$
- M is favorable for multigrid (due to damping term)
  M^{-1} is computed by one multigrid iteration $\longrightarrow \mathcal{O}(n^d)$ for preconditioning
- Computational cost: $n_{iter}\mathcal{O}(n^d)$

- Largest eigenvalues are bounded above by 1
- Still ill-conditioned (see small eigenvalues)



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Example with Marmousi velocity model

Frequency (Hz)	1	10	20	30
w/o M	17445	6623	14687	–
with M	39	54	98	144

Applications:

- 2D: [Riyanti, E., Plessix, Mulder, Vuik, Oosterlee, 2006]
  [Duff, Gratton, Pinel, Vasseur]
- 3D: [Riyanti, Kononov, Erlangga, Plessix, Mulder, Vuik, Oosterlee, 2006]

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Multilevel Krylov: tackling "ill-condition"

Problem remained with HM^{-1}: ill conditioning

Second step: make this system well-conditioned

Solution:

use operator which shifts small eigenvalues to the largest eigenvalue but keeps the upper bound the same

This is possible with the multilevel operator

$$Q = \overbrace{I - Z\widehat{H}^{-1}Z^T HM^{-1}}^{\text{shift small eigenvalues to 0}} + \overbrace{Z\widehat{H}^{-1}Z^T}^{\text{shift zero eigenvalues to 1}}, \quad \widehat{H} = Z^T HM^{-1}Z,$$

with Z linear mapping from coarse grid to fine grid.

$\widehat{H}$ implicitly contains information of small eigenvalues to be shifted.

[E., Nabben, 2007]

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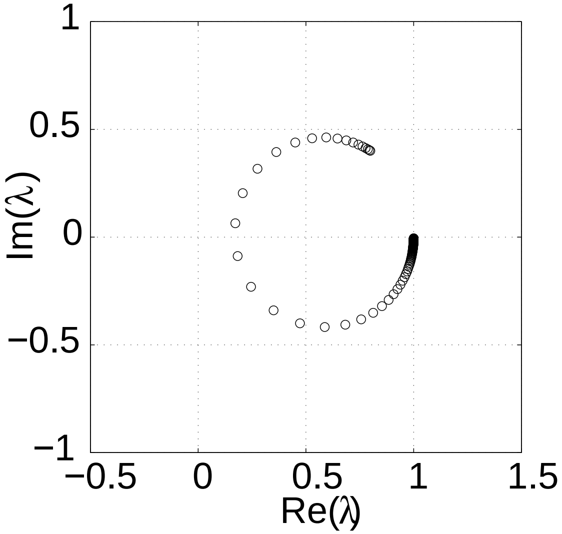
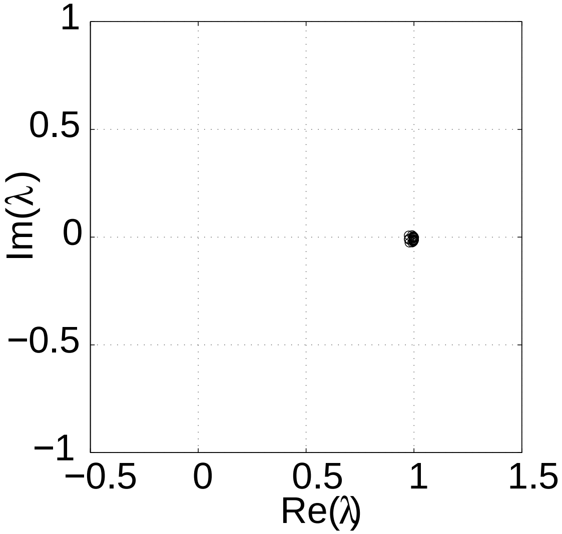
Example: 1D Helmholtz, $k := 2\pi fL/c = 50$.

HM^{-1}



$HM^{-1}Q$



Notice shift of small eigenvalues towards one.

This is very favorable for fast convergence of Krylov methods

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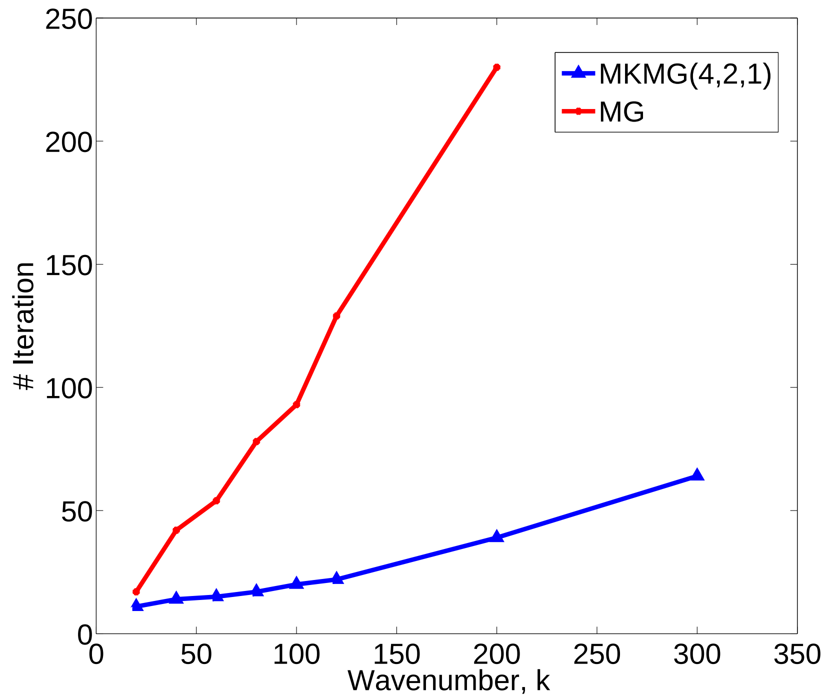
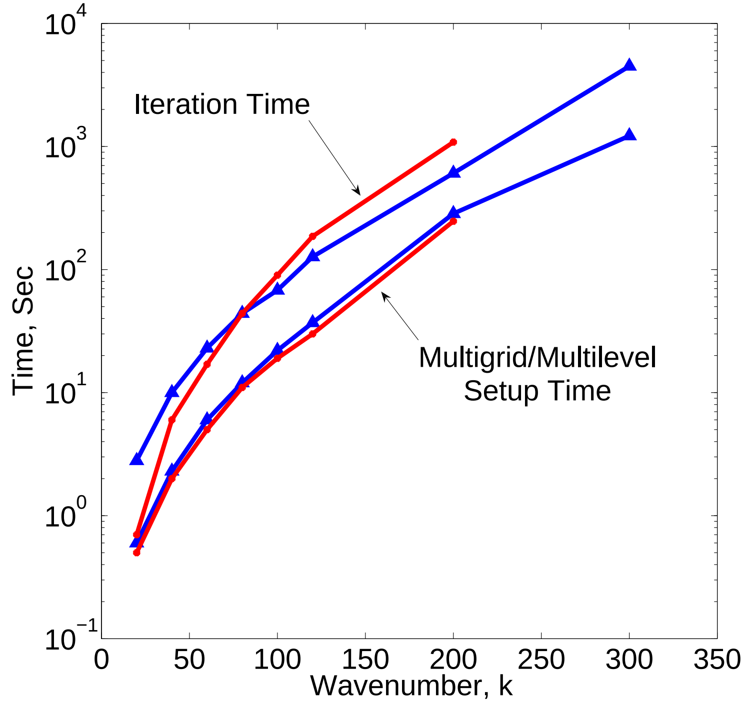
Numerical experiments: constant wavenumber

Improvement in #iterations and CPU time
High wavenumbers $k := 2\pi fL/c$
Grid resolution: 15 points per wavelength





$k = 300$: equivalent to $f = 60$ Hz for typical 1000 m depth. 750^2 grid points.

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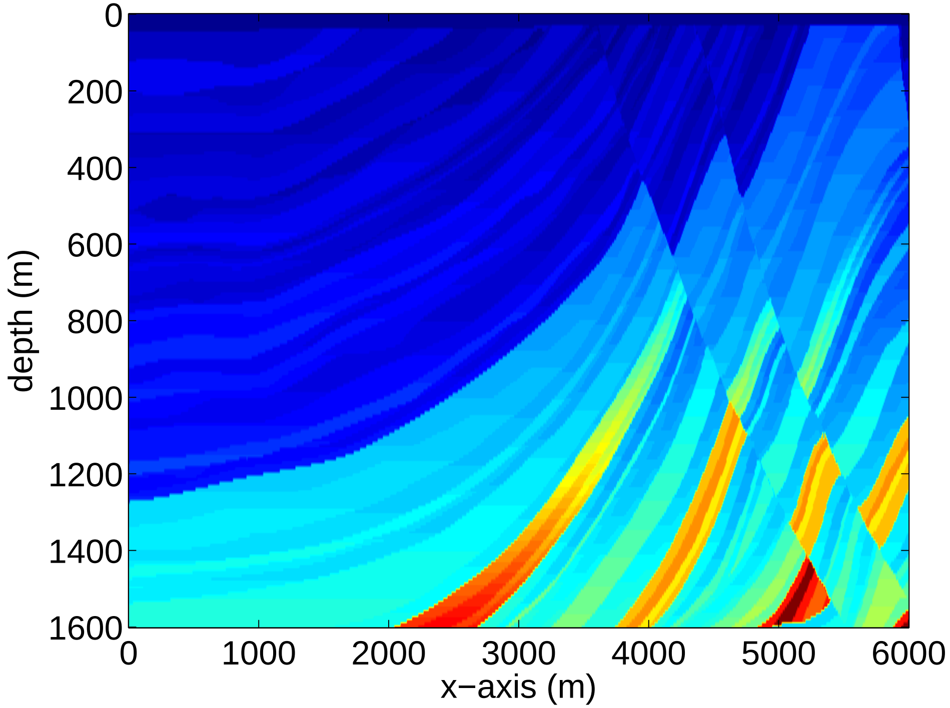
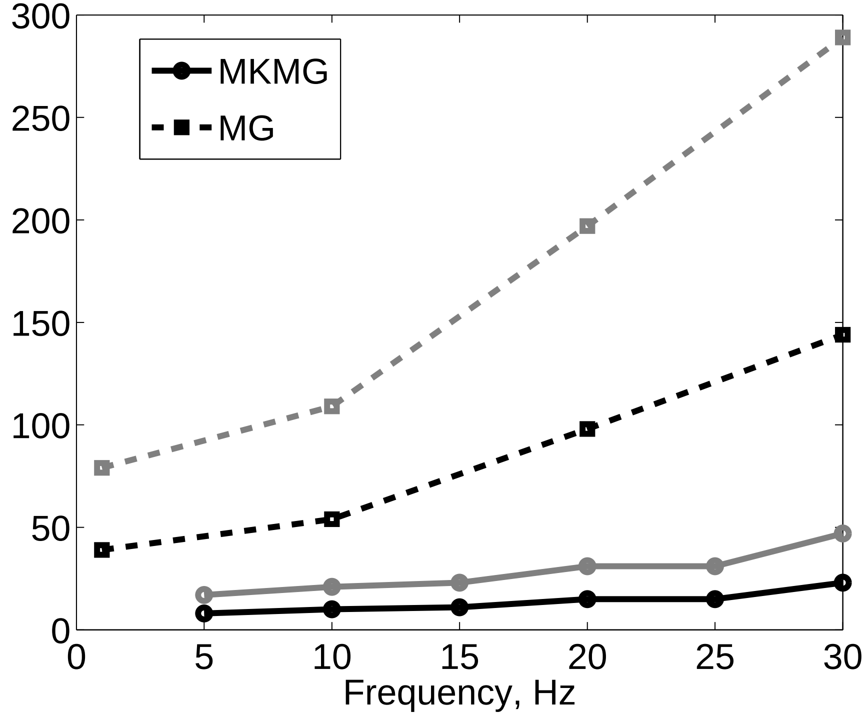
Numerical experiments: Marmousi

(—): No. of iterations. (—): No. of matrix-vector multiplies.





MKMG : GMRES on $HM^{-1}Q\widetilde{u} = b$.

MG : Bi-CGSTAB on $HM^{-1}\widehat{u} = b$ [E., Oosterlee, Vuik, 2006]

Convergence can be made less independent of frequencies

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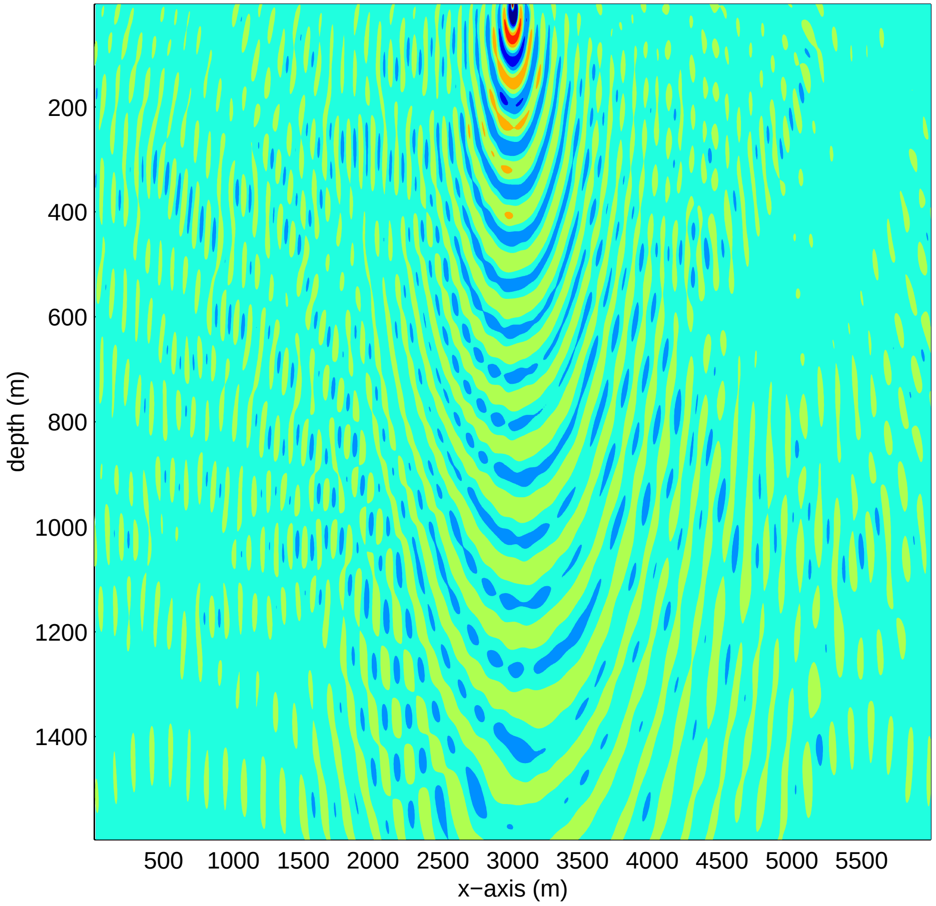
Numerical experiments: Marmousi

Computed wavefield, $f = 20$ Hz



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Influence of gridsize to the convergence

Frequency (Hz)	5	10	15	20	25	30
Grid adapted to f	8	10	11	15	15	23
Grid fixed at $f = 30$ Hz	8	8	11	12	15	23

Convergence can be made independent of grid size

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Impact on the computational complexity

Frequency-domain:

- For one frequency, one shot: $n_{iter} = \mathcal{O}(1) \quad \rightarrow \mathcal{O}(n^d)$ flops.
- For multiple frequencies and shots: $n_{freq} n_{shot} \mathcal{O}(n^d)$ flops.
- Assuming $\mathcal{O}(n_{freq}) = \mathcal{O}(n)$ and $\mathcal{O}(n_{shot}) = \mathcal{O}(n^{d-1}) \rightarrow \mathcal{O}(n^{2d})$

Time-domain:

- Simulation with n_t time levels: $n_t n_{shot} \mathcal{O}(n^d) = \mathcal{O}(n^{2d}) \qquad (n_t = \mathcal{O}(n))$.

Frequency- and time-domain wavefield computations are at the same order of computational complexity!

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Impact on the computational complexity

Frequency-domain wavefield simulation:

- conducive to frequency subsampling [Lin, Lebed, E., Herrmann, this conference]
  - use of $n_{freq} \ll n$ (or $n_{freq} = \mathcal{O}(1)$)
  - full wavefield recovered by (ℓ_1 minimization) sparsity-promoting program

Frequency-domain imaging:

- no "time history" – less memory requirement

  in time-domain, check-pointing [Symes, 2008]
- conducive to frequency subsampling
  - [Plessix, Mulder, 2004], ...
  - (ℓ_1 minimization) sparsity-promoting program

Conducive to simultaneous shots simulation [Herrmann, E., Lin, 2008]

Refer to talk by Lin.

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Conclusion

- key aspect of successful iterative methods for Helmholtz equation: tackling indefiniteness and ill-conditioning
- indefiniteness: use shifted-Laplacian (damped Helmholtz)
- ill-conditioning: use multilevel Krylov method

- combined, convergence less independent of frequencies and grid size

- paving the way to
  - compressive full wavefield computation (use of subsets of frequencies)
  - more rigorous compressive seismic imaging

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