Compressed Wavefield Extrapolation with Curvelets

SLIM
Seismic Laboratory for Imaging and Modeling

Tim T.Y. Lin and Felix J. Herrmann
University of British Columbia

SEG 2007
San Antonio, Sept 25

Introduction

- Concerned with *explicit* forms of wavefield propagator **W** of the linearized forward model

$$P^- = \sum_{x_3 > 0} W^- R^+ W^+ s^+ \Delta x_3$$

- Would like to find explicit **W** suitable for wave-equation migration:
  - simultaneously operates on sets of traces
  - fully incorporates velocity information of medium
  - no parabolic approximations

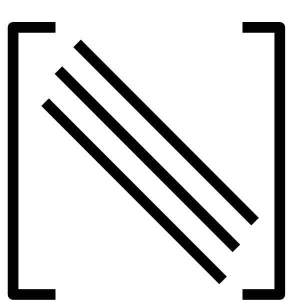
Introduction

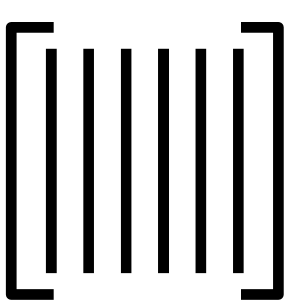
- Goal: employ the complete 1-Way Helmholtz operator for W

Grimbergen, J., F. Dessing, and C. Wapenaar, 1998, Modal expansion of one-way operator on laterally varying media: Geophysics, **63**, 995–1005.

$$\mathbf{W}^{\pm} = e^{\mp j \Delta x \mathbf{H}_1} \qquad \mathbf{H}_2 = \mathbf{H}_1 \mathbf{H}_1$$

- Problem: computation & storage complexity
  - creating and storing $\mathbf{H}_2$ is trivial
  - however $\mathbf{H}_1$ is *not* trivial to compute and store

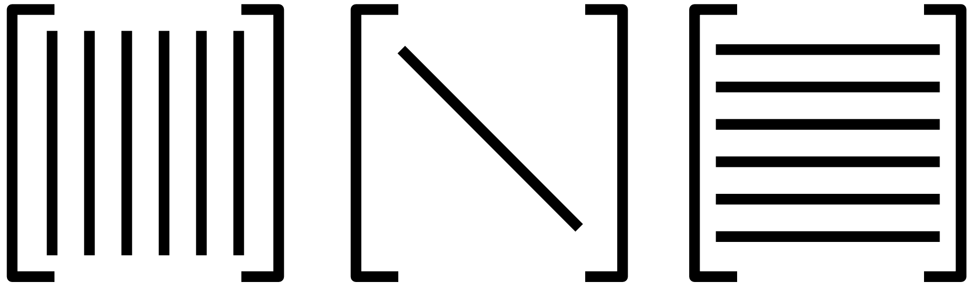
$\mathbf{H}_2 =$ 

$\mathbf{H}_1 =$ 

Introduction

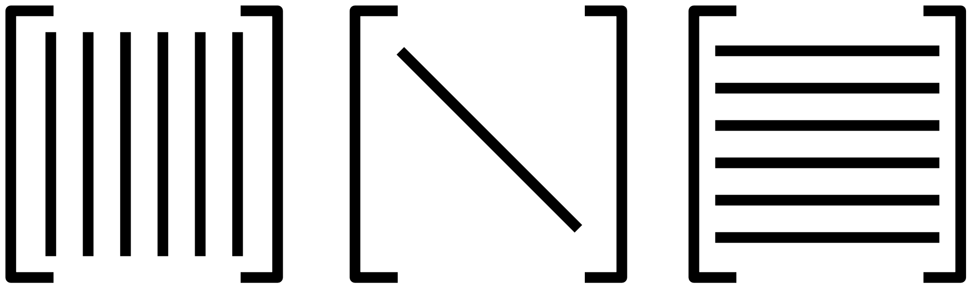
- In this case **W** is computed by eigenvalue decomposition

$$\mathbf{H}_2 = \mathbf{L}\mathbf{\Lambda}\mathbf{L}^{\mathrm{T}} =$$



$$\mathbf{L} \quad \mathbf{\Lambda} \quad \mathbf{L}^{\mathrm{T}}$$

$$\mathbf{W}^{\pm} =$$



$$\mathbf{L} \quad e^{-j\sqrt{\mathbf{\Lambda}}\Delta x_3} \quad \mathbf{L}^{\mathrm{T}}$$

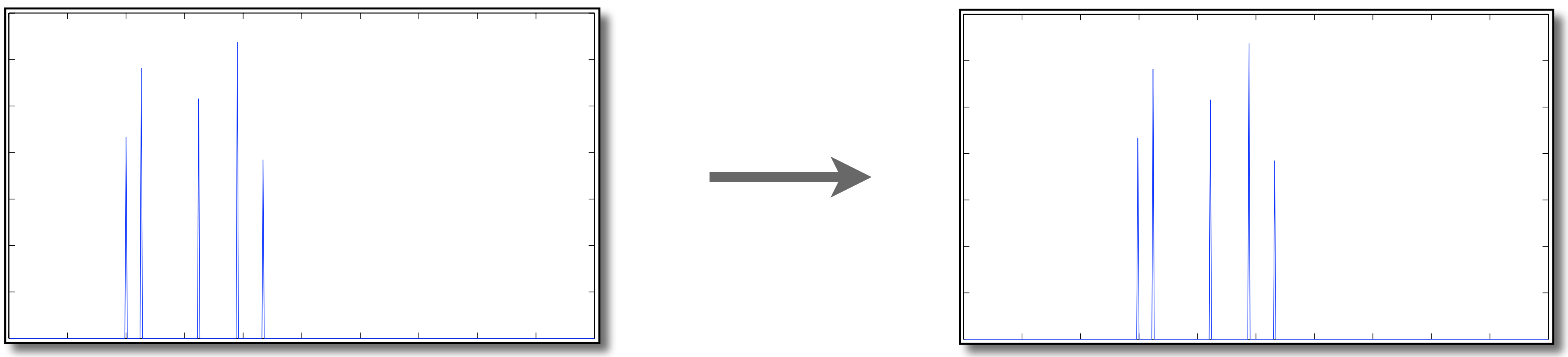
- requires, per frequency:
  - 1 eigenvalue problem ($O(n^4)$)
  - 2 full matrix-vector for eigenspace transform ($O(n^2)$)

Introduction

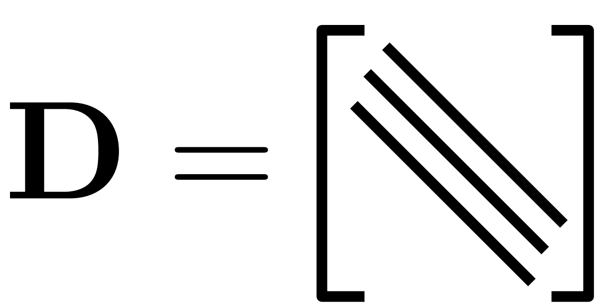
- Band-diagonalization techniques like parabolic approximation trades for speed with approximations
- *Is there another way?*

Our approach

- Consider a related, but simpler problem: shifting (or translating) signal



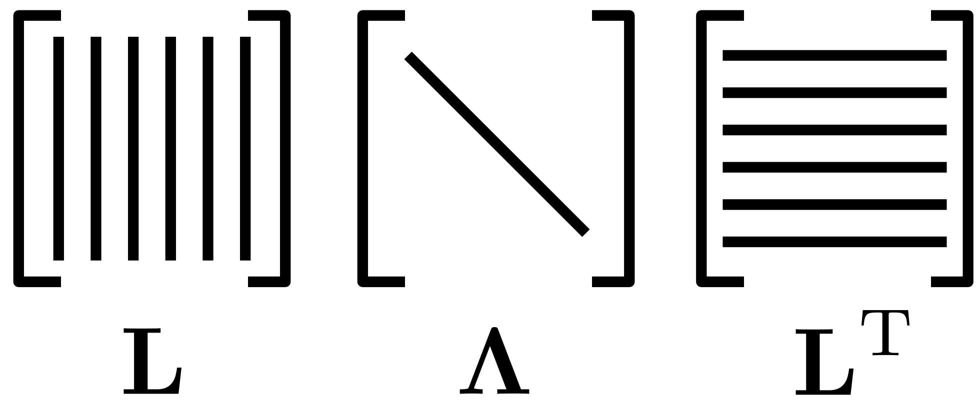
- operator is $\mathbf{S} = e^{-j\frac{\Delta x}{2\pi}\mathbf{D}}$
- $\mathbf{D}$ is differential operator



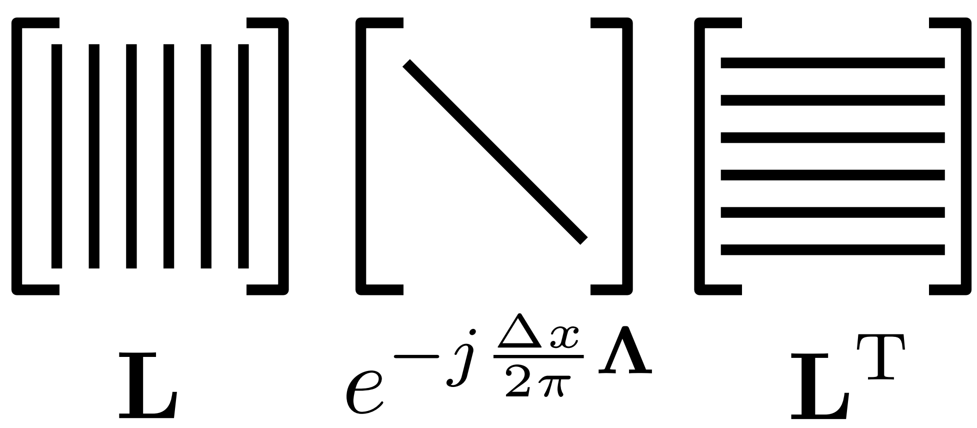
Our approach

- Computation requires similar approach to $\mathbf{W}^{\pm}$

$$\mathbf{D} = \mathbf{L}\mathbf{\Lambda}\mathbf{L}^{\mathrm{T}} =$$



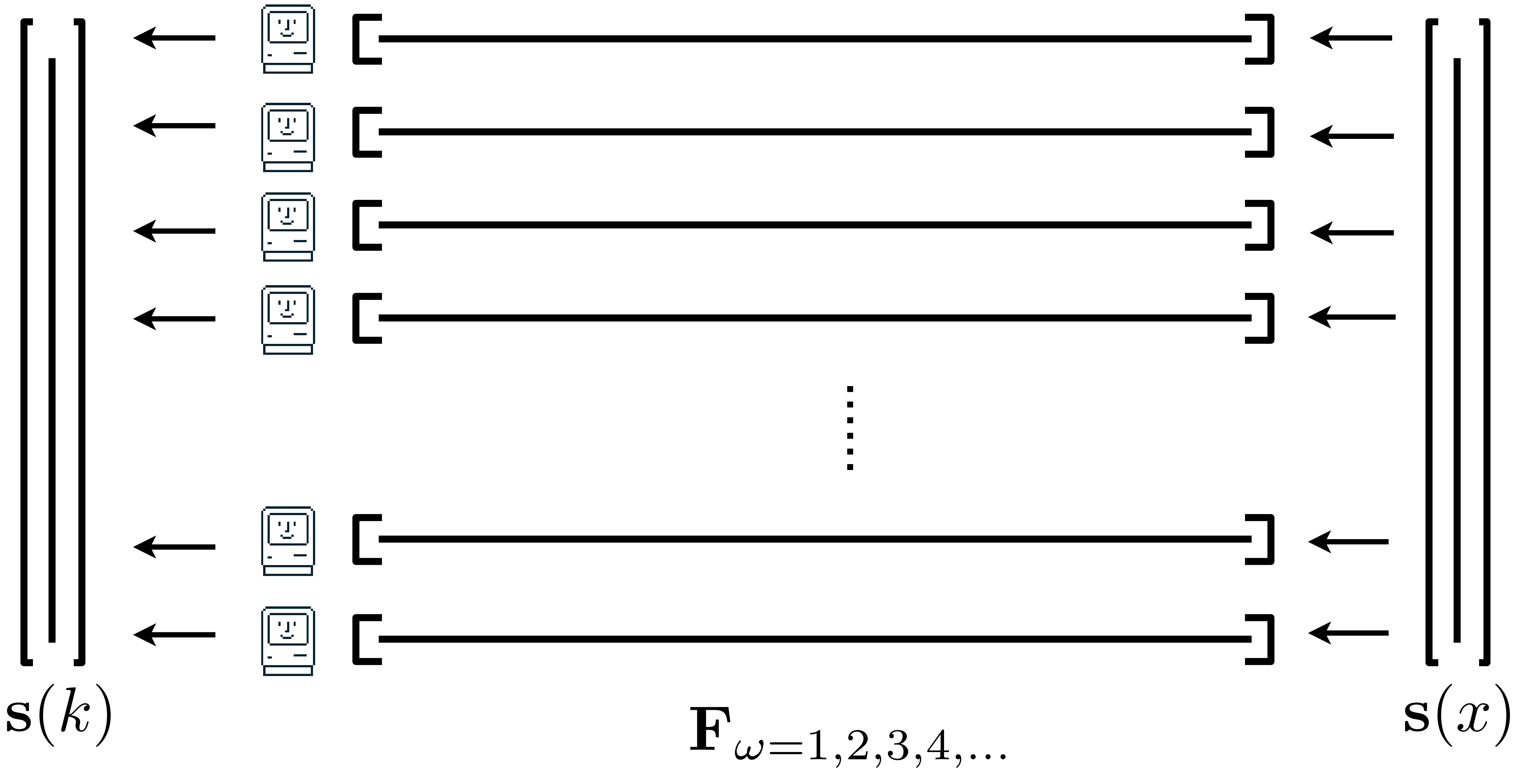
$$\mathbf{S} =$$



- However, for $\mathbf{D}$, $\mathbf{L} = \mathrm{DFT}$, so computation trivial with FFT

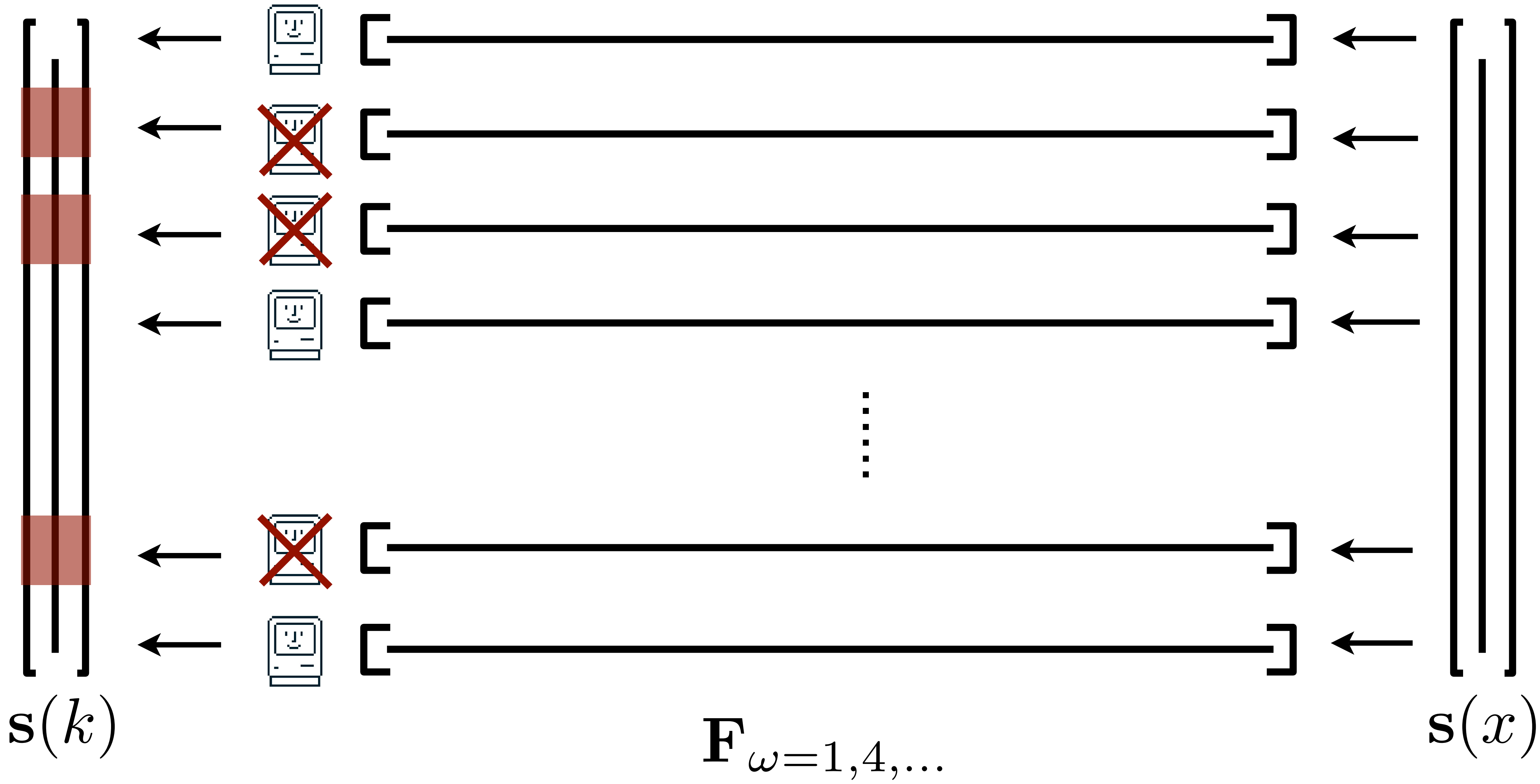
Our approach

- Suppose FFT does not exist yet



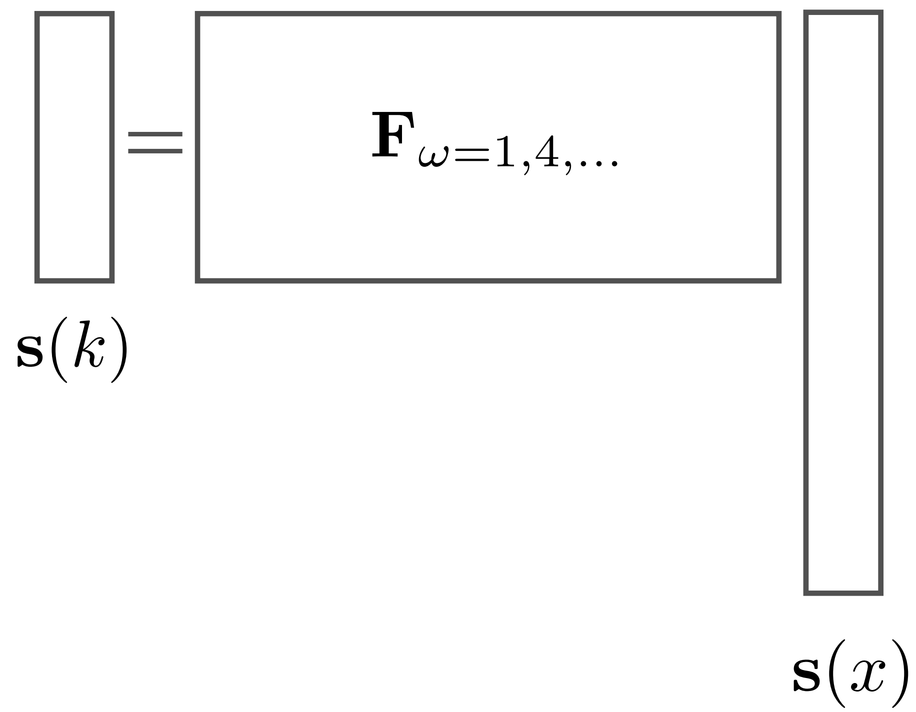
Our approach

- suppose some nodes didn't finish their jobs



Our approach

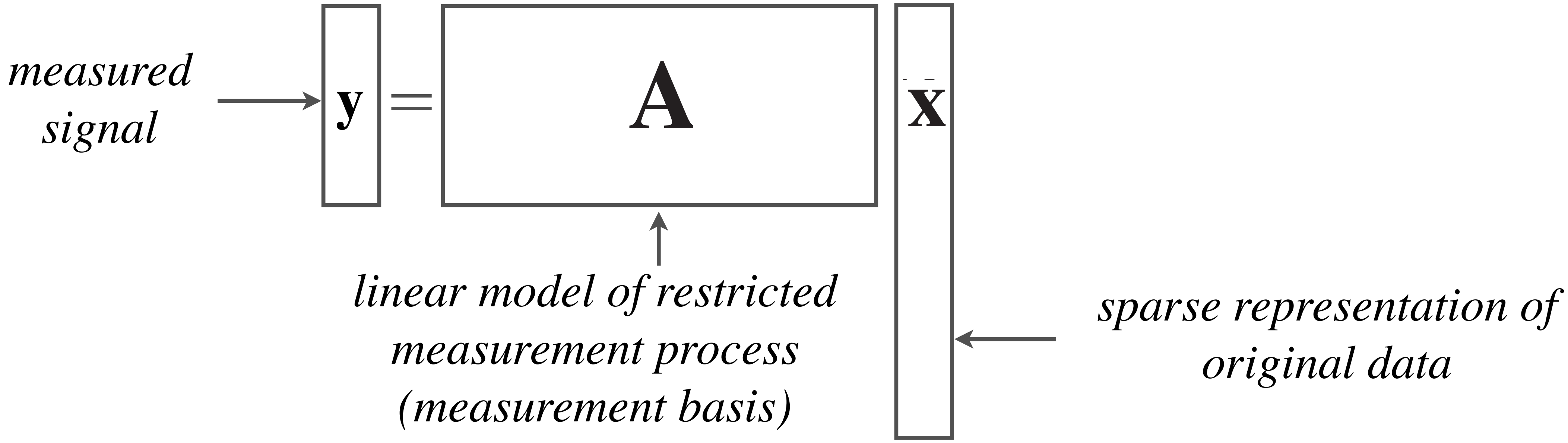
- mathematically, the system is incomplete



- evidently some information of original $\mathbf{s}(x)$ is invariably lost. *Or is it?*

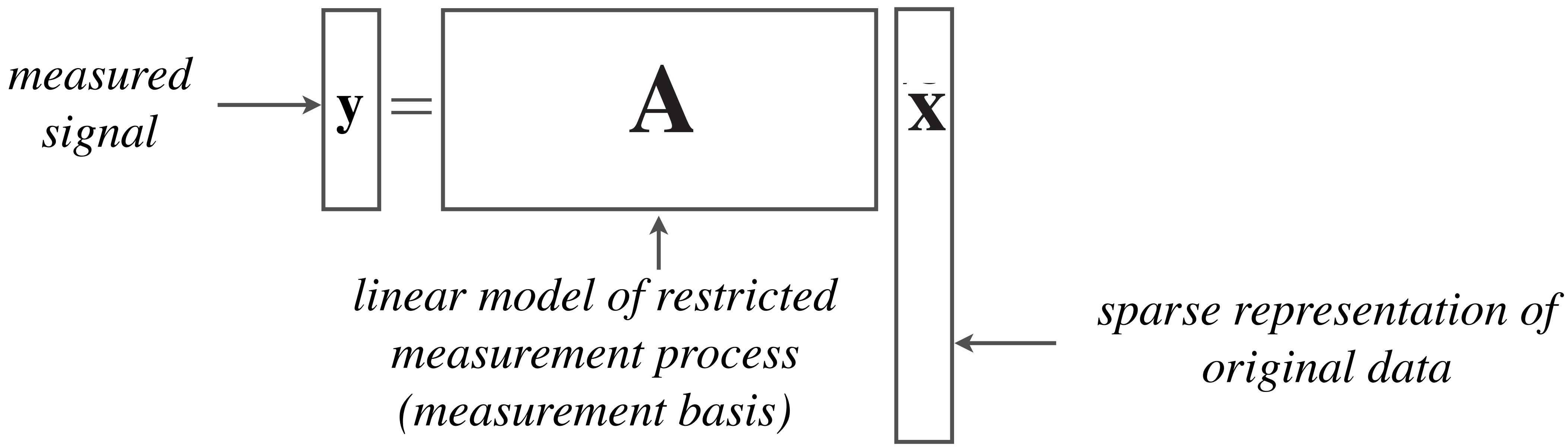
Compressed Sensing

- states that given system of the form



Compressed Sensing

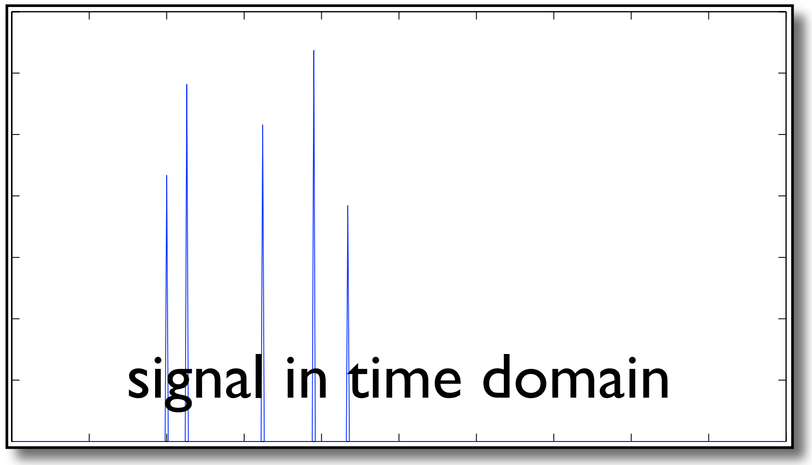
- states that given system of the form

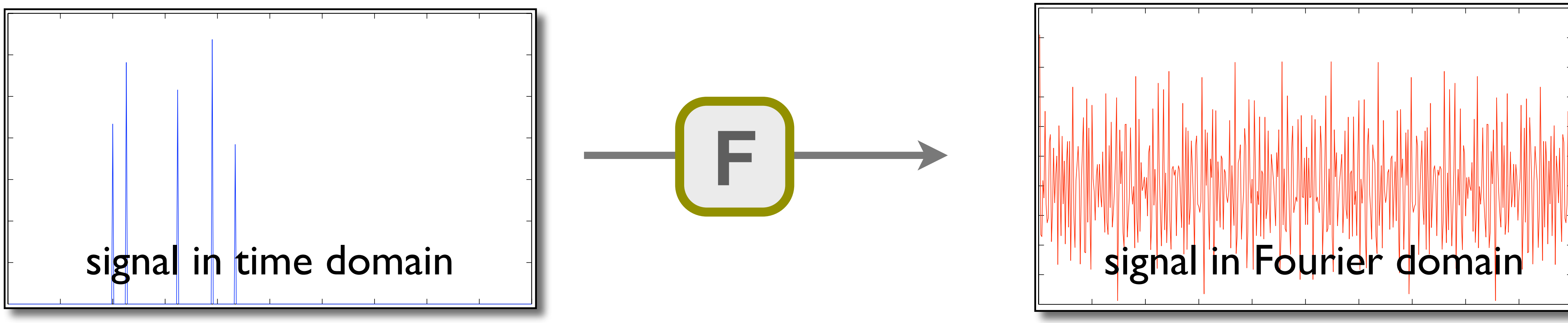


- can exactly "recover" x from y by solving L1 problem

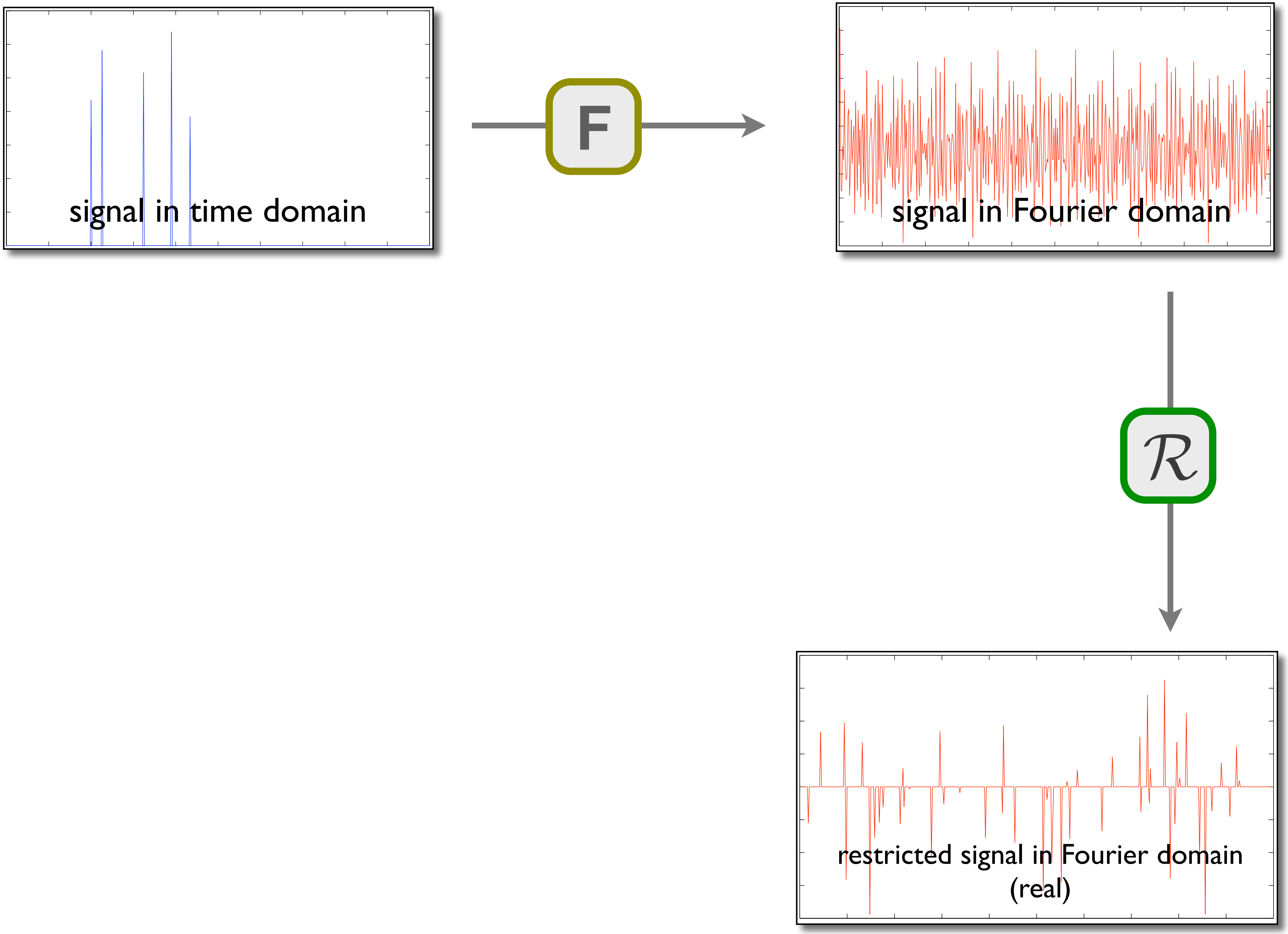
$$\tilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{x}\|_1 = \sum_{i=1}^{N} |x_i| \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{y},$$

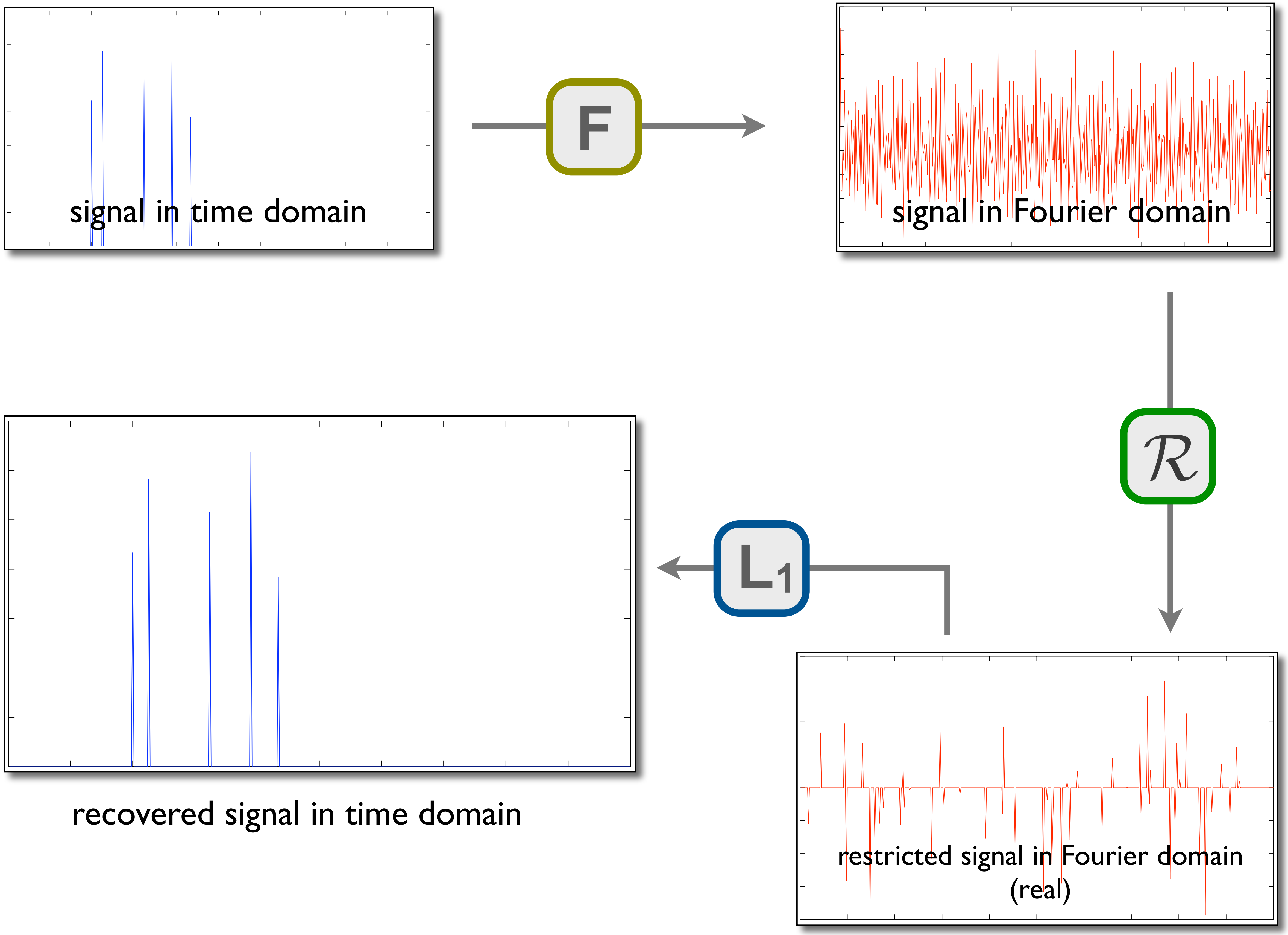
Candès, E., J. Romberg, and T. Tao, 2006b, Stable signal recovery from incomplete and inaccurate measurements: Communications On Pure and Applied Mathematics, **59**, 1207–1223.


signal in time domain



SLIM
Seismic Laboratory for Imaging and Modeling


signal in time domain
F
signal in Fourier domain
$\mathcal{R}$
restricted signal in Fourier domain
(real)


signal in time domain
F
signal in Fourier domain
R
L1
restricted signal in Fourier domain
(real)
recovered signal in time domain
SLIM
Seismic Laboratory for
Imaging and Modeling

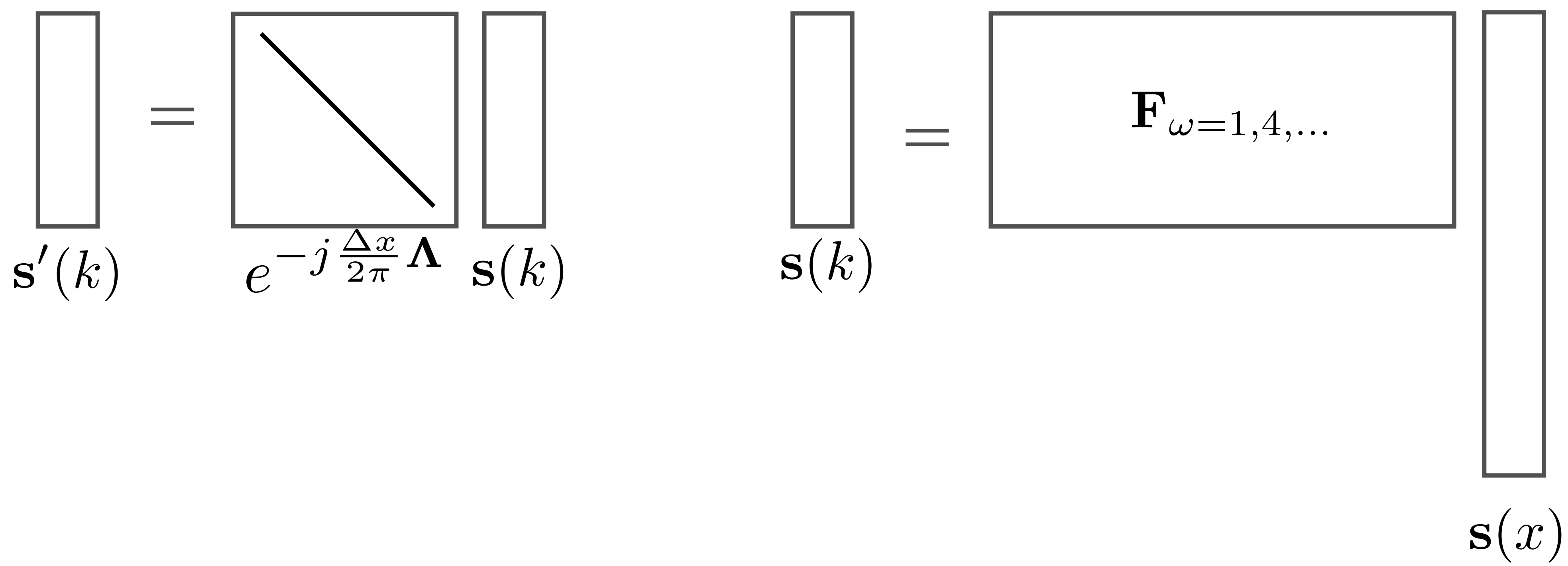
Compressed Sensing

- x has to be sparse
- **A** has to be Fourier transform

- Compressed sensing theory gives us strict bounds on regions of recoverability
- Enables deliberate *incomplete computations*

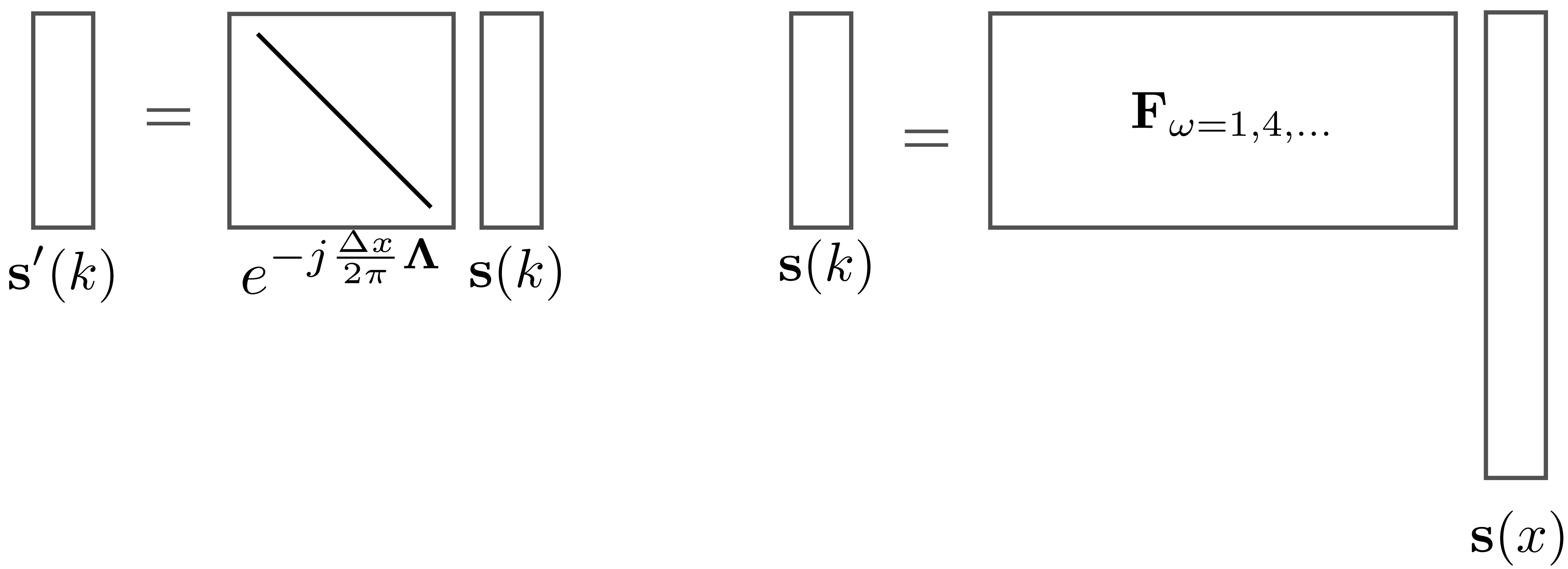
Compressed ~~Sensing~~ "Computation"

- if we "shift" s(k) with $e^{-j\frac{\Delta x}{2\pi}\mathbf{\Lambda}}$, what happens when we recover s(x) using s'(k)?



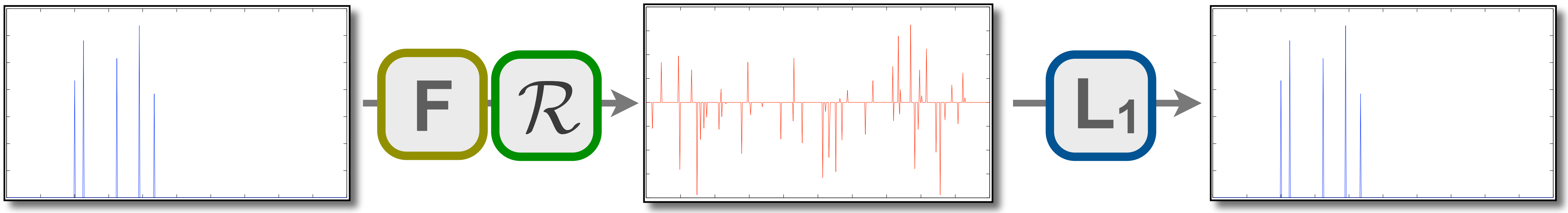
Compressed ~~Sensing~~ "Computation"

- if we "shift" s(k) with $e^{-j\frac{\Delta x}{2\pi}\mathbf{\Lambda}}$, what happens when we recover s(x) using s'(k)?



- Answer: we recover a shifted s(x)!

Compressed Sensing

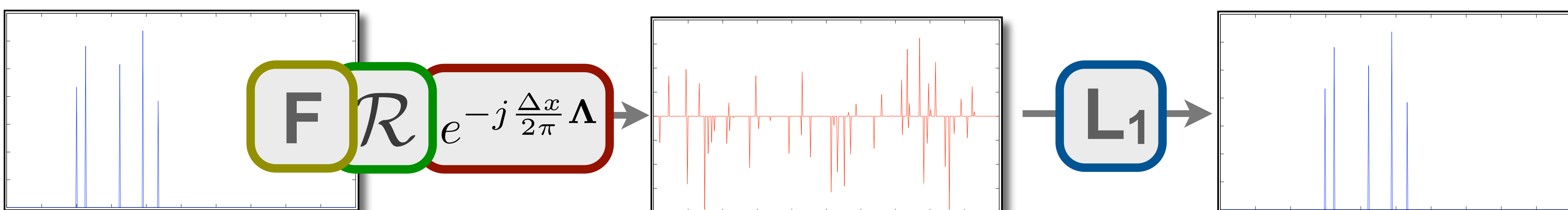


signal in space domain

incomplete signal in Fourier domain

signal in space domain

Compressed Processing

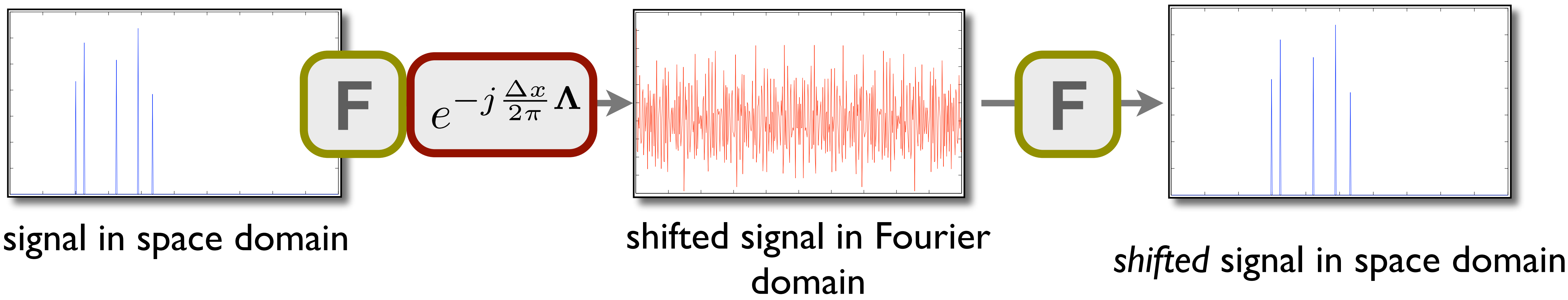


signal in space domain

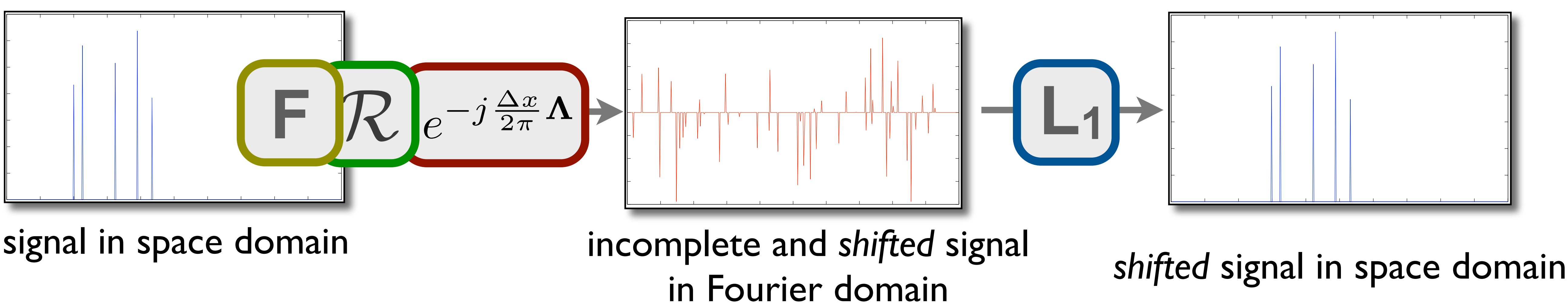
incomplete and *shifted* signal in Fourier domain

shifted signal in space domain

Straightforward Computation



Compressed Processing



Compressed ~~Sensing~~ "Computation"

- In a nutshell:
  - Trades the cost of L1 solvers for a compressed operator that is cheaper to compute, store, and synthesize

- L1 solver research is currently a hot topic in applied mathematics

Tibshirani, R., 1996, Least absolute shrinkage and selection operator, Software: http://www-stat.stanford.edu/~tibs/lasso.html.

Candès, E. J., and J. Romberg, 2005, ℓ_1-magic. Software: http://www.acm.caltech.edu/limagic/.

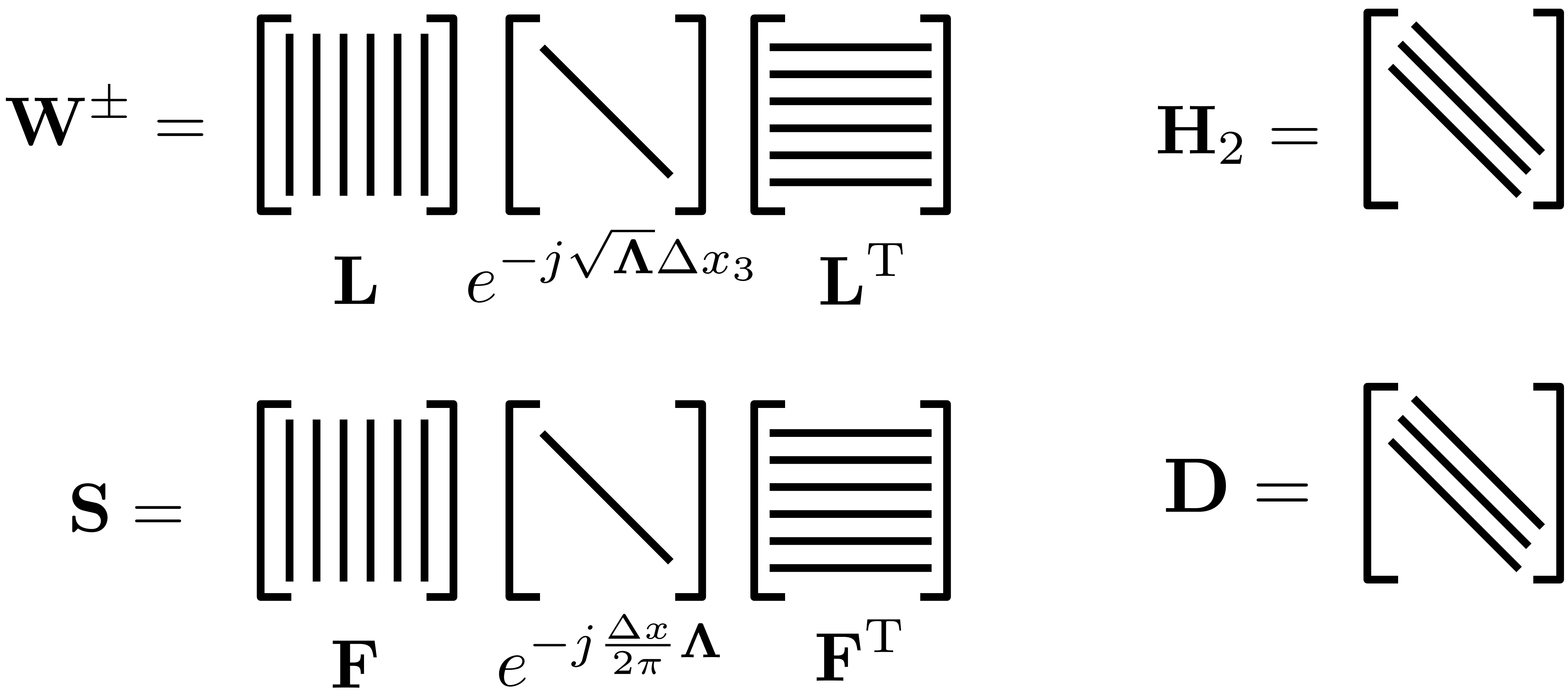
Donoho, D. L., I. Drori, V. Stodden, and Y. Tsaig, 2005, SparseLab, Software: http://sparselab.stanford.edu/.

Figueiredo, M., R. D. Nowak, and S. J. Wright, 2007, Gradient projection for sparse reconstruction, Software: http://www.lx.it.pt/~mtf/GPSR/.

Koh, K., S. J. Kim, and S. Boyd, 2007, Simple matlab solver for l1-regularized least squares problems, Software: http://www-stat.stanford.edu/~tibs/lasso.html.

Compressed Wavefield Extrapolation

- Recall the similarity between $\mathbf{W}^{\pm}$ and $\mathbf{S}$



Compressed Wavefield Extrapolation

- Structure of $\mathbf{H}_1$ $\quad \mathbf{H}_2 = \mathbf{L}\boldsymbol{\Lambda}\mathbf{L}^{\mathrm{T}} \quad \mathbf{H}_1 = \mathbf{L}\boldsymbol{\Lambda}^{1/2}\mathbf{L}^{\mathrm{T}}$
  - analytically

    $$\mathcal{H}_2 = \mathcal{H}_1\mathcal{H}_1$$

    $$\mathcal{H}_2 = k^2(\boldsymbol{x}) + \partial_\mu\partial_\mu$$

  - discretely

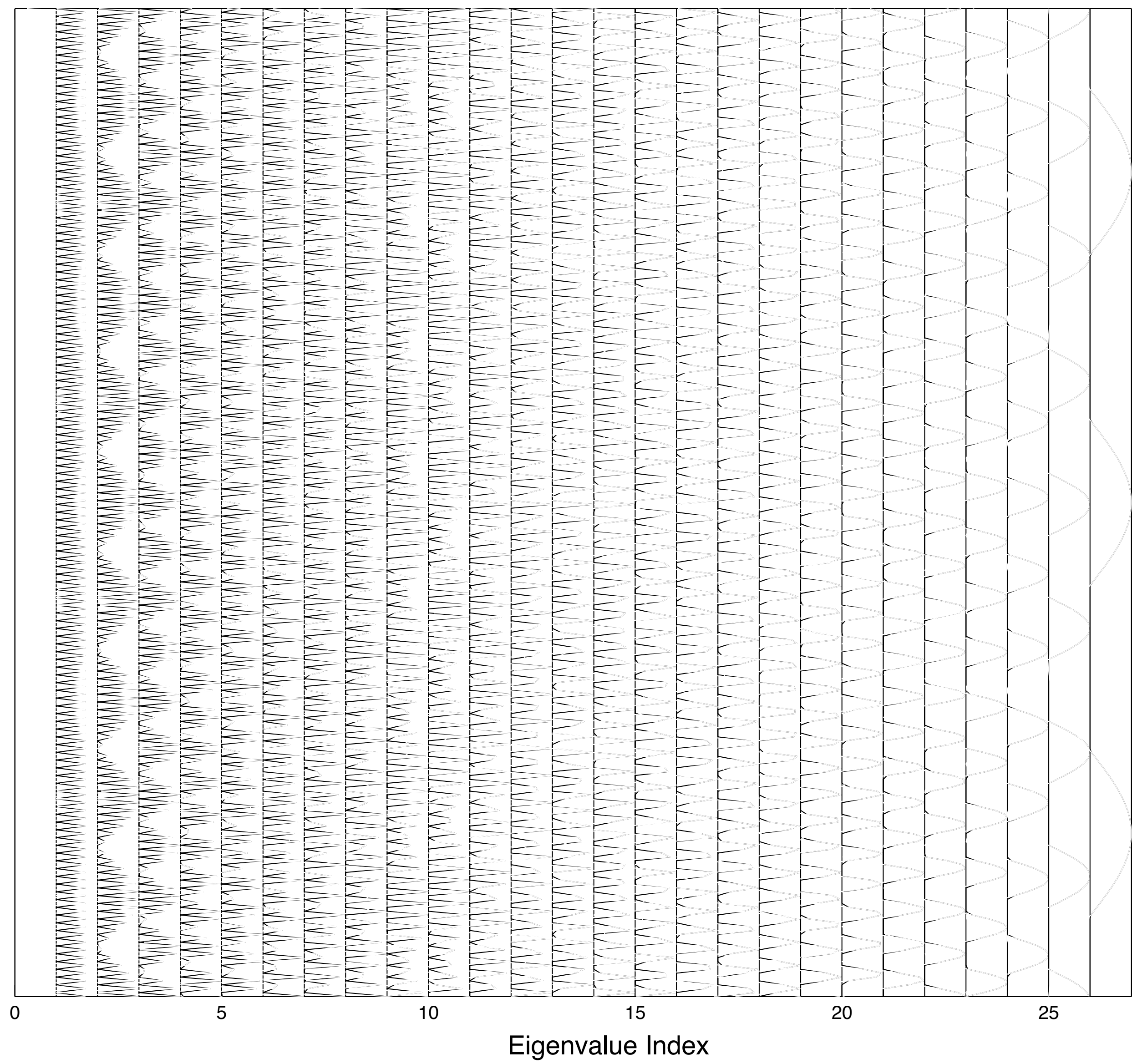
    $$\mathbf{H}_2 = \mathbf{C} + \mathbf{D}_2$$

    $$\mathbf{H}_2 = \begin{bmatrix} \left(\frac{\omega}{c_1}\right)^2 & 0 & \cdots & 0 \\ 0 & \left(\frac{\omega}{c_2}\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\frac{\omega}{c_{n_1}}\right)^2 \end{bmatrix} + \mathbf{D}_2$$

SLIM Seismic Laboratory for Imaging and Modeling

Compressed Wavefield extrapolation

eigenfunctions of $\mathbf{H}_2$ at 30 Hz for constant velocity medium

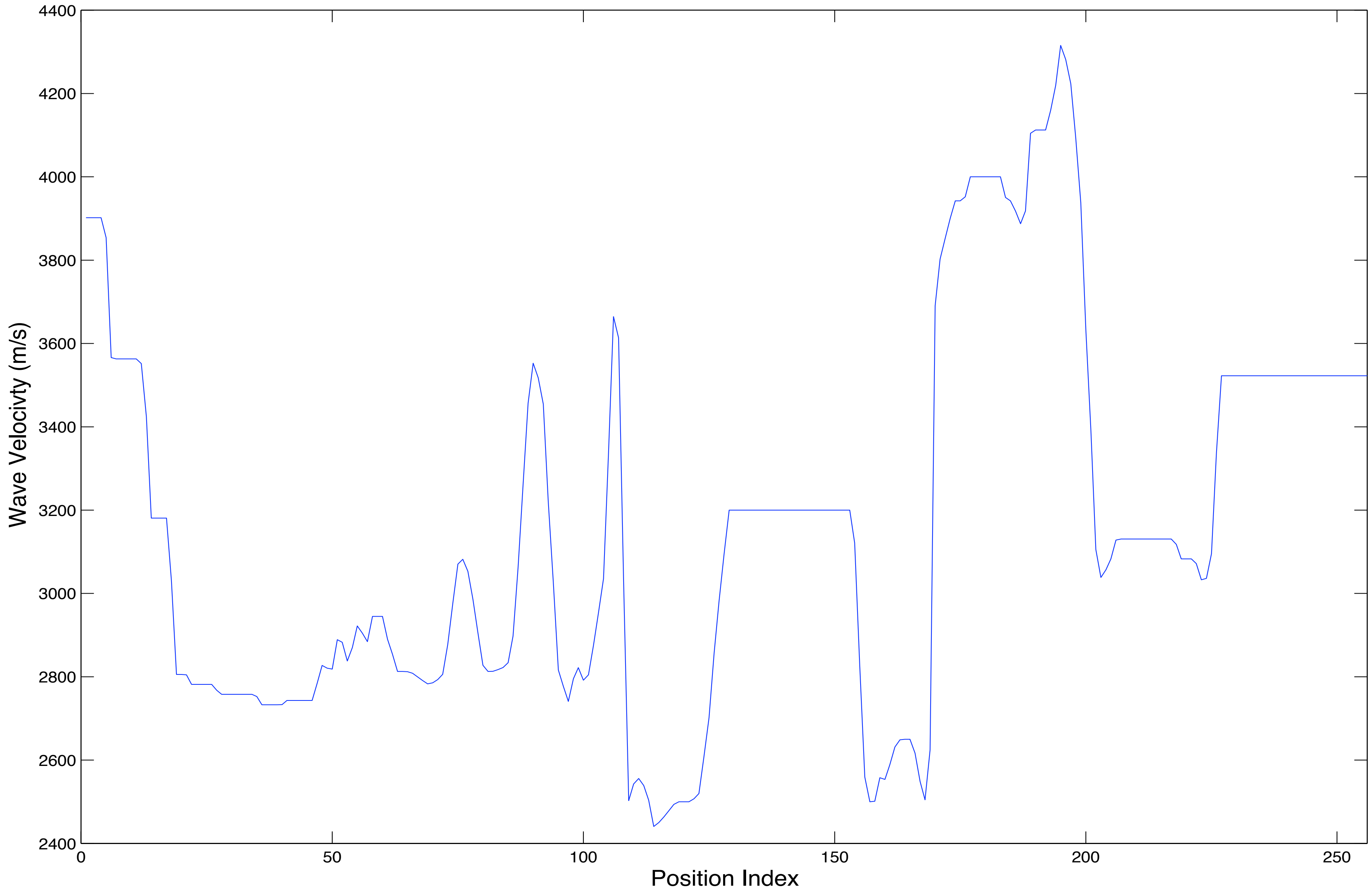


Asymptotically identical to the Cosine transform

SLIM
Seismic Laboratory for Imaging and Modeling

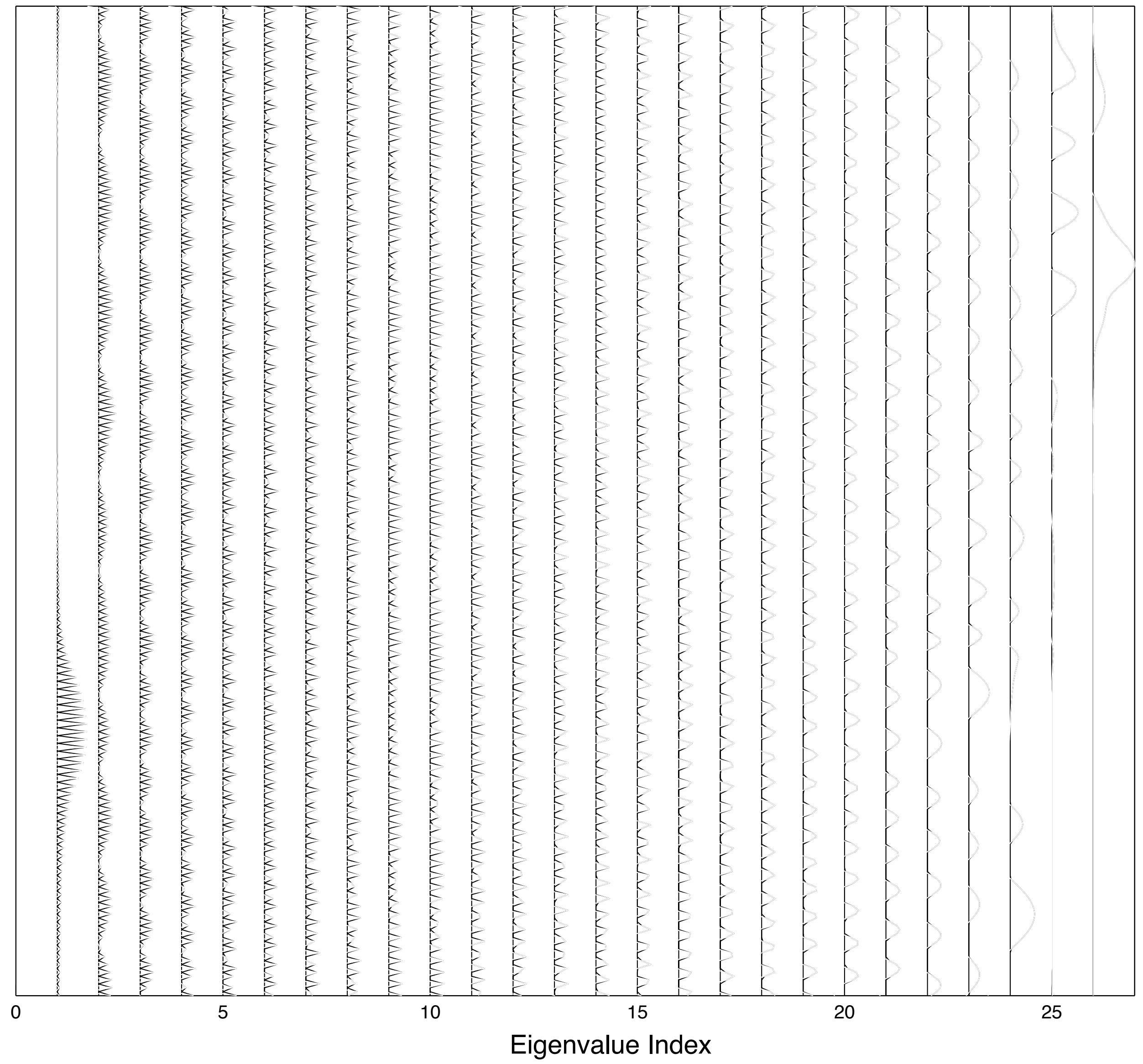
Compressed Wavefield extrapolation

eigenfunctions of $\mathbf{H}_2$ at 30 Hz for Marmousi velocity medium



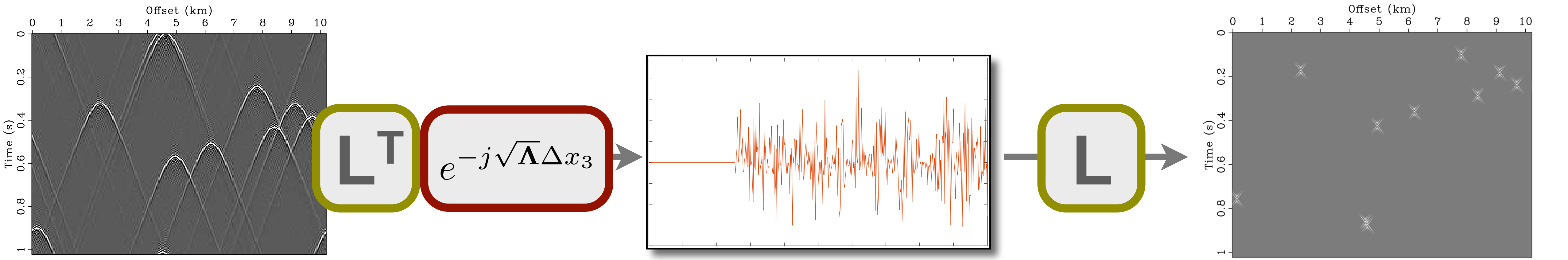
Compressed Wavefield extrapolation

eigenfunctions of $\mathbf{H}_2$ at 30 Hz for Marmousi velocity medium



fairly close to the Cosine transform

Straightforward 1-Way inverse Wavefield Extrapolation

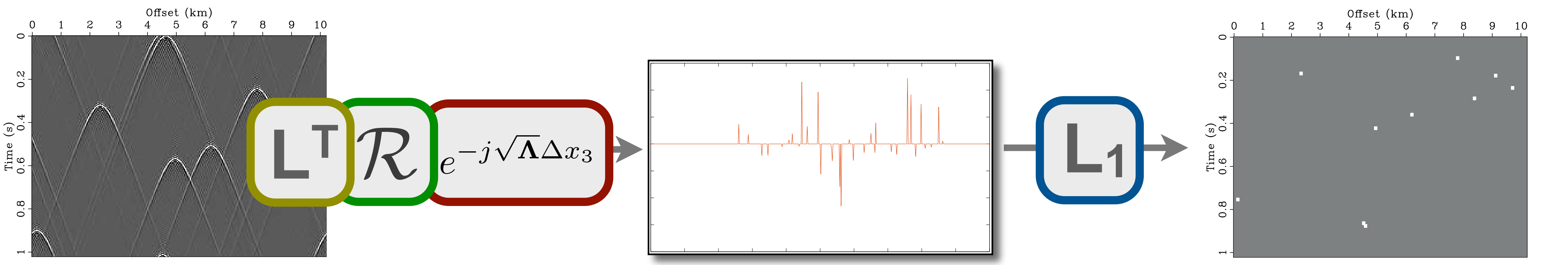


wavefield in space-time domain

back-extrapolated wavefield in H2 domain

back-extrapolated to impulse source in space-time domain

Compressed 1-Way Wavefield Extrapolation



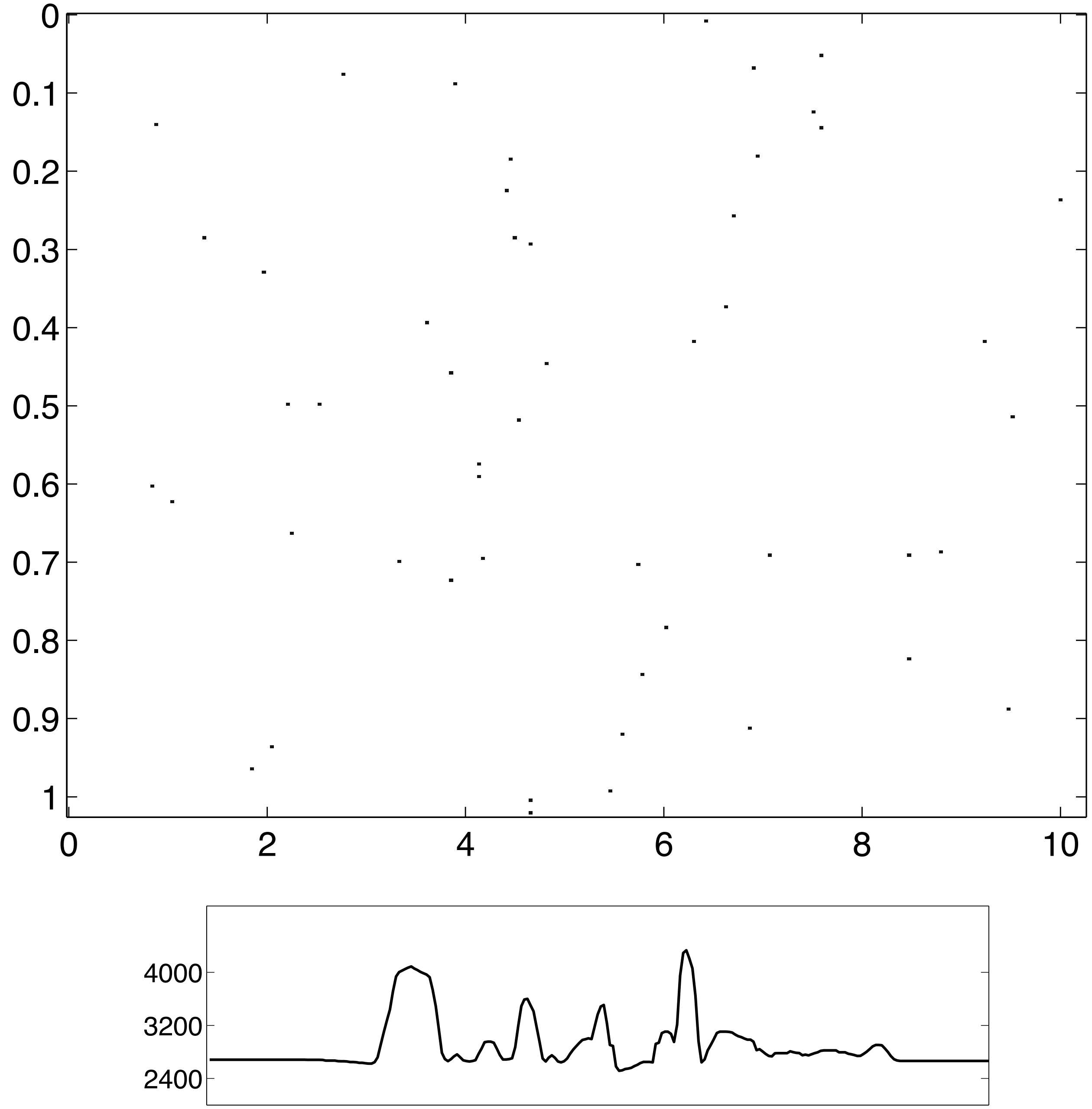
wavefield in space-time domain

incomplete back-extrapolated wavefield in H2 domain

back-extrapolated to impulse source in space-time domain

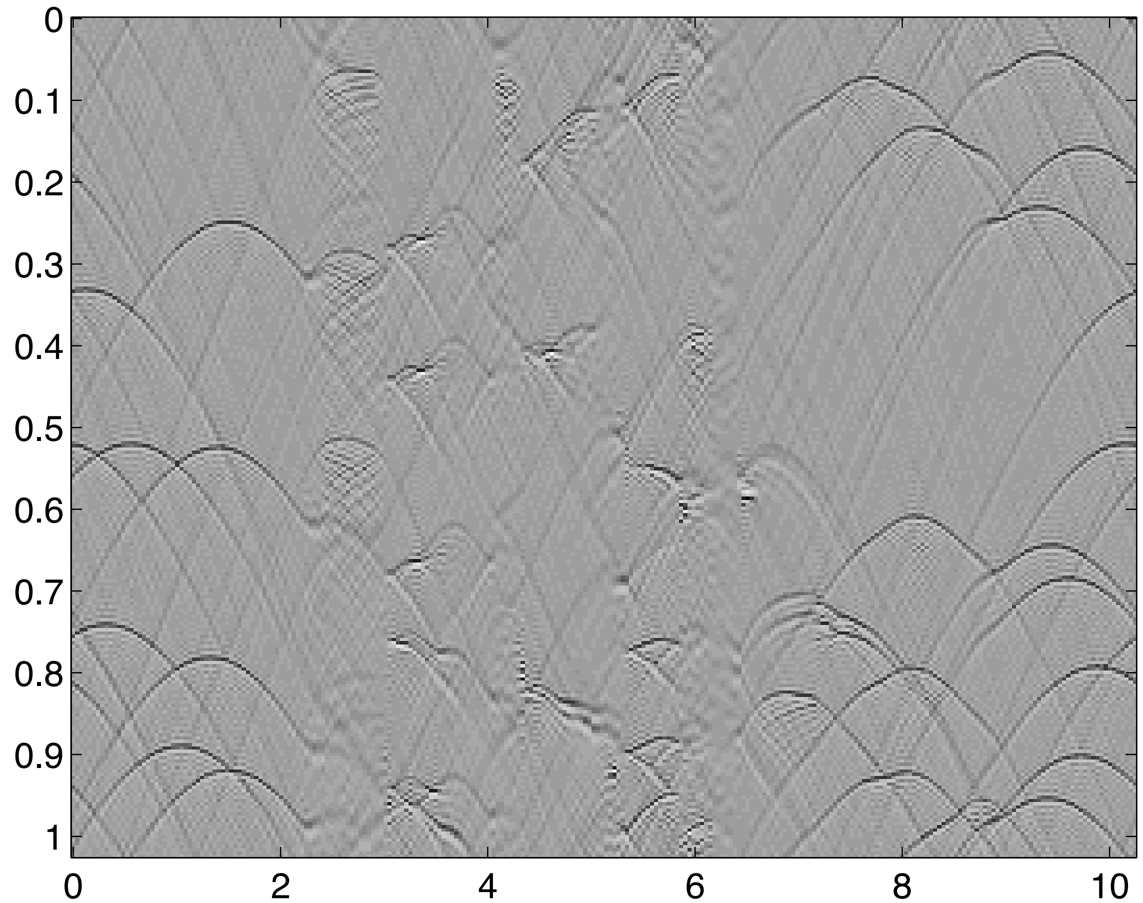
Compressed wavefield extrapolation

simple 1-D space/time propagation example with point scatters

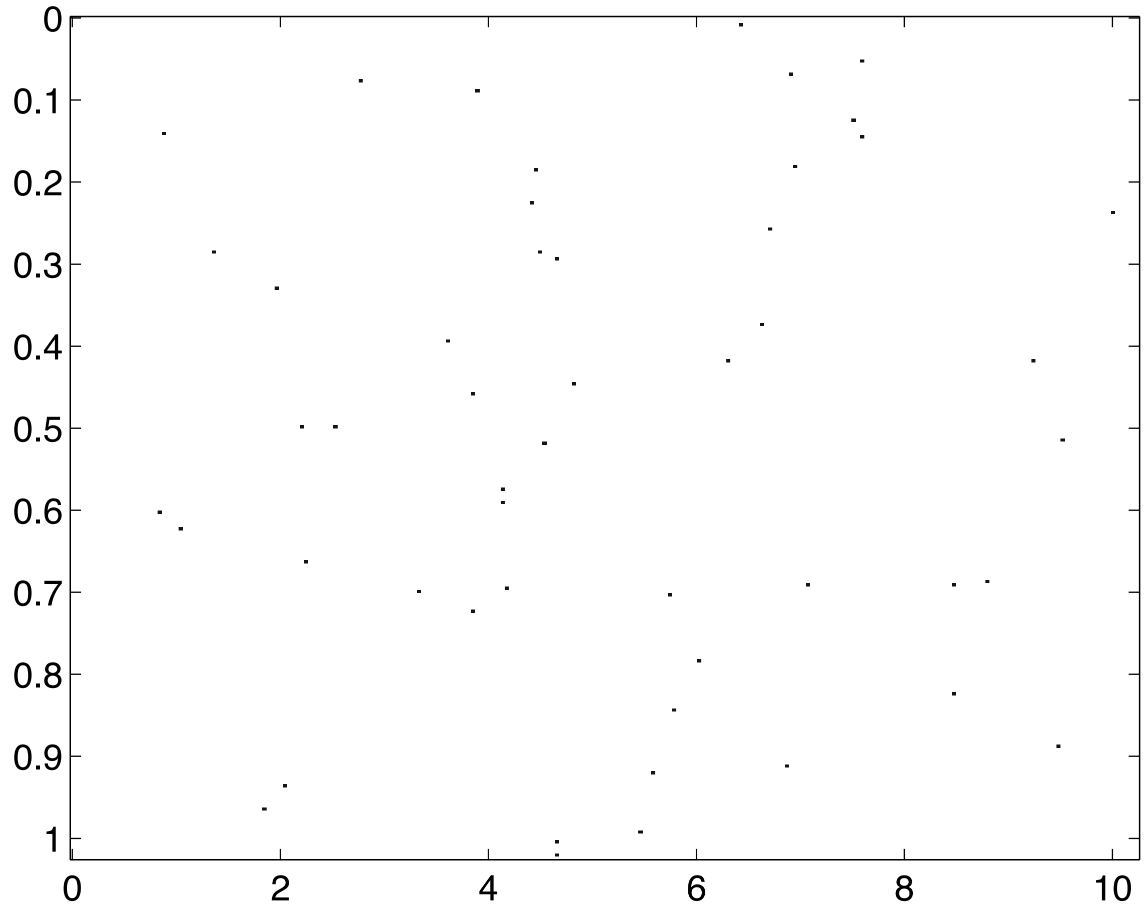


Compressed wavefield extrapolation

simple 1-D space/time propagation example with point scatters



propagated 1.5km down



recovered though L1 inverson

Restricted L transform to ~0.01 of original coefficients

SLIM
Seismic Laboratory for Imaging and Modeling

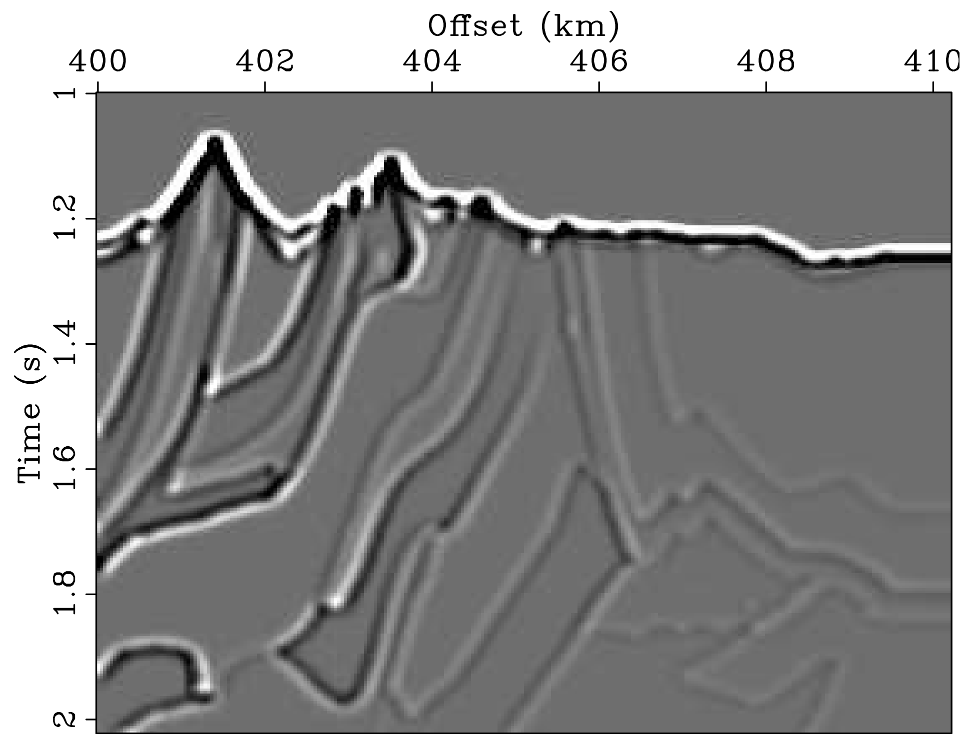
Sparsity through curvelets

- for extrapolation to reflectivity, we first transform signal into a sparsifies reflectivity
- we know reflectivity are sparse in curvelets

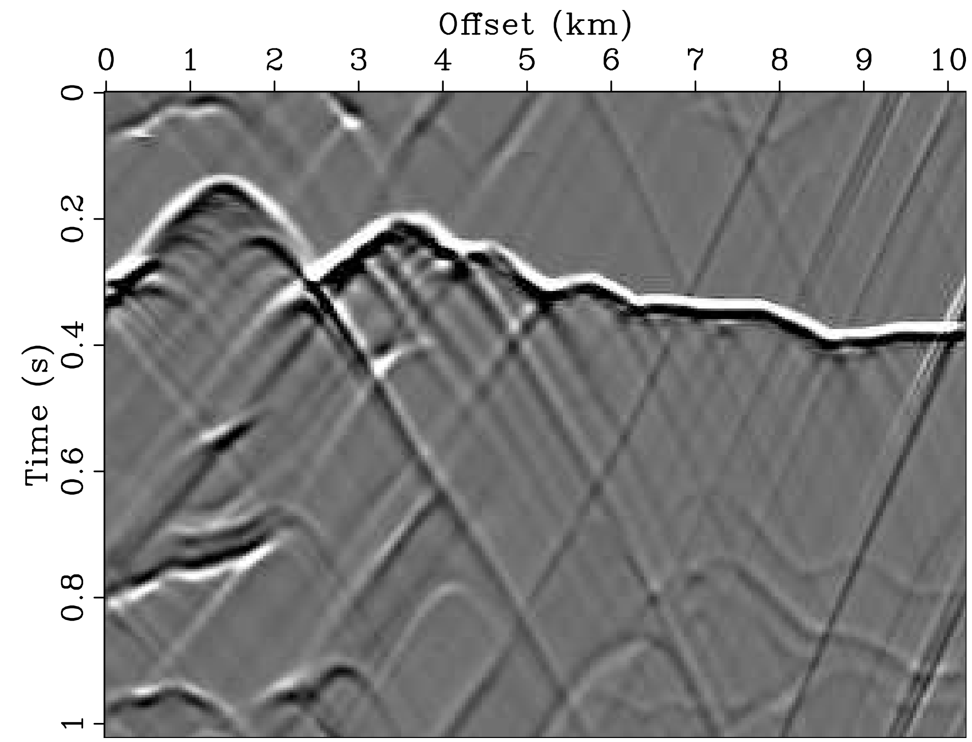
Candès, E. J., and L. Demanet, 2005, The curvelet representation of wave propagators is optimally sparse: Communications on Pure and Applied Mathematics, **58**, 1472–1528.

SLIM
Seismic Laboratory for Imaging and Modeling

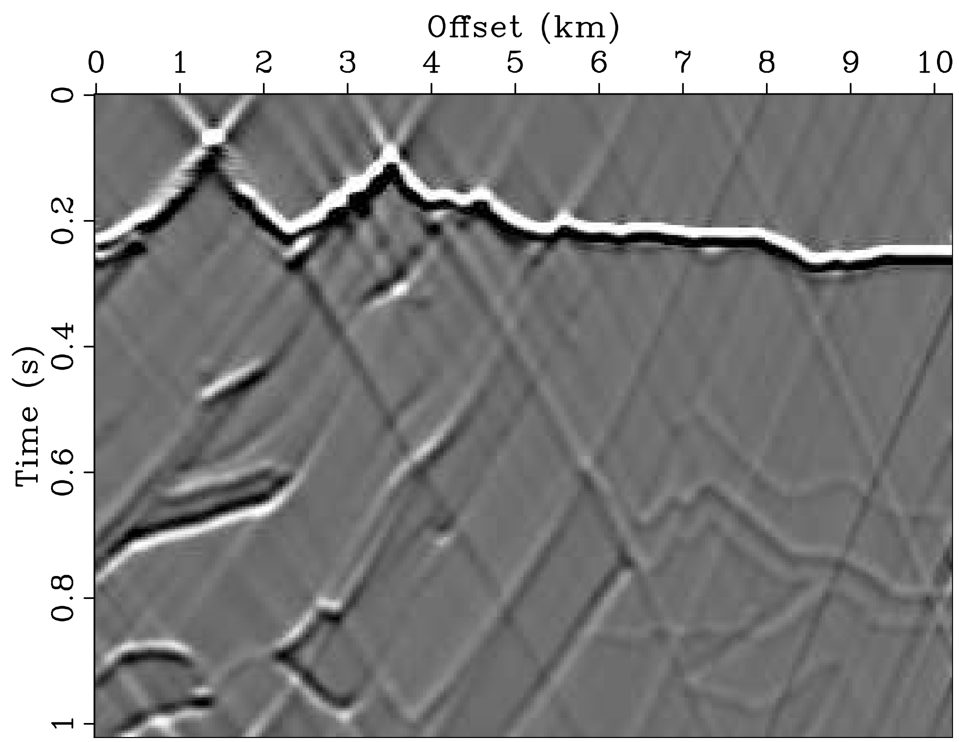
Example (Canadian overthrust)



original reflectivity



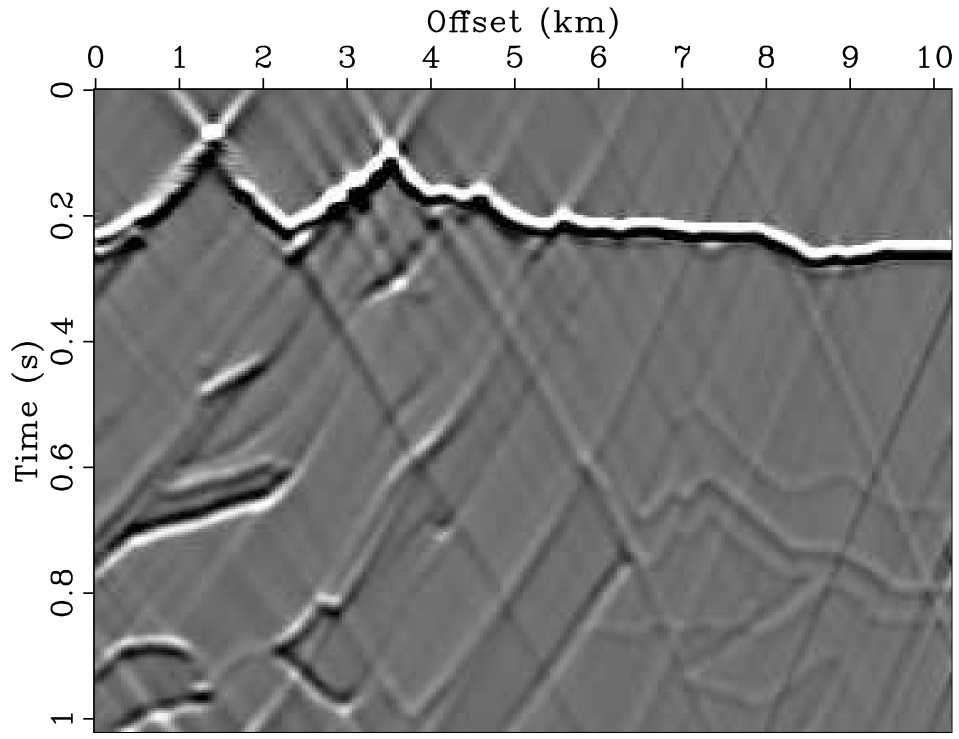
downward extrapolated 50m



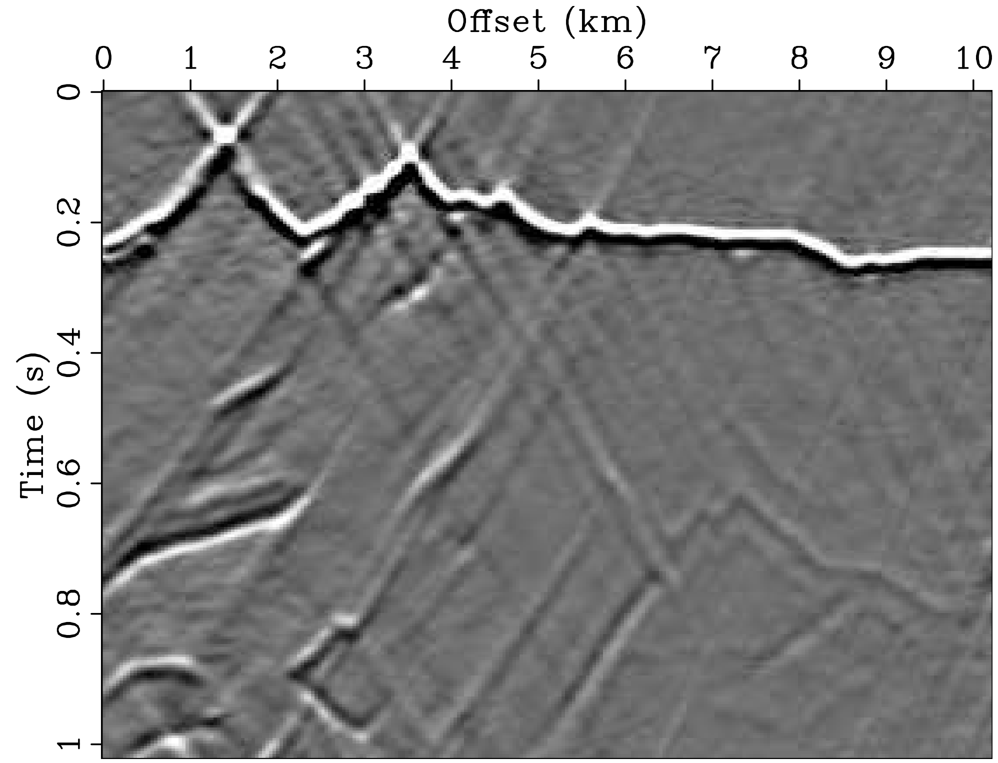
inverse extrapolated explicitly

SLIM Seismic Laboratory for Imaging and Modeling

Example (Canadian overthrust)



inverse extrapolated explicitly



inverse extrapolated with compressed computation

~15% coefficients used

SLIM Seismic Laboratory for Imaging and Modeling

Discussions

- Bottom line: synthesis, operation, and storage cost savings *versus* L1-solver cost
- require good sparsity-promoting basis (ie Curvelets)
- potential to apply same technique to a variety of different operators

Conclusions

- 1) Take linear operator with suitable structure for compressed sensing, having a diagonalizing basis which is incoherent with the signal basis
- 2) Compressed sensing theory tells us how much computation we can throw away while still recovering full signal with L1 solver
- 3) Then we can take advantage of results in compressed sampling for compressed computation

- Take home point:
  - *Exploit compressed sensing theory for gains in scientific computation*

Still awake?

- Check-out the full paper at:

Lin, T.T.Y. and F. Herrmann, 2007, Compressed wavefield extrapolation: Geophysics, 72, SM77-SM93

Compressed wavefield extrapolation

$$\begin{cases} \mathbf{y} & = \mathbf{R} e^{-j\omega\sqrt{\mathbf{\Lambda}}\Delta x_3} \mathbf{L}^{\mathrm{T}} \mathbf{u} \\ \tilde{\mathbf{x}} & = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{R}\mathbf{L}^{\mathrm{T}}\mathbf{x} = \mathbf{y} \\ \tilde{\mathbf{u}}' & = \tilde{\mathbf{x}} \end{cases}$$

- Randomly subsample in the Modal domain
- Recover by norm-one minimization
- Capitalize on
  - the incoherence between modal functions and impulse sources
  - reduced explicit matrix size

Compressed wavefield extrapolation with curvelets

$$\begin{cases} \mathbf{y} & = \mathbf{R} e^{-j\omega\sqrt{\mathbf{\Lambda}}\Delta x_3} \mathbf{L}^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{u} \\ \tilde{\mathbf{x}} & = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{R}\mathbf{L}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{x} = \mathbf{y} \\ \tilde{\mathbf{u}}' & = \tilde{\mathbf{x}} \end{cases}$$

- Original and reconstructed signals remain in the curvelet domain
- Original curvelet transform must be done outside of the algorithm