Curvelet-based non-linear adaptive subtraction with sparseness constraints

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Context

Spiky/minimal structure deconvolution (Claerbout, Ulrych, Oldenburg, Sacchi, Trad etc.)

Sparse Radon (Ulrych, Sacchi, Trad, etc.)

FFT/Focus transform-based Interpolation (Duyndam, Zwartjes, Verschuur, Berkhout)

Redundant dictonaries/Morphological component separation/Pursuits (Mallat, Chen, Donoho, Starck, Elad)

L2-migration (Nemeth, Chavent, de Hoop, Hu, Kuehl)

2-D/3-D Curvelets Non-linear synthesis (Durand, Starck, Candes, Demanet, Ying)

Anisotropic Diffusion (Osher)

Goals

Processing & imaging scheme

- **increases resolution & SNR**
- **preserves edges = freq. content**
- **works with and extends existing**
  - **noise removal/signal separation**
  - **imaging schemes**

Develop the right *language* to deal with SNR ≤ 0 ....

General Framework

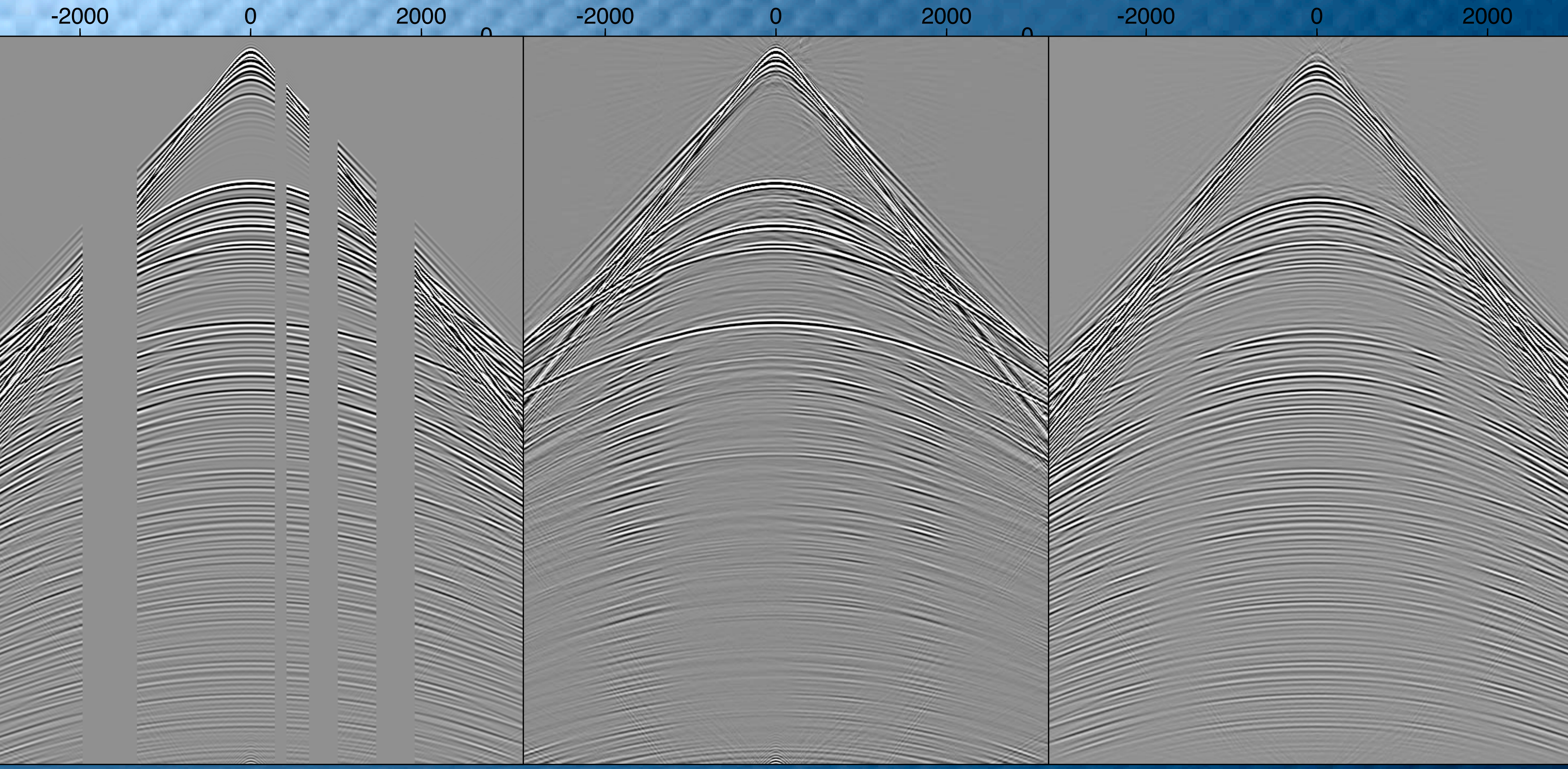
Divide-and-conquer approach:

1. **Sparseness with thresholding**
2. **Continuity with constrained optimization**

Main focus:

- **use of Curvelets as optimal Frames**
- **(iterative) thresholding**

Appetizer



missing data with multiples **estimated primaries** **estimated multiples**

Wish list

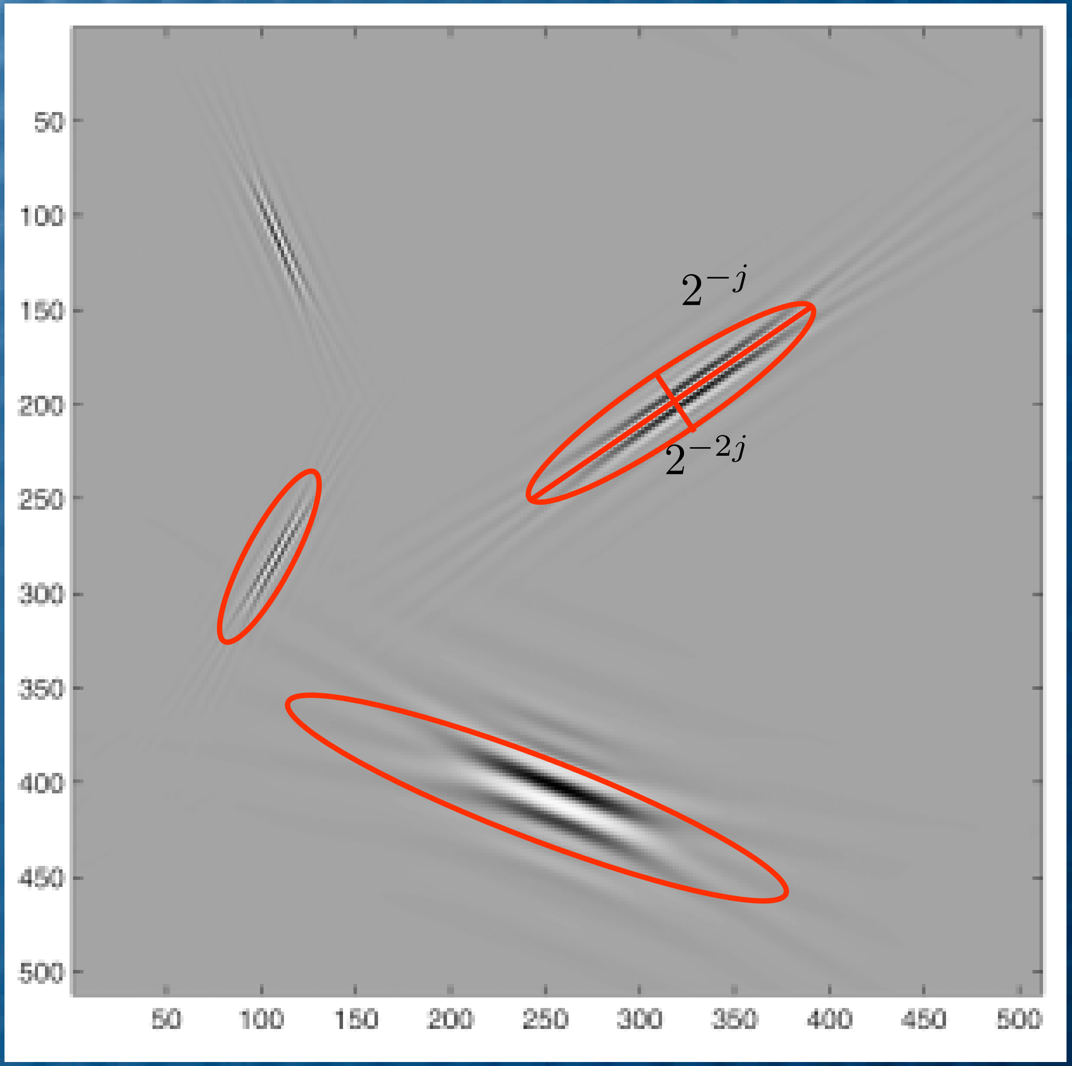
Seek a *transform domain* that is

- ***relative insensitive* to *local phase***
- ***sparse & local* (position/dip)**
- ***optimal* for *curved* events**
- ***well-behaved* under *operators***
- ***near diagonalizes Covariance***

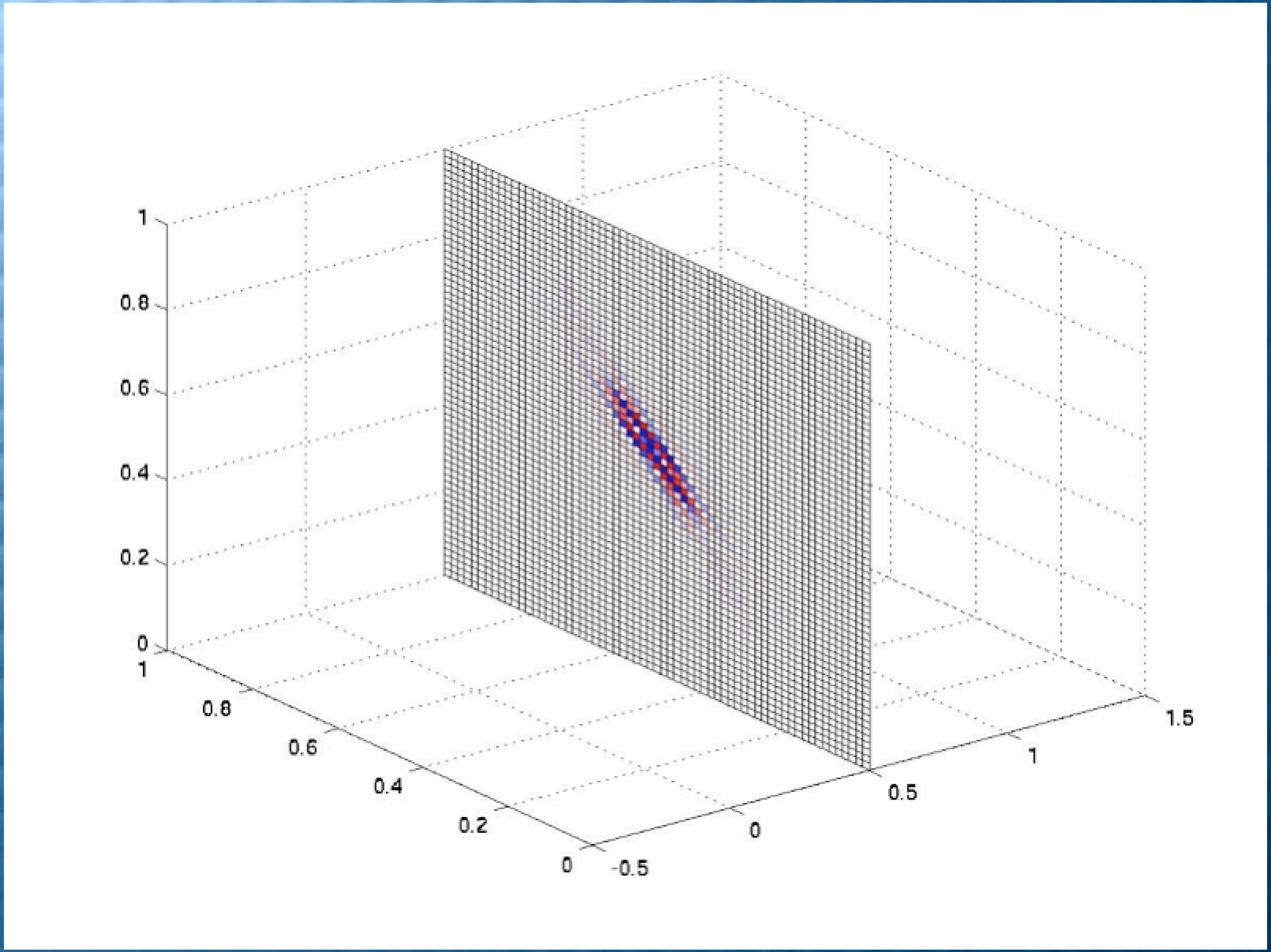
Aim to bring out those high frequencies with *ultra-low* SNR<0!

Why curvelets

- Nonseparable
- Local in 2-D space
- Local in 2-D Fourier
- Anisotropic
- Multiscale
- Almost orthogonal
- Tight frame $\mathrm{B}^T\mathrm{B} = \mathrm{I}$
- Optimal

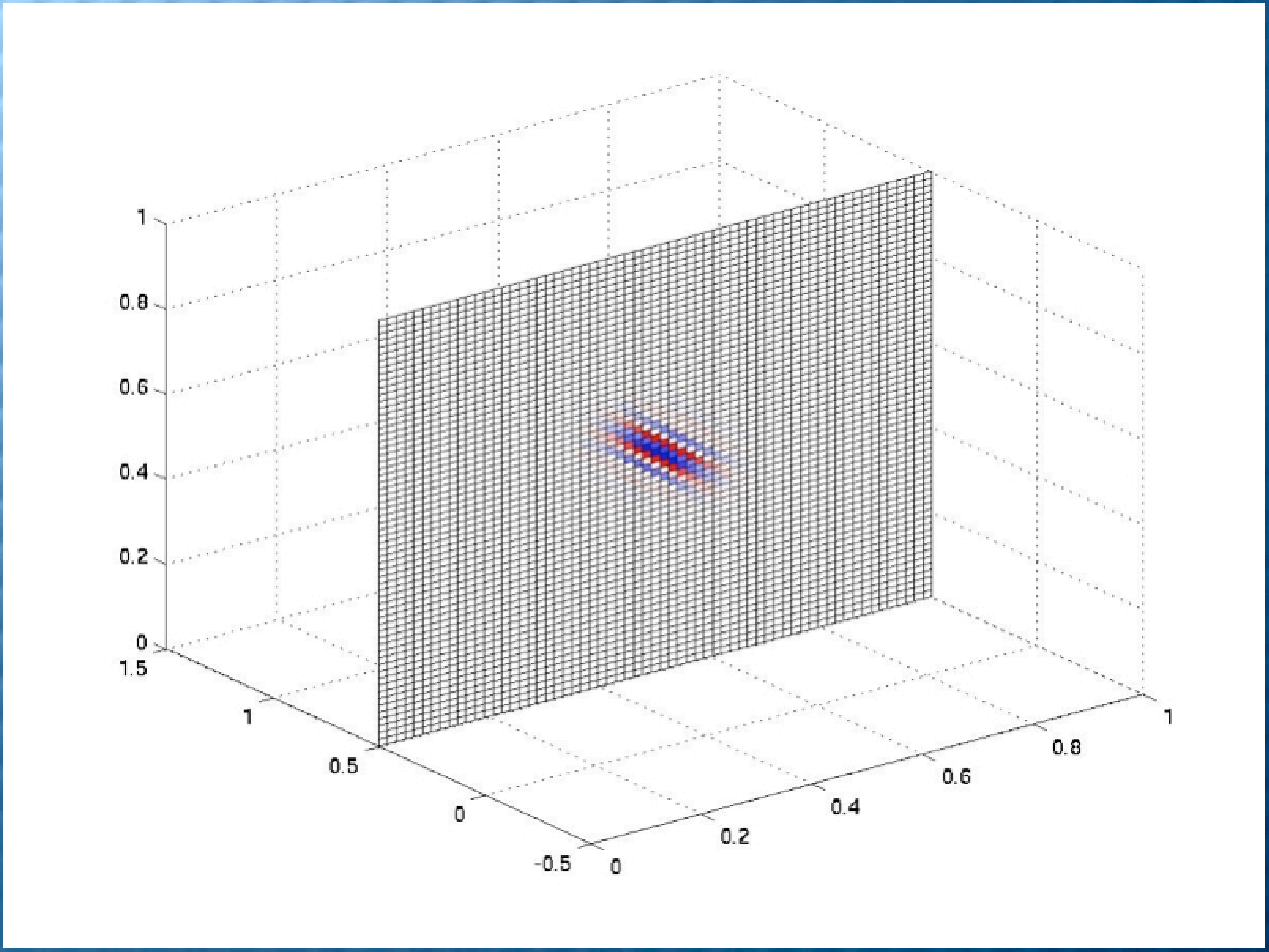


3-D Curvelets



Thanks to Demanet & Ying

3-D Curvelets



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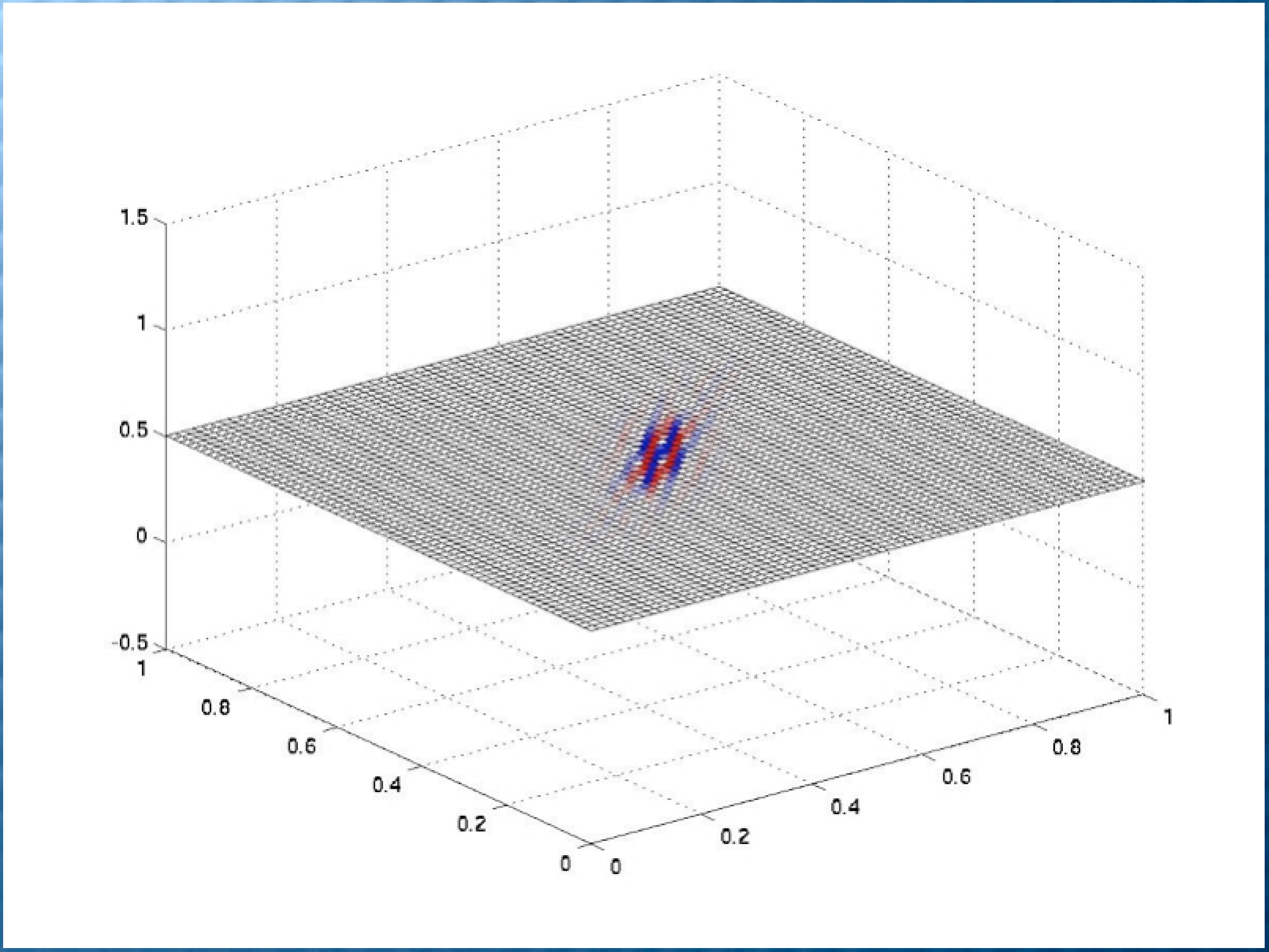
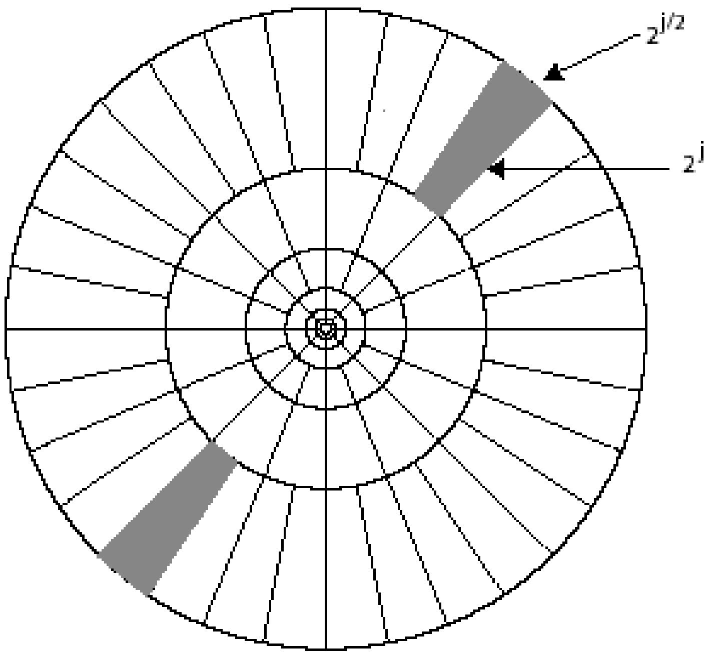
3-D Curvelets



Thanks to Demanet & Ying

Why curvelets

$$W_j = \{\zeta, \quad 2^j \le |\zeta| \le 2^{j+1}, |\theta - \theta_J| \le \pi \cdot 2^{\lfloor j/2 \rfloor}\}$$



second dyadic partitioning

Fourier/SVD/KL

$$||f - \tilde{f}_m^F|| \propto m^{-1/2}, \; m \to \infty$$

Wavelet

$$||f - \tilde{f}_m^W|| \propto m^{-1}, \; m \to \infty$$

Optimal data adaptive

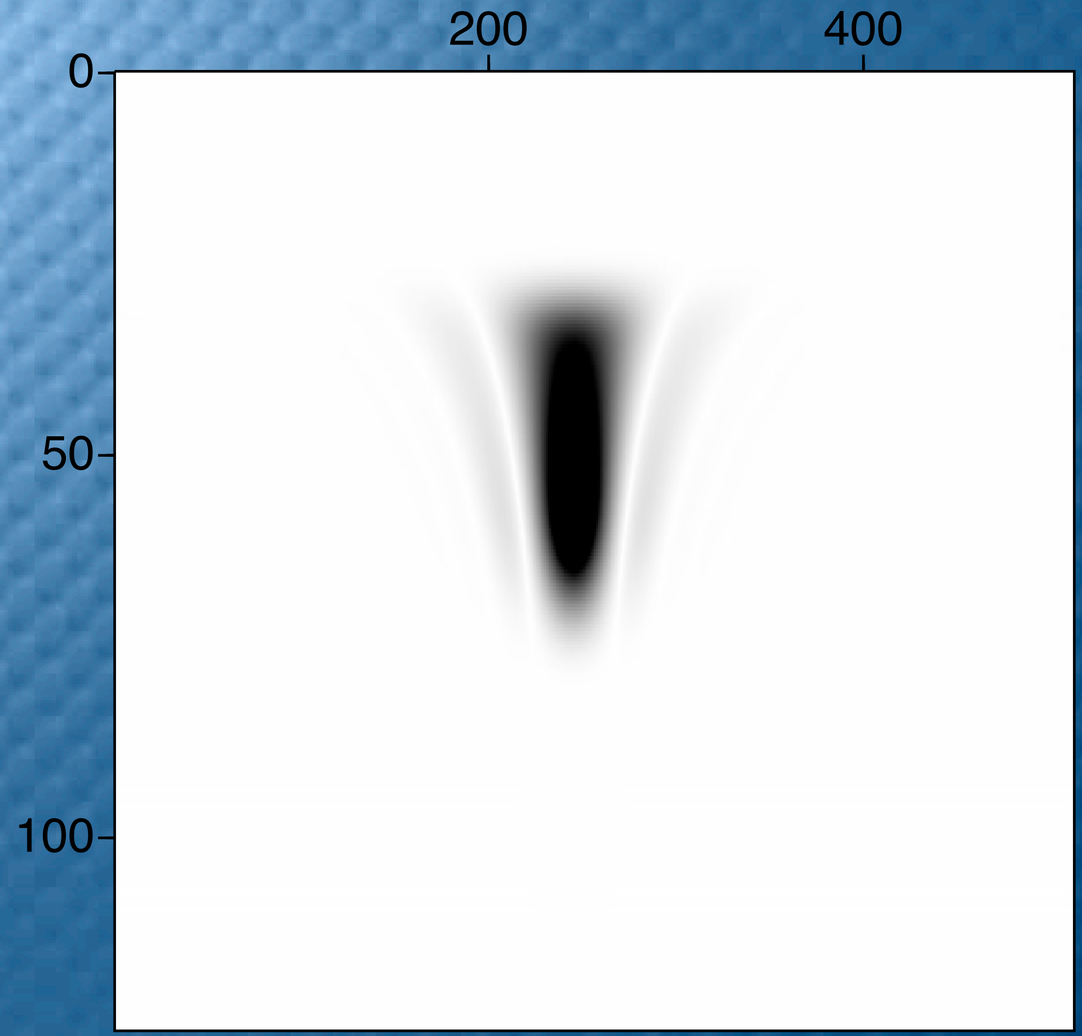
$$||f - \tilde{f}_m^A|| \propto m^{-2}, \; m \to \infty$$

Close to optimal Curvelet

$$||f - \tilde{f}_m^C|| \le C \cdot m^{-2} (\log m)^3, \; m \to \infty$$

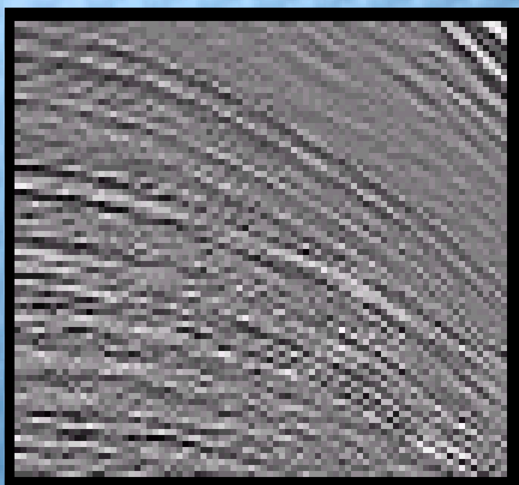
source: Candes'01, Stein '90

Why curvelets



Curvelet in FK-domain

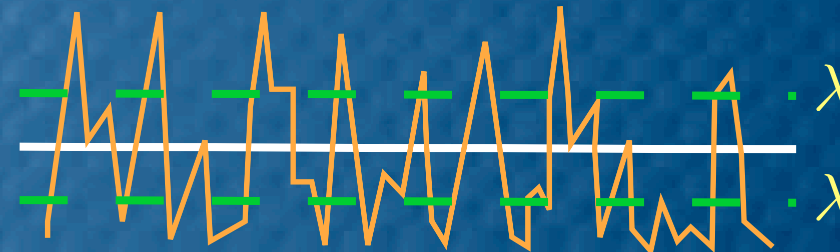
Denoising



Input data with noise

Curvelet transform

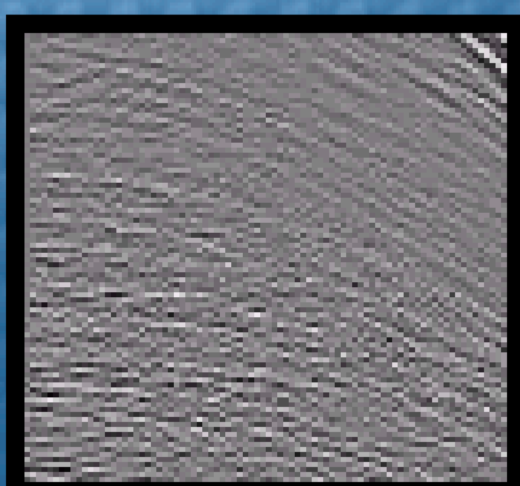
$\mathbf{Bd}$



Curvelet coeff. data

Threshold

$S_\lambda(\mathbf{Bd})$

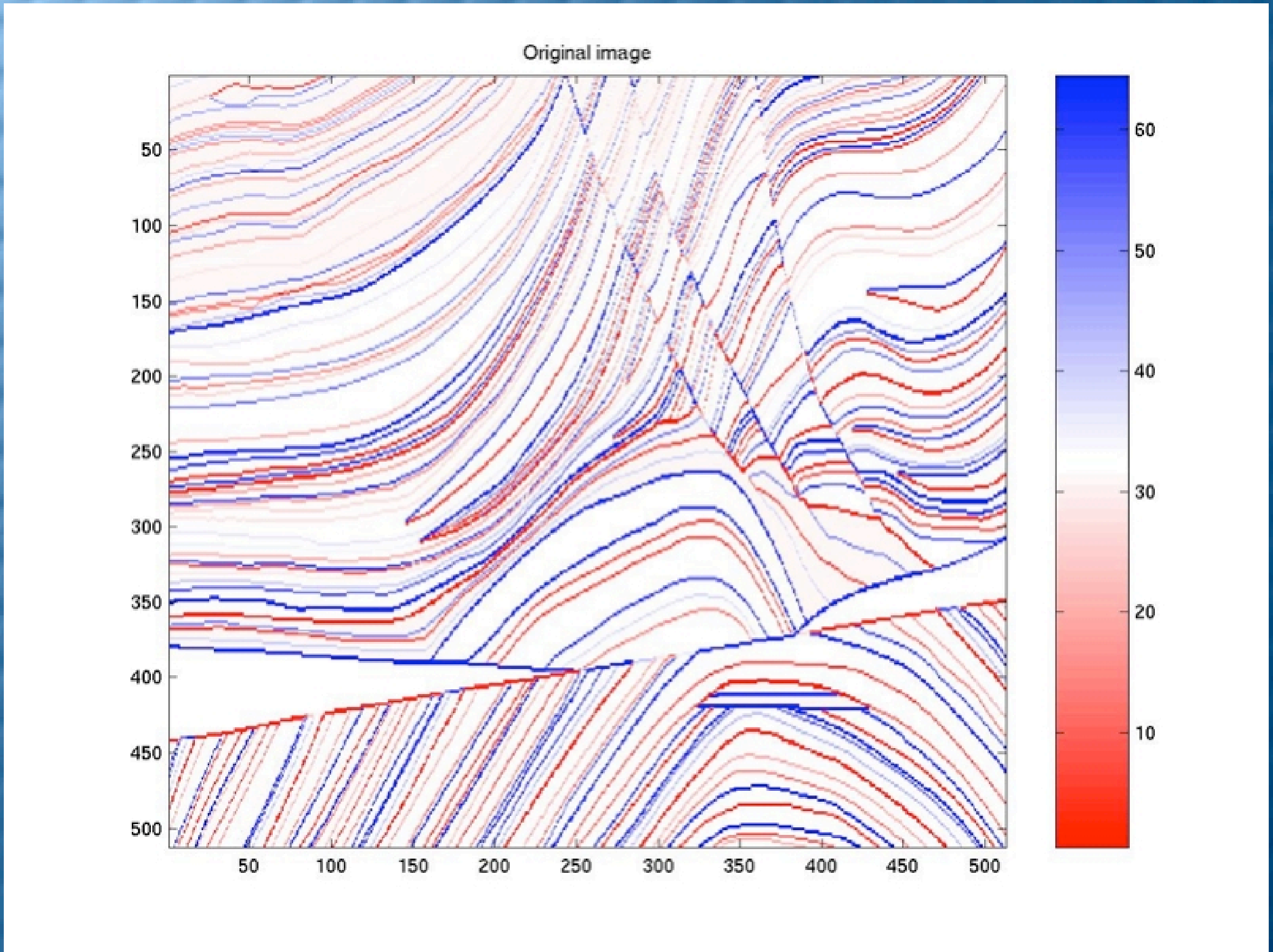


Filtered input data

Inv. curvelet transform

denoised Curvelet coeff.

Denoising



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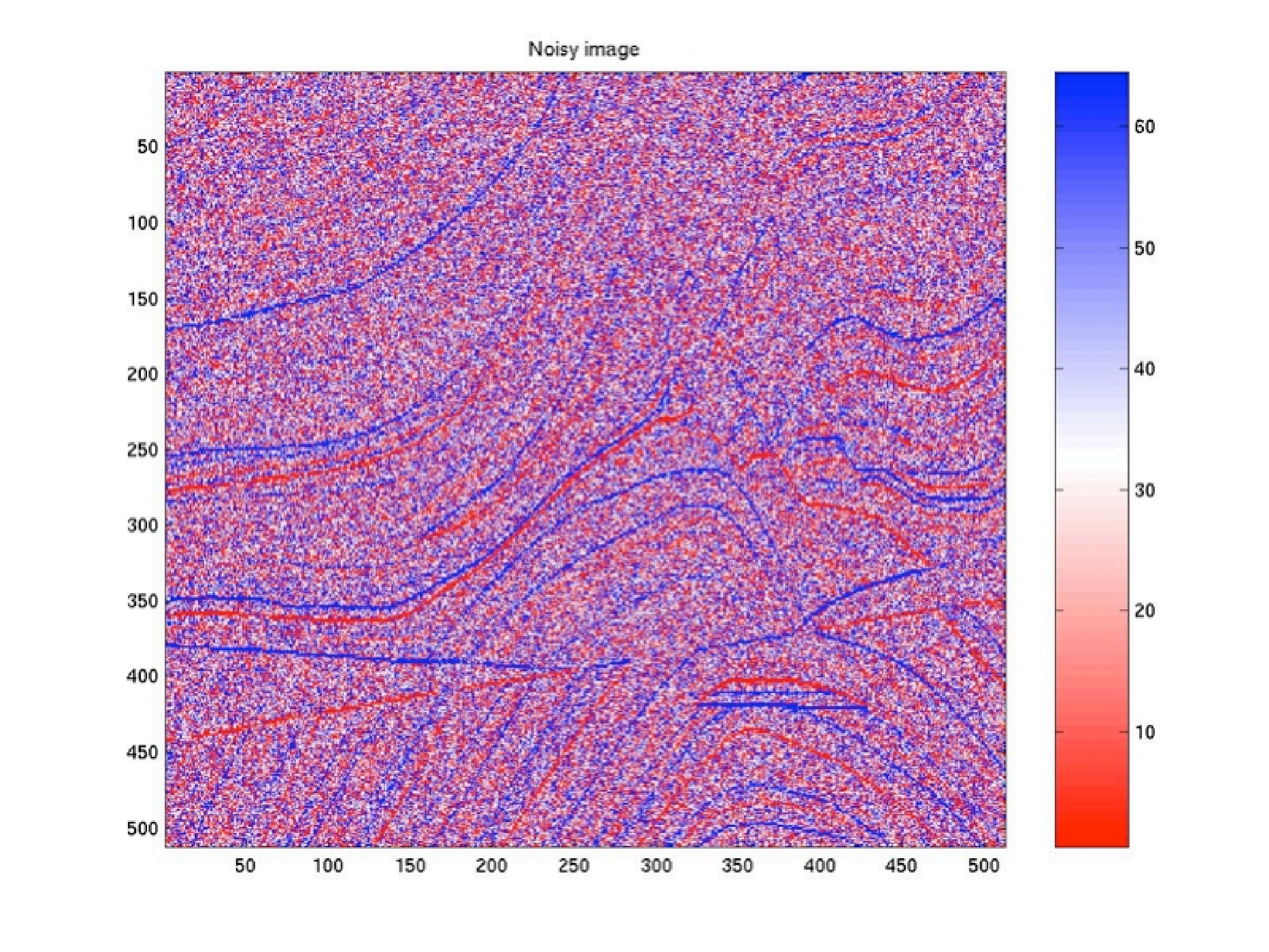
Denoising


Noisy image
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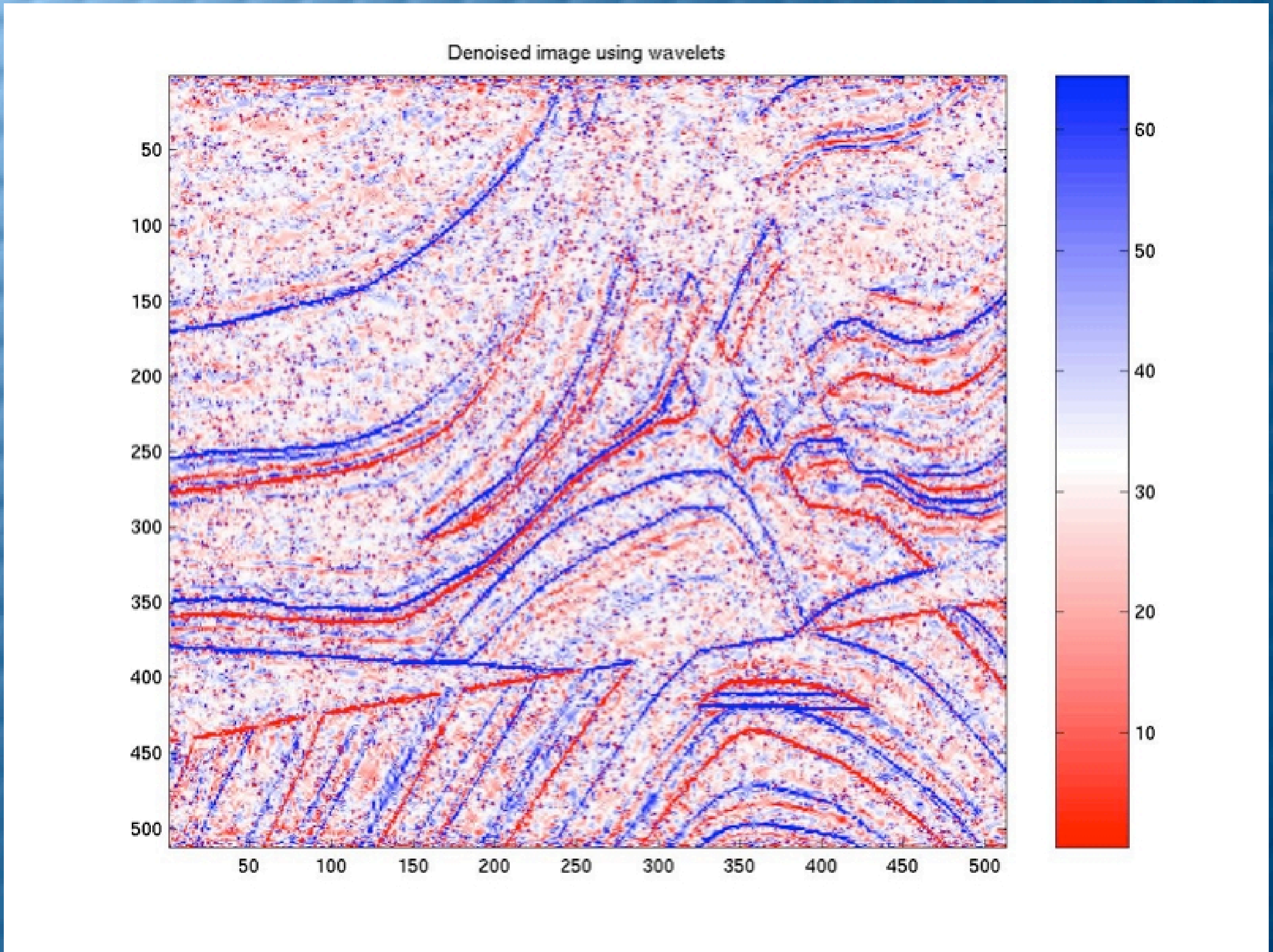
Wavelets



Denoised image using wavelets
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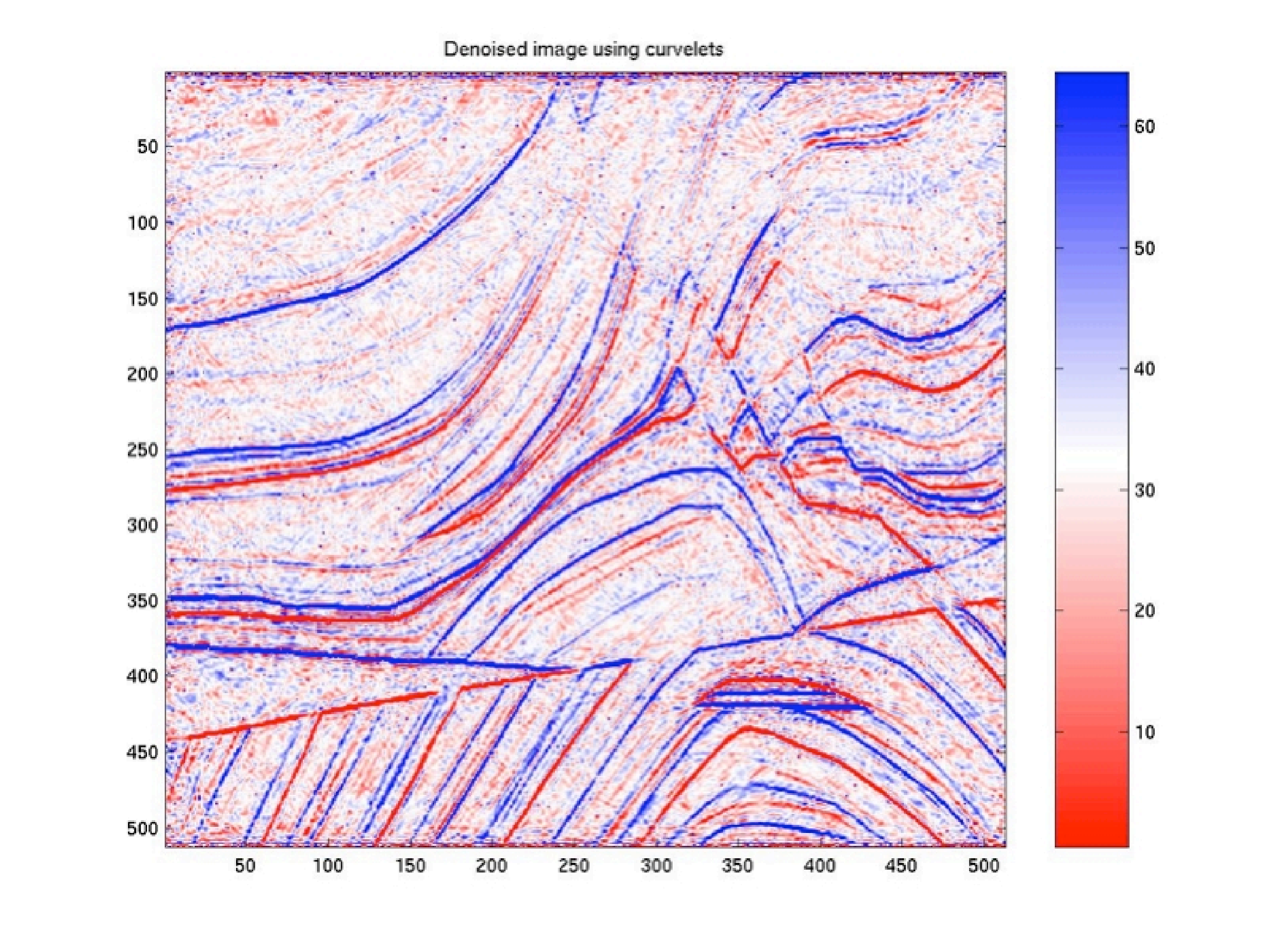
Curvelets


Denoised image using curvelets
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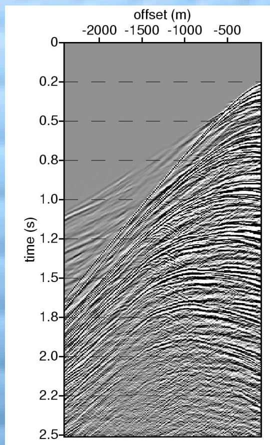
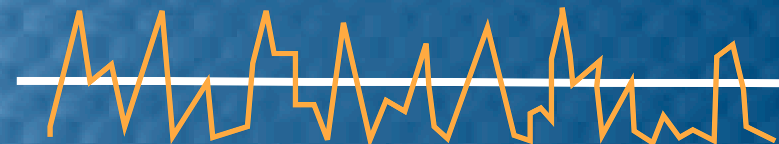
Coherent noise suppression with curvelets



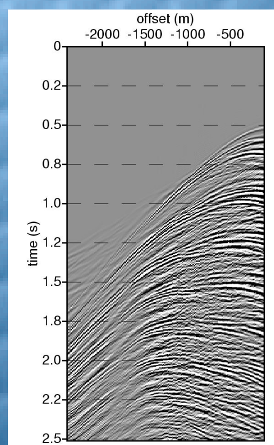
Input data with noise

Curvelet transform



Curvelet coeff. data

$\mathbf{Bd}$



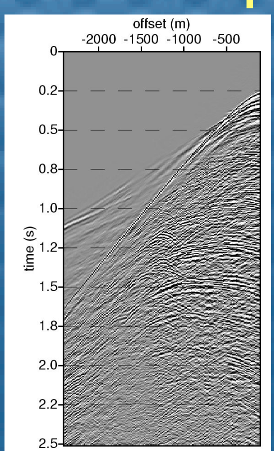
predicted noise

Threshold

Γ

$|\mathbf{B}\hat{\mathbf{n}}|$

Curvelet coeff. pred. noise



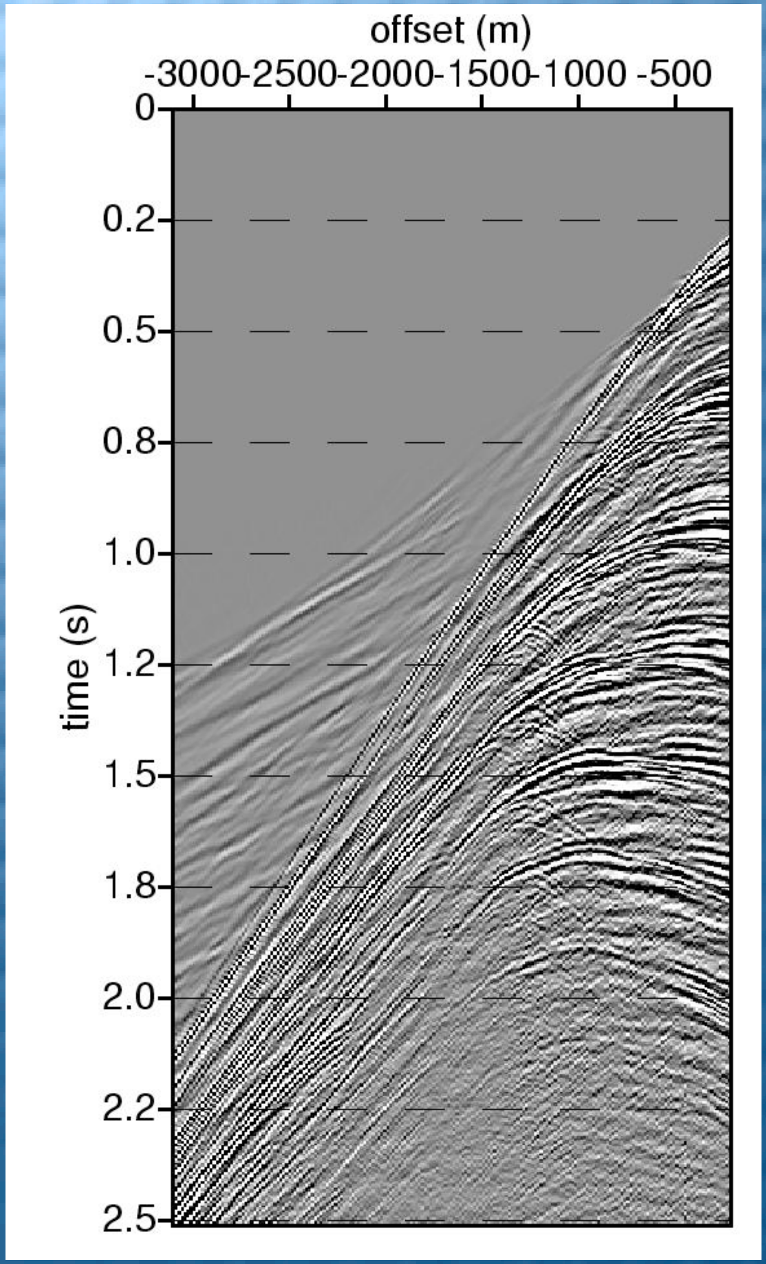
Filtered input data

Inv. curvelet transform

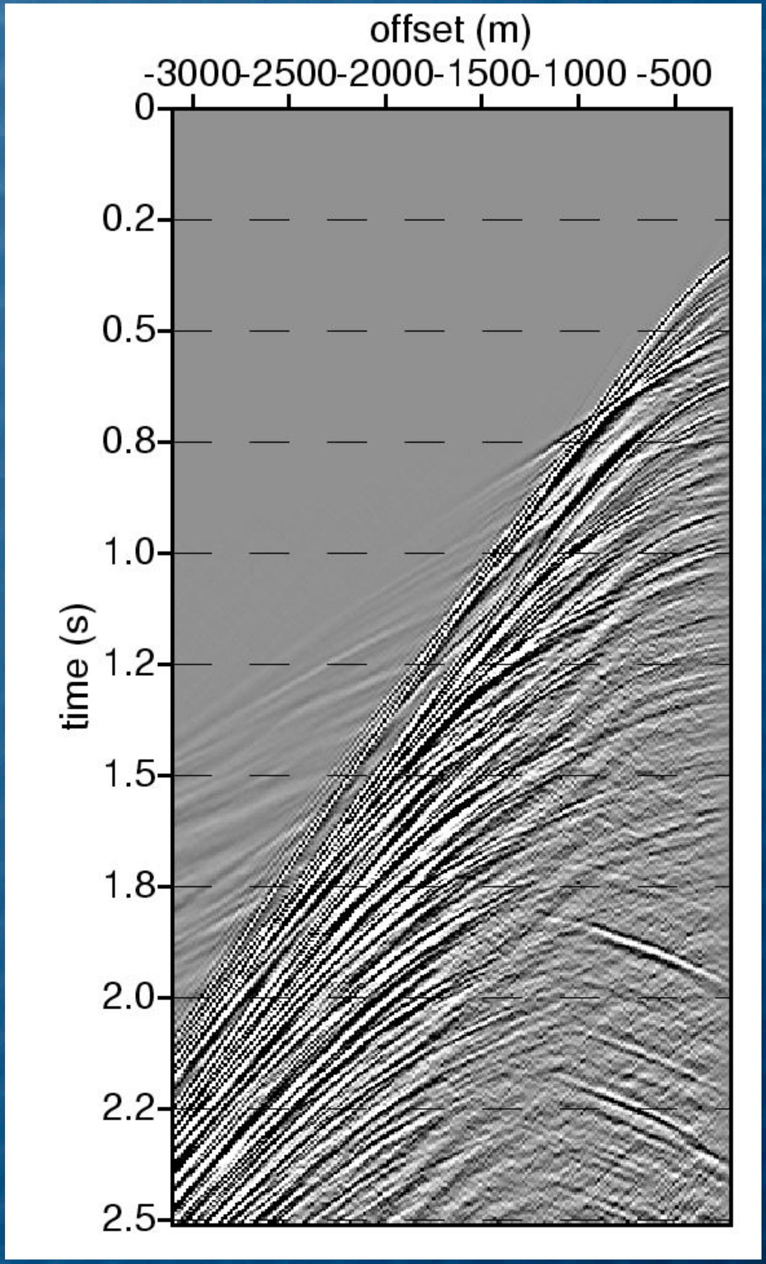
$S_{\lambda\Gamma}(\mathbf{Bd})$

Curvelet coeff. model

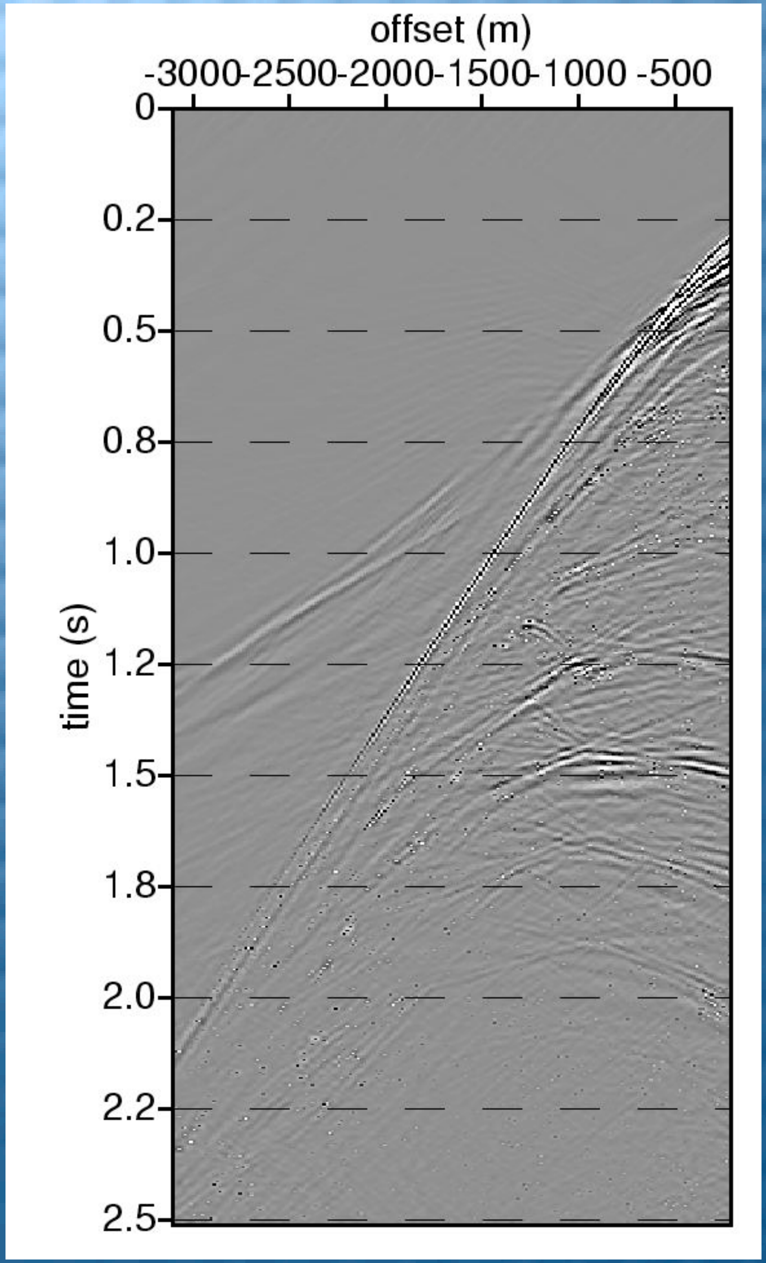
Multiple suppression with curvelets



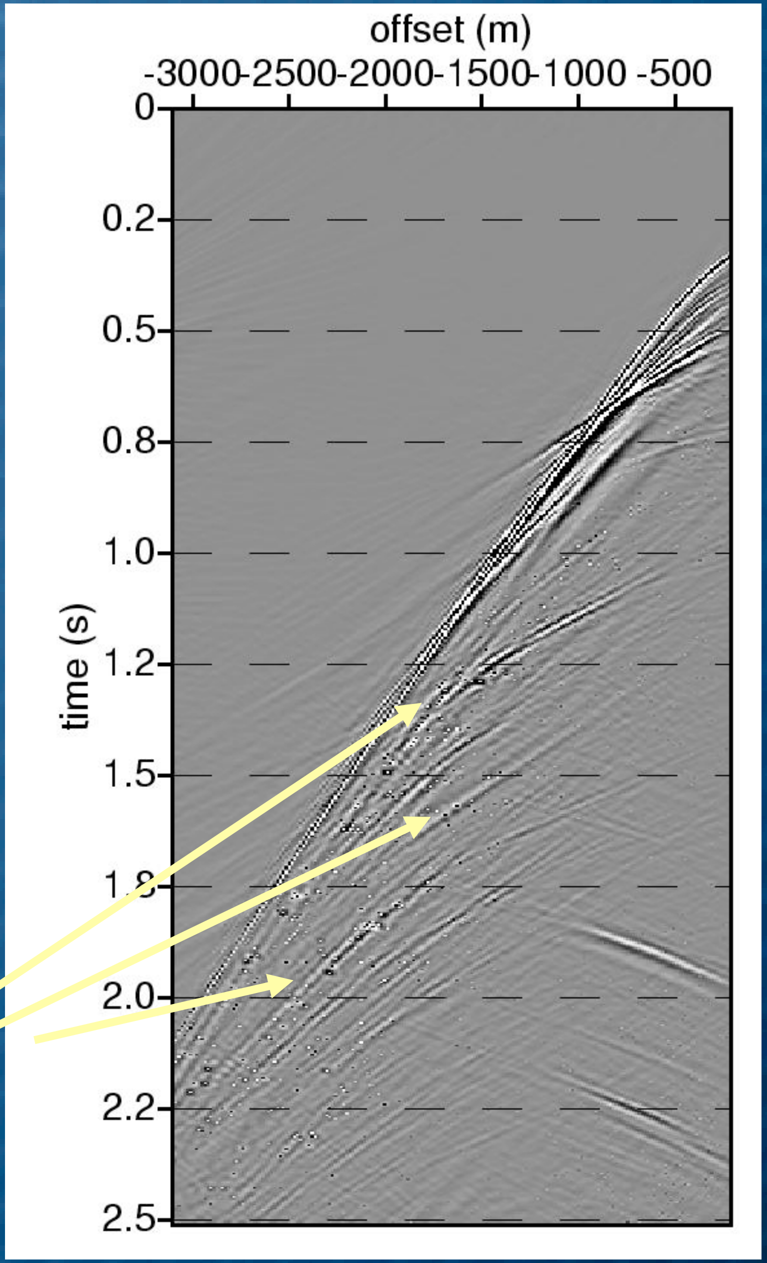
Input with multiples



Multiple suppression with curvelets



Output curvelet filtering with stronger threshold



Preserved primaries

Denoising

Denoising ⇔ *signal separation*

- **Multiple & Ground-roll removal**
- **Migration denoising (H & M '04)**
- **4-D difference cubes**

Model "noise"

Strategies:

- **Weighted thresholding**
- **Iterated weighted thresholding**

L2/L1-matched filter

Matched filter:

$$\overbrace{\hat{\mathbf{n}}}^{\text{denoised}} : \min_{\mathbf{\Phi}} = \| \underbrace{\mathbf{d}}_{\text{noisy data}} - \overbrace{\mathbf{\Phi}}^{\text{matched filter}} \overset{t}{*} \underbrace{\mathbf{m}}_{\text{pred. noise}} \|_p$$

- **p=1 enhances sparseness**
- **residue is the denoised data**
- **risk of over fitting**

Loose primary reflection events ...

Guitton&
Verschuur'04

Colored denoising

$$\overbrace{\mathbf{d}}^{\text{noisy data}} = \underbrace{\mathbf{m}}_{\text{noise-free}} + \overbrace{\mathbf{n}}^{\text{col. noise}}$$

Denoising:

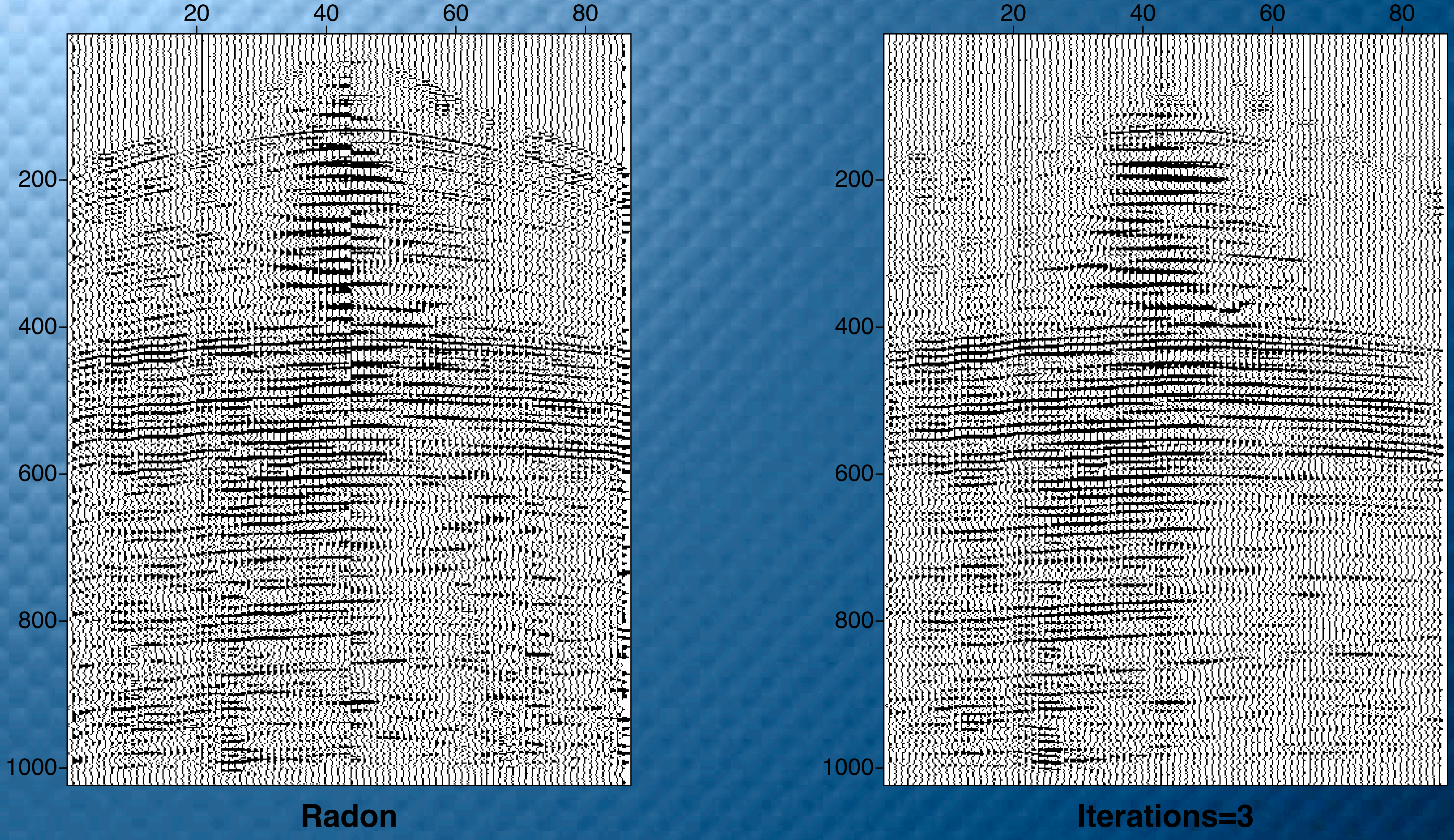
$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \frac{1}{2}\|\mathbf{C}_n^{-1/2}(\mathbf{d}-\mathbf{m})\|_2^2 + J(\mathbf{m})$$

with covariance

$$\mathbf{C}_n \equiv \mathbf{E}\{\mathbf{n}\mathbf{n}^T\}$$

and both m, n **related to PDE**

Ground-roll removal with curvelets



Weighted thresholding

Covariance model & noise near diagonal:

$$\mathbf{B}\mathbf{C}_{n\,\mathrm{or}\,m}\mathbf{B}^T \approx \mathbf{\Gamma}^2 \quad \text{near diagonal}$$

For ortho basis and app. noise prediction:

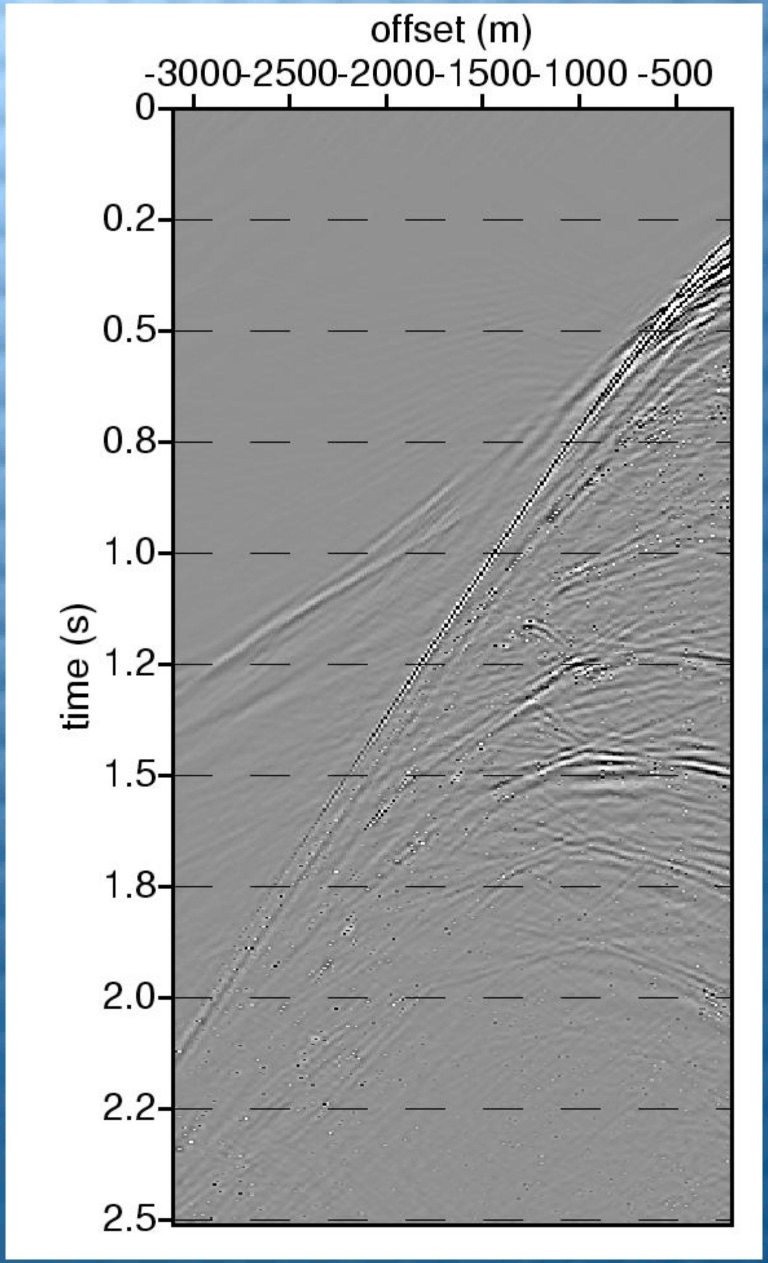
$$\boxed{\hat{m} = \mathbf{B}^T S_{\lambda\Gamma}\left(\mathbf{Bd}\right)}$$

equivalent to

$$\boxed{\hat{\mathbf{m}} = \mathbf{B}^T \arg\min_{\tilde{m}} \frac{1}{2}\|\mathbf{\Gamma}^{-1}\left(\tilde{\mathbf{d}} - \tilde{\mathbf{m}}\right)\|_2^2 + \|\tilde{\mathbf{m}}\|_{1,\lambda}}$$

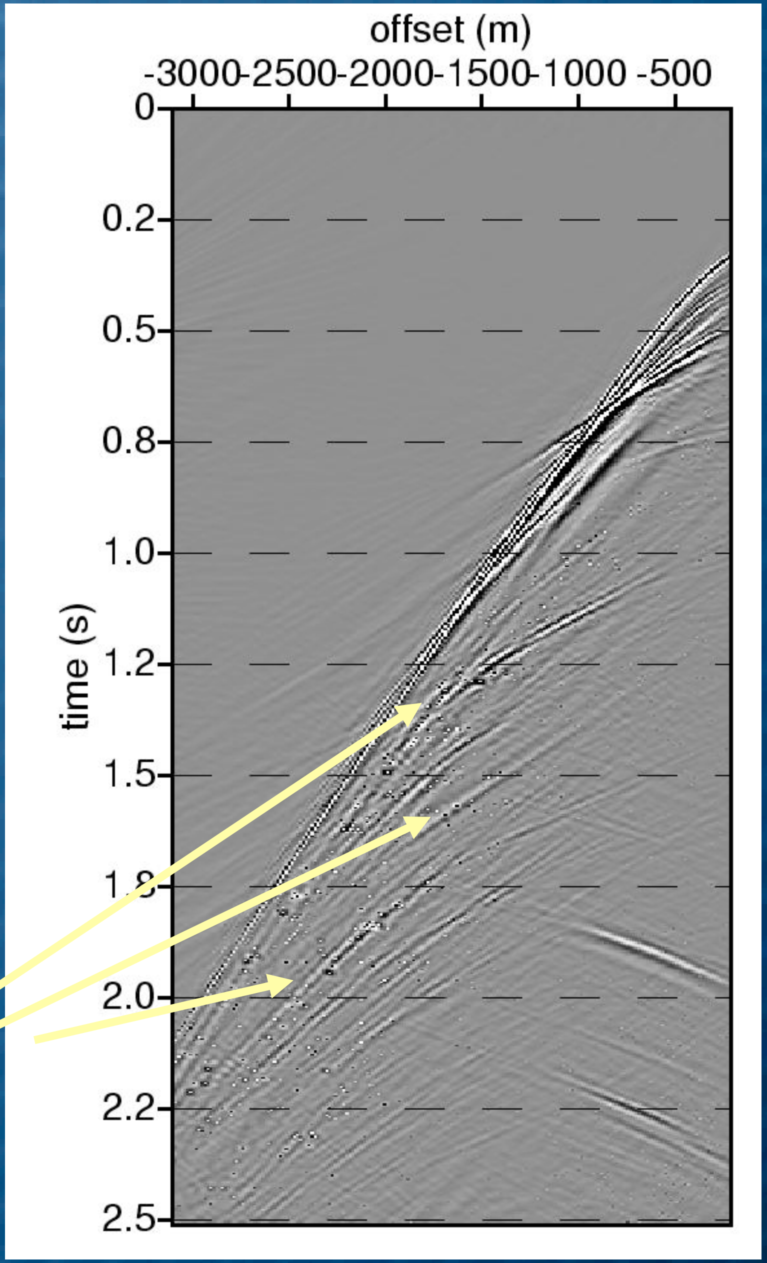
$$\tilde{\mathbf{d}} = \mathbf{Bd}, \quad \tilde{\mathbf{m}} = \mathbf{Bm} \quad \text{and} \quad \mathbf{\Gamma} = [\mathrm{diag}\{\mathrm{diag}\{\mathbf{B}\hat{\mathbf{n}}\}\}]^{1/2}$$

Multiple suppression with curvelets

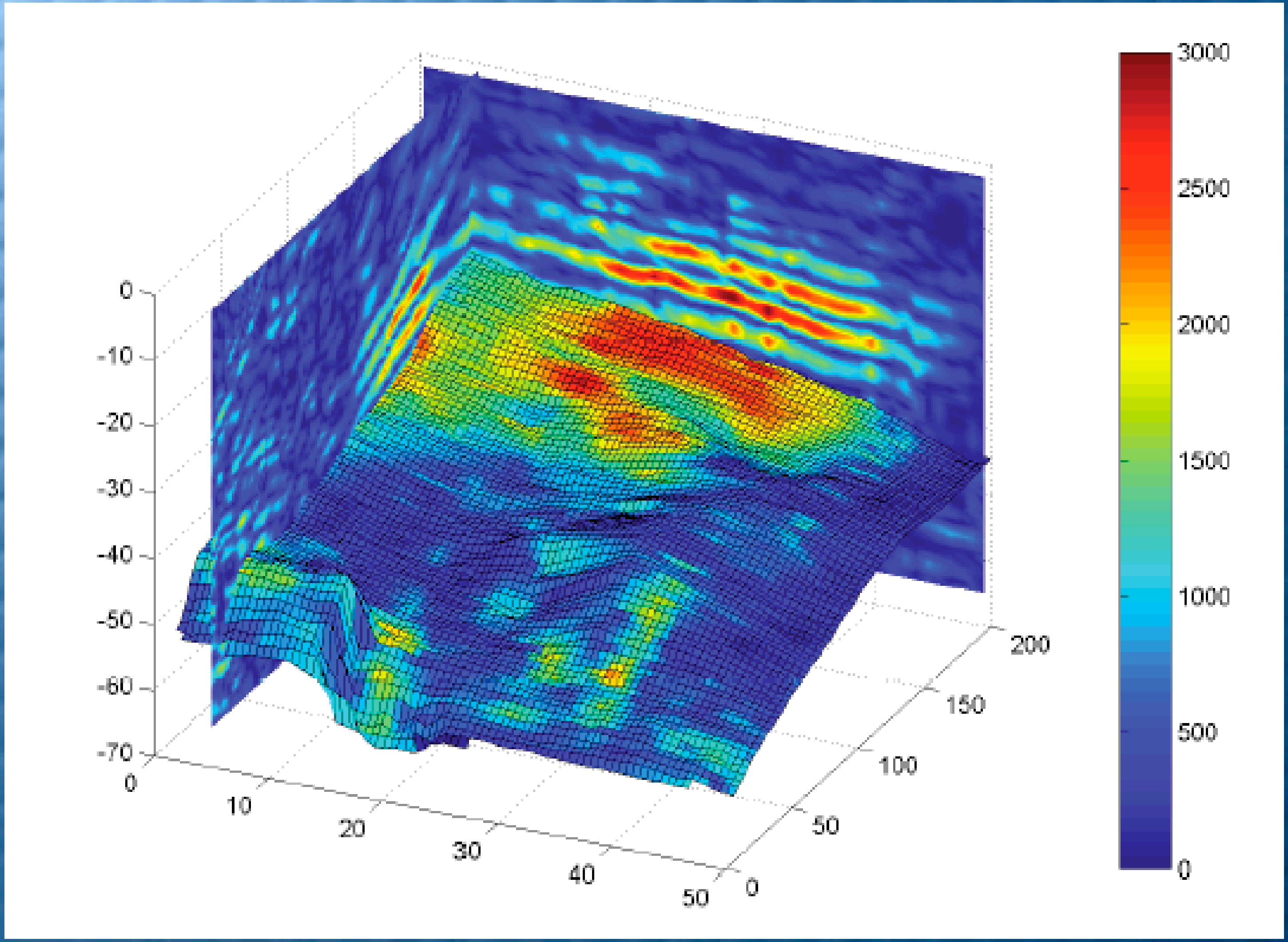


Output curvelet filtering with stronger threshold

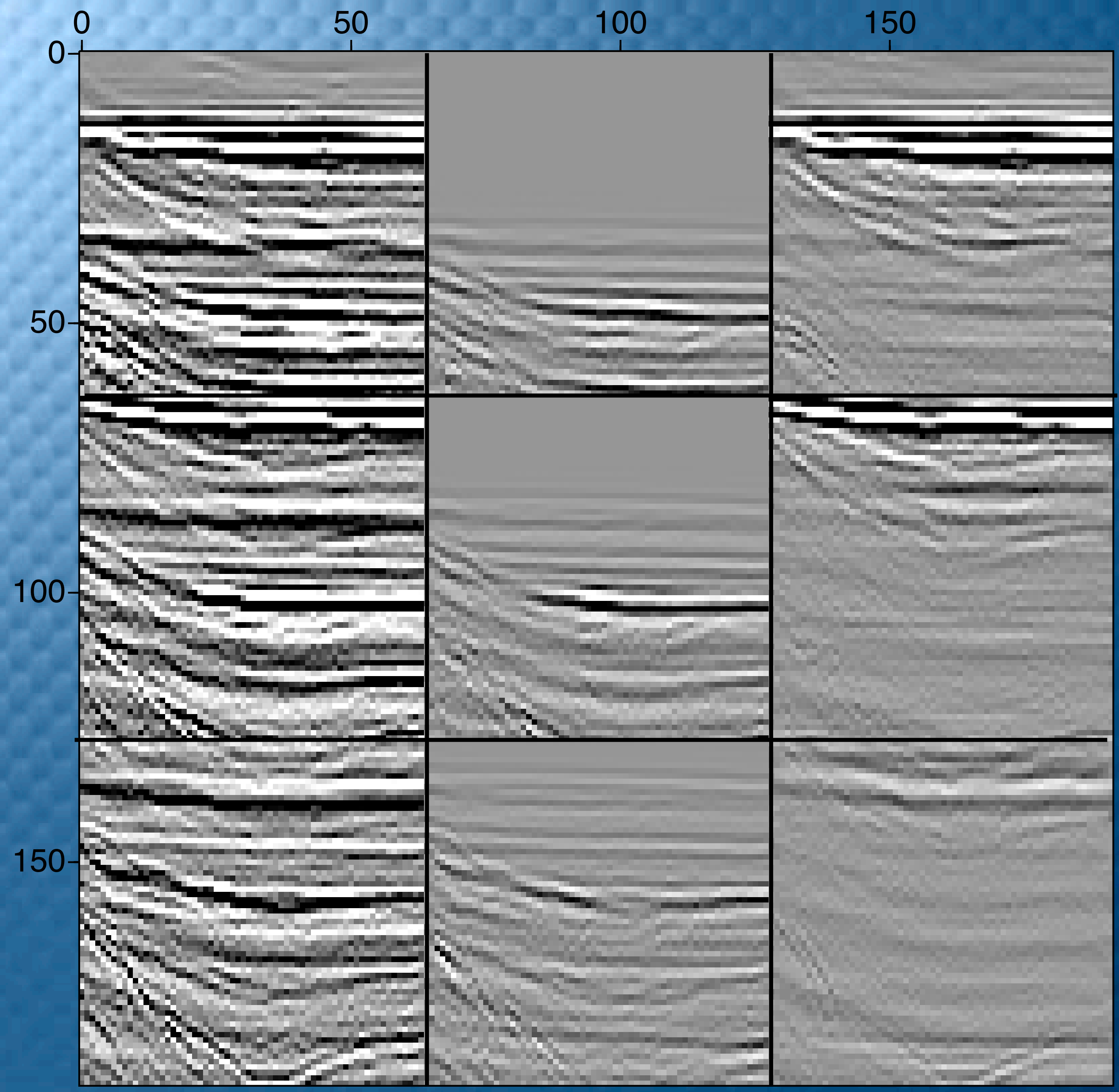
Preserved primaries



4-D difference



3-D Curvelets



Iterative thresholding

Curvelets are Frames:

- **redundant (factor 7.5-4)**
- **thresholding does *not* solve:**

$$\hat{\mathbf{m}} = \mathrm{B}^T \arg\min_{\tilde{m}} \frac{1}{2} \|\boldsymbol{\Gamma}^{-1} \left(\tilde{\mathbf{d}} - \tilde{\mathbf{m}}\right) \|_2^2 + \|\tilde{\mathbf{m}}\|_{1,\lambda}$$

Alternative formulation by iterative thresholding!

Iterative thresholding

Solves signal-separation problem:

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{n} \quad \text{and} \quad \mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$$

by minimizing LP-program

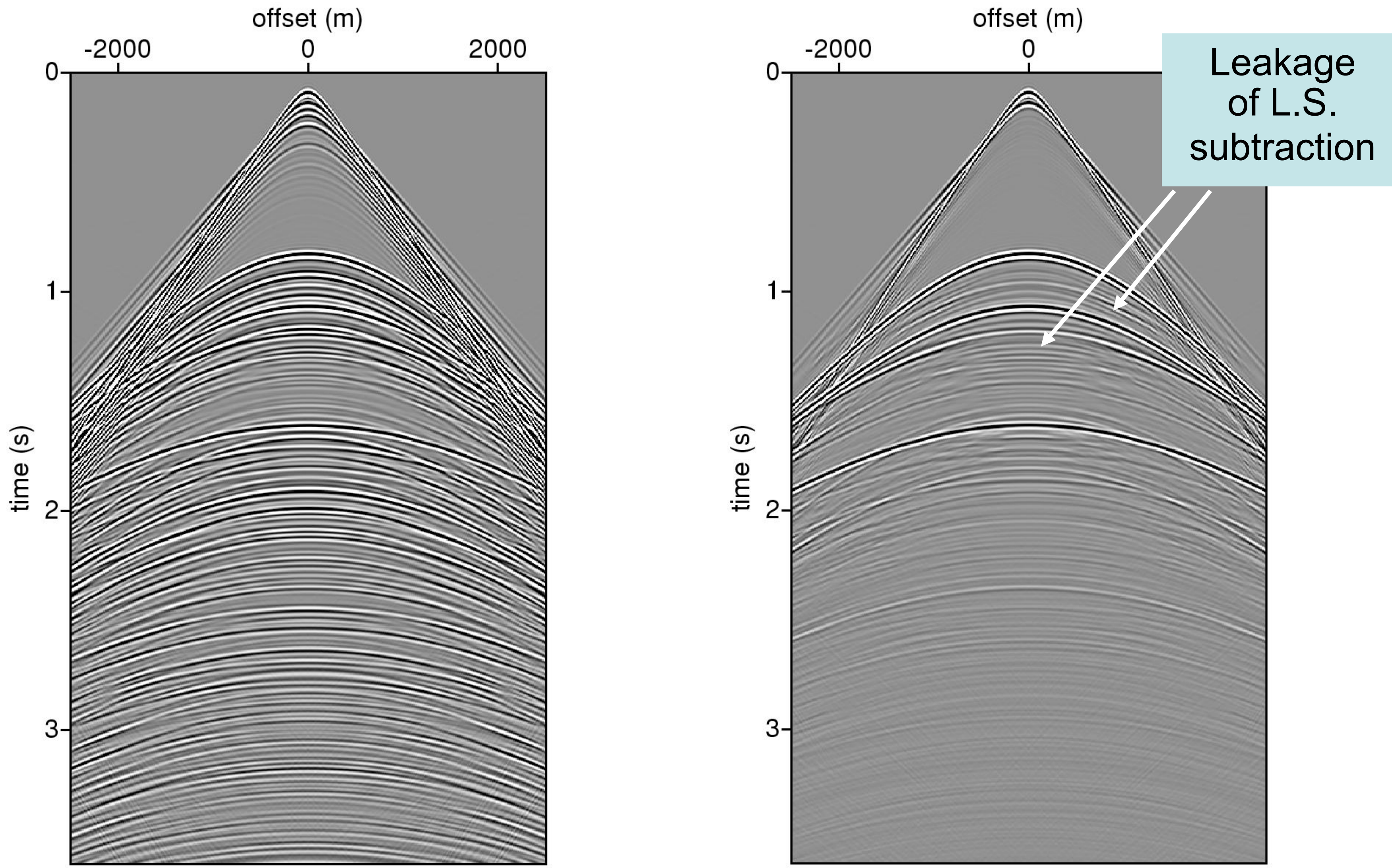
$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{s} - \mathbf{\Phi}\mathbf{x}\|_2^2 + \|\mathbf{x}_1\|_{w_1,1} + \|\mathbf{x}_2\|_{w_2,1}$$

with

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}^T \quad \text{and} \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \end{bmatrix}$$

Starck *et al.;* Daubechies *et al., Donoho 2004*
(also Zwartjes, Sacchi, Trad & Ulrich)

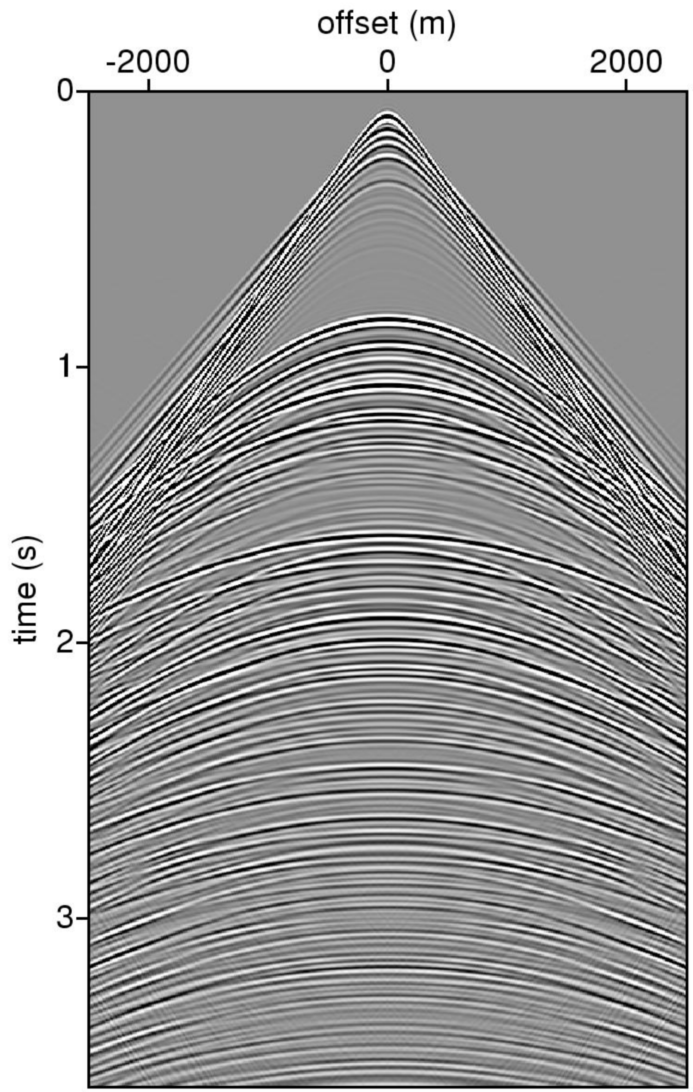
Example shallow water environment



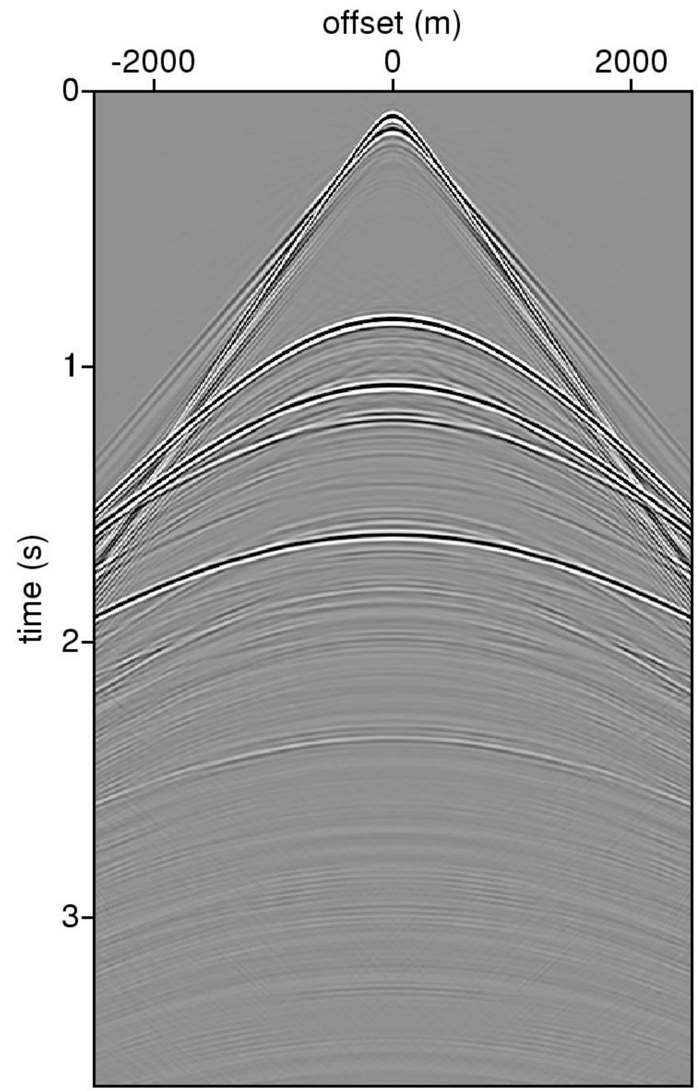
Input data with multiples

Result L.S. subtraction

Example shallow water environment

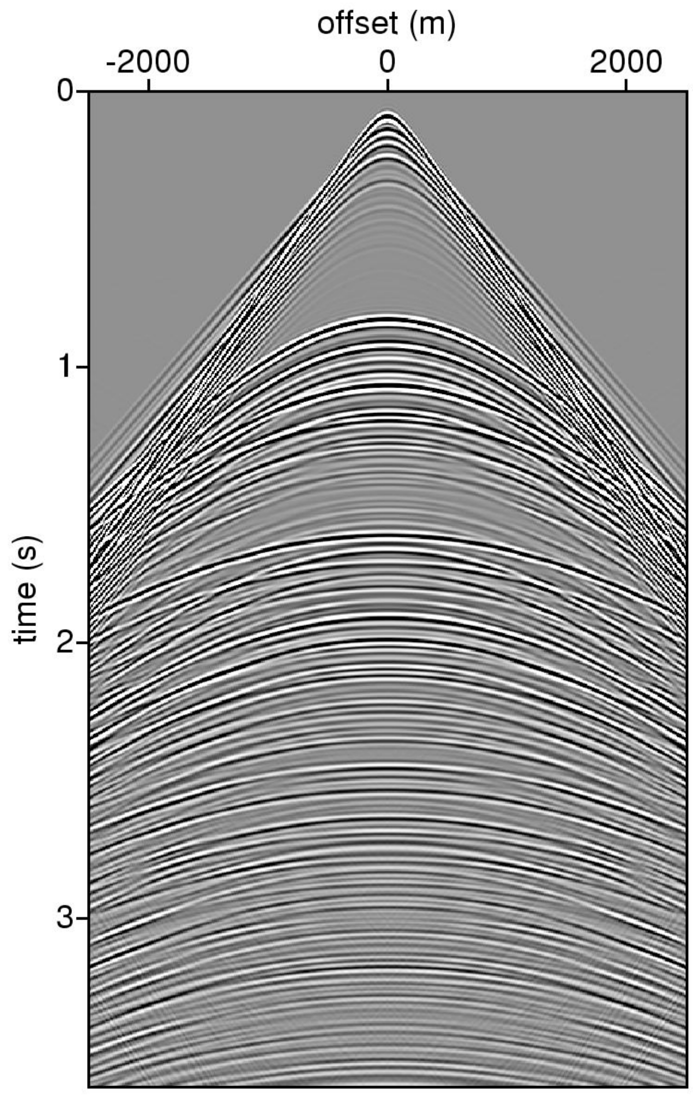


Input data with multiples

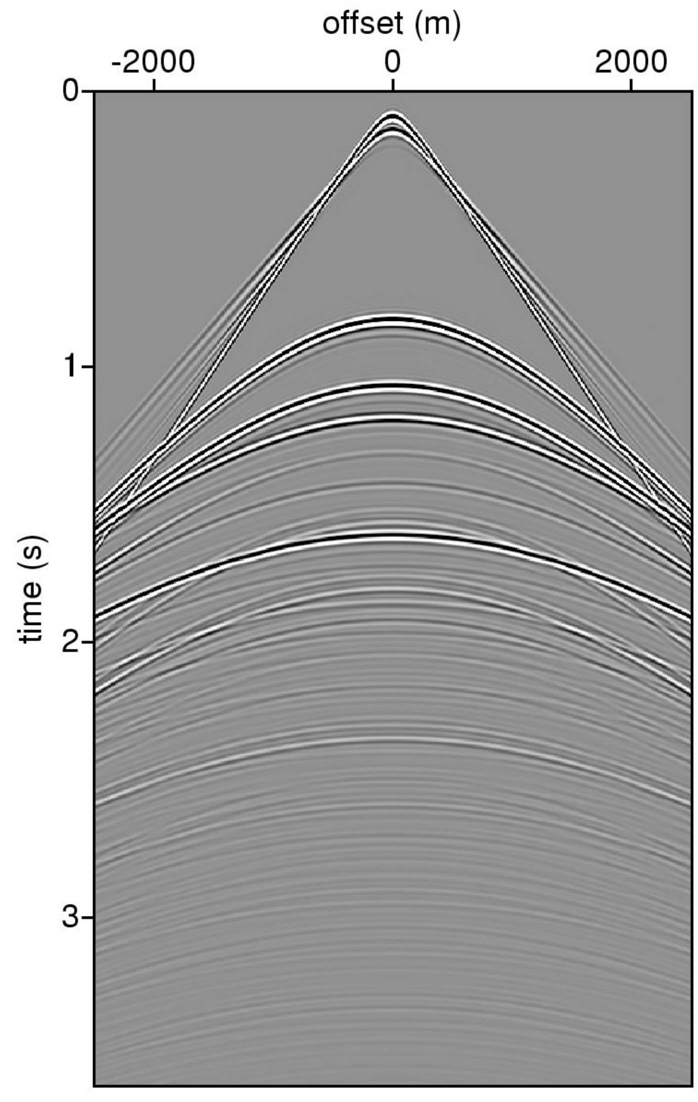


Result curvelet-based subtraction

Example shallow water environment

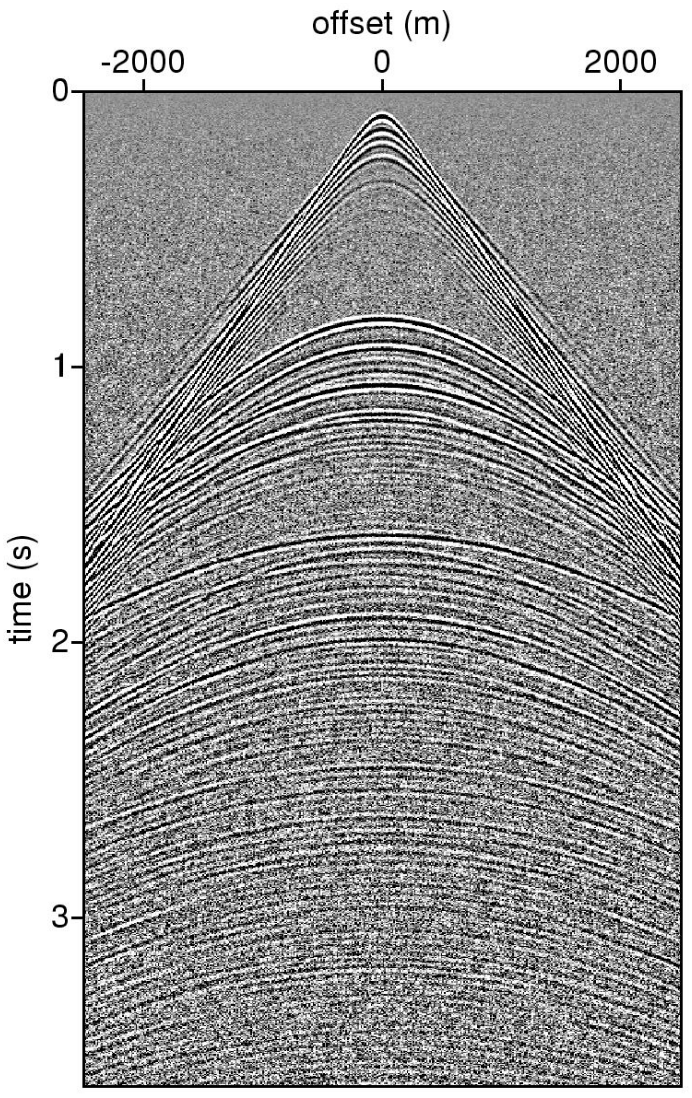


Input data with multiples

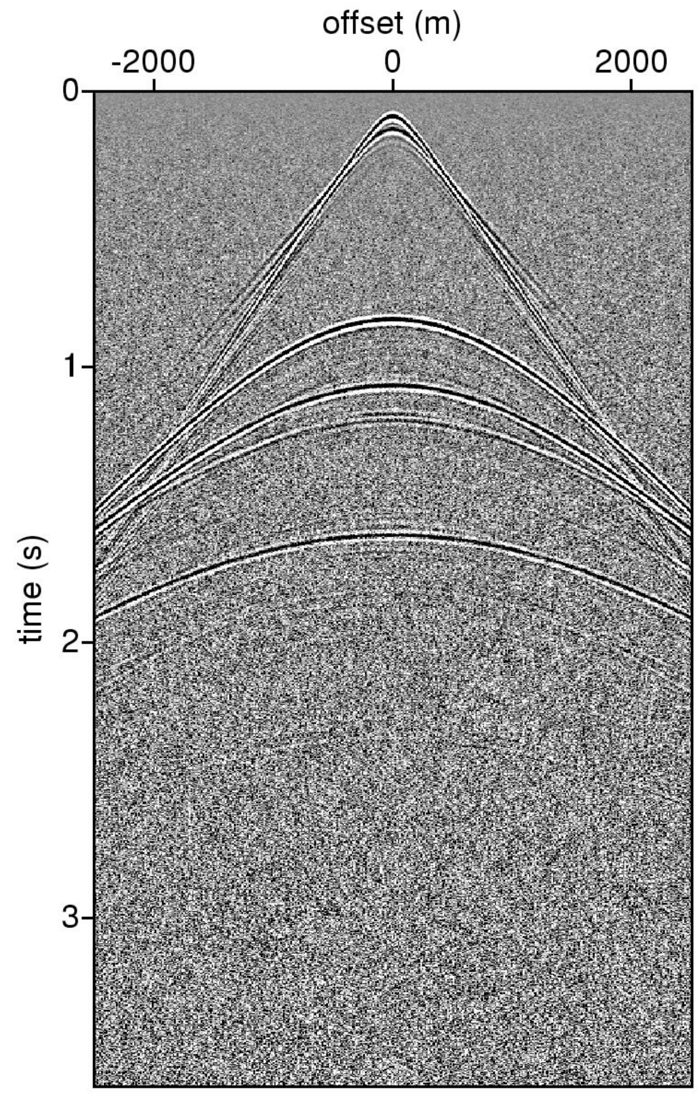


True primaries

Example shallow water environment



Input data with multiples and noise



Result curvelet-based subtraction

Global optimization

After thresholding:

- **remove artifacts & ‘miss fires’**
- **normal operator (inversion)**

Impose additional *penalty functional*

- ***prior* information**
- ***sparseness & continuity***

Defining the right norm is crucial ...

Global optimization

Formulate constrained optimization:

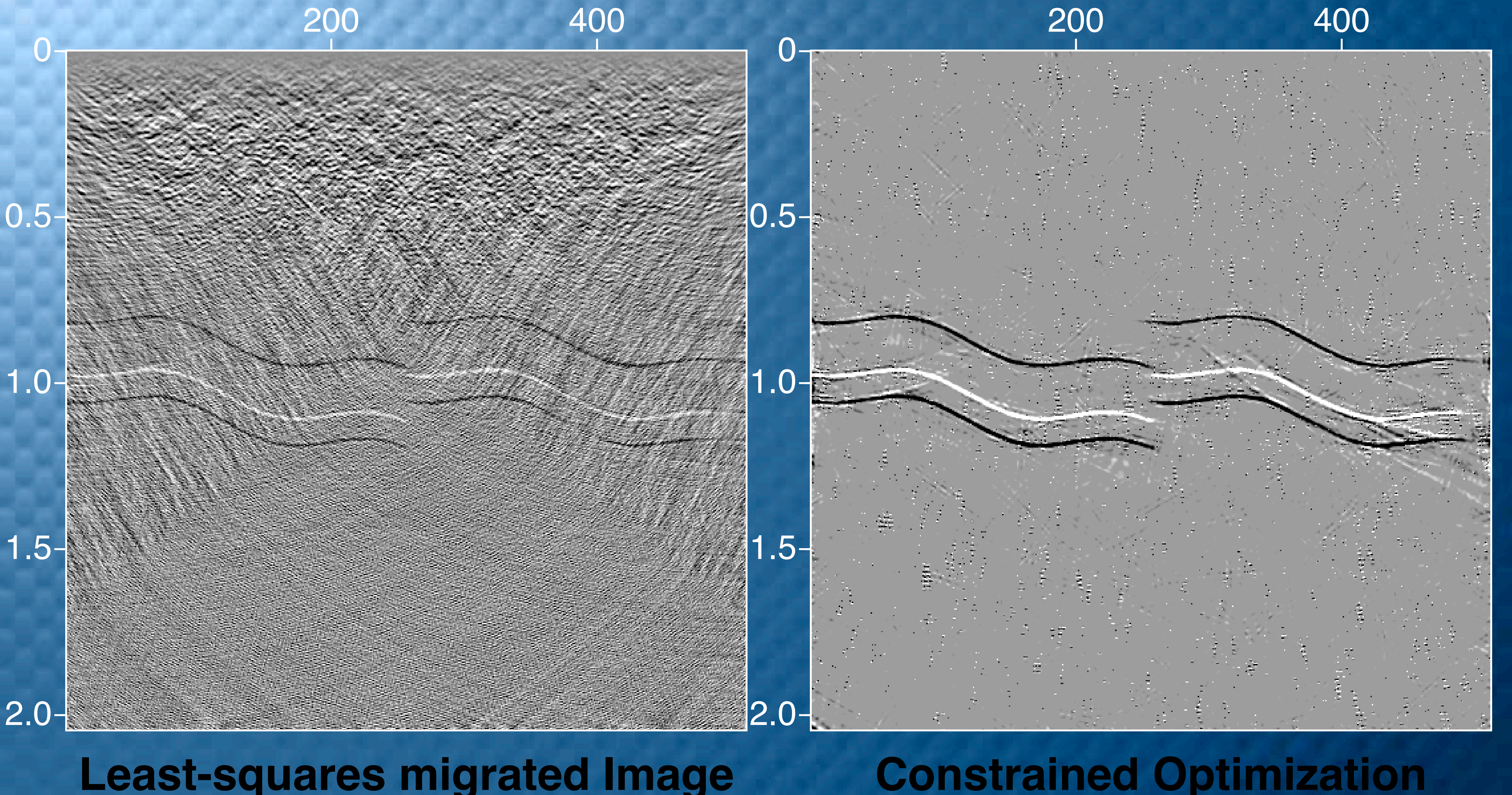
$$\hat{\mathbf{m}} : \quad \min_{m} J(\mathbf{m}) \quad \text{s.t.} \quad |\tilde{\mathbf{m}} - \hat{\tilde{\mathbf{m}}}_0|_\mu \leq \mathbf{e}_\mu, \quad \forall \mu$$

with

$$\hat{\mathbf{m}}_0 = \mathbf{B}^\dagger \Theta_{\boldsymbol{\lambda}\boldsymbol{\Gamma}} \left(\tilde{\mathbf{d}} \right)$$

and with e_μ threshold and noise-dependent *tolerance* on curvelet coeff.

Imaging with Curvelets



Penalty functionals

Anisotropic diffusion for imaging:

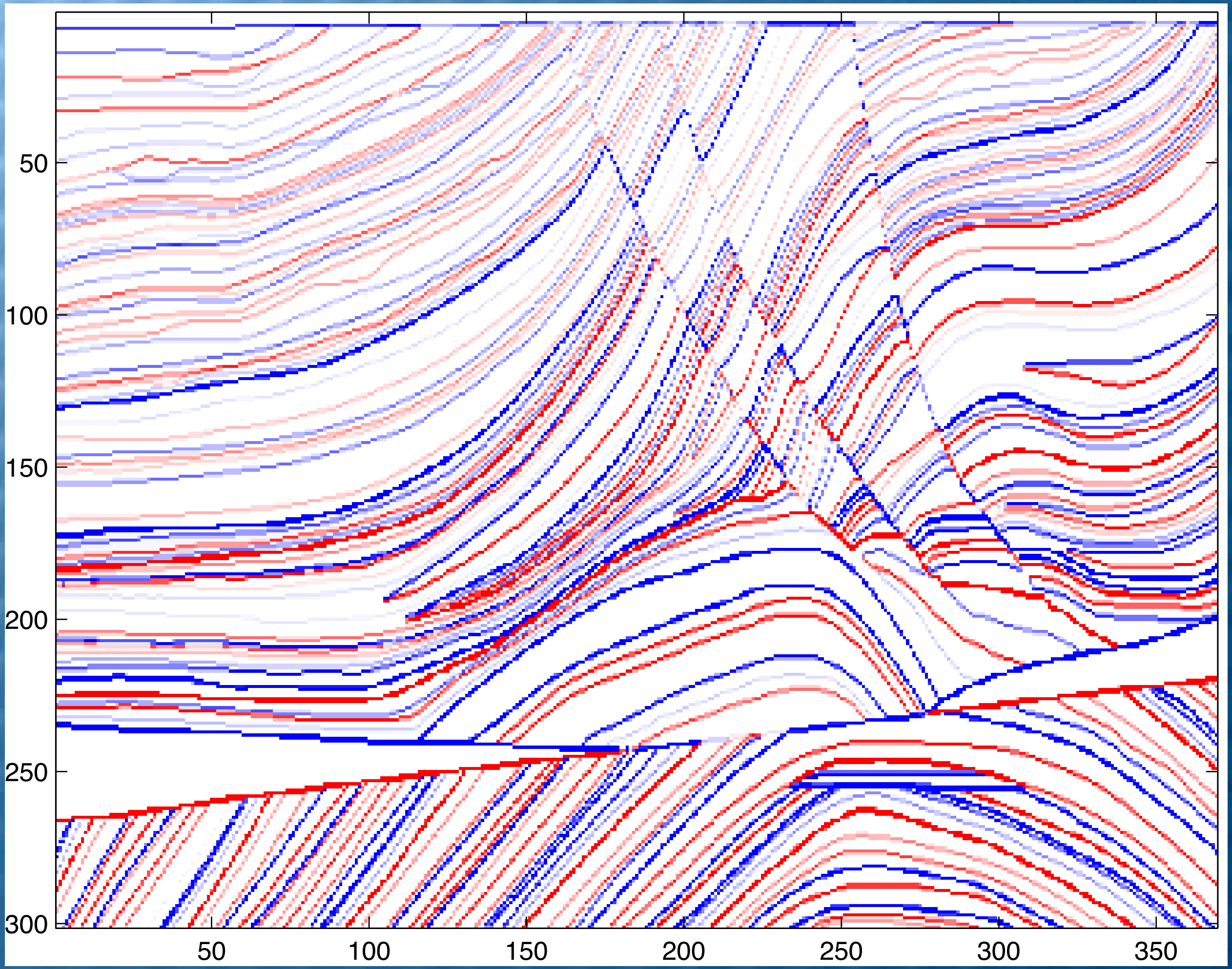
$$J(\mathbf{m}) = \|\mathbf{\Lambda}^{1/2}\mathbf{\nabla}\mathbf{m}\|_p$$

with

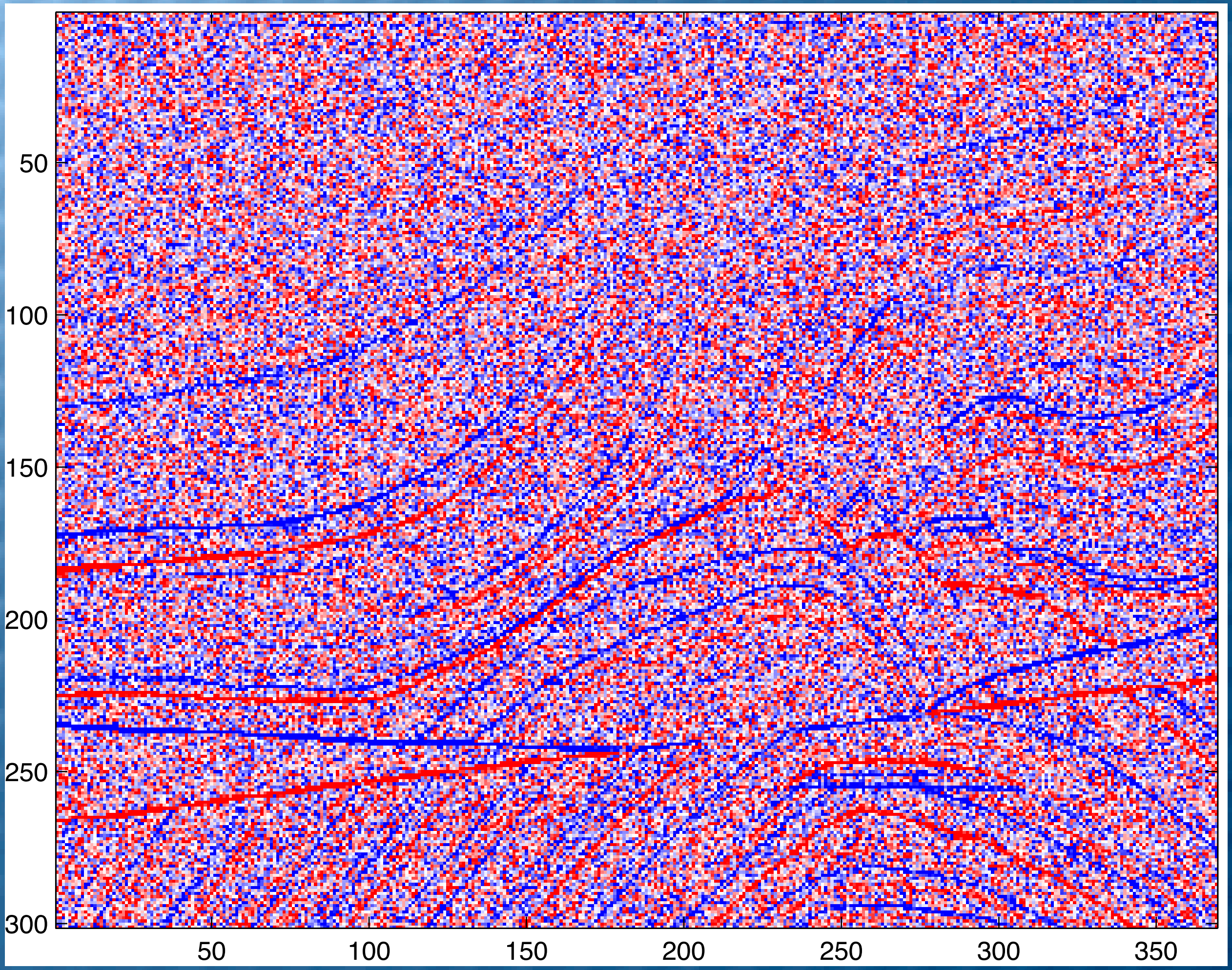
$$\mathbf{\Lambda}[\bar{c}] = \frac{1}{|\nabla\bar{c}|^2 + 2\beta_2}\left\{\begin{pmatrix}\partial_{x_2}\bar{c}\\ -\partial_{x_1}\bar{c}\end{pmatrix}\begin{pmatrix}\partial_{x_2}\bar{c} & -\partial_{x_1}\bar{c}\end{pmatrix} + \beta_2\mathbf{I}\right\}.$$

brings out wavefront set.

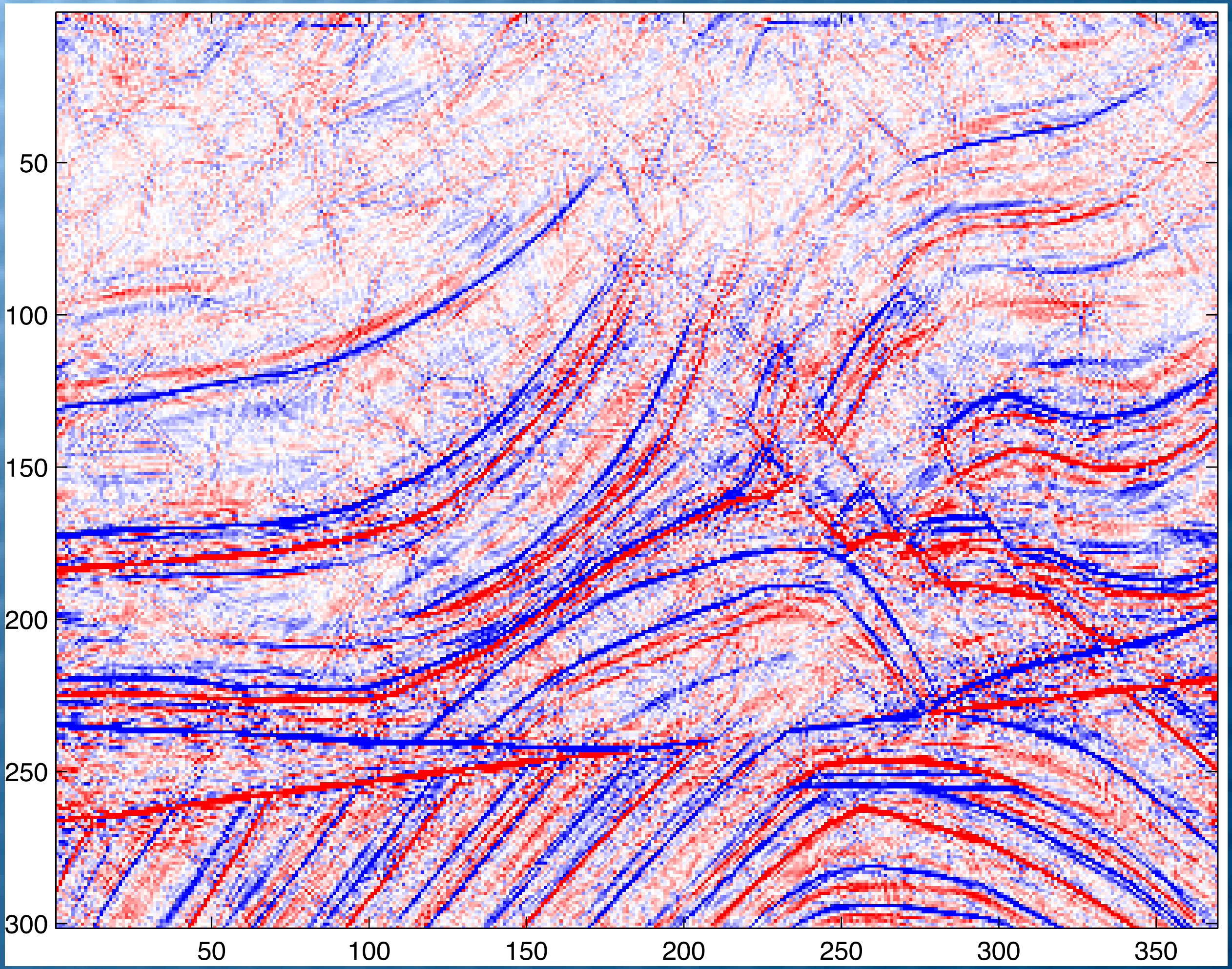
Denoising



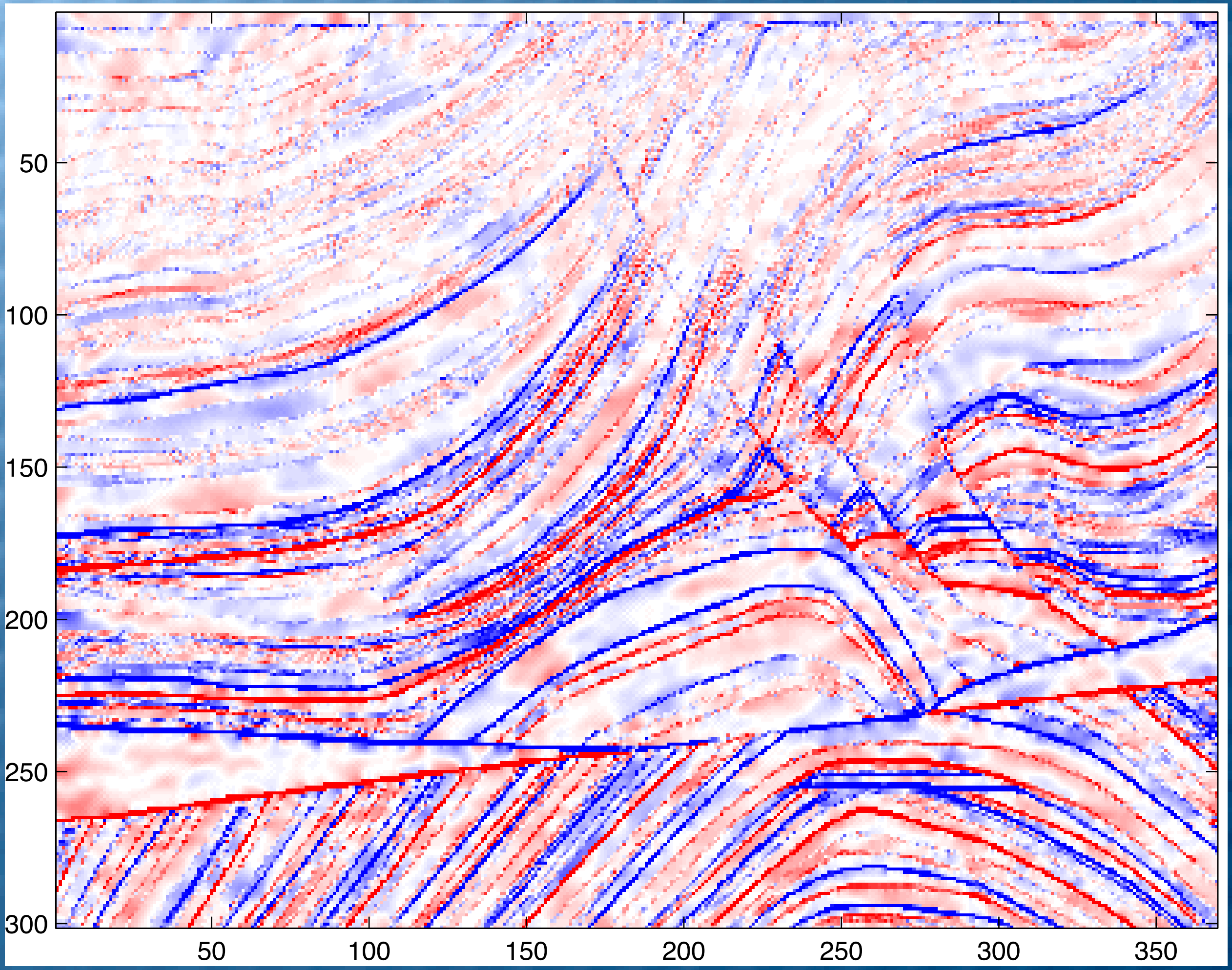
Denoising



Denoising



Denoising



Conclusions

Presented a framework that

- **Stable signal separation *via* thresholding:**
  - Ground-roll & Multiple removal (Wednesday)
  - Compute 4-D difference Cubes
- **improves imaging & inversion:**
  - deals with incoherent noise & missing data
  - sparseness constrained imaging (H & P ‘04)
  - extended to 3-D

Acknowledgements

Candes, Donoho, Demanet, Ying for making their Curvelet code available.

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NSERC Investing in people, discovery and innovation
CRSNG Investir dans les gens, la découverte et l'innovation