"Optimal" imaging with curvelets

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thanks to: Minh Do, Mauricio Sacchi, WaveLab

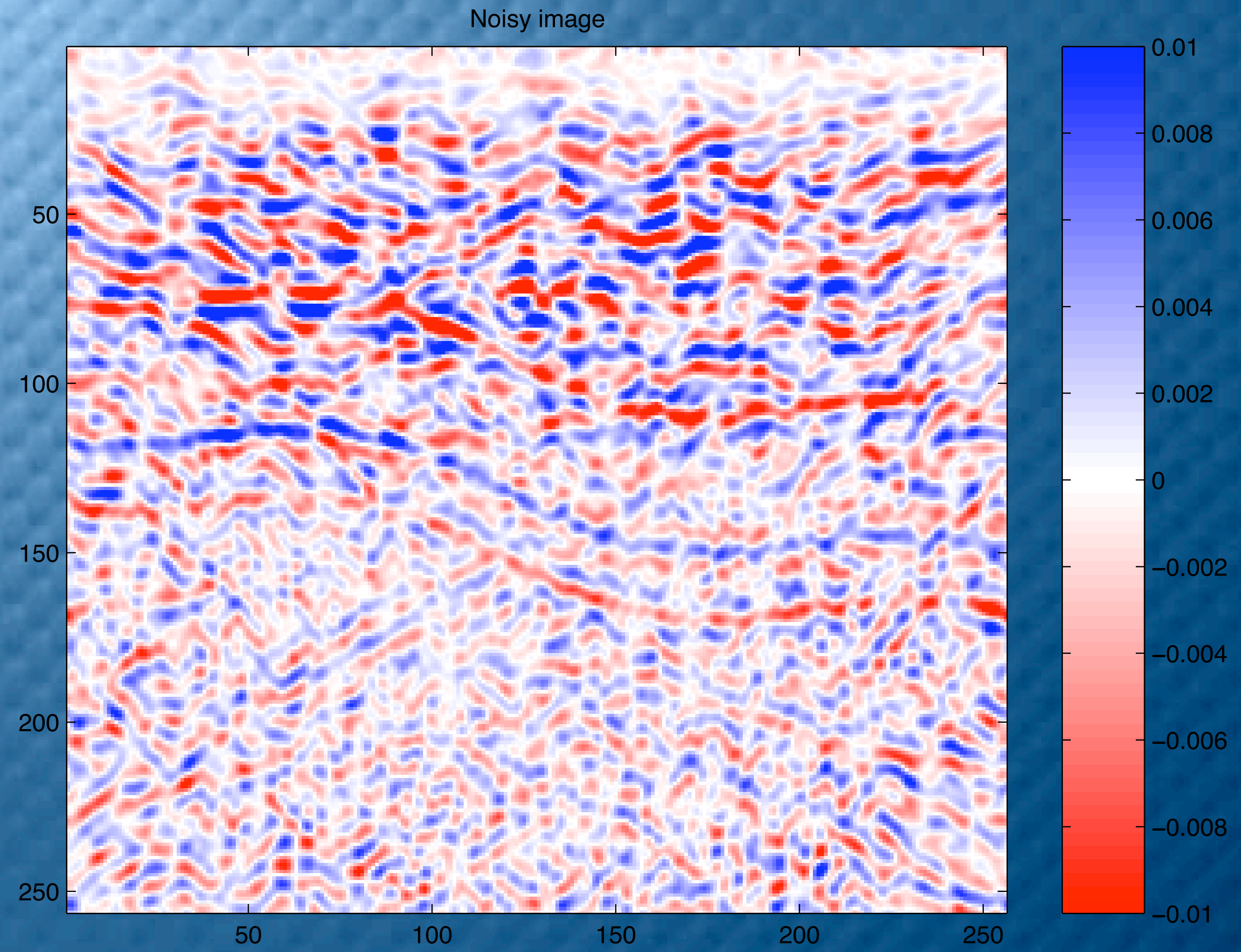
Optimal Seismic imaging

We are in the business of

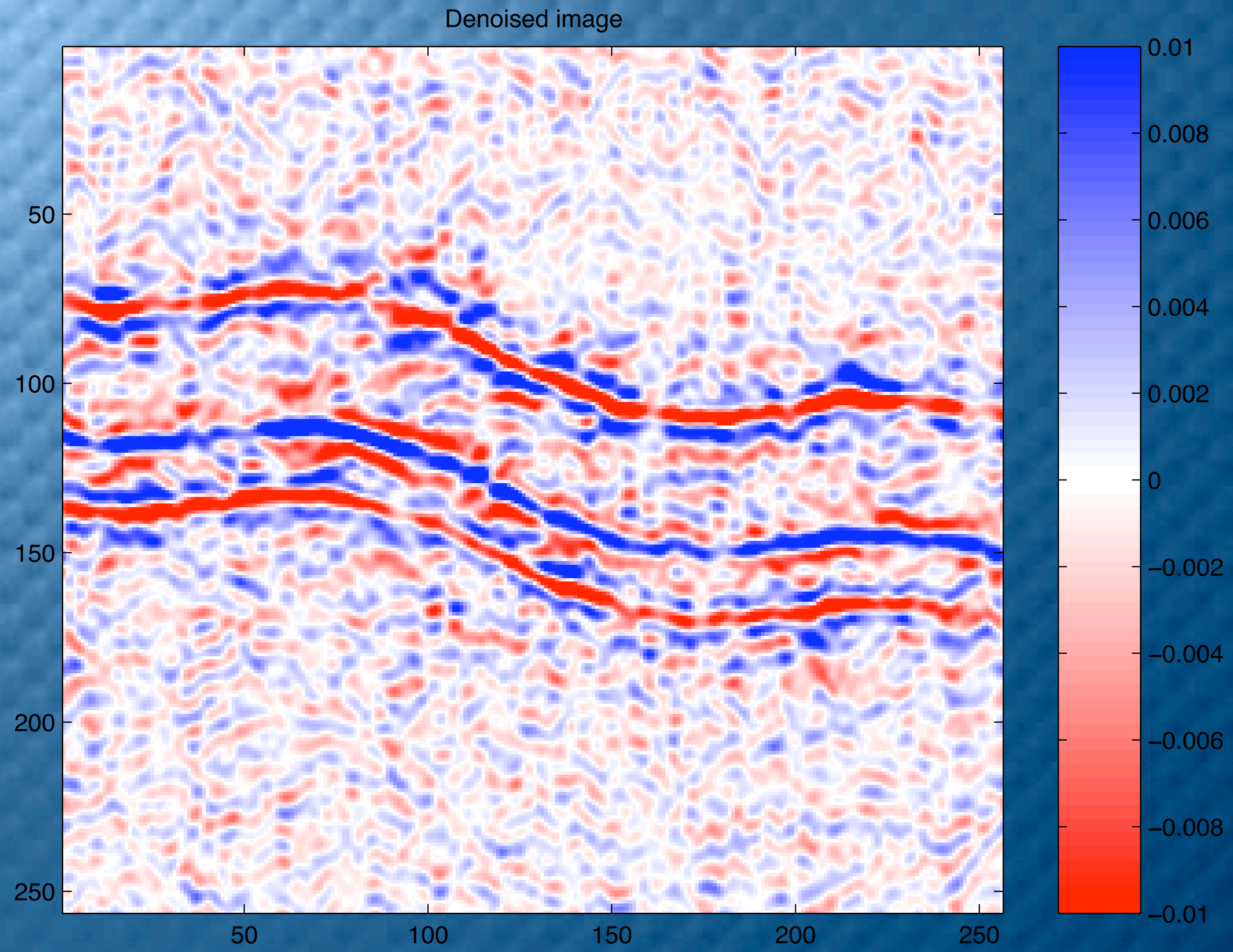
- ***Improving the signal-to-noise ratio (SNR)***
- ***Preserving edges on the model space***
- ***Sparsifying (de)-migration operators***

In the presence of noise ... Lots of it!

Seismic imaging



Seismic imaging



Basic idea

Build on the premise that you stand a better chance of solving a denoising and/or inversion problem when the model is represented optimally by basis functions ...

- **local**
- **sparse**
- **multi-scale and multi-directional**

Well behaved under migration!

Basic imaging problem

$$\mathbf{d} = \mathbf{Km} + \mathbf{n}$$

$$\begin{aligned}
\mathbf{d} &= \text{measured data.}\\
\mathbf{K} &= \text{demigration operator.}\\
\mathbf{m} &= \text{model.}\\
\mathbf{n} &= \text{white Gaussian noise.}
\end{aligned}$$

Basic imaging problem

$$\hat{\mathbf{m}}: \quad \arg\min_{\mathbf{m}} \|\mathbf{d} - \mathbf{K}\mathbf{m}\|_2 + \nu J(\mathbf{m})$$

with ***prior*** **info**

$$J(\mathbf{m}) \quad = \quad \textbf{Global penalty function}$$

- **Tikhonov regularization**
- $L^2 - \text{norms}$ **smooth too much...**

Preserve *singularities* in the presence of noise ...

Main questions

Effectively (sparsely) ***represent*** **(de)-migration** operators: $\mathbf{K}\,,\mathbf{K}^*$?

Effectively ***estimate*** the model **m?**

- preserving the edges
- improving SNR

Address both issues with emphasis on estimation!

Estimation

$$\hat{\mathbf{m}} : \quad \arg\min_{\mathbf{m}} \|\mathbf{d} - \mathbf{m}\|_2 + \nu \|m\|_1$$

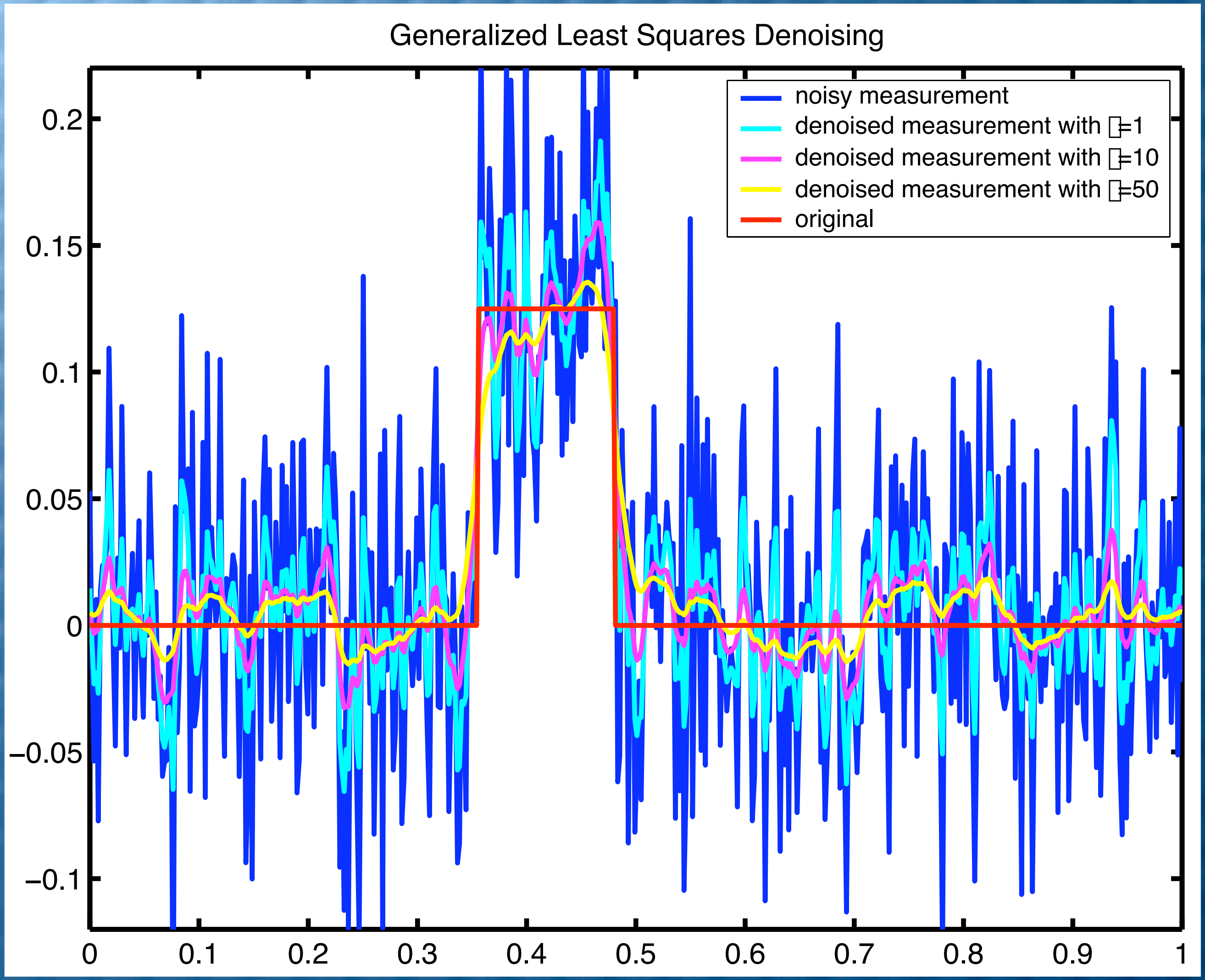
simply solved by shrinkage

$$\hat{\mathbf{m}} = \mathbf{B}^* \Theta_t \left(\mathbf{Bm} \right)$$

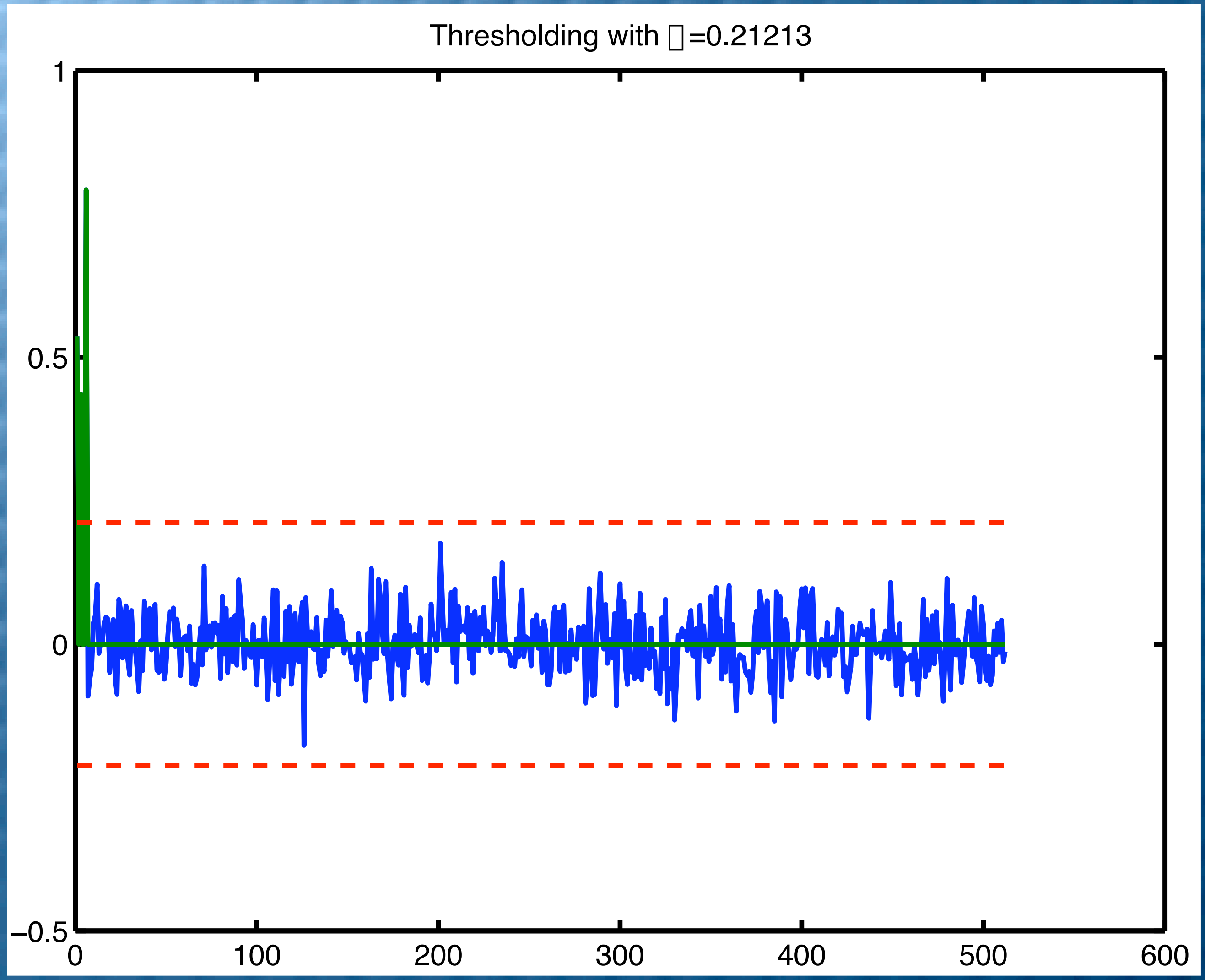
$$\begin{aligned} \mathbf{B} &= \textbf{Basis-function decomp.} \\ \mathbf{B}^* &= \textbf{Basis-function comp.} \\ \Theta &= \textbf{shrinkage/threshold operator.} \\ t &= \textbf{Threhold level (related to } \nu\textbf{).} \end{aligned}$$

Mallat '97
Donoho '92

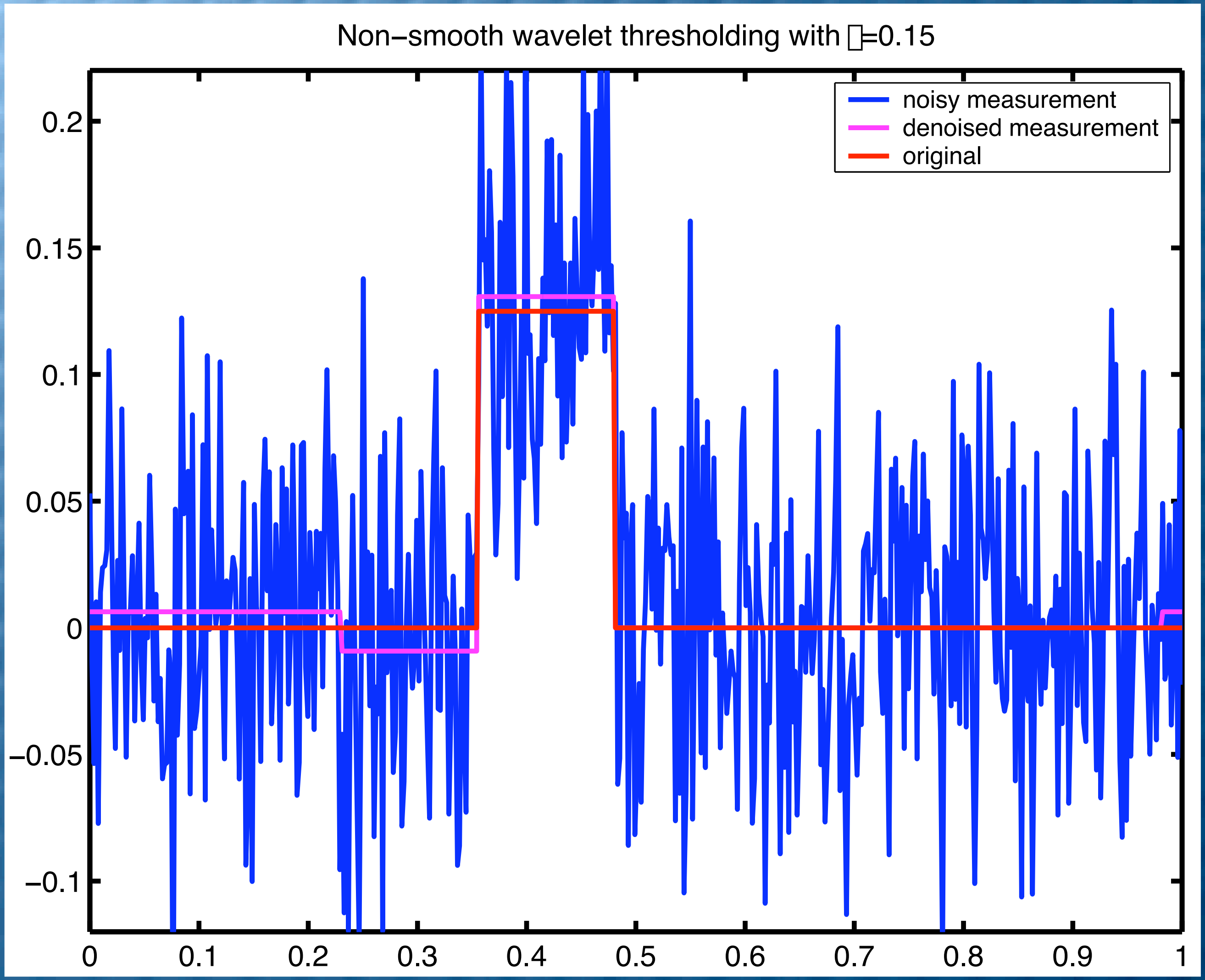
Estimation


Generalized Least Squares Denoising
noisy measurement
denoised measurement with □=1
denoised measurement with □=10
denoised measurement with □=50
original
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Estimation



Estimation



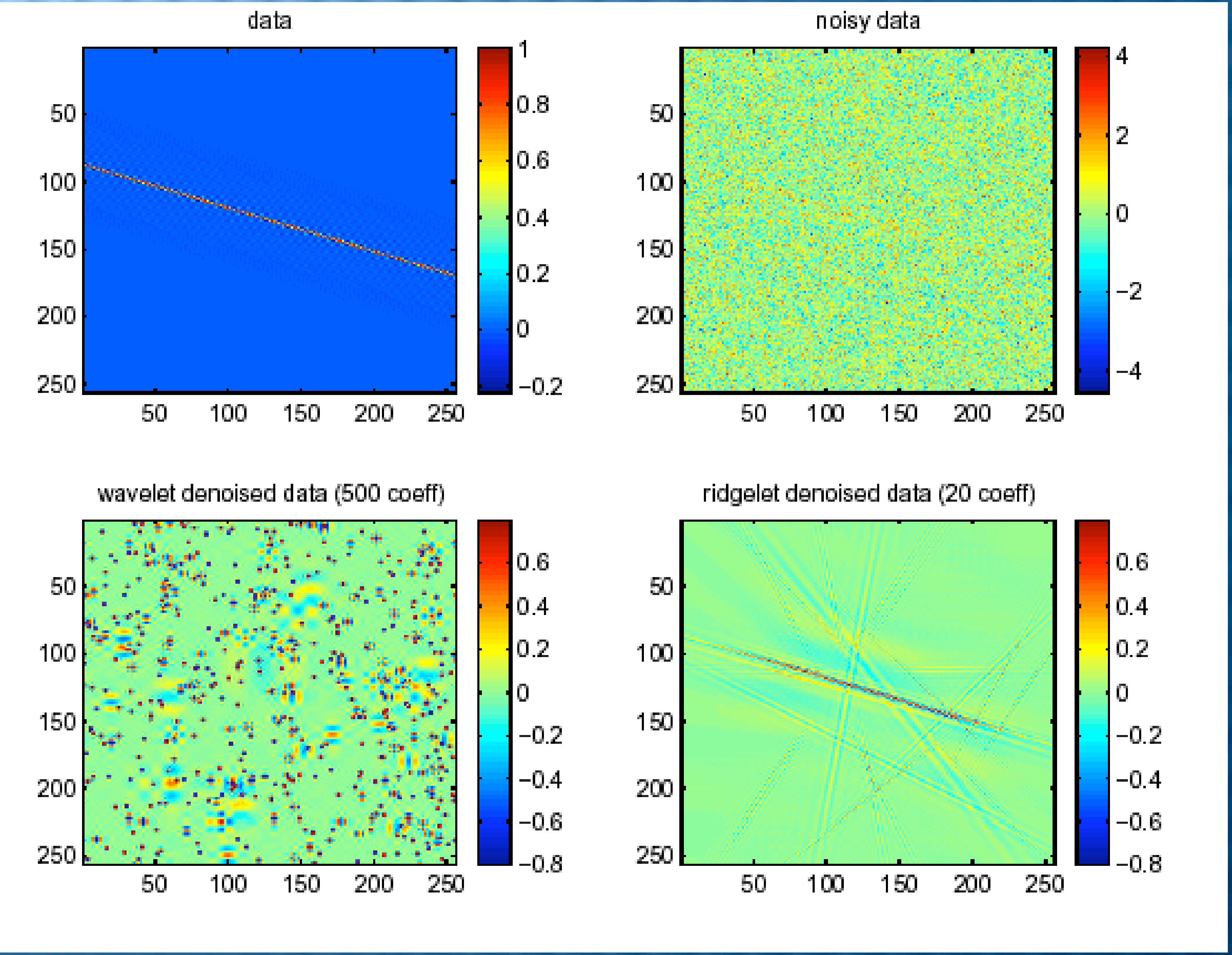
Non−smooth wavelet thresholding with □=0.15
noisy measurement
denoised measurement
original
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Estimation

Wavelets:

- represent **piece-wise smooth** functions at “**no**” additional cost
- do **not** have to know where the **singularities** are
- **only** good for **point-scatterers** or **horizon/vertically-aligned reflectors**

Estimation

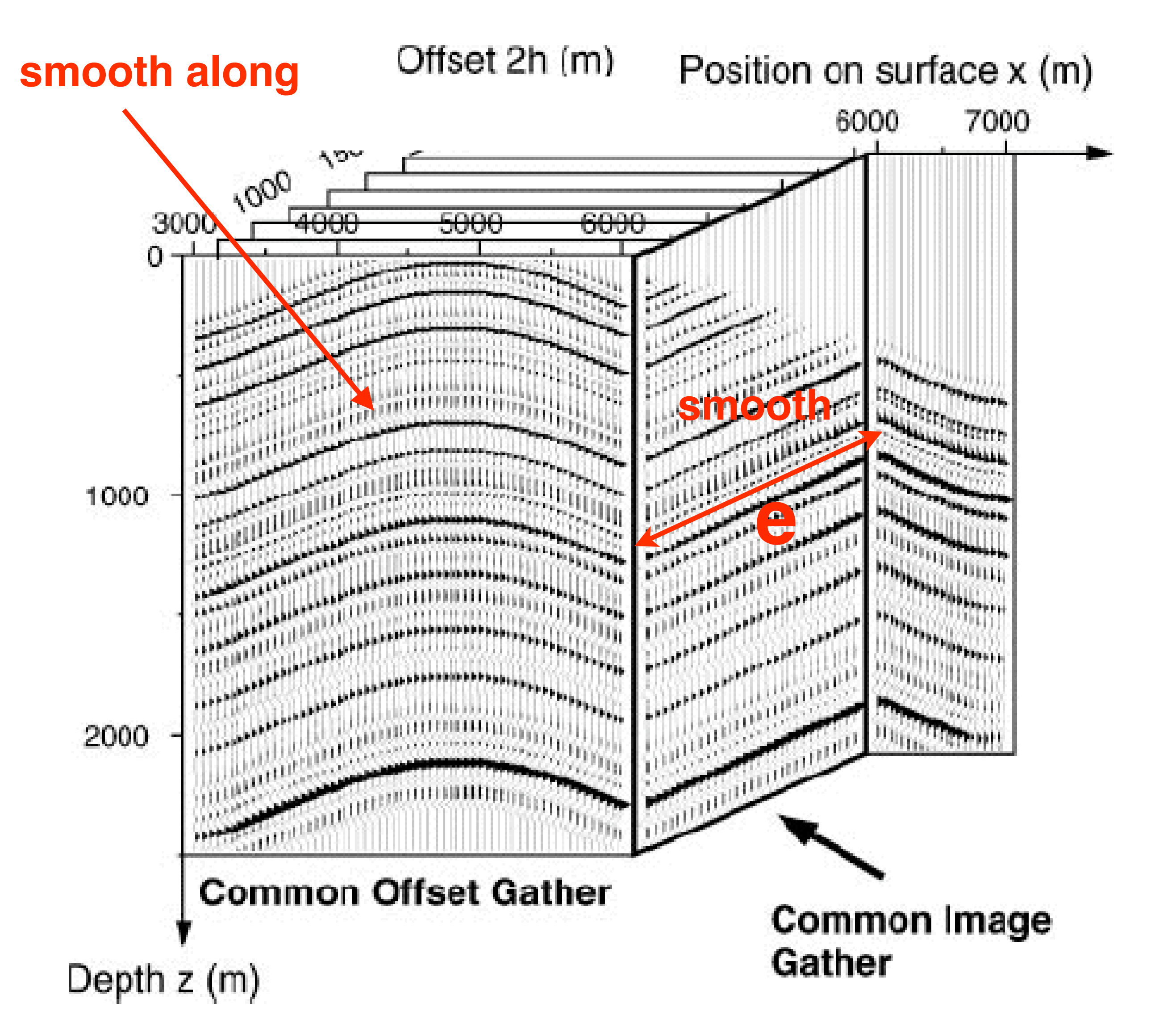


courtesy WaveleLab + Beamlab '02

Seismic imaging

- Use local basis-functions to
  - *sparsely* represent seismic images.
  - exploit redundancy seismic images.
  - almost diagonalize operators.
  - approximate the normal operator.
- Identify (de)-migration & normal operators.

Seismic imaging



$$K^* d$$

Seismic imaging

Image:

$$\underbrace{\mathbf{K}^*\mathbf{d}}_{\text{noisy\&blurry pre-stack image}} = \overbrace{\mathbf{K}^*\mathbf{K}\mathbf{m}}^{\text{"blurry" refl.}} + \underbrace{\mathbf{K}^*\mathbf{n}}_{\text{colored noise}}$$

- **how to project on a new basis**
  - **sparse on m.**
  - **makes it easy to get rid of the coloring.**
  - **sparsifies** $\mathbf{K}^*\mathbf{K}$

Seismic imaging

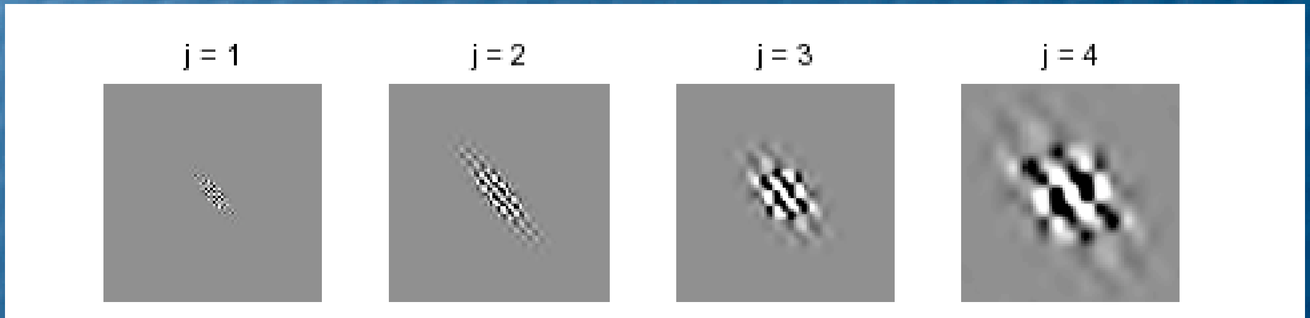
$$\hat{m} = \underbrace{\left(K^*K\right)^{-1}}_{\square\text{DO}} \overbrace{K^*d}^{\text{FIO}}$$

	□DO	FIO	d & m	**e**
Wavelets	✔	✗	✗	✔
Curvelets	✔	✔	✔	✔

Directional wavelets

Construct basis functions that are

- Local in 2-D space
- Local in 2-D Fourier space (angle)

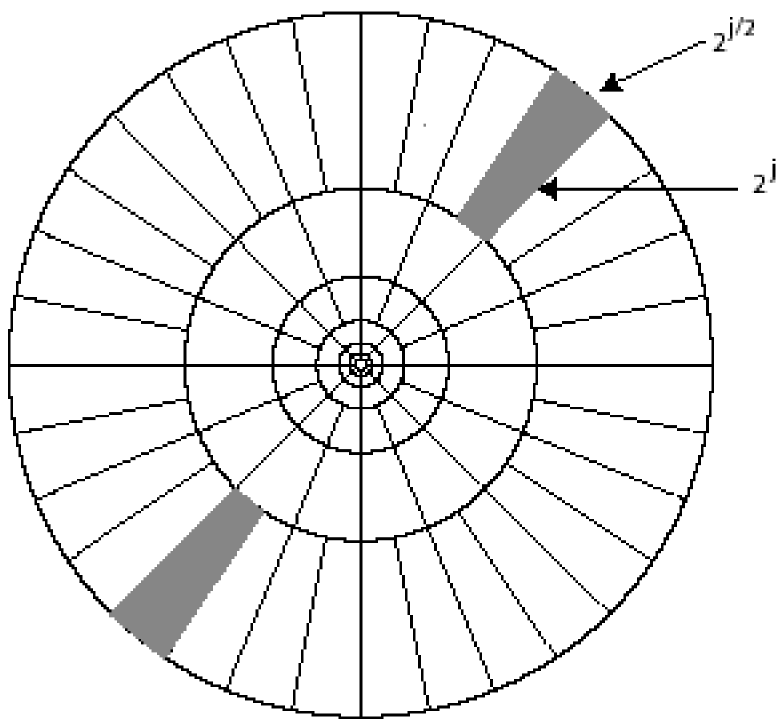


multiscale

Do '02

Directional wavelets

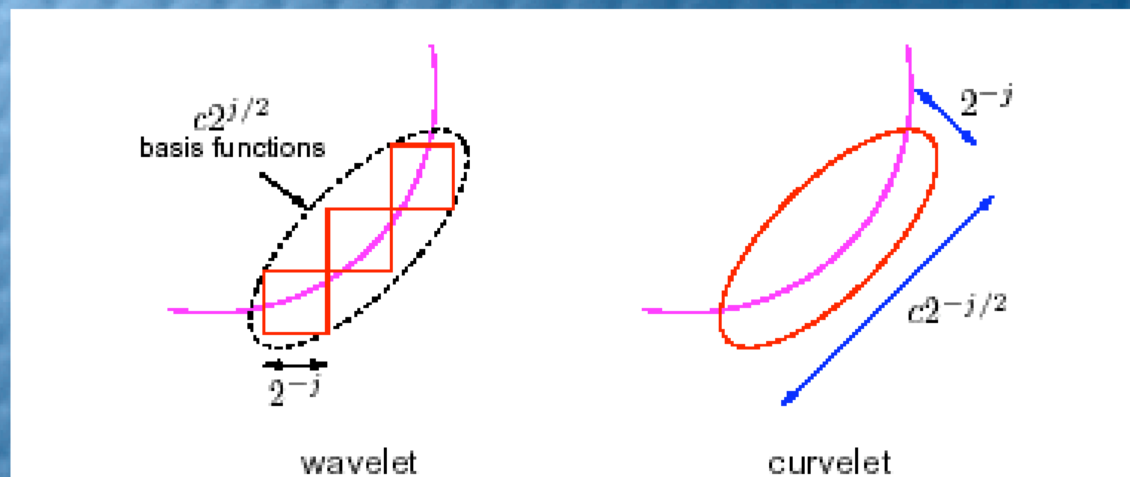
$$W_j = \{\square, \quad 2^j \le |\square| \le 2^{j+1}, \ |\square - \square_J| \le \square \cdot 2^{\lfloor j/2 \rfloor}\}$$



second dyadic partitioning

Stein '93
Candes '02

Directional wavelets



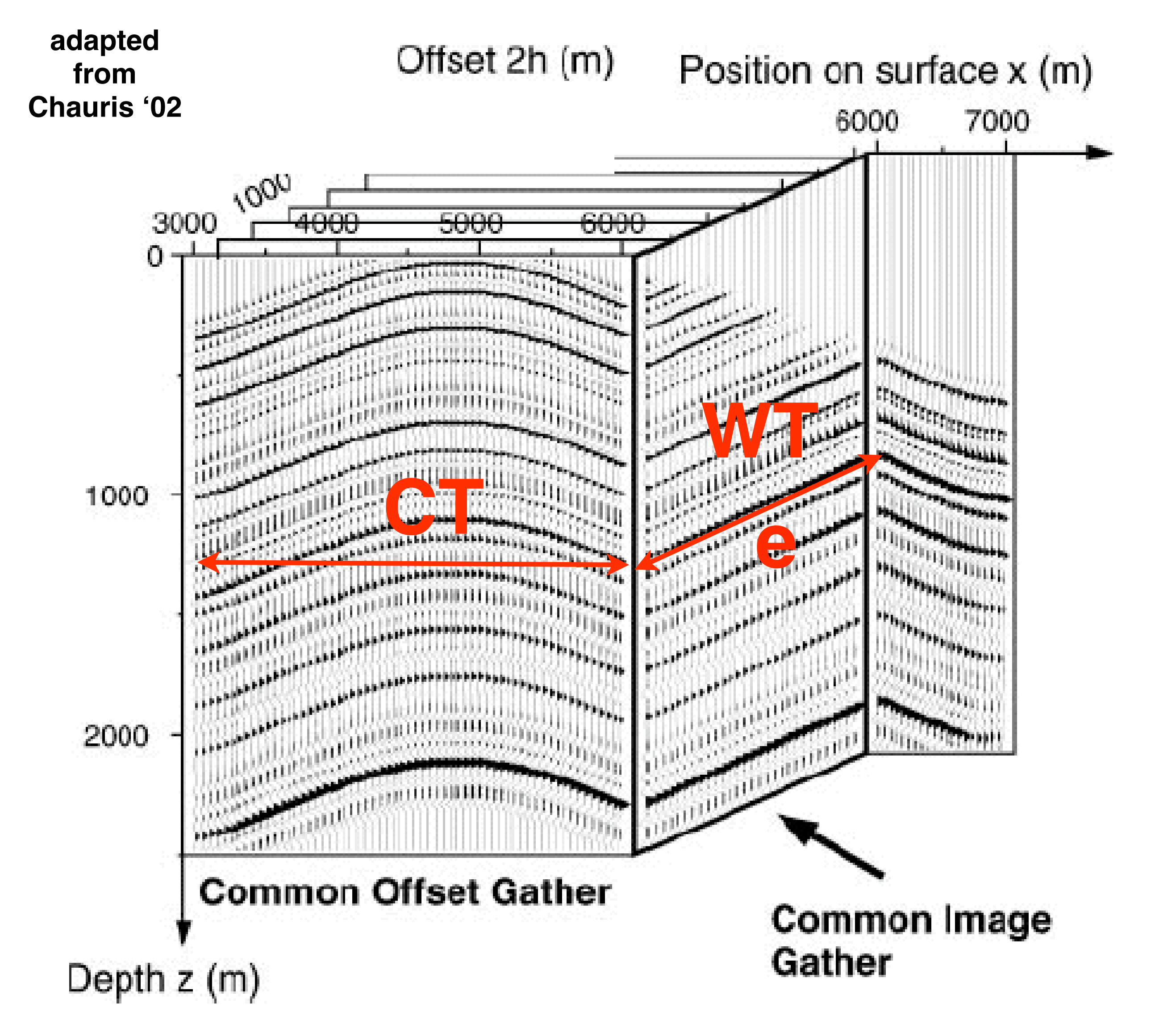
Optimal app. rate

$$\|m - \tilde{m}_m^{\text{wavelet}}\|_2 \quad m^{-1} \qquad\qquad \|m - \tilde{m}_m^{\text{curvelet}}\| \quad m^{-2}$$

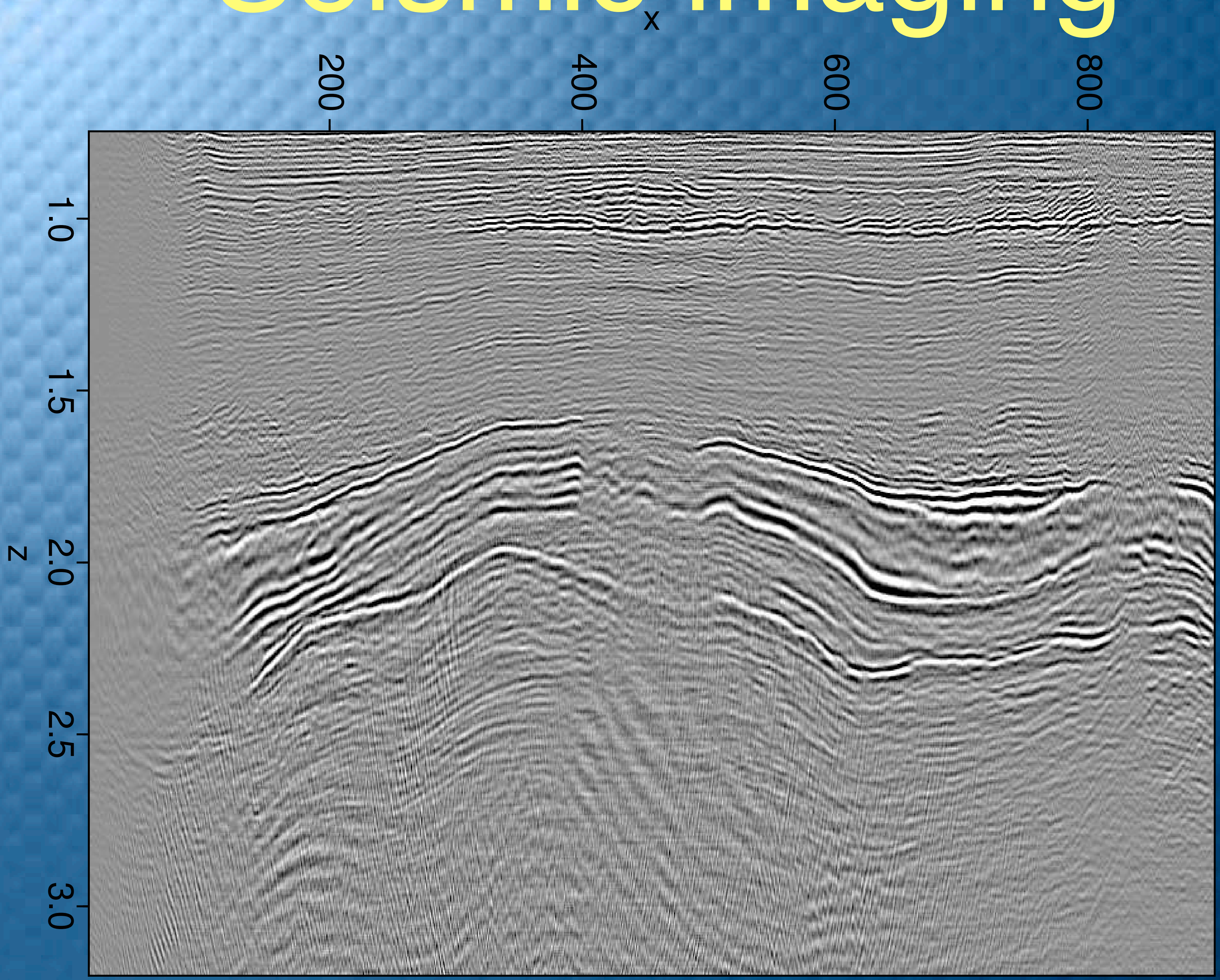
Non-linear approximation rate is ruler of the game!

Do '02

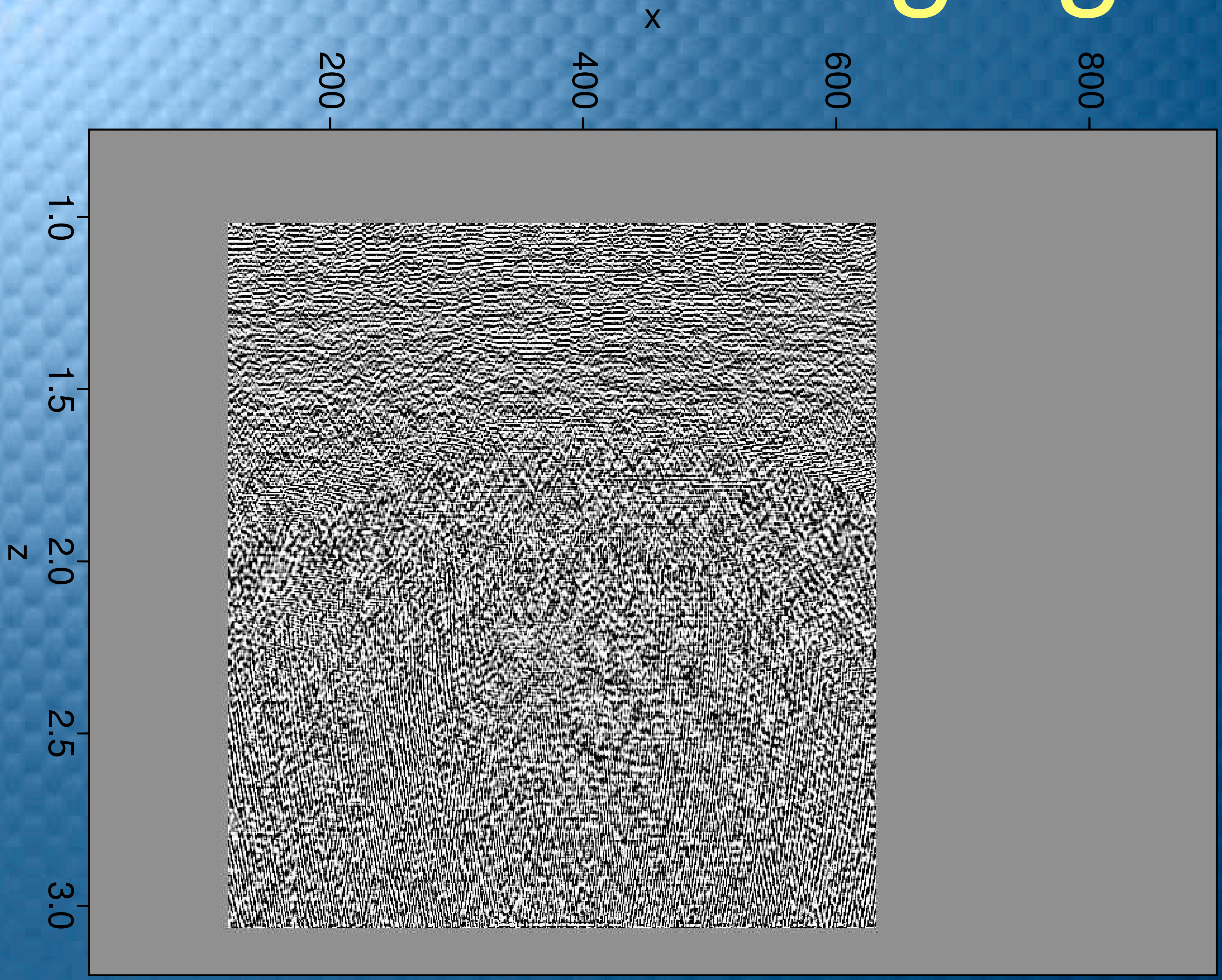
Seismic imaging



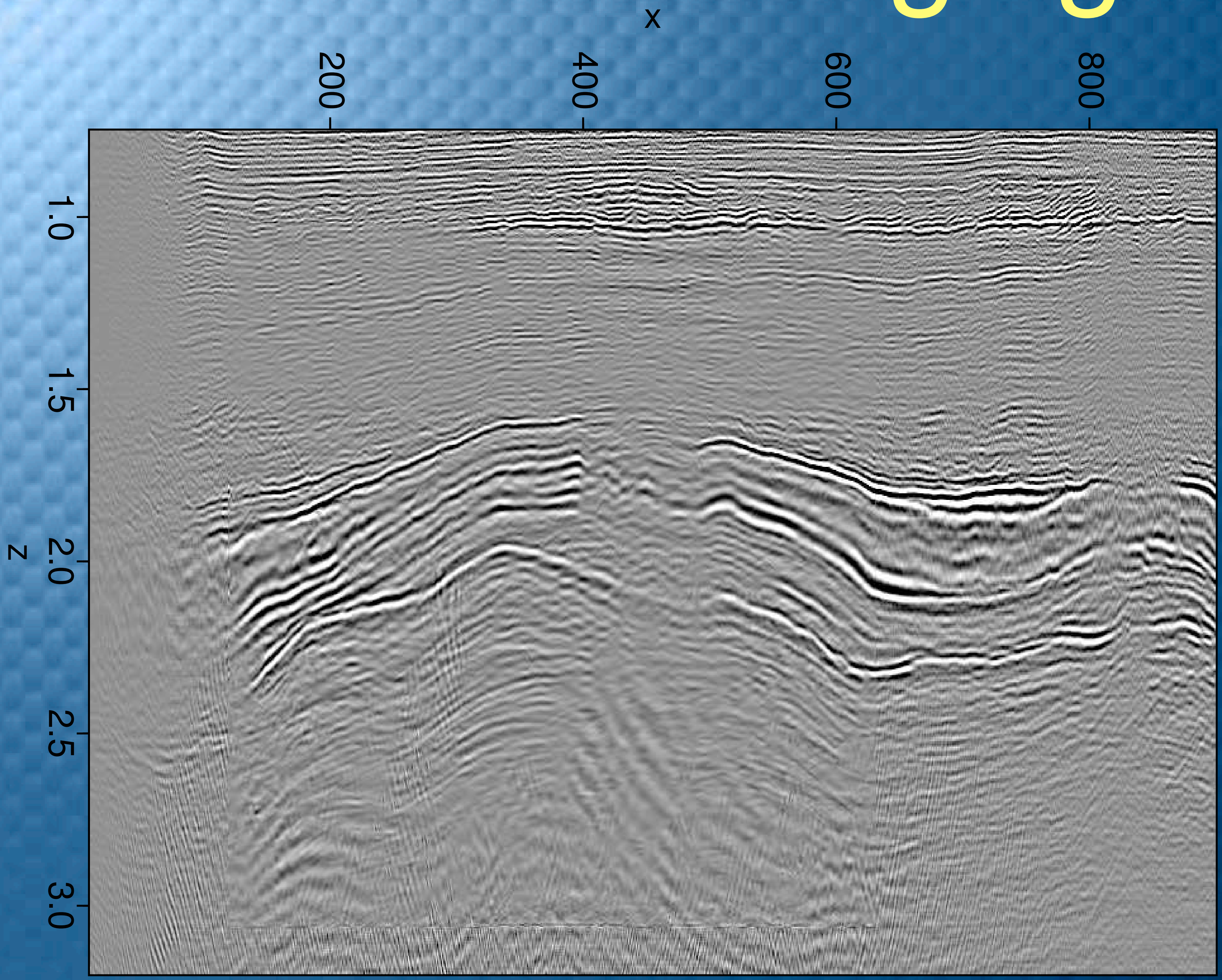
Seismic imaging



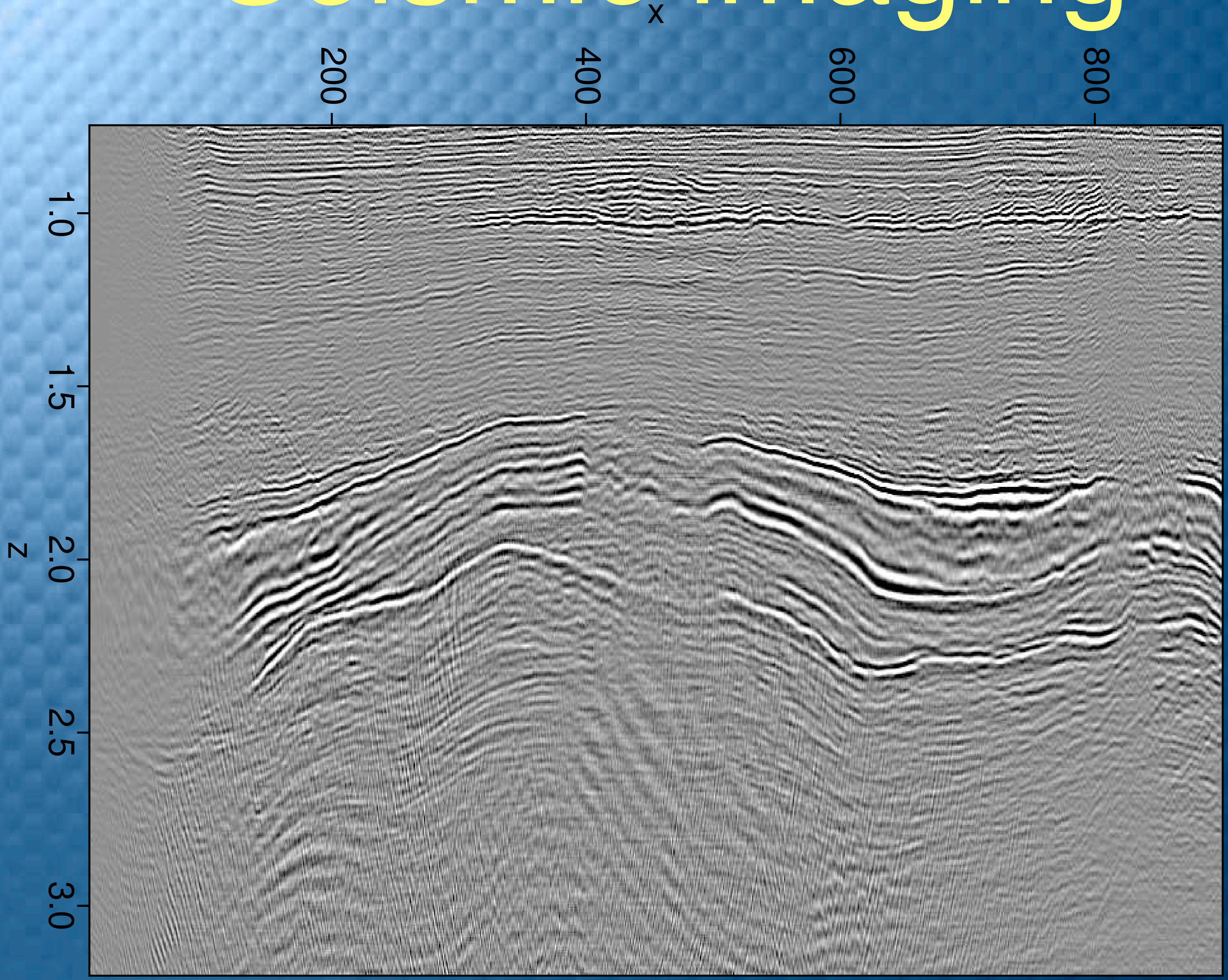
Seismic imaging



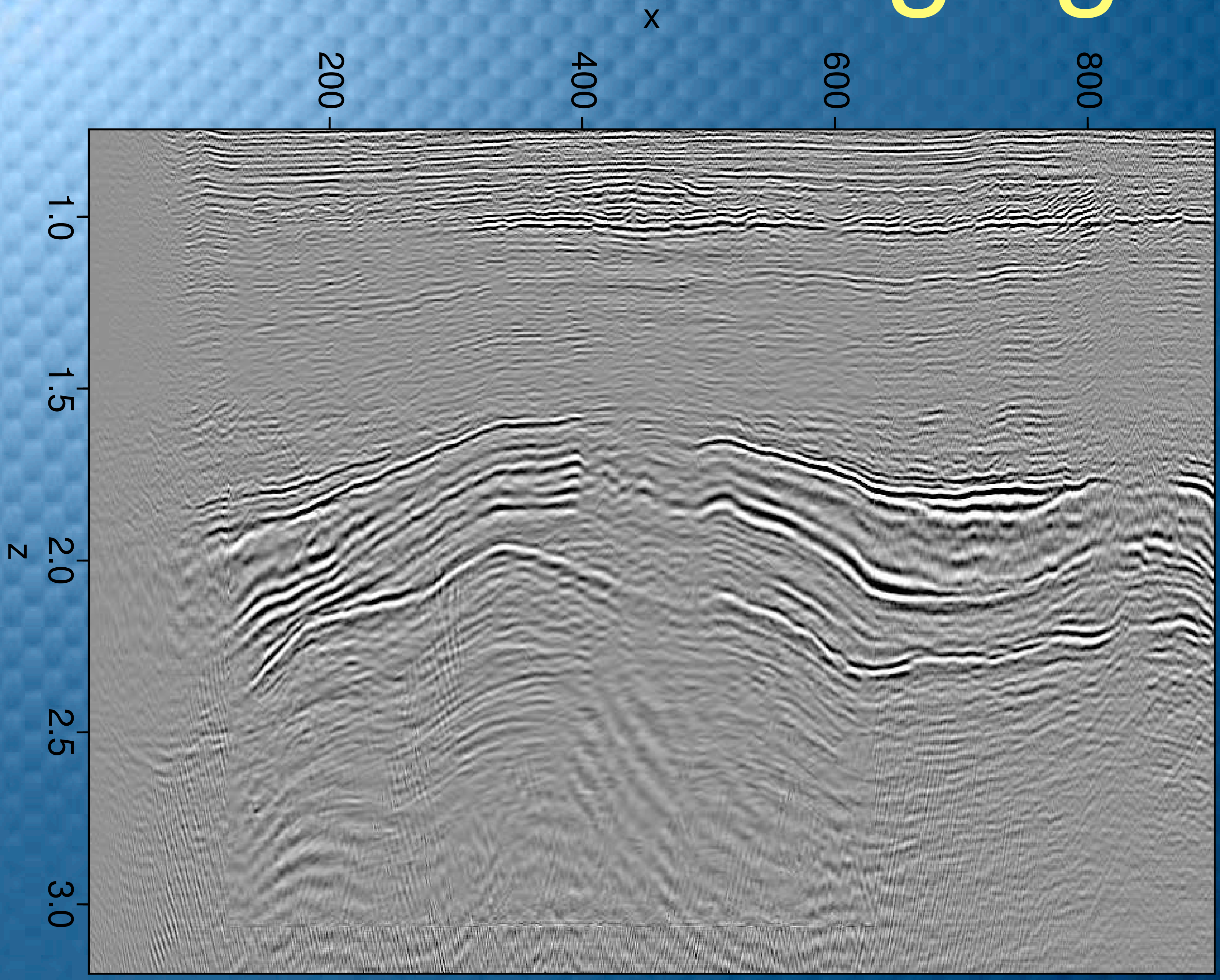
Seismic imaging



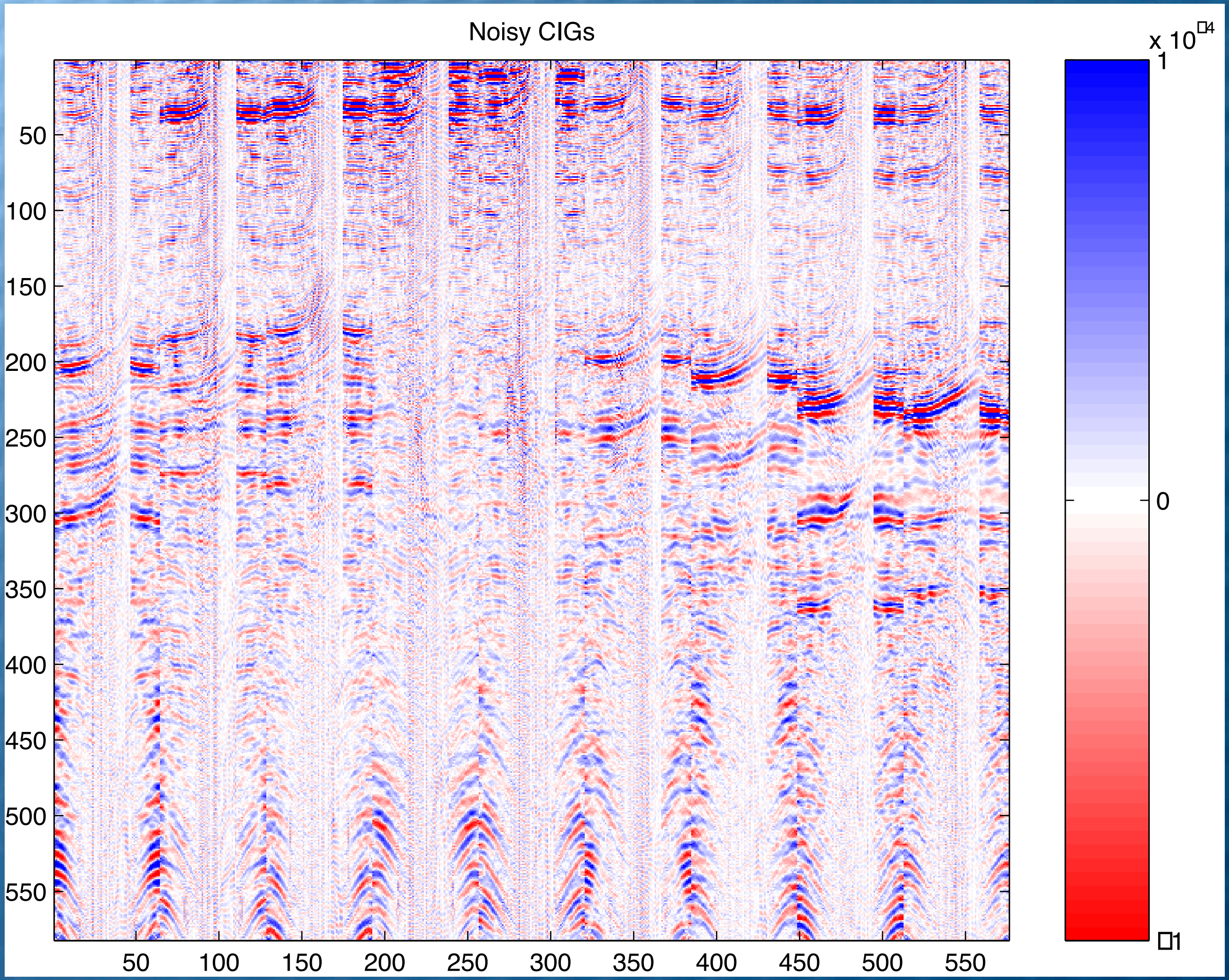
Seismic imaging



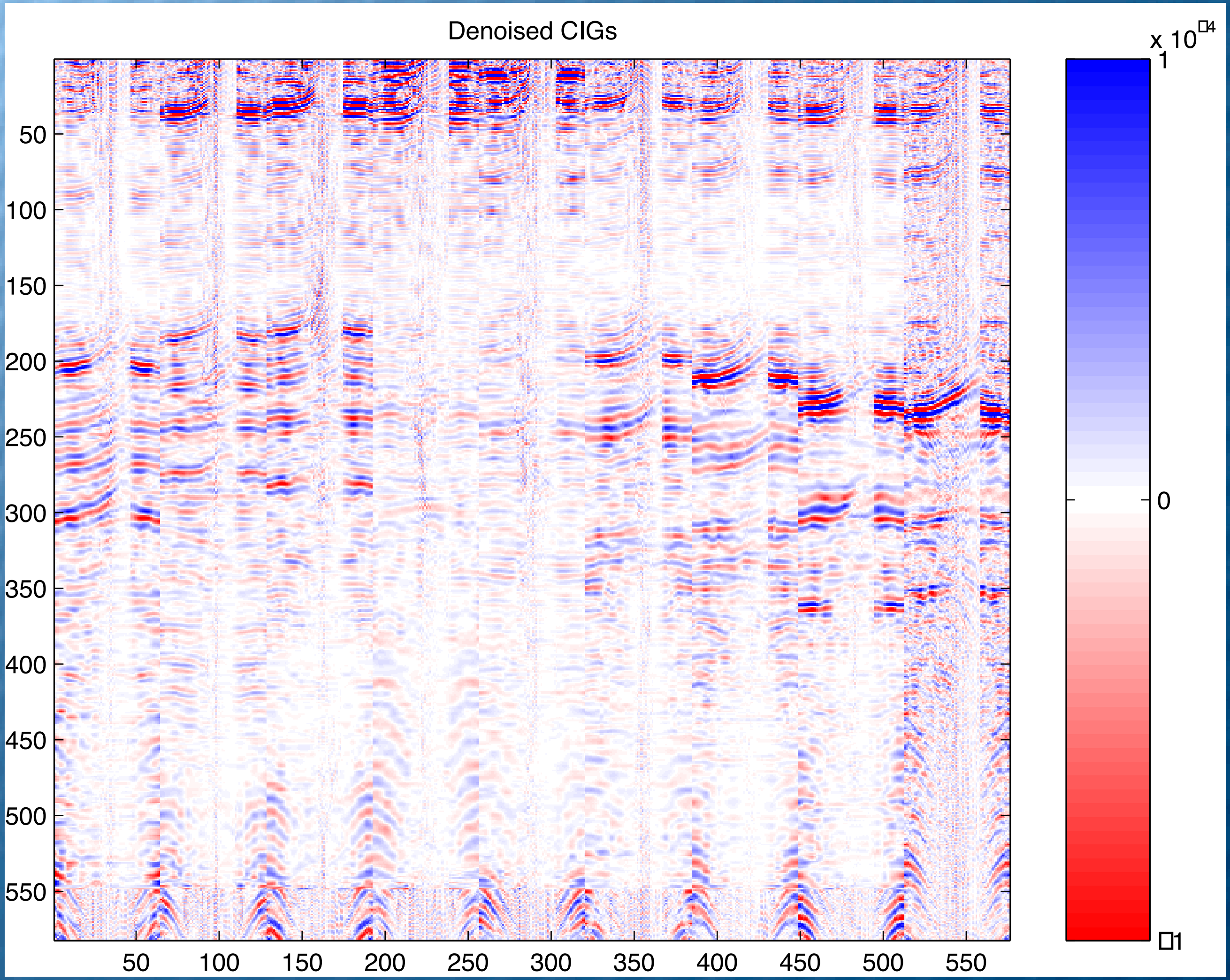
Seismic imaging



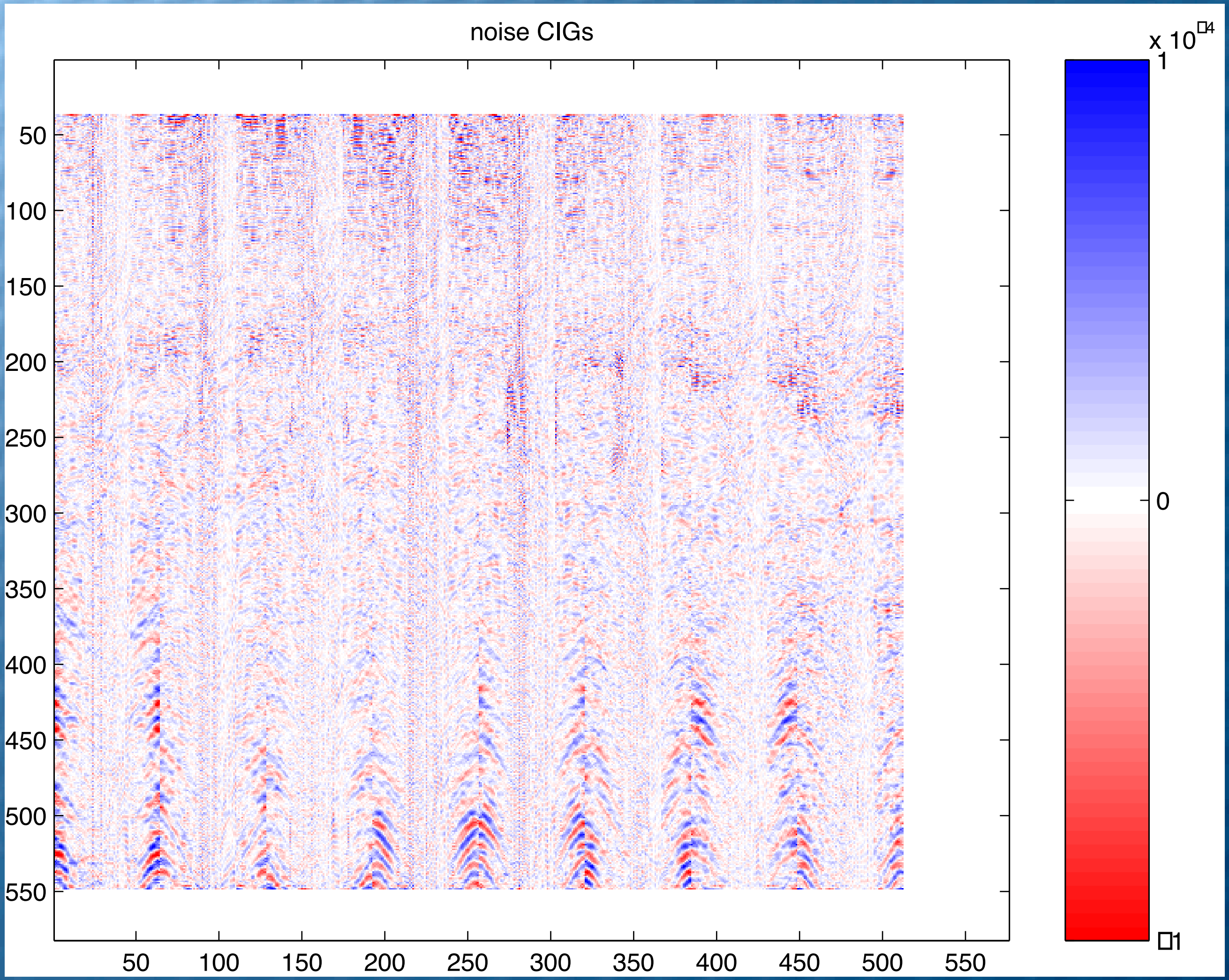
Seismic imaging



Seismic imaging



Seismic imaging



Seismic imaging

Works so well because we exploit

- **continuity *along* reflectors**
- smoothness **e-direction** (off-set)
- *adaptive* **local** smoothing

Remaining challenges:

- deal with the operator/coloring
- compensate for the normal operator

Imaging

Insert basis-function (de)-composition:

$$\mathbf{d} = \mathbf{K}\overbrace{\mathbf{B}^*\mathbf{B}}^{\mathbf{I}}\mathbf{m} + \mathbf{n}$$
$$= \tilde{\mathbf{K}}\tilde{\mathbf{m}} + \mathbf{n}$$

where

$$\tilde{\mathbf{K}}\cdot = \mathbf{K}\mathbf{W}^*\mathbf{C}^*\cdot \qquad \tilde{\mathbf{K}}^*\cdot = \mathbf{W}\mathbf{C}\mathbf{K}^*\cdot$$

Seismic imaging

Use $\tilde{\mathrm{K}}, \tilde{\mathrm{K}}^*$ in a CG-scheme.

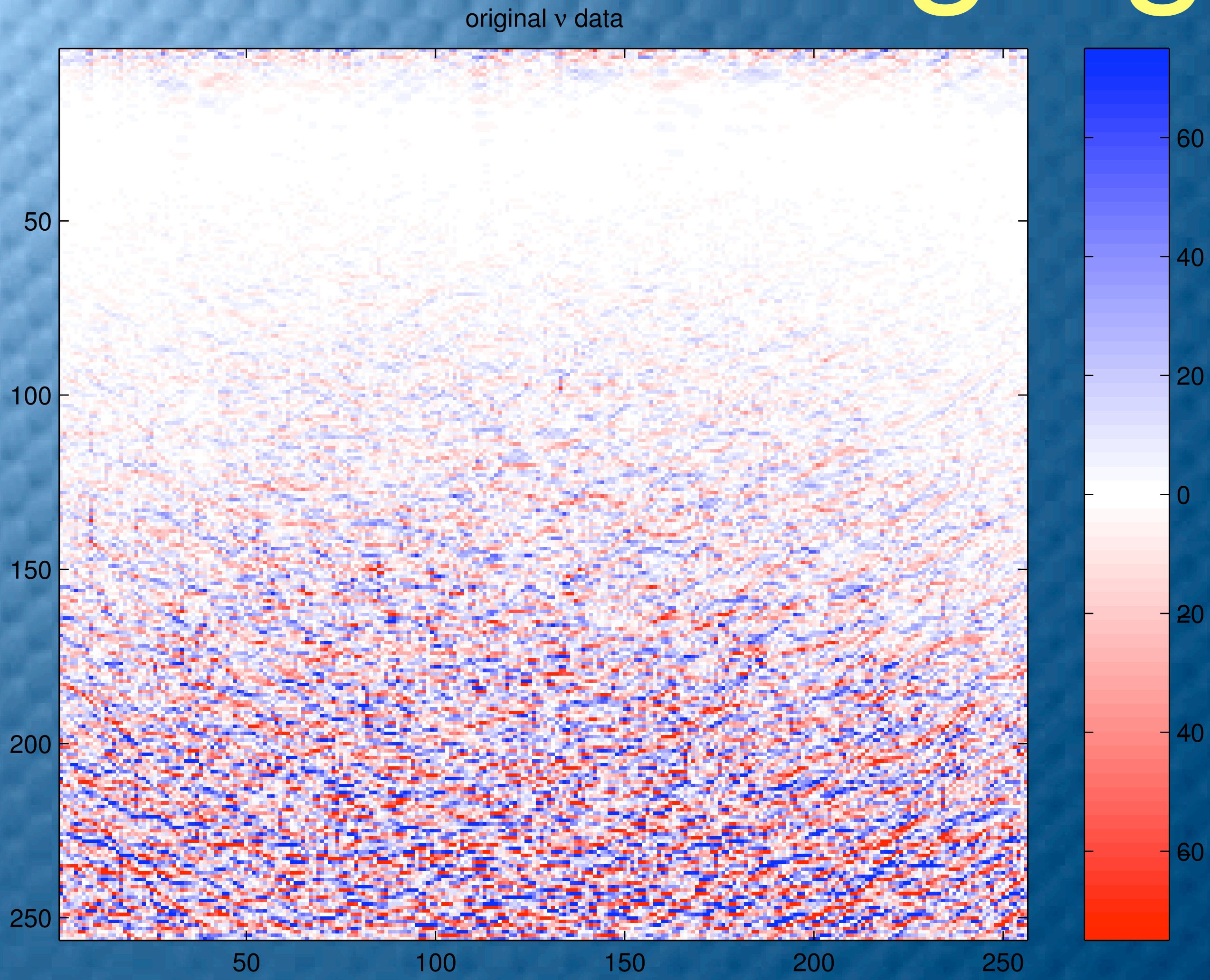
Without regularization CG fits noise ...

- no image
- no reliable estimates for **m**

Our solution

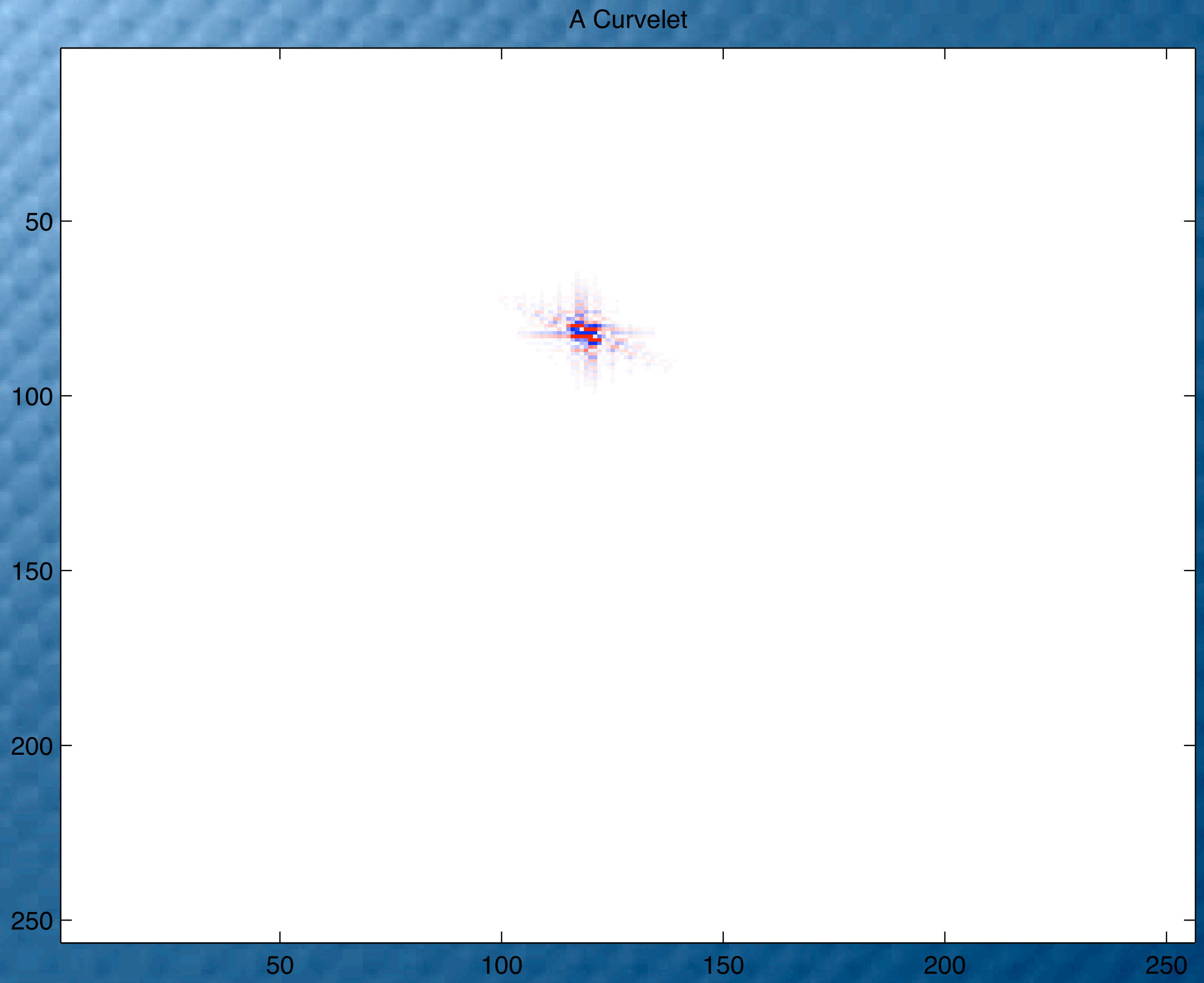
- correct for coloring noise
- apply thresholding
- correct for normal operator

Seismic imaging



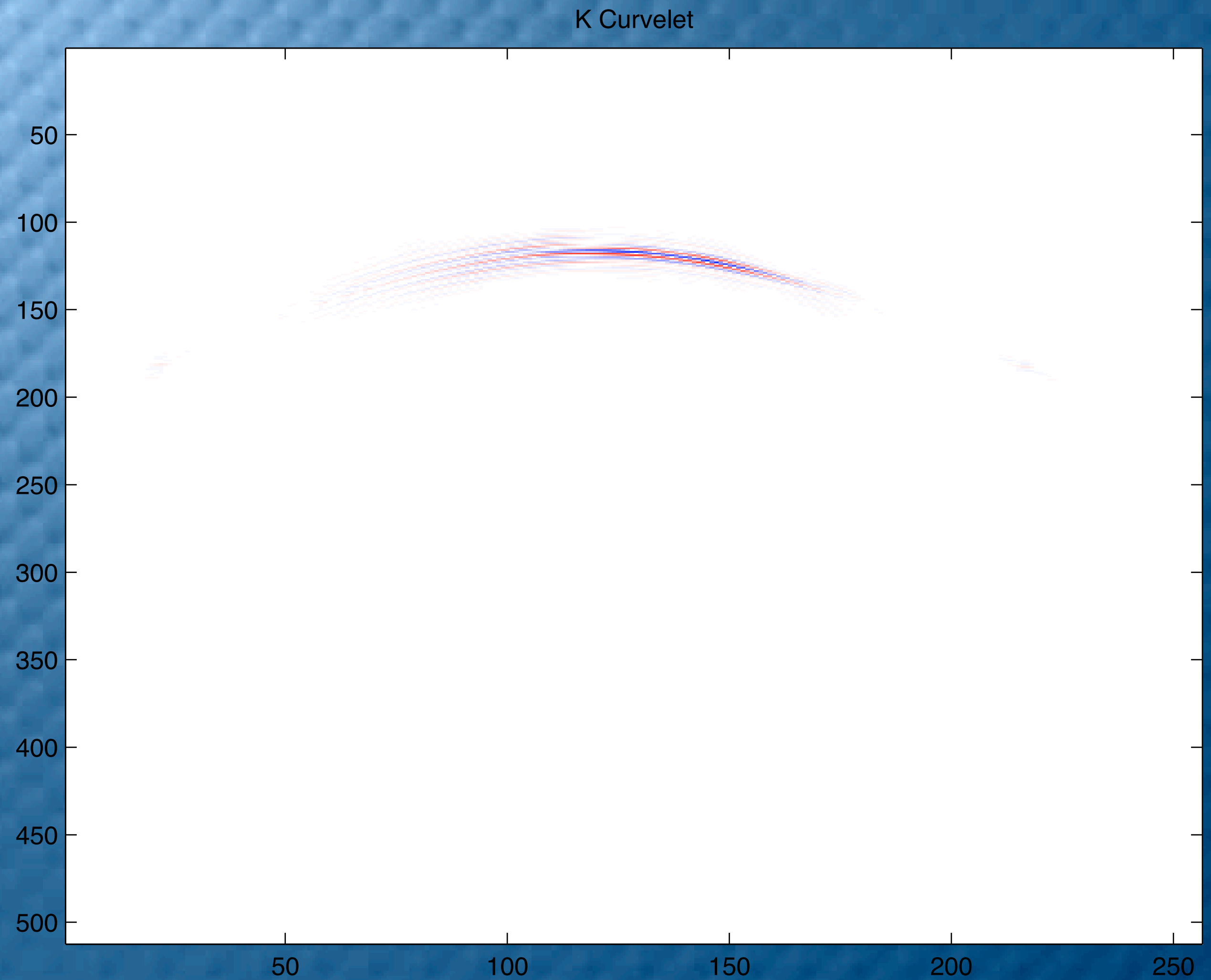
$$\mathbf{K}^*\mathbf{n}$$

Seismic imaging



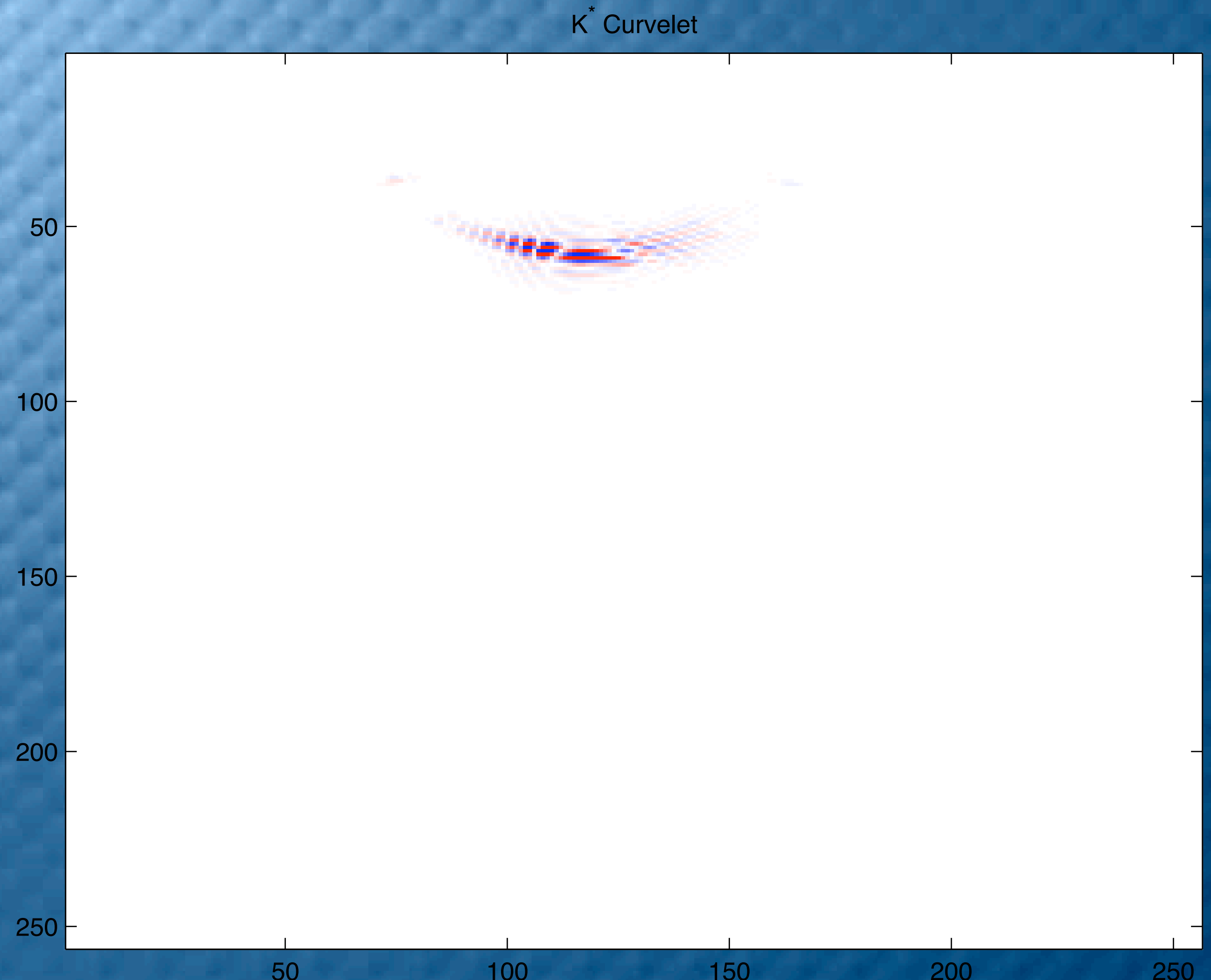
$$\mathbf{C}^* \begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}^*$$

Seismic imaging



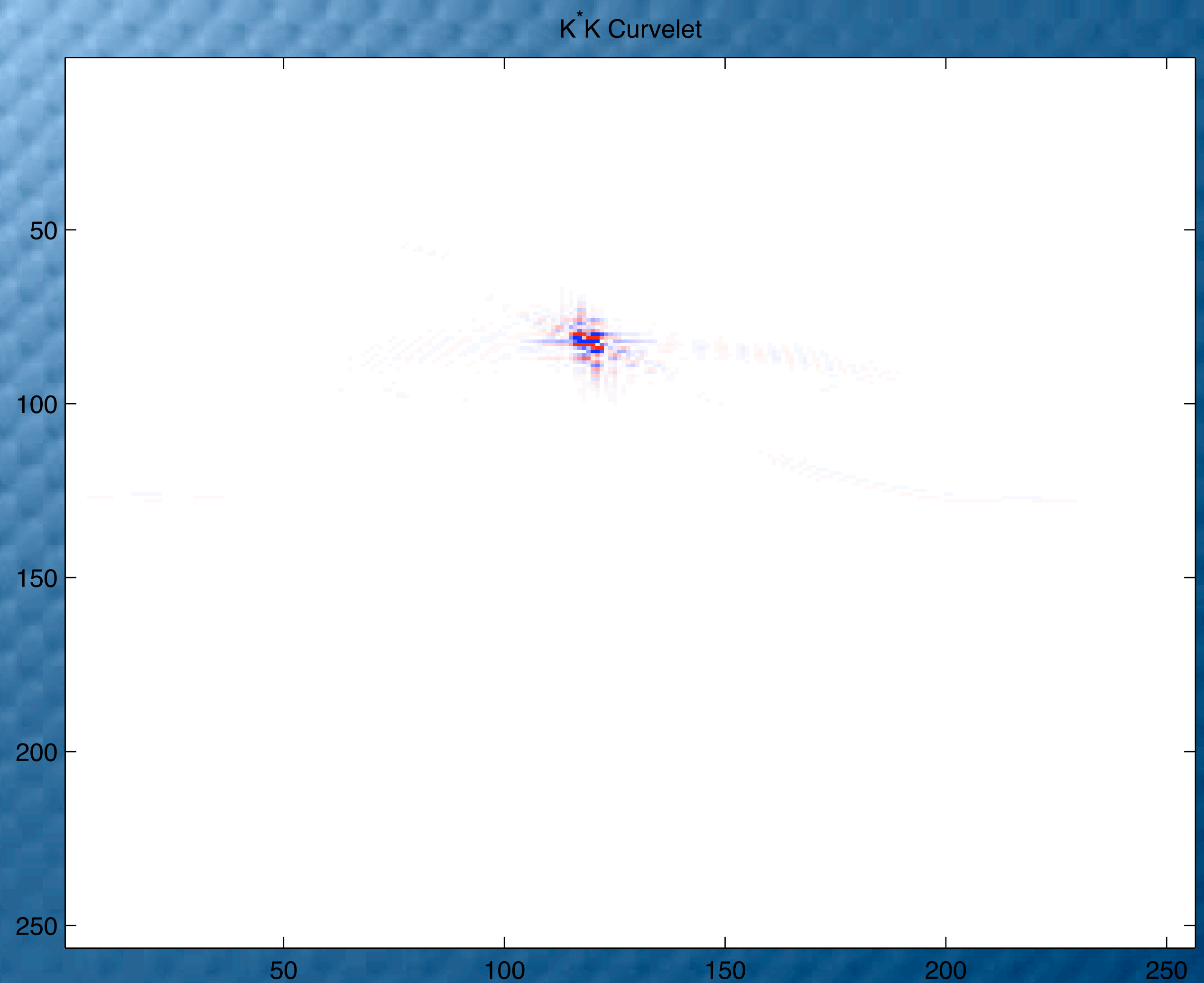
$$\mathbf{KC}^* \begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}^*$$

Seismic imaging



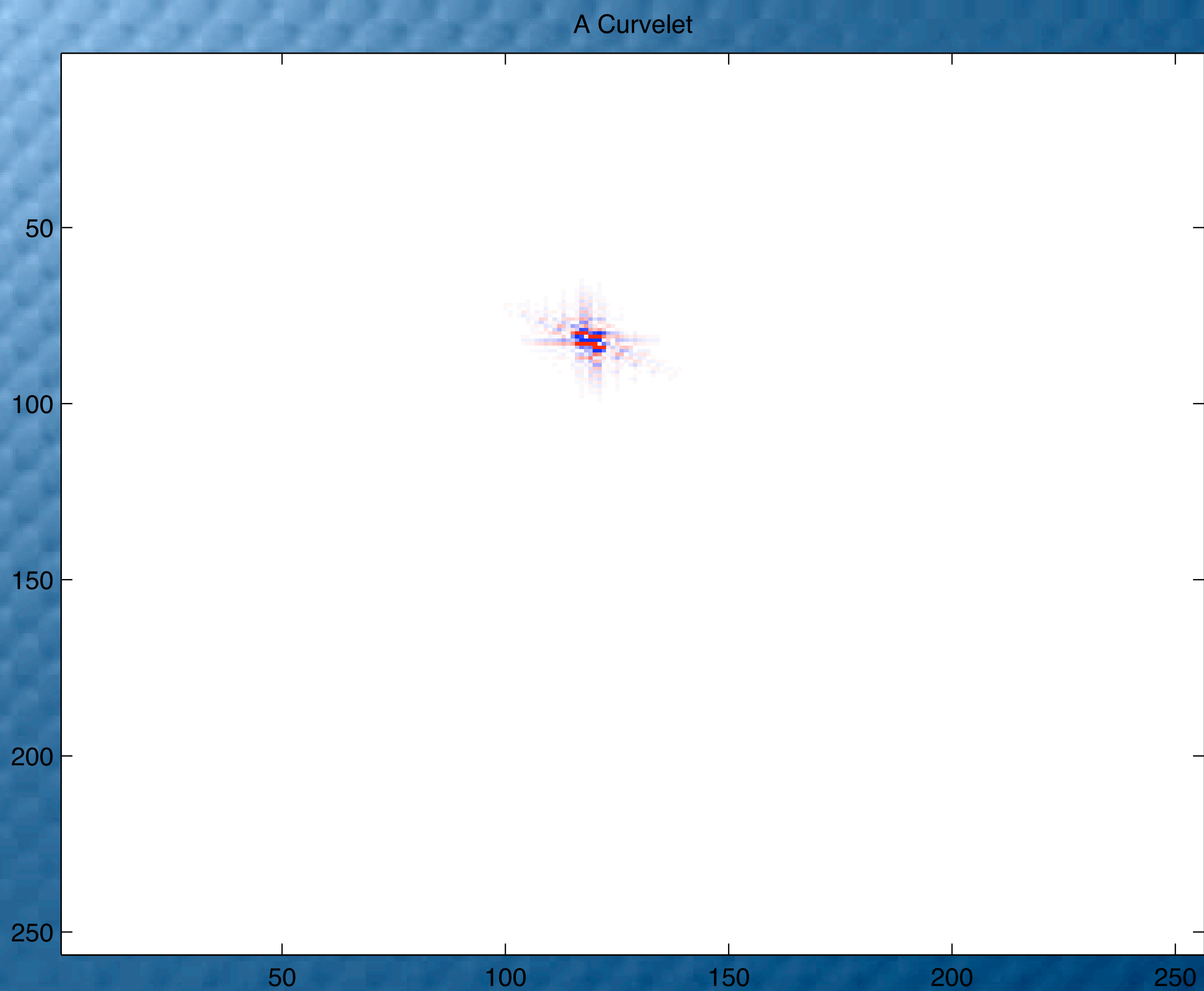
$$\mathbf{K}^*\mathbf{C}^* \begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}^*$$

Seismic imaging



$$\mathbf{K}^*\mathbf{K}\mathbf{C}^*\begin{pmatrix}\cdots & 0 & 0 & 1 & 0 & 0 & \cdots\end{pmatrix}^*$$

Seismic imaging



Estimation

Estimate with WV/Quasi-SVD

$$\hat{\mathbf{m}} = \sum_{\text{off-sets}} \mathbf{B}^{*}\Gamma^{\dagger}\Theta_{t(\Gamma)}\left(\tilde{\mathbf{K}}^{*}\mathbf{d}\right)$$

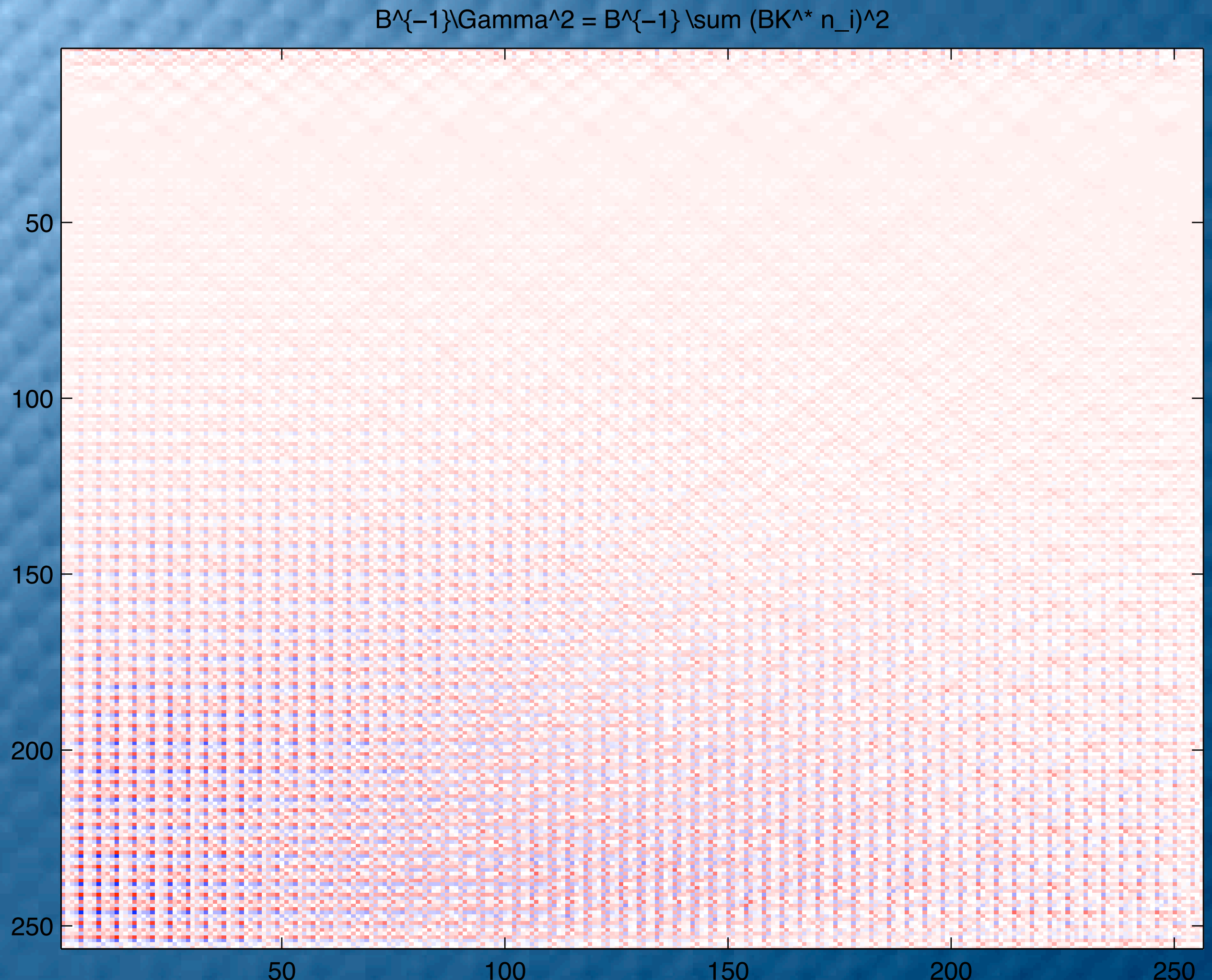
Curvelets almost diagonalize $\mathbf{K}^{*}\mathbf{K}$

$$\mathbf{K}^{*}\mathbf{K}\cdot \approx \mathbf{B}^{-1}\Gamma\mathbf{B}\cdot \quad \Longleftrightarrow \quad \tilde{\mathbf{K}}^{*}\tilde{\mathbf{K}}\cdot \approx \Gamma\cdot$$

- **correct for coloring n via Γ**
- **correct for the normal operator**

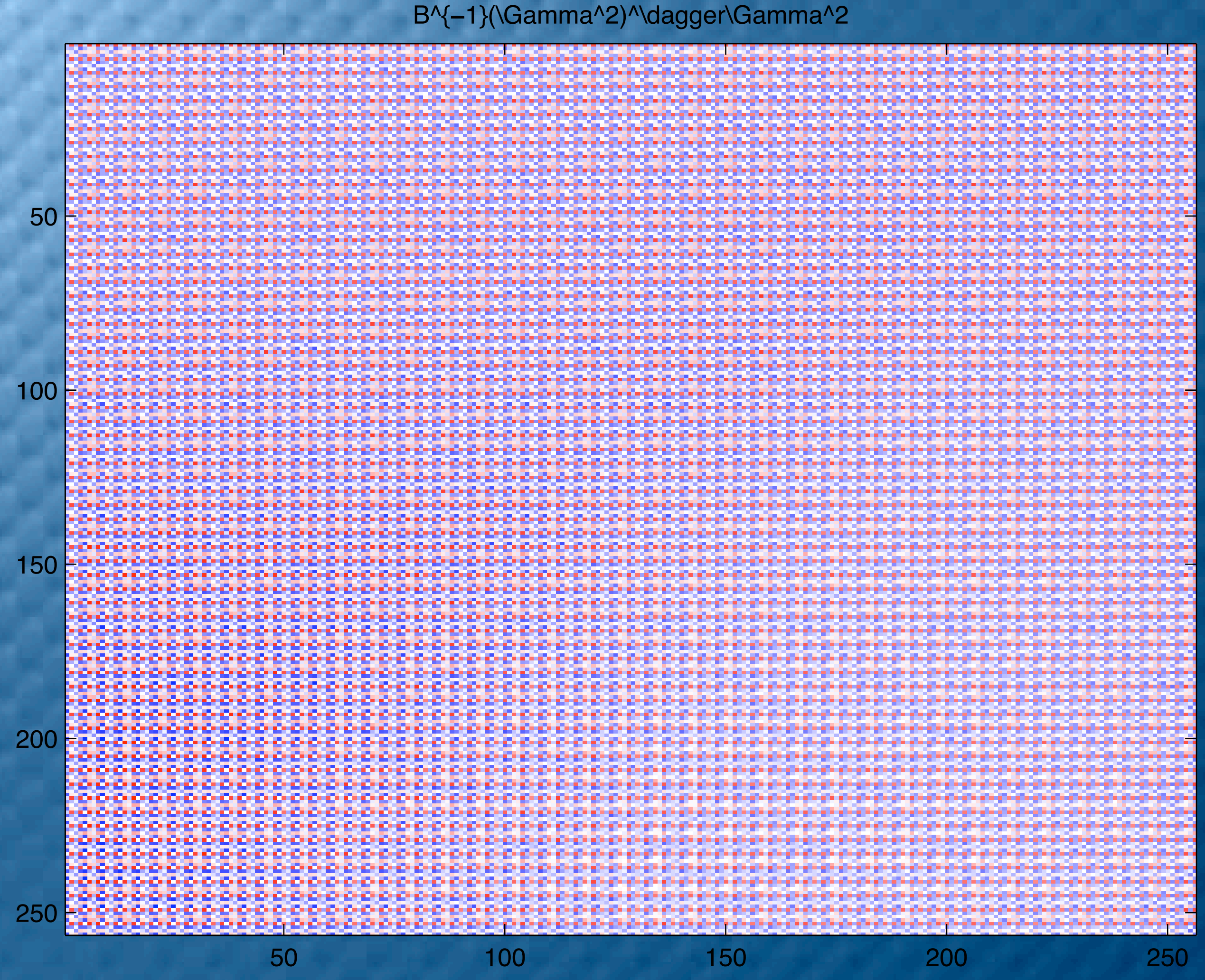
Donoho,Candes, Lucier, Kane, H.

Seismic imaging



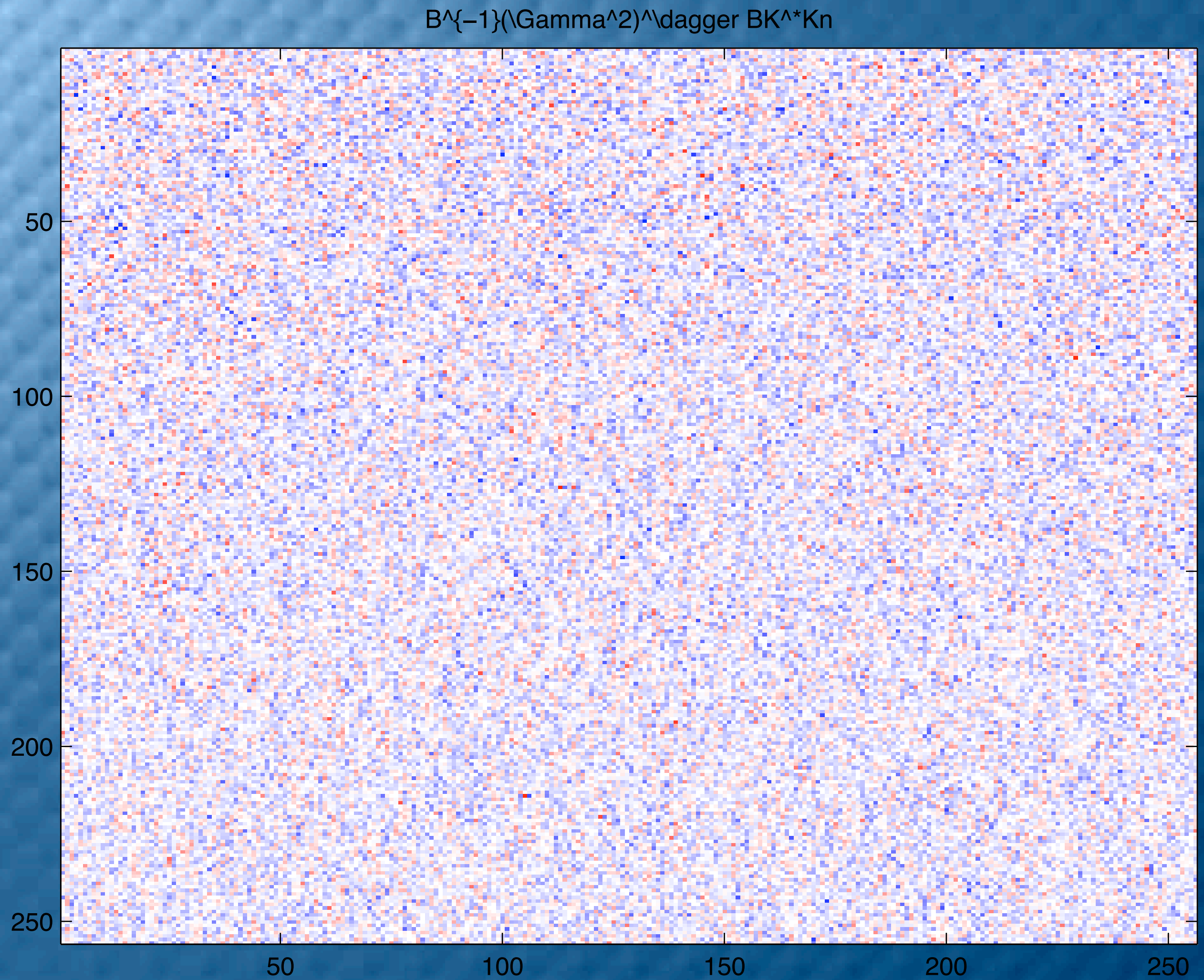
$\mathbf{B}^{*}\Gamma$

Seismic imaging



$$\mathbf{B}^*\Gamma^\dagger\Gamma$$

Seismic imaging



$$\mathbf{B}^*\Gamma^\dagger\tilde{\mathbf{K}}^*\mathbf{n}$$

Estimation

Compute **Γ** with MC-sampling:

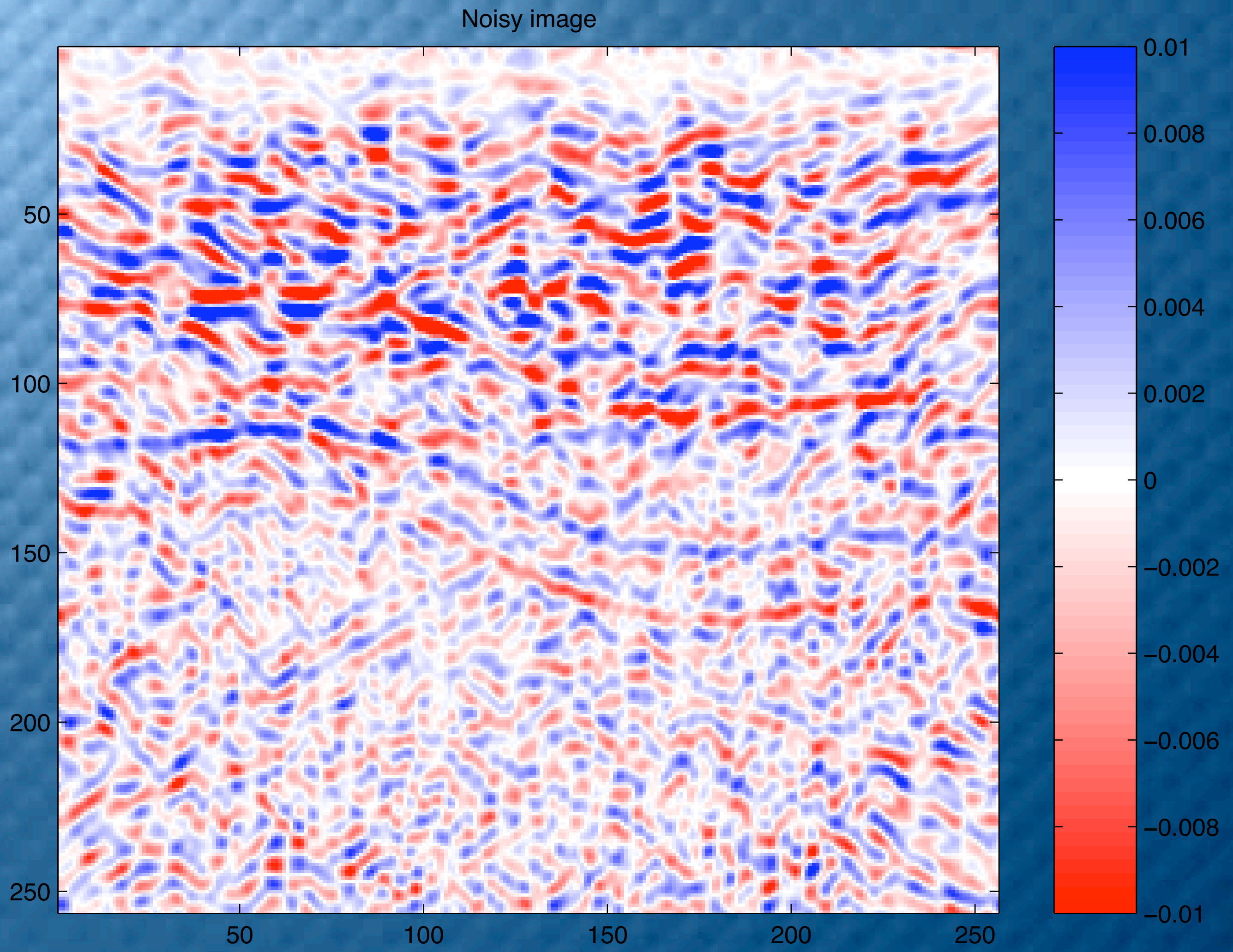
$$\Gamma = diag\{Cov_{\tilde{\mathbf{n}}\tilde{\mathbf{n}}}\} \approx \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{\mathbf{K}}\tilde{\mathbf{n}}\right)^2$$

Set the threshold

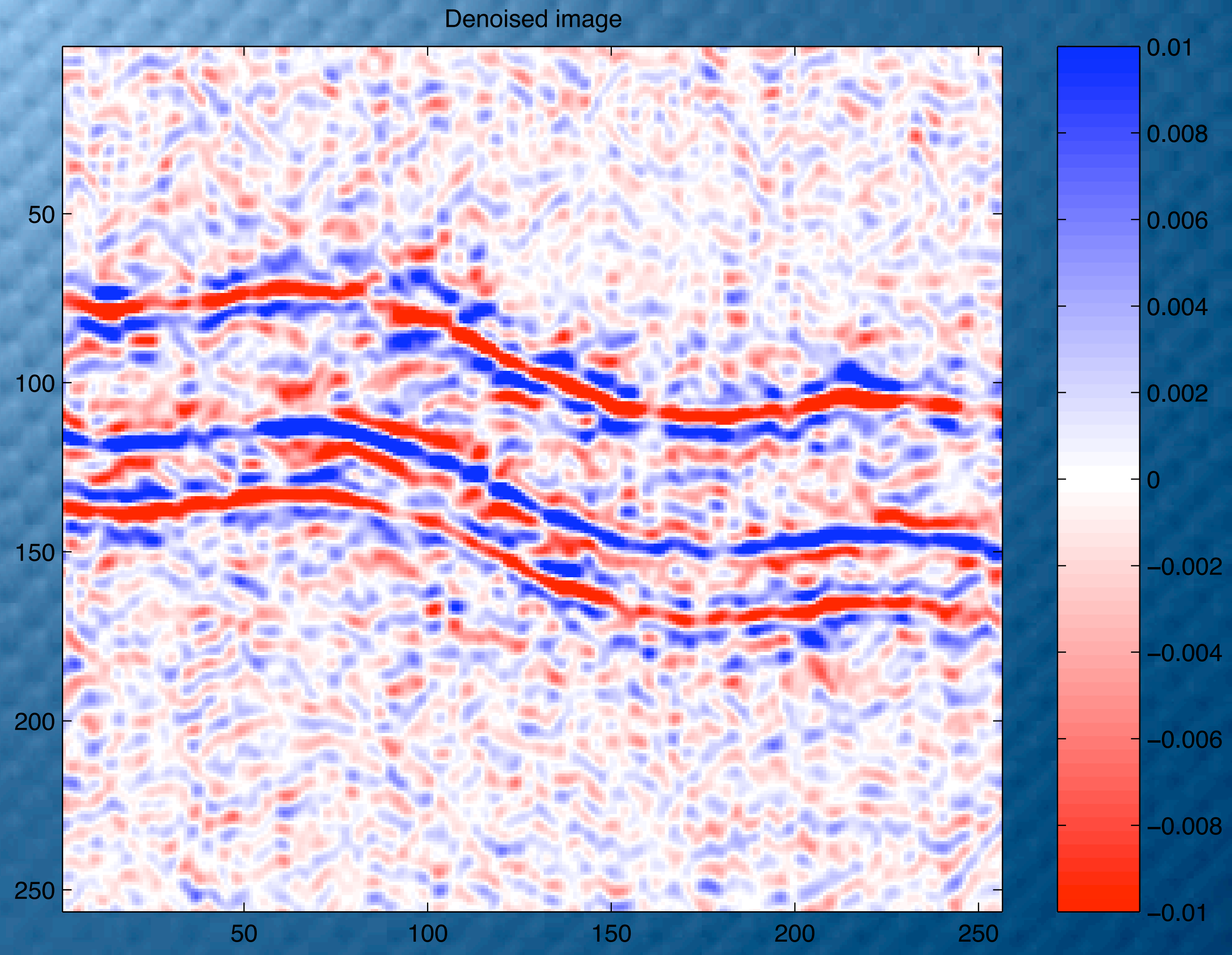
$$t(\Gamma) = \begin{cases} 3\sigma\Gamma & \text{coarsest scales} \\ 4\sigma\Gamma & \text{finest scales} \end{cases}$$

Candes, Do

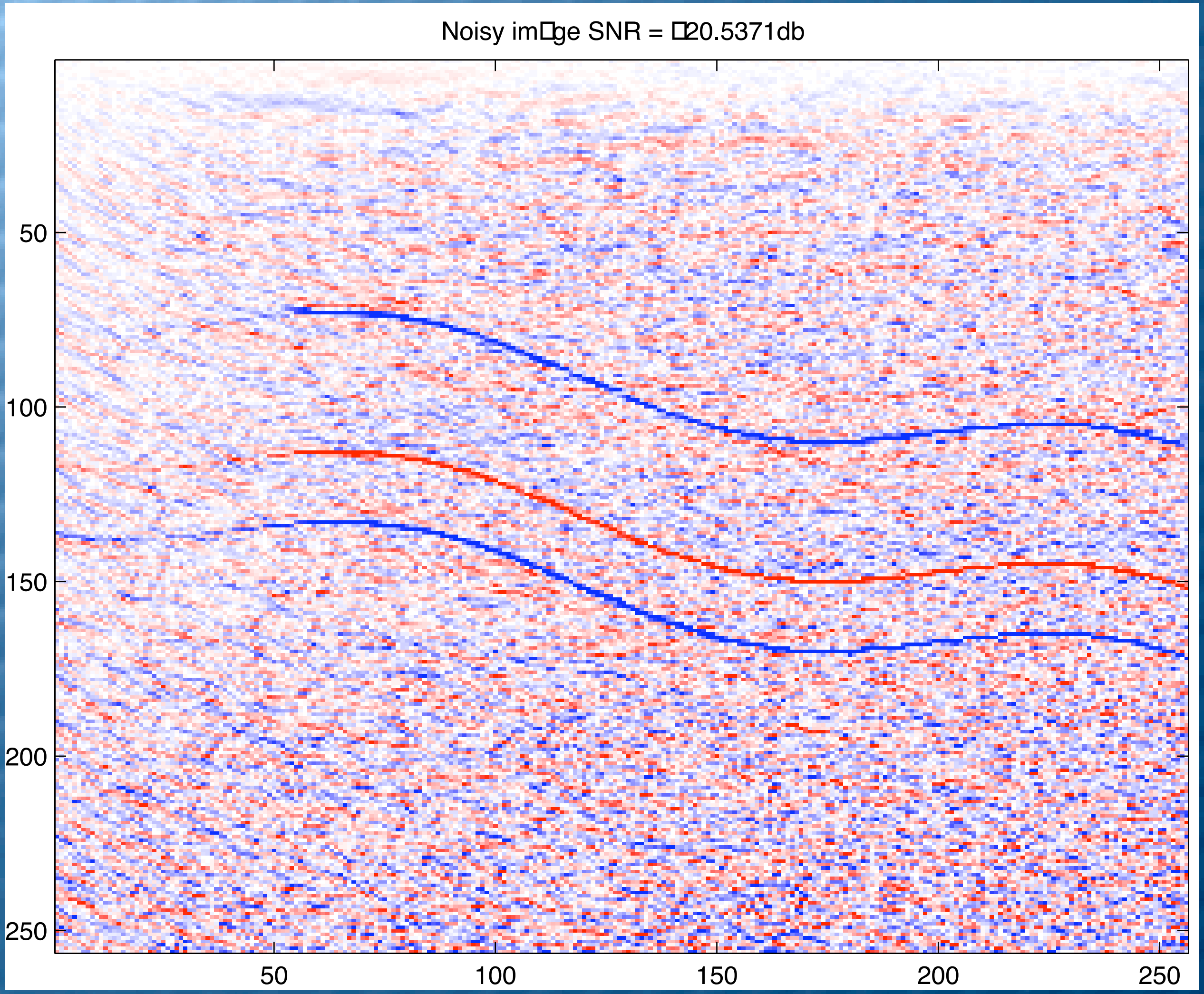
Optimal imaging



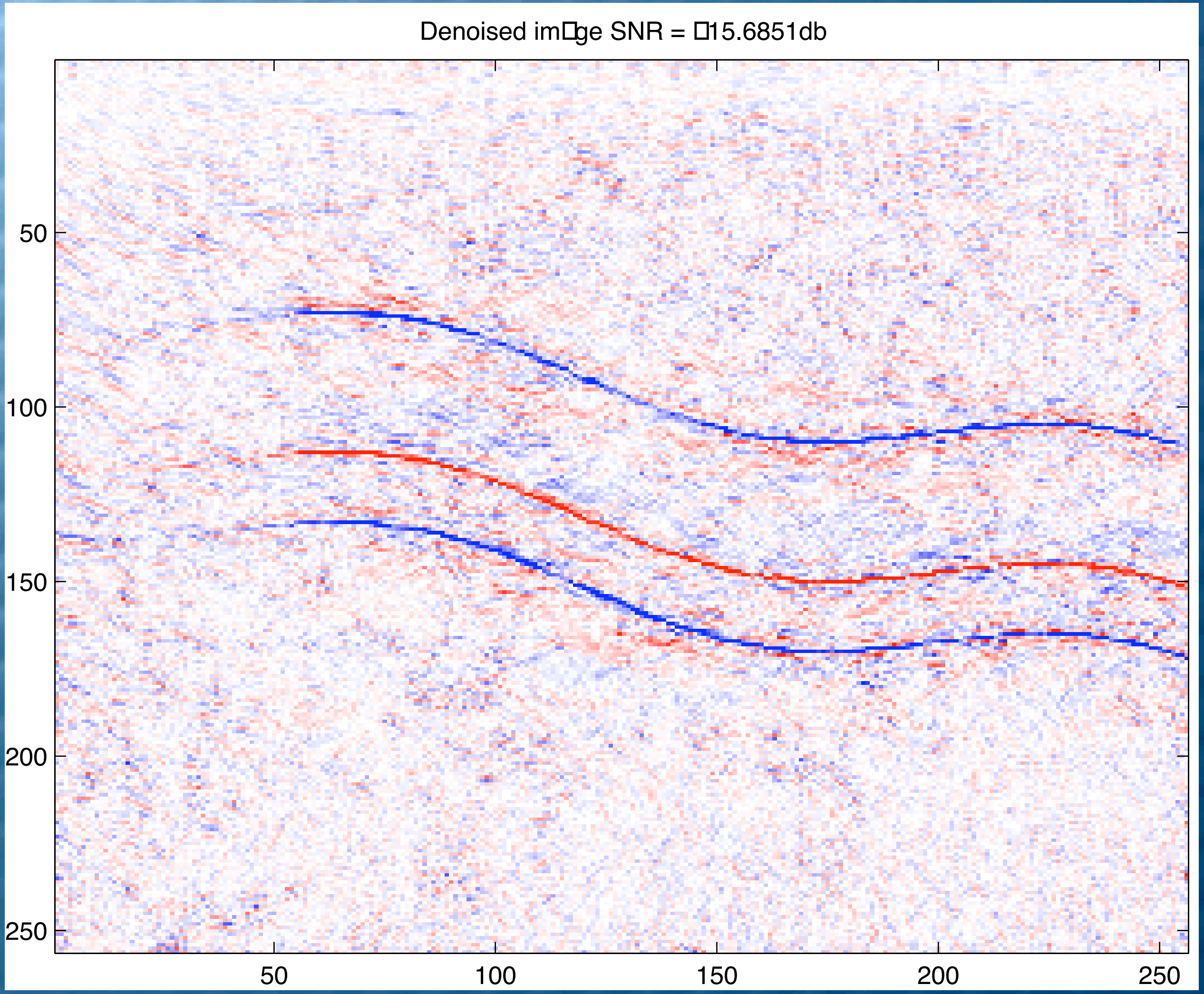
Optimal imaging



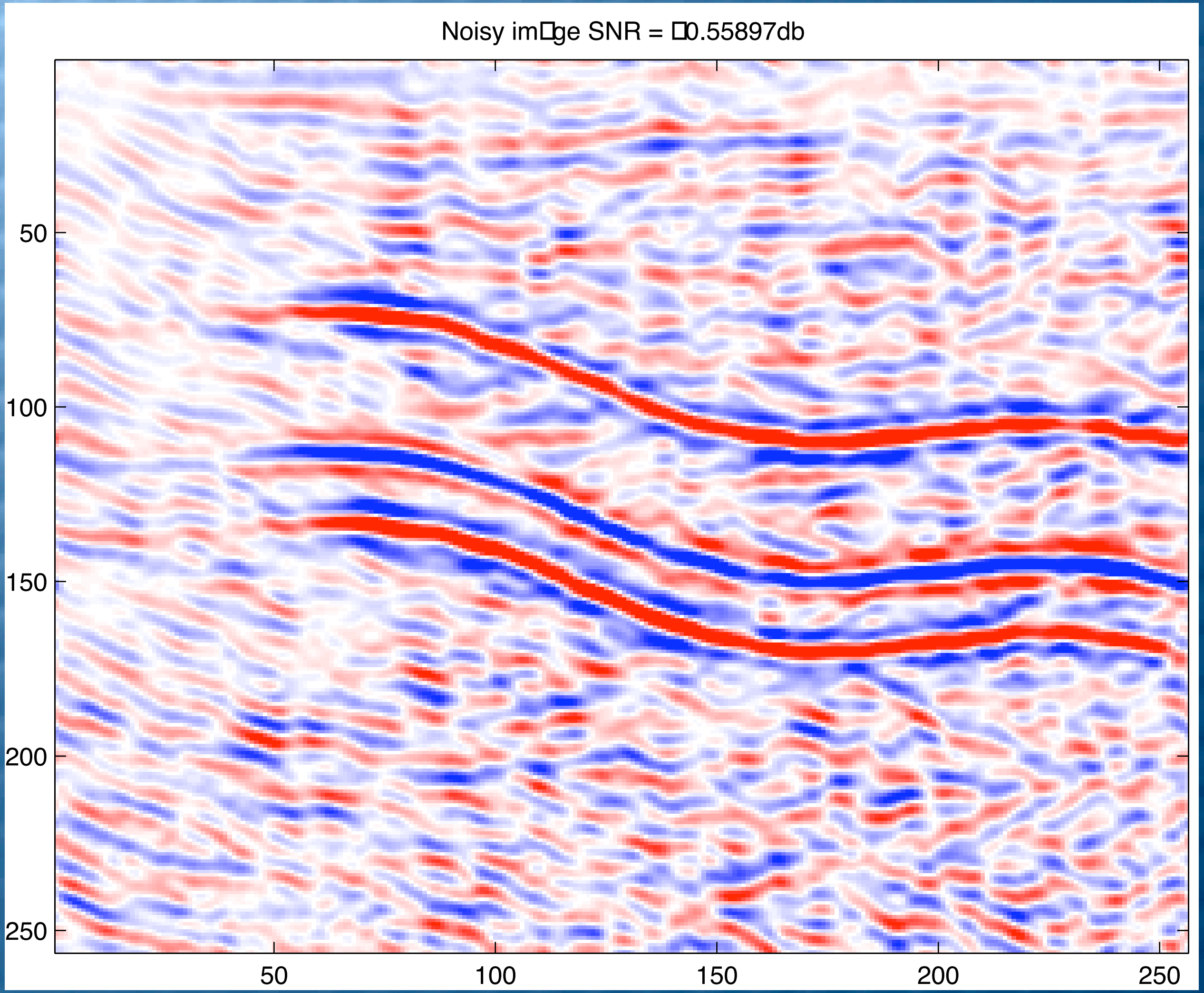
Optimal imaging



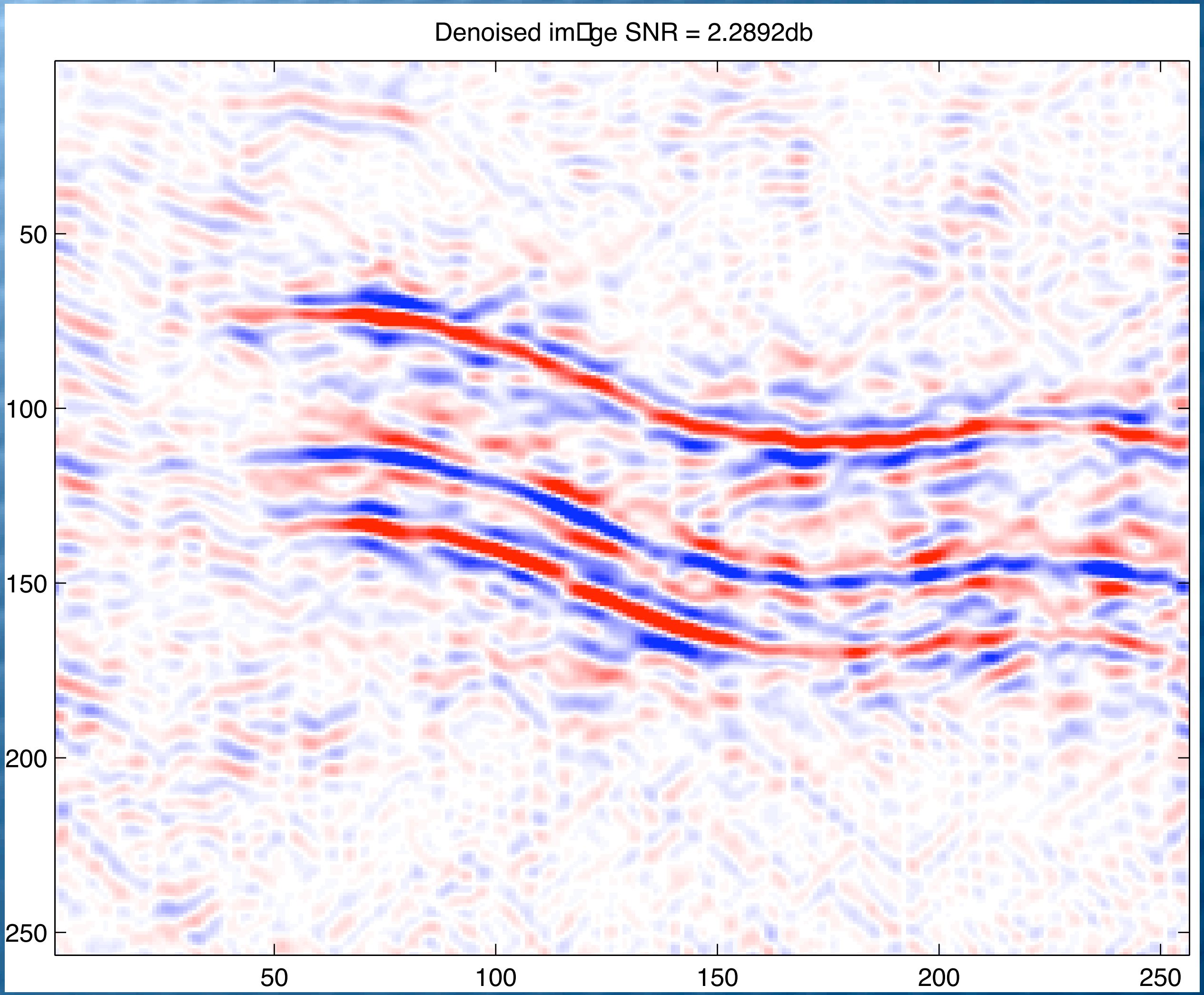
Optimal imaging



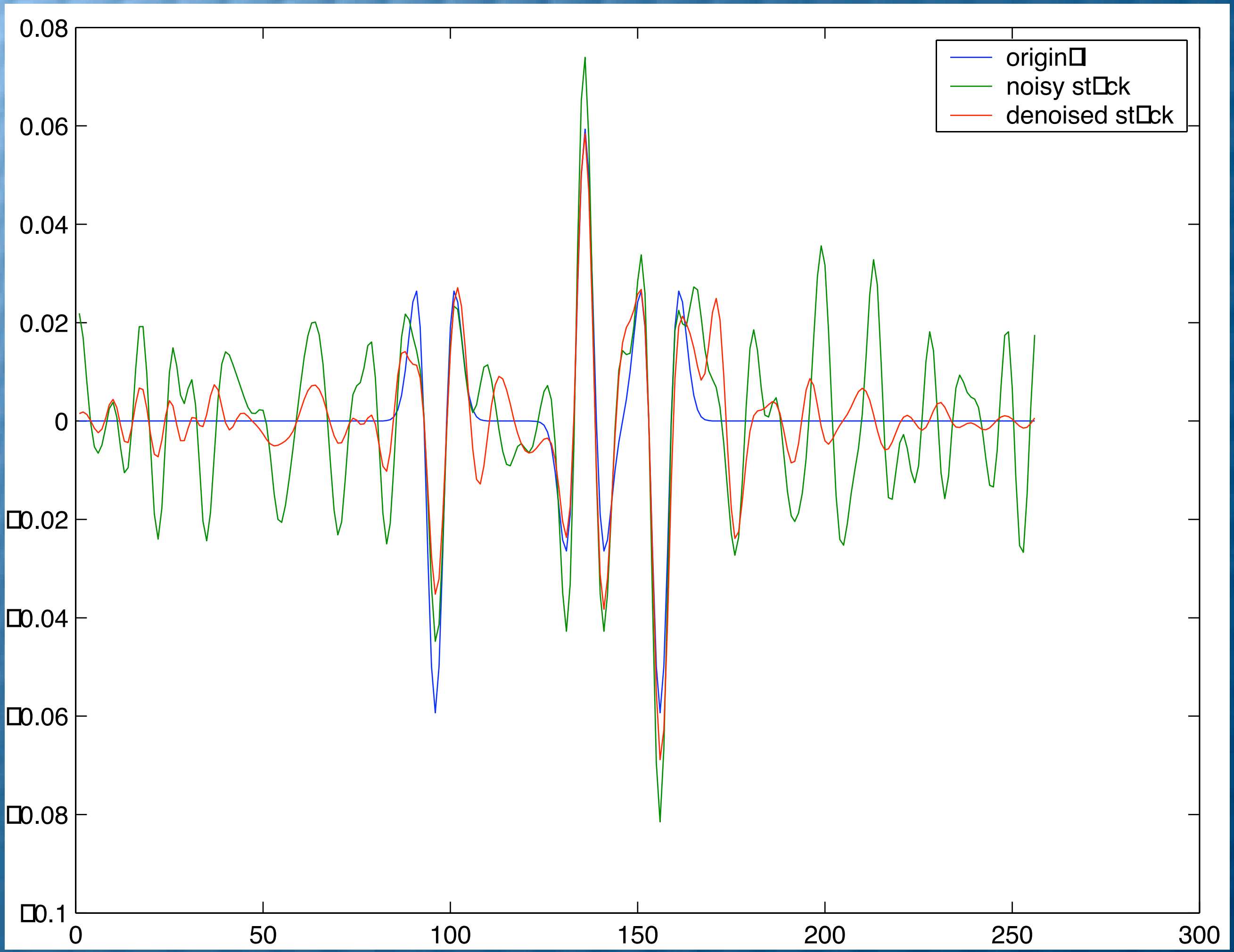
Optimal imaging



Optimal imaging



Optimal imaging



Conclusions

- *Optimal* representation for m
- *Diagonal* (symbol) normal operator whitens n!
- ***Thresholding works***
- Exploiting redundancy pays off:
  - *smoothness* along reflectors
  - *smoothness* in e-direction
- **Improved the SNR!**

Conclusions

challenges remain:

- direct computation operators in basis-function domain is a challenge
- beyond L^1-norm
- global optimization on the coefficients

Singularities in m preserved!

- NSERC
- Valhall license partners BP, Shell, Total and Armada Hess