Migration with Implicit Solvers for the Time-harmonic Helmholtz

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## Time domain vs. Frequency domain

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<td><strong>Solution of wave equation</strong></td>
<td>explicit, easy</td>
<td>implicit, not easy</td>
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<td><strong>Imaging</strong></td>
<td>time history, checkpointing, not trivial</td>
<td>all frequencies, freq. subsampling, easy</td>
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<td><strong>Computational algorithm</strong></td>
<td>parallelizable via domain decomposition (DD)-type algorithm</td>
<td>embarrassingly parallel in frequency, no communication, DD-type can apply for very large problem (3D)</td>
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<tr>
<td><strong>Boundary condition and damping layer</strong></td>
<td>not trivial</td>
<td>trivial, use complex velocity</td>
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<td><strong>Modeling relaxation</strong></td>
<td>not trivial</td>
<td>trivial, use freq. dep. complex velocity</td>
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Our focus

Frequency domain

conducive to frequency subsampling and then imaging using non-linear inversion ...
Migration

Of interest:
Given data $\delta d$, compute

$$\delta m = K^T[m_0] \delta d$$

- $\delta m$: the "update" image
- $m_0$: smooth model
- $K^T[m_0]$: the \textit{migration} operator

Here: $m = m_0 + \delta m$

[Baysal, 1983], ..., [Plessis, Mulder, 2004], [De Roeck, 2004]
Adjoint-state method (1)

\[
\min_m \frac{1}{2} \|d - u\|_2^2 \quad \text{subject to} \quad H[\omega, m]u = f
\]

- \(H[\omega, m]\) : the Helmholtz operator
- \(u\) : wavefield at frequency \(\omega\)
- \(f\) : seismic source
- \(d - u =: \delta d\) : data misfit

Note: a penalty functional can be added

[Tarantola, 1984], [Pratt et al., 1998], [Pratt, 1999], Plessix [2006]
Adjoint-state method (2)

Gradient-based update (multi shots and freqs)

\[ \delta m = \Re \left( \sum_{i_\omega = 1}^{n_\omega} \sum_{i_s = 1}^{n_s} \mathbf{u}_{i_s, i_\omega} f[\mathbf{m}_0] \odot \mathbf{v}_{i_s, i_\omega} [\mathbf{m}_0] \right) \]

\[ = \Re (\text{diag}(\mathbf{UV}^*)) \sim \mathbf{K}^T \delta \mathbf{d} \]

- \( \mathbf{v} \): back-propagated wavefield, obtained from \( \mathbf{H}^*[\omega, \mathbf{m}_0] \mathbf{v} = \delta \mathbf{d} \)

Image is computed “implicitly” via \( \mathbf{u} \) and \( \mathbf{v} \). No explicit \( \mathbf{K} \) needed.
Implicit migration/waveform inversion

• compute subsequently \( u \) and \( v \)
• correlate: \( u \odot v \)
• sum over shot and frequency

Good facts:
- parallel over frequency and shots
- no storage needed for (de)migration operator
- conducive to freq & shot sampling (size reduction)

How to compute \( u \) and \( v \)
Today’s talk

- Iterative solver for computing wavefields (i.e., $u$ and $v$) in frequency-domain migration/waveform inversion
- Show example from migration or first step of gradient update
### Computational Imaging

**Multiple-shots (right-hand sides), multiple frequencies**

\[ n_s : \text{number of shots} \quad n_f : \text{number of frequencies} \]

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<th>2D</th>
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<td>( n_s n_f O(n^5) )</td>
<td>( n_s n_f n_{iter} O(n^3) )</td>
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IM can be competitive if \( n_{iter} \ll n^d \) (with, e.g., preconditioner)

For similar analysis for MUMPS, see [Virieux, The Leading Edge, 2009]
Computational Imaging

The previous results are only useful if the iterative methods converge.

For frequency-domain wave equation, convergence cannot be guaranteed.
Why difficult for Iterative Method

One-d example: not of practical interest but tells the story

\[
k = \frac{2\pi f L}{c} = 50
\]

- Small eigenvalues close to zero, large eigenvalues unbounded: ill-conditioned
- Real parts of eigenvalues change signs: indefinite

Having two properties, convergence is not guaranteed.

**Indefiniteness** the most difficult to handle. No iterative method for indefinite system
First step: tackling indefiniteness (1)

Use as preconditioner the damped Helmholtz op.:

\[ M \triangleq -\nabla \cdot (\nabla) - (1 - \frac{1}{2} \hat{j}) \left( \frac{\omega}{c} \right)^2, \quad \hat{j} = \sqrt{-1}. \]

Then solve using iterative method on the system

\[ HM^{-1}w = f, \quad u = M^{-1}w \]

(And similarly for back-propagated wavefield)

[Erlangga, Oosterlee, Vuik, 2006]
[Riyanti et al., 2006]
[Plessix et al., 2007]
First step: tackling indefiniteness (2)

- Real parts of eigenvalues have the same signs: \textit{definite}! 
  Iterative methods will converge easier \( n_{\text{iter}} < n^d \)

- To obtain \( O(n^d) \) method, \( M^{-1} \) computed by one multigrid iteration

- Large eigenvalue bounded by one, still some small eigenvalues ill-conditioned
Second step: tackling ill-condition

Multilevel operator:

\[
Q = I - Z \hat{H}^{-1} Z^T H M^{-1} + Z \hat{H}^{-1} Z^T ,
\]

with

\[
\hat{H} = Z^T H M^{-1} Z , \quad \text{dim} \hat{H} \ll \text{dim} H
\]

\( Z \) : sparse, interpolation operator

Then, solve

\[
H M^{-1} Qy = f , \quad u = M^{-1} Qy
\]

[Erlangga, Nabben, 2008]
[Erlangga, Herrmann, 2008]
Second step: tackling ill-condition

- Notice shift of eigenvalues towards one due to $Q$!
- The spectrum of $HM^{-1}Q$ is favorable for iterative methods
More on eigenvalues (1)

1D non-constant wavenumber $k$, smooth model $k = (50, 100)$

- $H$
- $HM^{-1}$
- $HM^{-1}Q$

Clustering around one
More on eigenvalues (2)

1D non-constant wavenumber $k$, hard model $k = (50, 100)$

For constant, smooth, or hard model, one can expect the same convergence rate.

Clustering around one
Example: forward modeling (1)

Forward modeling, one shot position, hard model

- Velocity contrast: 1500 - 4000 m/s
- Convergence is less dependent of frequency

\[
\begin{align*}
\text{MG:} & \quad H M^{-1} \\
\text{MKMG:} & \quad H M^{-1} Q
\end{align*}
\]
Example: forward modeling (2)

One shot position, hard model : wavefield

Real part of $u$, freq = 10 Hz, 9 grid/wavelength

Real part of $u$, freq = 10 Hz, 18 grid/wavelength
Example: forward modeling (3)

Traces

Trace at X = 3000 m, freq = 10 Hz

Trace at X = 800 m, freq = 10 Hz
Example: imaging (1)

Computational setup:
- part of Marmousi (shown before), 6 x 1.6 km2
- computational grid: 751 x 201 (18 gridpoint/wavelength)
  Twice more than time-domain grid, possible to use less
- frequency range: 0.5 - 5.0 Hz, 11 frequencies are used
- 188 shot positions, 751 receivers
- In case of Migration: 1 step gradient-based inversion

Speed-up:
- Parallel computation in frequency - each node computes one freq. case
- Use of less gridpoint per wavelength
- No communication cost: embarrassingly parallel
  In our case, 11 freqs, 11 nodes.

Est. 1 hour of CPU time <= approximate the same as Symes’ time-domain
finite difference code with checkpointing ...
Example: imaging (2)

Target model

After first gradient-based update
\[ m_1 = m_0 + \delta m \]

\( \delta m \) (not shown) is computed using data from 188 shots and 11 frequencies (0.5-5.0 Hz)
Conclusion

• Key of successful iterative methods for Helmholtz: handling indefiniteness and ill-conditioned

• In our method, both are handled by preconditioner and multilevel operator

• Computational example shows that in terms of memory and CPU time, an iterative method can be a viable alternative to direct method in frequency-domain waveform inversion or migration

• Extension general $d$-dimension is straightforward
Future direction

- 3D wave-modeler and inversion
  Use of domain-decomposition-type algorithm
  Iterative methods for multiple right-hand sides;
  (solve multiple shots for one frequencies)

- Waveform inversion with Gauss-Newton-Krylov methods
  Hessian is computed implicitly via forward/backward solves,
  faster convergence.
  The use of direct methods are too expensive; at every Gauss-Newton update, LU
  factors must be formed
  [Erlangga, Herrmann, SEG 2009]

- FD inversion - conducive to freq. sampling

  Alternative:
  Freq. and shot sampling & inversion using sparsity-promoting recovery
  [Herrmann, Erlangga, Lin, 2009]  
  [Herrmann, SEG 2009]
Further reading

- Y A Erlangga, C W Oosterlee and C Vuik  
  A novel multigrid-based preconditioner for the heterogeneous Helmholtz equation  

- Y A Erlangga and R Nabben  
  On a multilevel Krylov method for Helmholtz equation preconditioned by shifted Laplacian  
  To appear in Electronic Transaction on Numerical Analysis  
  http://slim.eos.ubc.ca/Publications/Public/Journals/erlangga08oam.pdf

- Y. Erlangga and F. J. Herrmann  
  An iterative multilevel method for computing wavefields in frequency-domain seismic inversion  
  http://slim.eos.ubc.ca/Publications/Public/Conferences/SEG/2008/erlangga08seg.pdf
Acknowledgments

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For other resources on frequency-domain compressive computation, visit

http://slim.eos.ubc.ca