SLIM Seismic Laboratory for Imaging and Modeling

Migration with Implicit Solvers for the Time-harmonic Helmholtz

Yogi A. Erlangga, Felix J. Herrmann

Seismic Laboratory for Imaging and Modeling, The University of British Columbia

{yerlangga,fherrmann}@eos.ubc.ca

Time domain vs. Frequency domain

	Time domain	Frequency domain
Solution of wave equation	explicit, easy	implicit, not easy
Imaging	time history, checkpointing, not trivial	all frequencies, freq. subsampling, easy
Computational algorithm	paralellizable via domain decomposition (DD)-type algorithm	embarrasingly parallel in frequency, no communication, DD-type can apply for very large problem (3D)
Boundary condition and damping layer	not trivial	trivial, use complex velocity
Modeling relaxation	not trivial	trivial, use freq. dep. complex velocity

Our focus

Frequency domain

conducive to frequency subsampling and then imaging using non-linear inversion ...

Migration

Of interest:
Given data $\delta \mathbf{d}$, compute

$$\delta \mathbf{m} = \mathbf{K}^T[\mathbf{m}_0]\delta \mathbf{d}$$

- $\delta \mathbf{m}$: the "update" image
- $\mathbf{m}_0$: smooth model
- $\mathbf{K}^T[\mathbf{m}_0]$: the *migration* operator

Here: $\mathbf{m} = \mathbf{m}_0 + \delta \mathbf{m}$

[Baysal, 1983], ..., [Plessix, Mulder, 2004], [De Roeck, 2004]

Adjoint-state method (1)

$$\min_{\mathbf{m}} \frac{1}{2}\|\mathbf{d}-\mathbf{u}\|_2^2 \quad \text{subject to} \quad \mathbf{H}[\omega,\mathbf{m}]\mathbf{u}=\mathbf{f}$$

- $\mathbf{H}[\omega,\mathbf{m}]$: the Helmholtz operator
- $\mathbf{u}$: wavefield at frequency ω
- $\mathbf{f}$: seismic source
- $\mathbf{d}-\mathbf{u} =: \delta\mathbf{d}$: data misfit

Note: a penalty functional can be added

[Tarantola, 1984], [Pratt et al., 1998], [Pratt, 1999], Plessix [2006]

Adjoint-state method (2)

Gradient-based update (multi shots and freqs)

$$\delta \mathbf{m} = \Re\left(\sum_{i_\omega=1}^{n_\omega}\sum_{i_s=1}^{n_s} \mathbf{u}_{i_s,i_\omega f}[\mathbf{m}_0] \odot \mathbf{v}_{i_s,i_\omega}[\mathbf{m}_0]\right)$$

$$= \Re\left(\mathrm{diag}(\mathbf{U}\mathbf{V}^*)\right) \sim \mathbf{K}^T \delta\mathbf{d}$$

- $\mathbf{v}$: back-propagated wavefield, obtained from

$$\mathbf{H}^*[\omega, \mathbf{m}_0]\mathbf{v} = \delta\mathbf{d}$$

Image is computed "implicitly" via $\mathbf{u}$ and $\mathbf{v}$.
No explicit $\mathbf{K}$ needed.

Implicit migration/waveform inversion

- compute subsequently $\mathbf{u}$ and $\mathbf{v}$
- correlate : $\mathbf{u} \odot \mathbf{v}$
- sum over shot and frequency

Good facts:

- parallel over frequency and shots
- no storage needed for (de)migration operator
- conducive to freq & shot sampling (size reduction)

How to compute $\mathbf{u}$ and $\mathbf{v}$

Today's talk

- Iterative solver for computing wavefields (i.e., $\mathbf{u}$ and $\mathbf{v}$) in frequency-domain migration/ waveform inversion
- Show example from migration or first step of gradient update

Computational Imaging

Multiple-shots (right-hand sides), multiple frequencies

n_s : number of shots n_f : number of frequencies

2D	Direct methods	Iterative Methods
LU factors	$n_f \mathcal{O}(n^4)$	-
Solution	$n_s n_f \mathcal{O}(n^3)$	$n_s n_f n_{iter} \mathcal{O}(n^2)$

3D	Direct methods	Iterative Methods
LU factors	$n_f \mathcal{O}(n^9)$	-
Solution	$n_s n_f \mathcal{O}(n^5)$	$n_s n_f n_{iter} \mathcal{O}(n^3)$

IM can be competitive if $n_{iter} \ll n^d$ (with, e.g., preconditioner)

For similar analysis for MUMPS, see [Virieux, The Leading Edge, 2009]

Computational Imaging

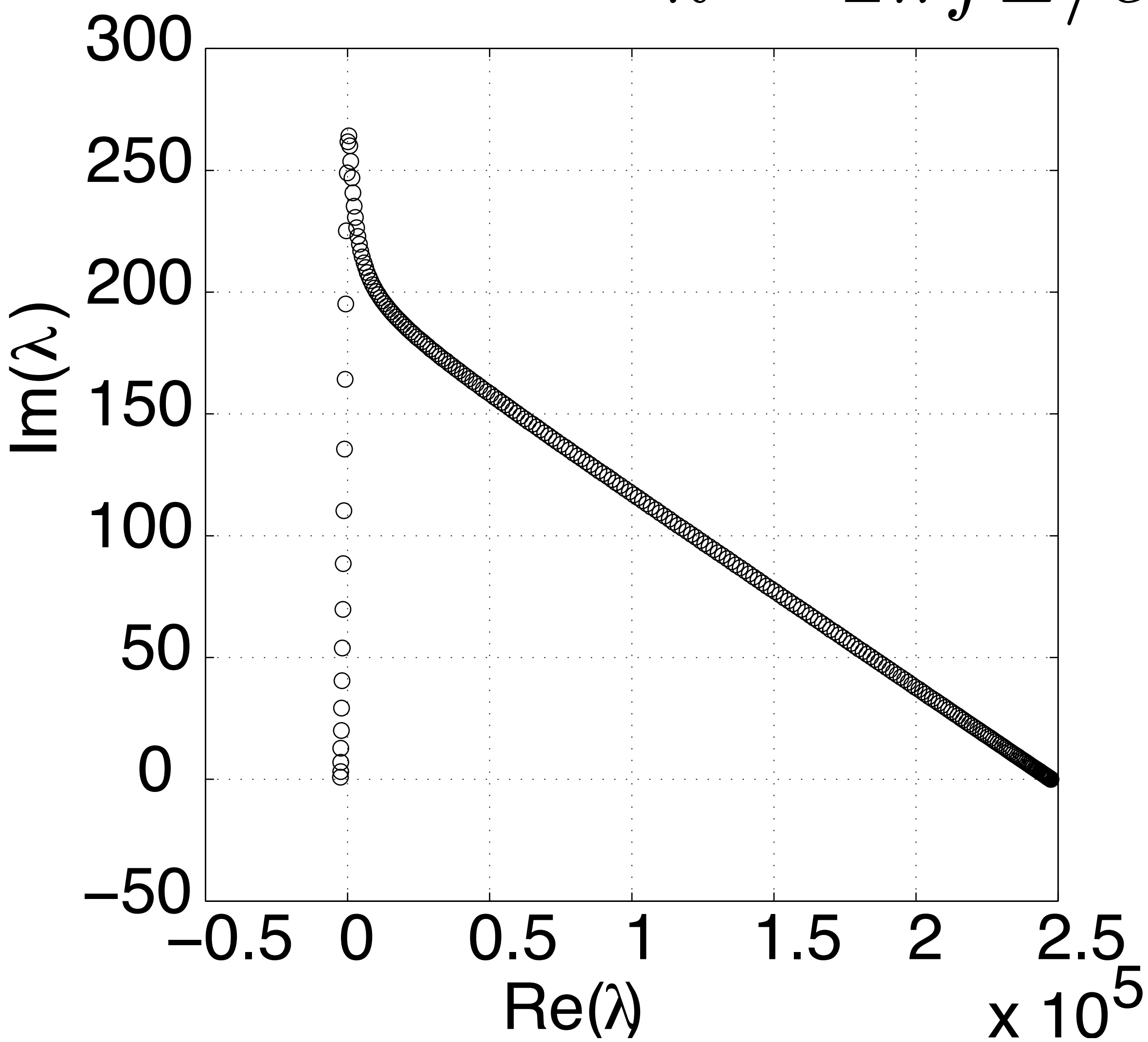
The previous results are only useful if the iterative methods converge.

For frequency-domain wave equation, convergence can not be guaranteed.

Why difficult for Iterative Method

One-*d* example: not of practical interest but tells the story

constant $k = 2\pi f L/c = 50$



- Small eigenvalues close to zero, large eigenvalues unbounded: ill-conditioned
- Real parts of eigenvalues change signs: indefinite

Having two properties, convergence is not guaranteed.

Indefiniteness the most difficult to handle. No iterative method for indefinite system

First step: tackling indefiniteness (1)

Use as preconditioner the damped Helmholtz op.:

$$\mathbf{M} \triangleq -\nabla \cdot (\nabla) - (1 - \frac{1}{2}\hat{j}) \left(\frac{\omega}{c}\right)^2, \quad \hat{j} = \sqrt{-1}.$$

Then solve using iterative method on the system

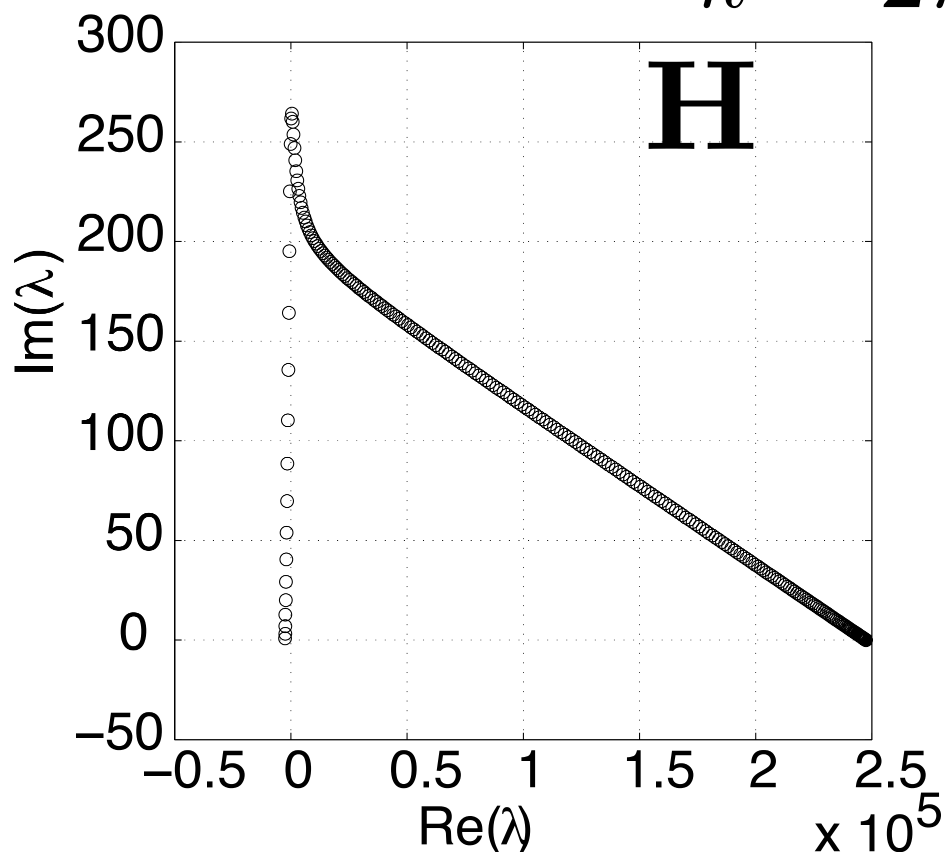
$$\boxed{\mathbf{H}\mathbf{M}^{-1}\mathbf{w} = \mathbf{f}, \quad \mathbf{u} = \mathbf{M}^{-1}\mathbf{w}}$$

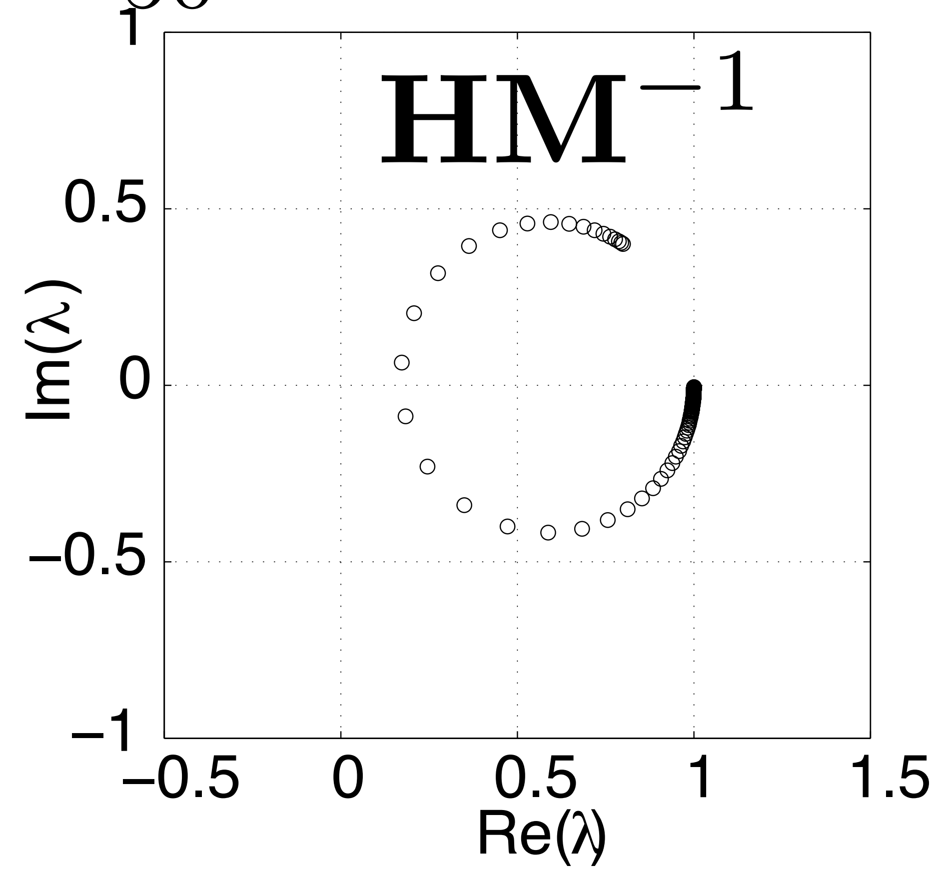
(And similarly for back-propagated wavefield)

[Erlangga, Oosterlee, Vuik, 2006]
[Riyanti et al., 2006]
[Plessix et al., 2007]

First step: tackling indefiniteness (2)

$$k = 2\pi f L/c = 50$$





- Real parts of eigenvalues have the same signs: definite! Iterative methods will converge easier $n_{iter} < n^d$
- To obtain $\mathcal{O}(n^d)$ method , $\mathbf{M}^{-1}$ computed by one multigrid iteration
- Large eigenvalue bounded by one, still some small eigenvalues ill-conditioned

Second step: tackling ill-condition

Multilevel operator:

$$\mathbf{Q} = \overbrace{\mathbf{I} - \mathbf{Z}\widehat{\mathbf{H}}^{-1}\mathbf{Z}^T\mathbf{H}\mathbf{M}^{-1}}^{\text{shift small eigenvalues to 0}} + \overbrace{\mathbf{Z}\widehat{\mathbf{H}}^{-1}\mathbf{Z}^T}^{\text{shift zero eigenvalues to 1}},$$

with

$$\widehat{\mathbf{H}} = \mathbf{Z}^T\mathbf{H}\mathbf{M}^{-1}\mathbf{Z}, \quad dim\widehat{\mathbf{H}} \ll dim\mathbf{H}$$

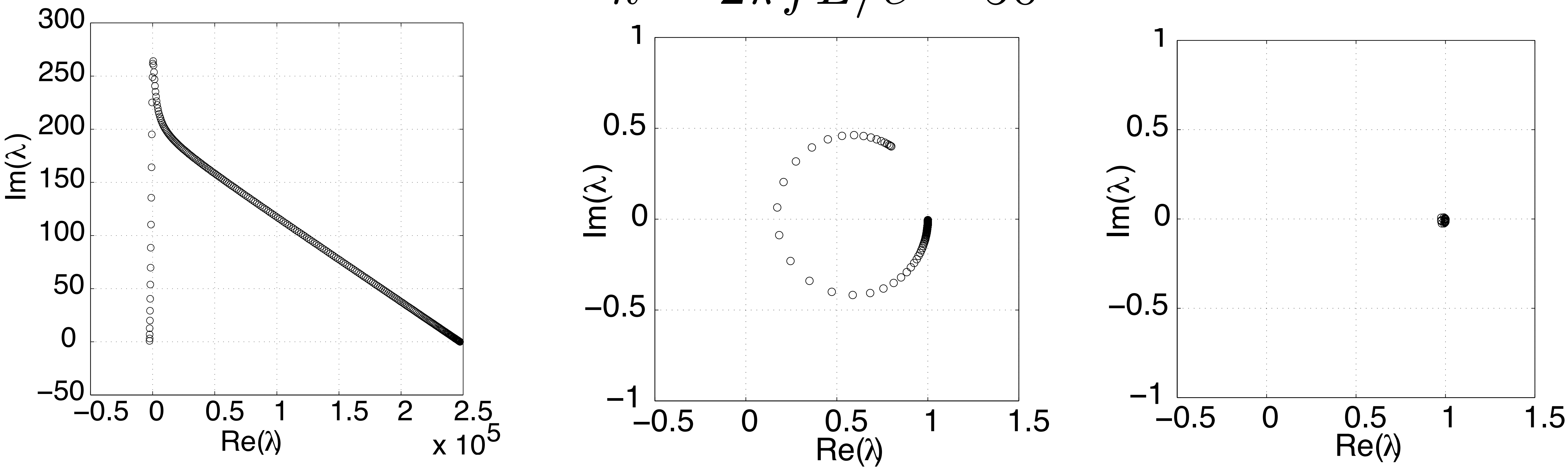
$\mathbf{Z}$: sparse, interpolation operator

Then, solve $$\boxed{\mathbf{H}\mathbf{M}^{-1}\mathbf{Q}\mathbf{y} = \mathbf{f}, \quad \mathbf{u} = \mathbf{M}^{-1}\mathbf{Q}\mathbf{y}}$$

[Erlangga, Nabben, 2008]
[Erlangga, Herrmann, 2008]

Second step: tackling ill-condition

$$k = 2\pi f L / c = 50$$



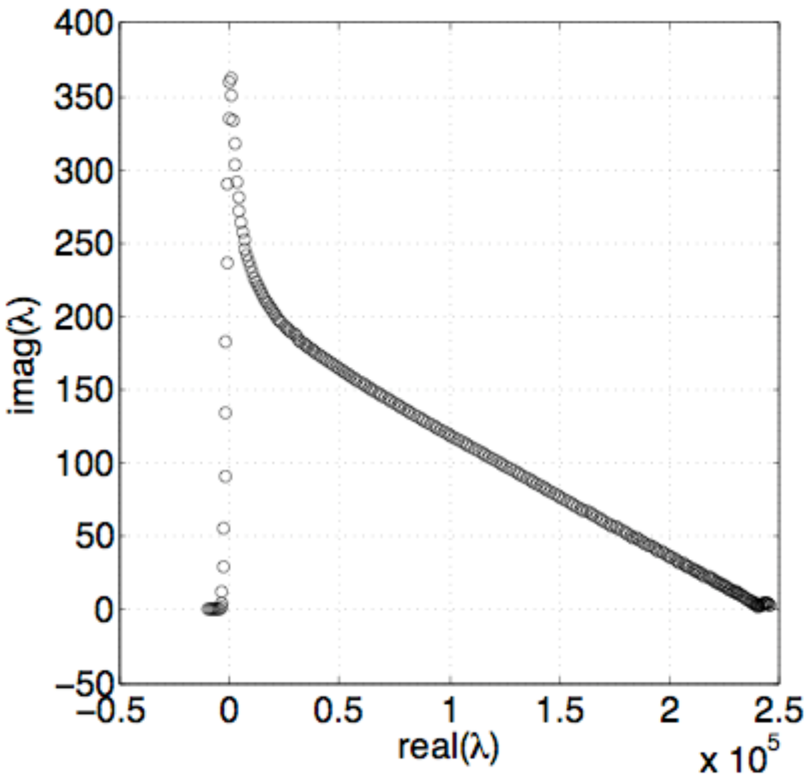
$\mathbf{H}$ $\qquad$ $\mathbf{HM^{-1}}$ $\qquad$ $\mathbf{HM^{-1}Q}$

- Notice shift of eigenvalues towards one due to Q!
- The spectrum of $\mathbf{HM^{-1}Q}$ is favorable for iterative methods

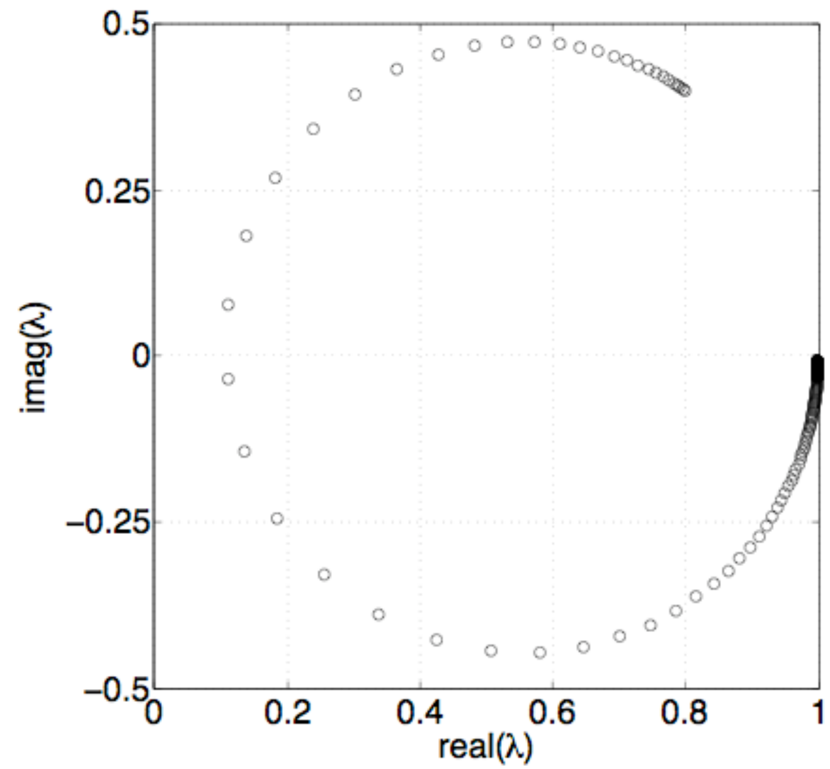
More on eigenvalues (1)

1D non-constant wavenumber k, smooth model $k = (50, 100)$

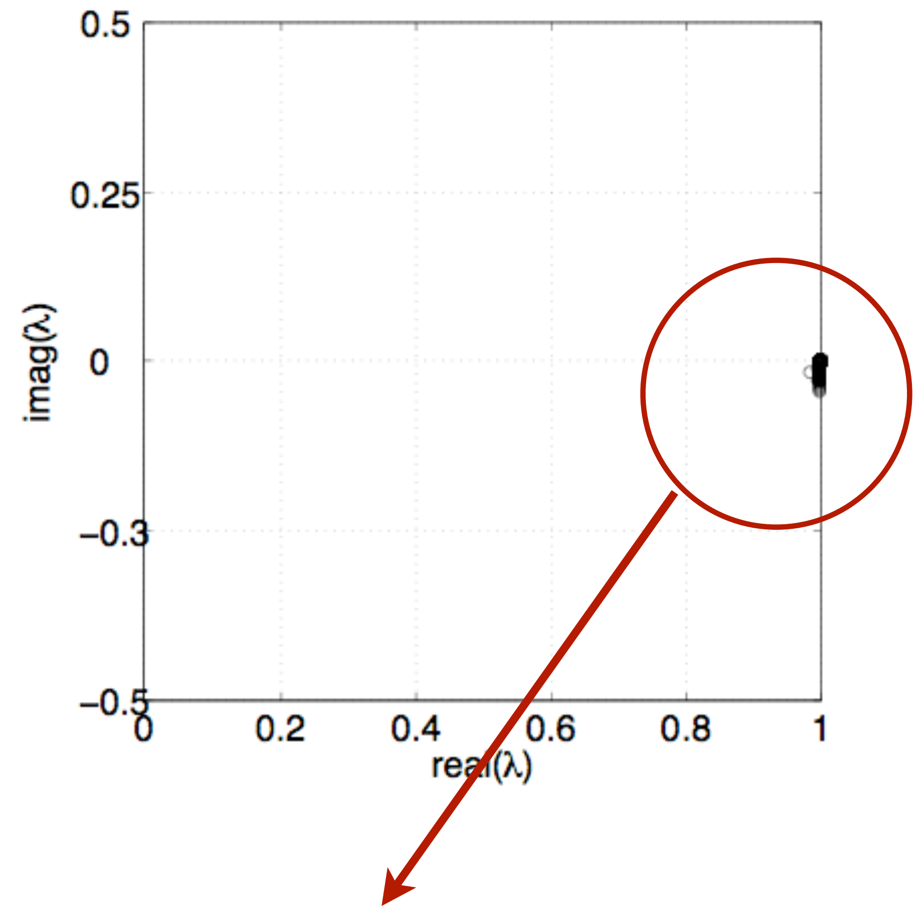
$\mathbf{H}$

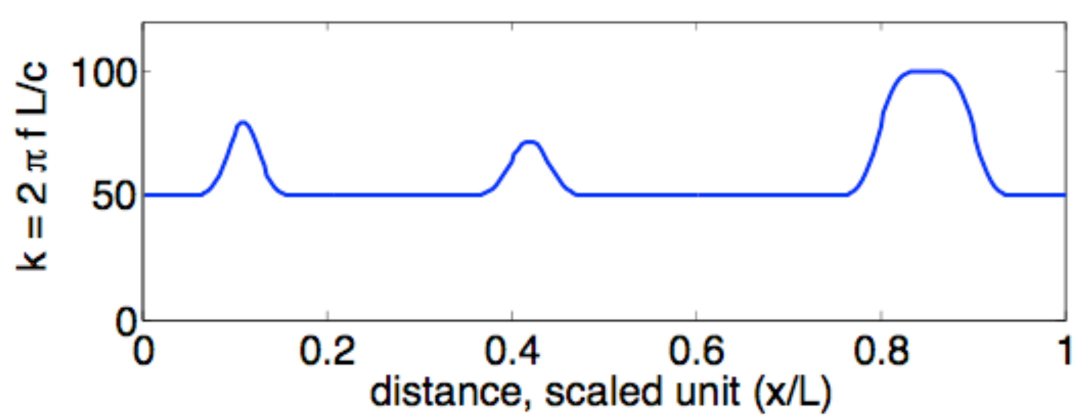


$\mathbf{HM}^{-1}$



$\mathbf{HM}^{-1}\mathbf{Q}$



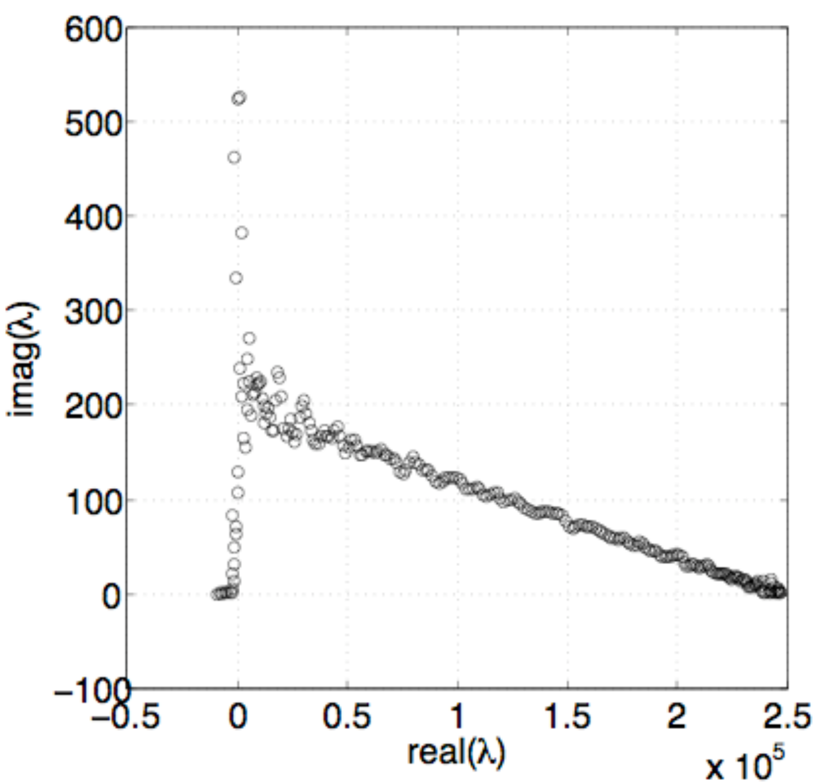


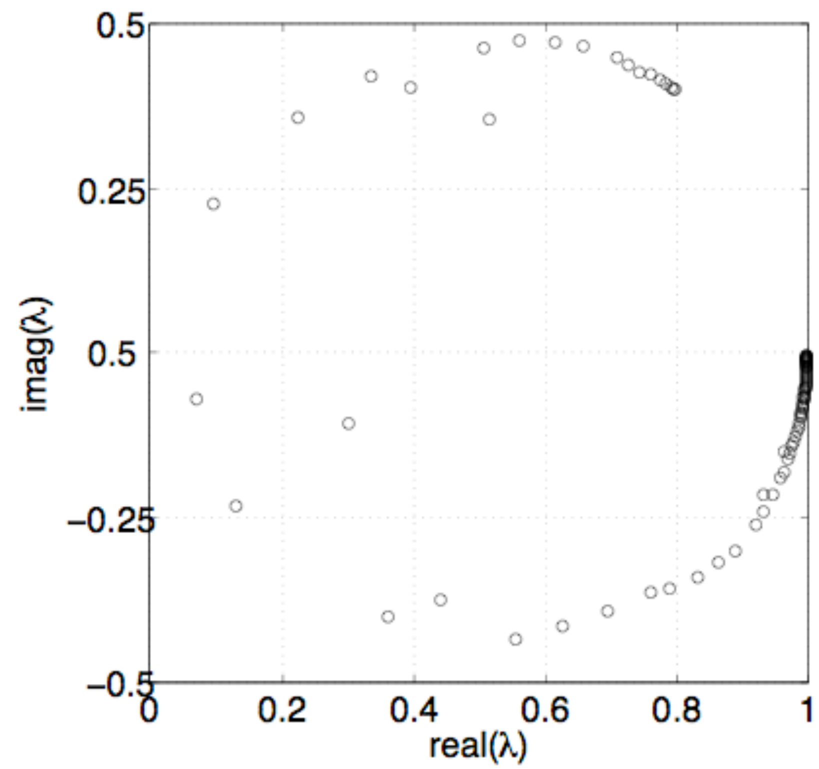
Clustering around one

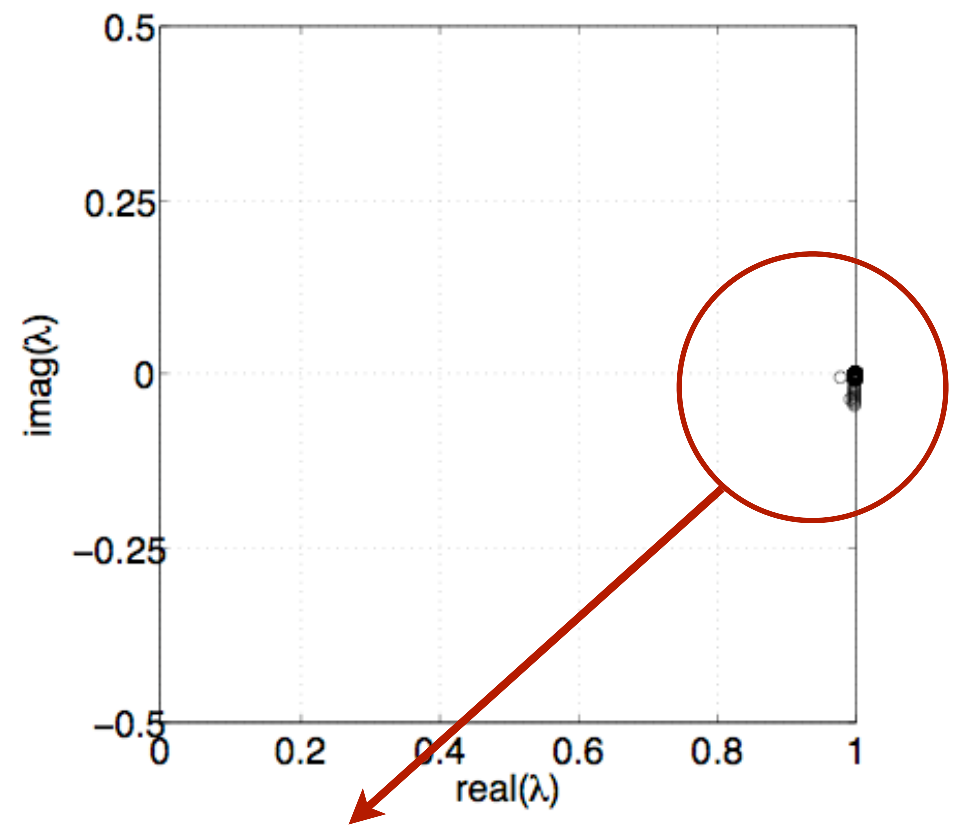
More on eigenvalues (2)

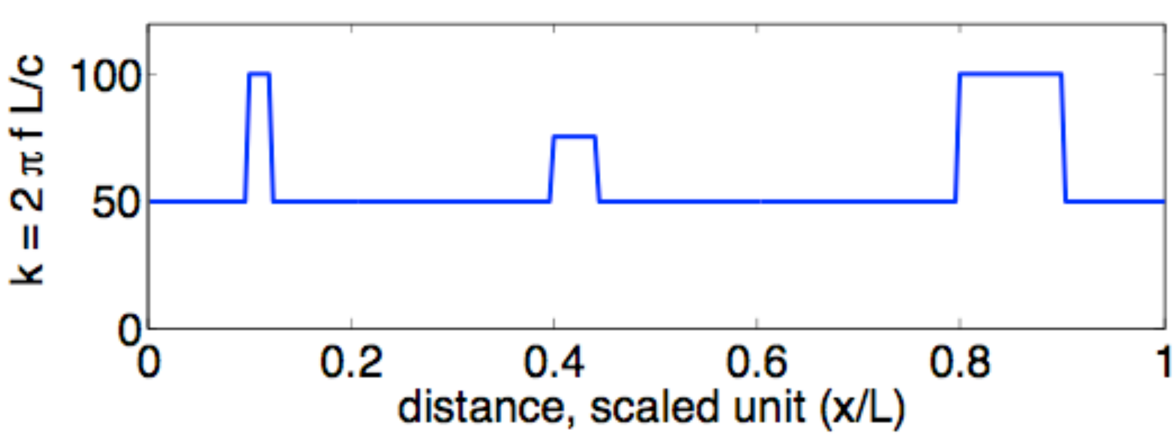
1D non-constant wavenumber k, hard model $k = (50, 100)$

$\mathbf{H}$ $\quad\quad$ $\mathbf{HM^{-1}}$ $\quad\quad$ $\mathbf{HM^{-1}Q}$







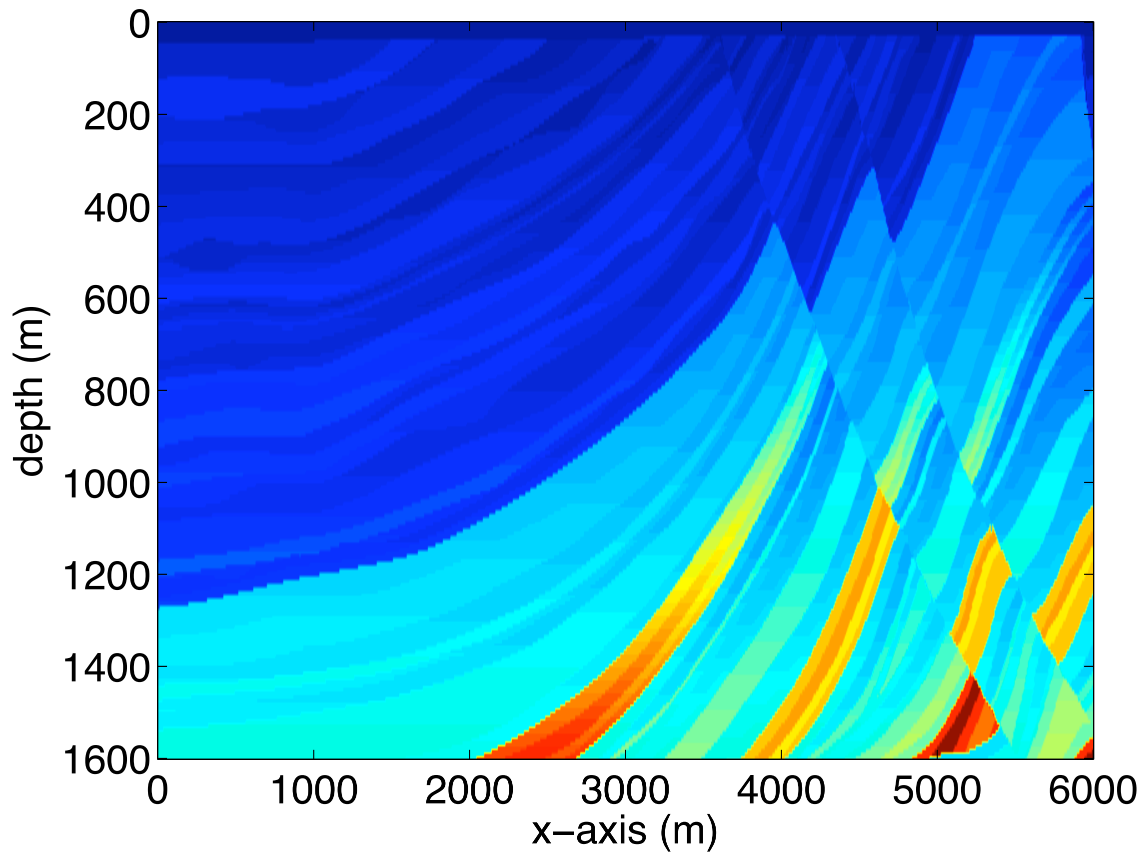


Clustering around one

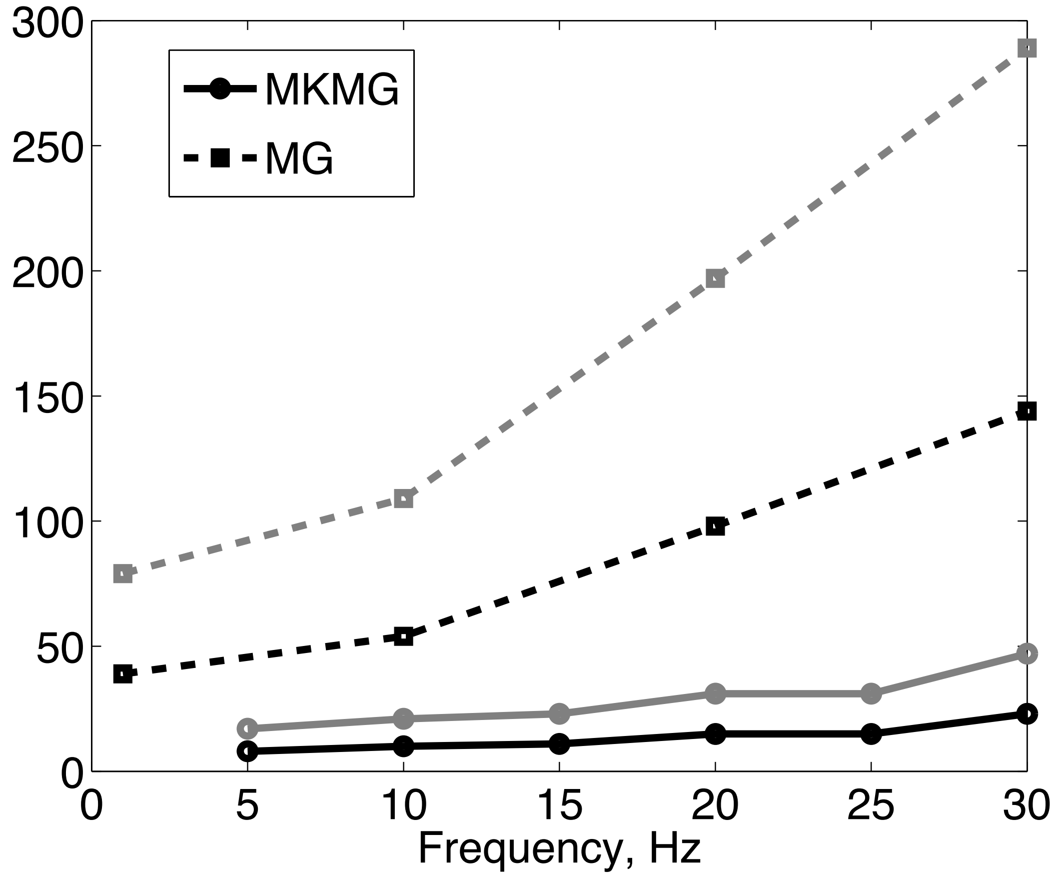
For constant, smooth, or hard model, one can expect the same convergence rate

Example: forward modeling (1)

Forward modeling, one shot position, hard model



- Velocity contrast: 1500 - 4000 m/s
- Convergence is less dependent of frequency



Number of iteration

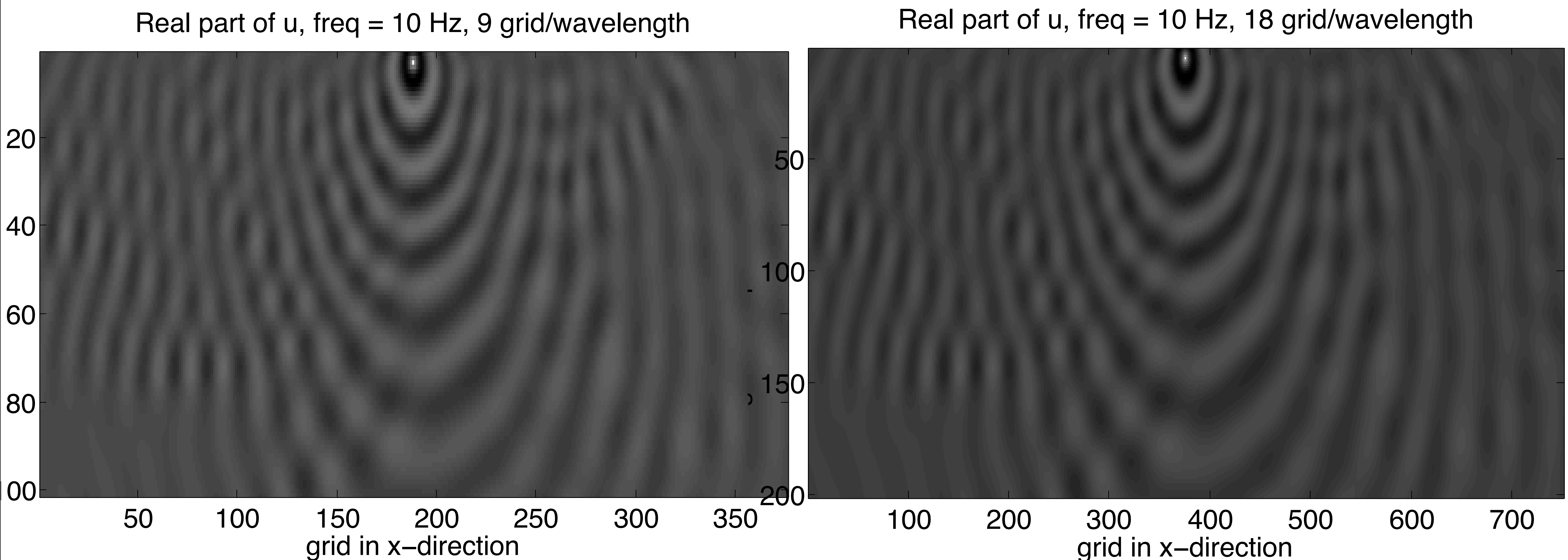
Number of matrix-vec multiplies

MG: $\mathbf{HM}^{-1}$

MKMG: $\mathbf{HM}^{-1}\mathbf{Q}$

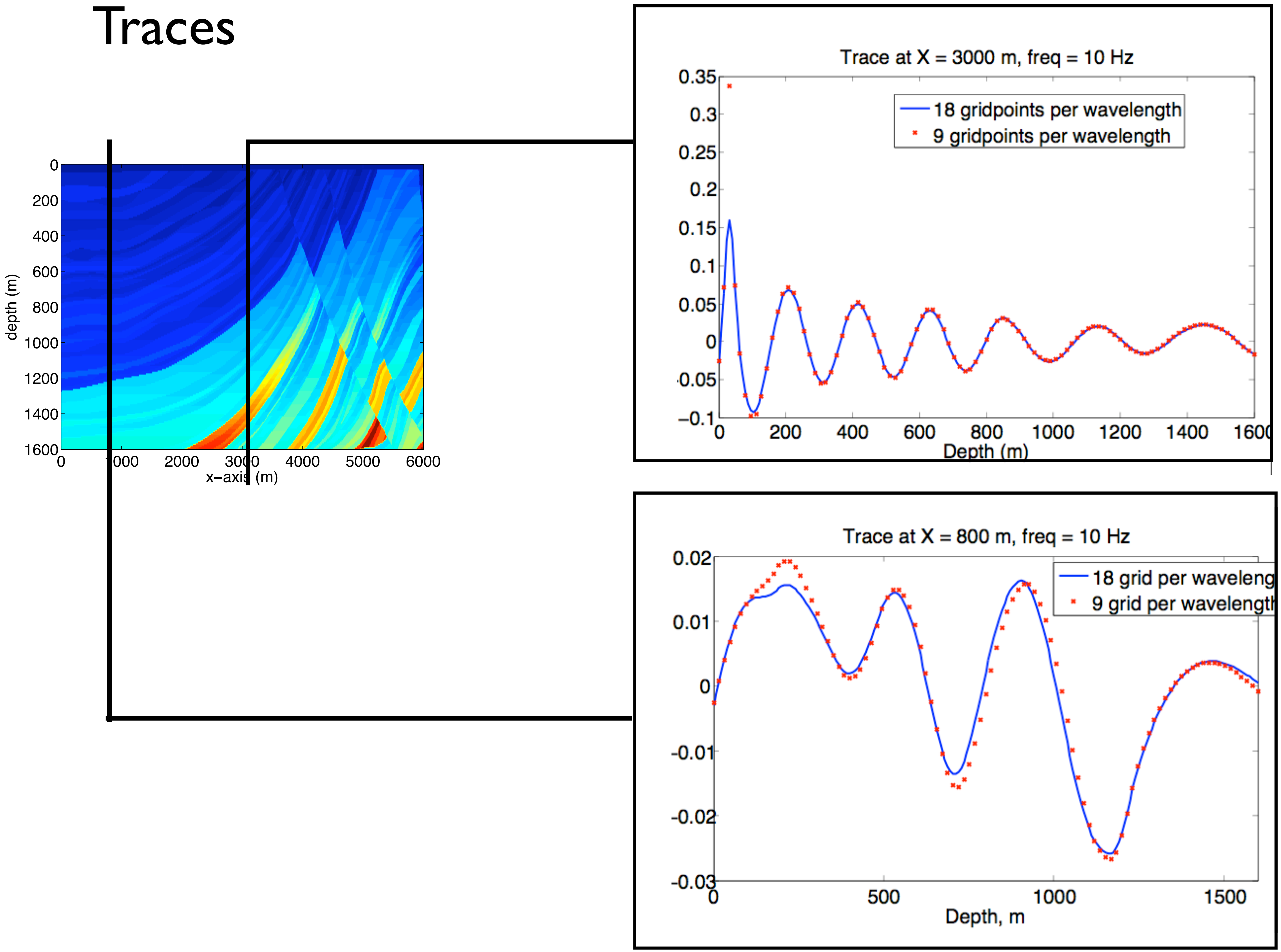
Example: forward modeling (2)

One shot position, hard model : wavefield



Example: forward modeling (3)

Traces



Example: imaging (1)

Computational setup:

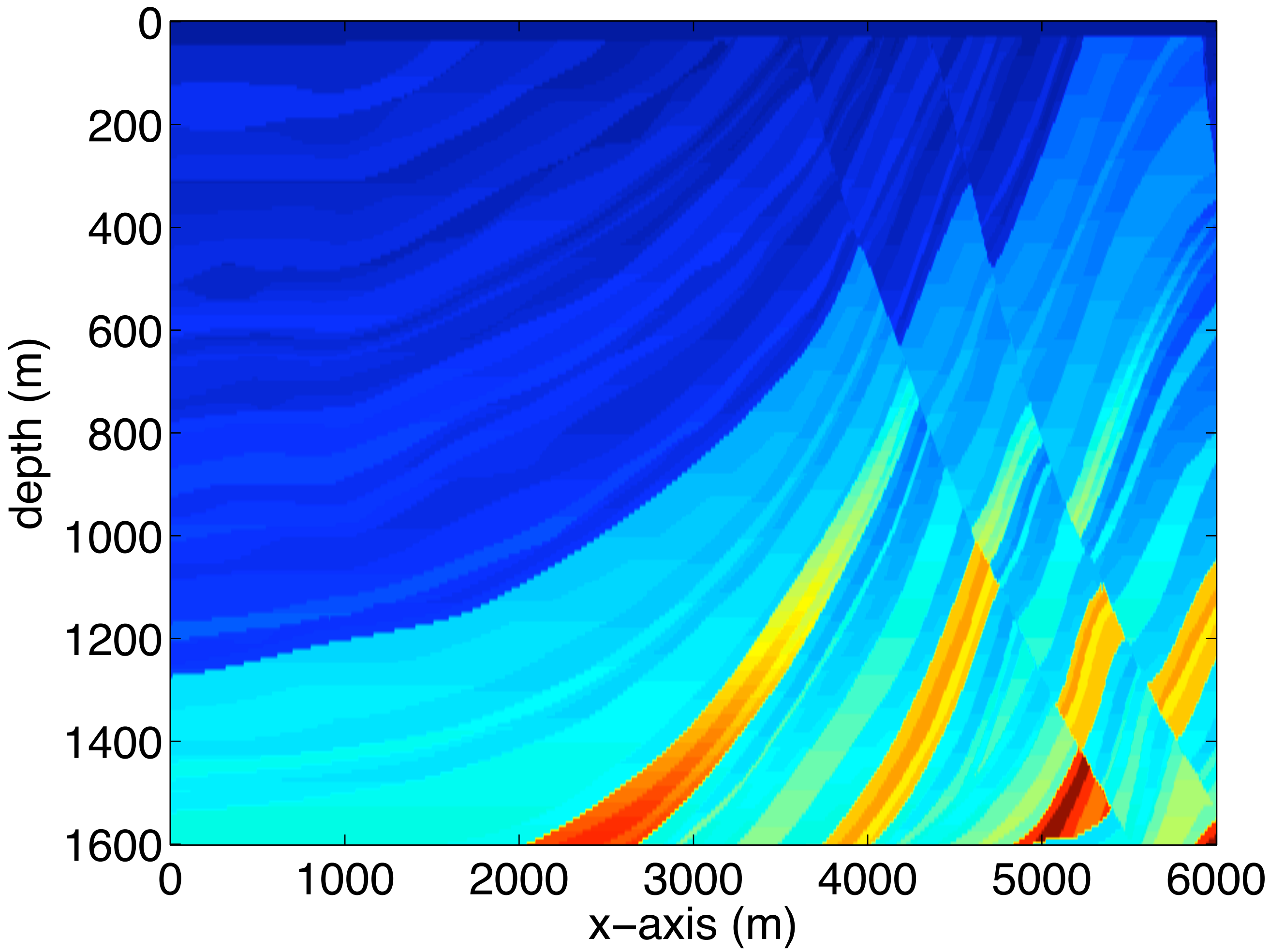
- part of Marmousi (shown before), 6 x 1.6 km2
- computational grid: 751 x 201 (18 gridpoint/wavelength)
  Twice more than time-domain grid, possible to use less
- frequency range: 0.5 - 5.0 Hz, 11 frequencies are used
- 188 shot positions, 751 receivers
- In case of Migration: 1 step gradient-based inversion

Speed-up:

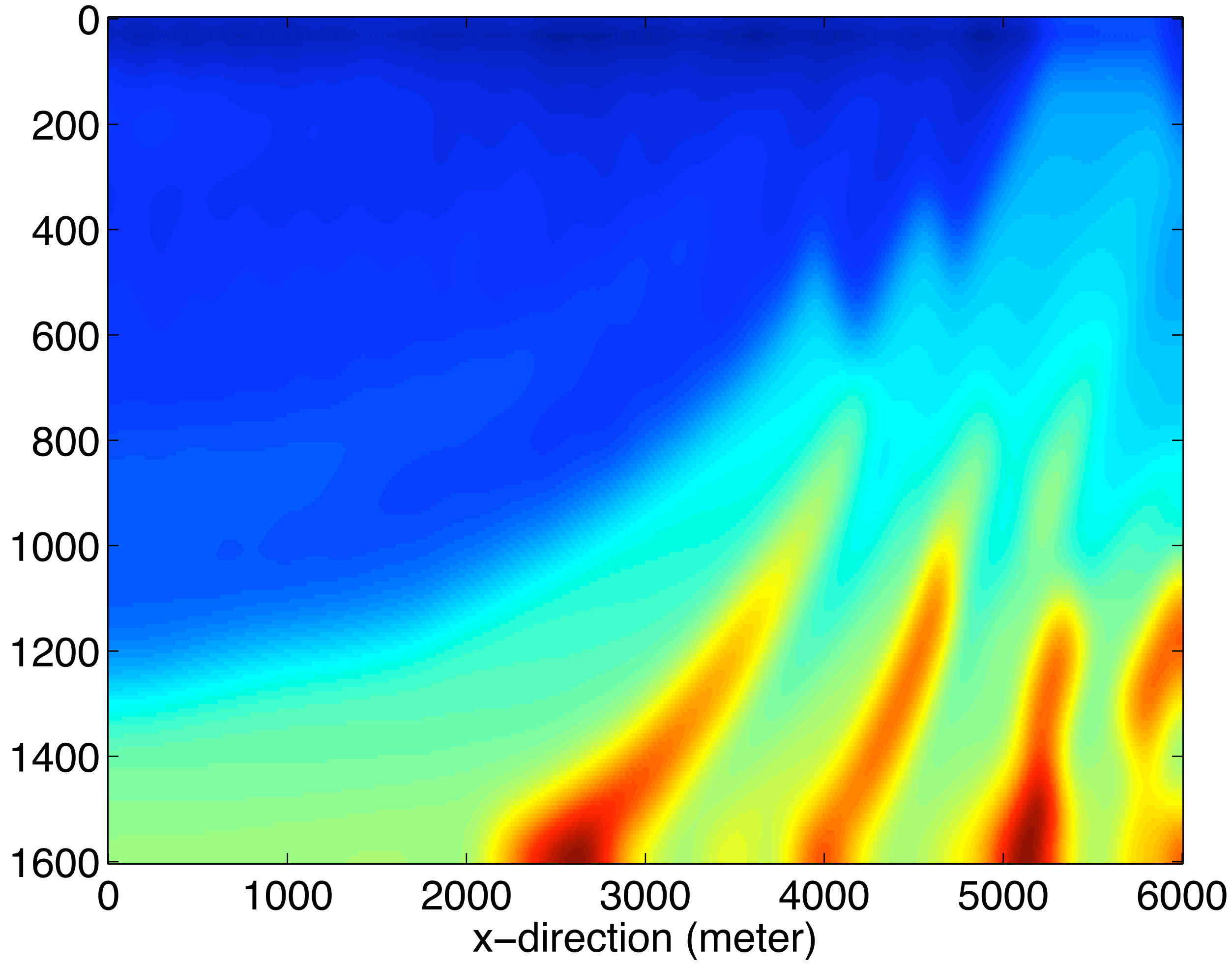
- Parallel computation in frequency - each node computes one freq. case
- Use of less gridpoint per wavelength
- No communication cost: embarrassingly parallel
  In our case, 11 freqs, 11 nodes.

Est. 1 hour of CPU time <=> approximately the same as Symes' time-domain finite difference code with checkpointing ...

Example: imaging (2)



Target model



After first gradient-based update

$$\mathbf{m}_1 = \mathbf{m}_0 + \delta\mathbf{m}$$

$\delta\mathbf{m}$ (not shown) is computed using data from 188 shots and 11 frequencies (0.5-5.0 Hz)

Conclusion

- Key of successful iterative methods for Helmholtz: handling indefiniteness and ill-conditioned
- In our method, both are handled by preconditioner and multilevel operator
- Computational example shows that in terms of memory and CPU time, an iterative method can be a viable alternative to direct method in frequency-domain waveform inversion or migration
- Extension general *d*-dimension is straightforward

Future direction

- 3D wave-modeler and inversion

  Use of domain-decomposition-type algorithm
  Iterative methods for multiple right-hand sides;
  (solve multiple shots for one frequencies)

- Waveform inversion with Gauss-Newton-Krylov methods

  Hessian is computed implicitly via forward/backward solves,
  faster convergence.
  The use of direct methods are too expensive; at every Gauss-Newton update, LU factors must be formed

  [Erlangga, Herrmann, SEG 2009]

- FD inversion - conducive to freq. sampling

  [Mulder, Plessix, 2004, Sirgue, Pratt, 2009]

  Alternative:
  Freq. and shot sampling & inversion using sparsity-promoting recovery

  [Herrmann, Erlangga, Lin, 2009]
  [Herrmann, SEG 2009]

Further reading

- Y A Erlangga, C W Oosterlee and C Vuik
  A novel multigrid-based preconditioner for the heterogeneous Helmholtz equation
  SIAM J. Sci. Comput., 27, 1471-1492, 2006.

- Y A Erlangga and R Nabben
  On a multilevel Krylov method for Helmholtz equation preconditioned by shifted Laplacian
  To appear in Electronic Transaction on Numerical Analysis
  http://slim.eos.ubc.ca/Publications/Public/Journals/erlangga08oam.pdf

- Y. Erlangga and F. J. Herrmann
  An iterative multilevel method for computing wavefields in frequency-domain seismic inversion
  SEG Technical Program Expanded Abstracts, SEG, 2008.
  http://slim.eos.ubc.ca/Publications/Public/Conferences/SEG/2008/erlangga08seg.pdf

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For other resources on frequency-domain compressive computation, visit

http://slim.eos.ubc.ca