Seismic Data Interpolation

- With each 3D seismic experiment, we acquire a massive 5D data volume
  - Volumes easily have dimensions 100 x 100 (srcx, srcy) x 400 x 400 (recx, recy) x 1024 (time)
- Physical and budgetary constraints limit the number of available sources and receivers

Seismic Data Interpolation

- Knowledge of the low-rank structure of certain *matricizations* of the underlying (full) tensor allows us to recover unknown entries from the subsampled tensor
- Matrix Completion - Jellyfish
- Hierarchical Tucker Tensor Completion

Matricization

- The *matricization* of a tensor X with dimensions $1,\ldots,d$ along the dimensions $t=(t_1,\ldots,t_r)$ is the matrix formed by placing the dimensions t along the rows and dimensions t^c along the columns
- Denoted $X^{(t)}$

Hierarchical Tucker Format

- A *dimension tree* T for dimensions $\{1, \ldots, d\}$ is a non-trivial binary tree such that
  - the root, t_{root}, is has the label $\{1, \ldots, d\}$
  - each non-leaf node, t, can be written as $t = t_l \cup t_r$, $t_l \cap t_r = \emptyset$ where t_l is the left child of t, t_r is the right child of t

Hierarchical Tucker Format

- A tensor X can be written in the *Hierarchical Tucker format* corresponding to a dimension tree T and a vector of hierarchical ranks

$$(k_t)_{t \in T}, k_{\text{root}} = 1$$

if it can be written as

SLIM

Hierarchical Tucker Format

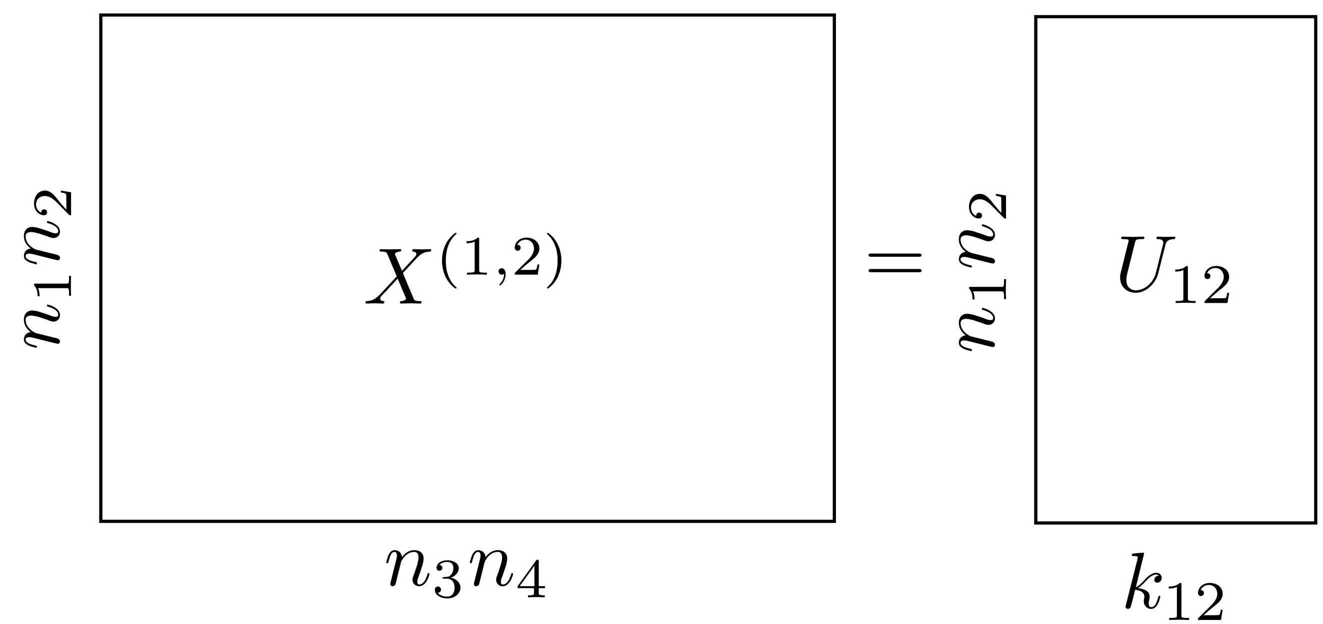
$$\mathrm{vec}(X) = (U_{t_l} \otimes U_{t_r}) B_{t_{\mathrm{root}}}^{(1,2)} \quad t = t_{\mathrm{root}}$$

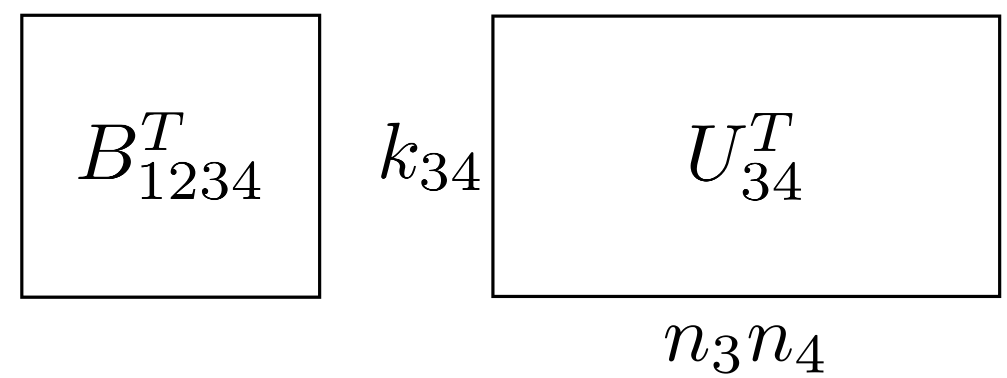
$$U_t = (U_{t_l} \otimes U_{t_r}) B_t^{(1,2)} \quad t \text{ not a leaf}$$

$$U_t \in \mathbb{C}^{n_t \times k_t} \qquad B_t \in \mathbb{C}^{k_l \times k_r \times k_t}$$

Hierarchical Tucker Format

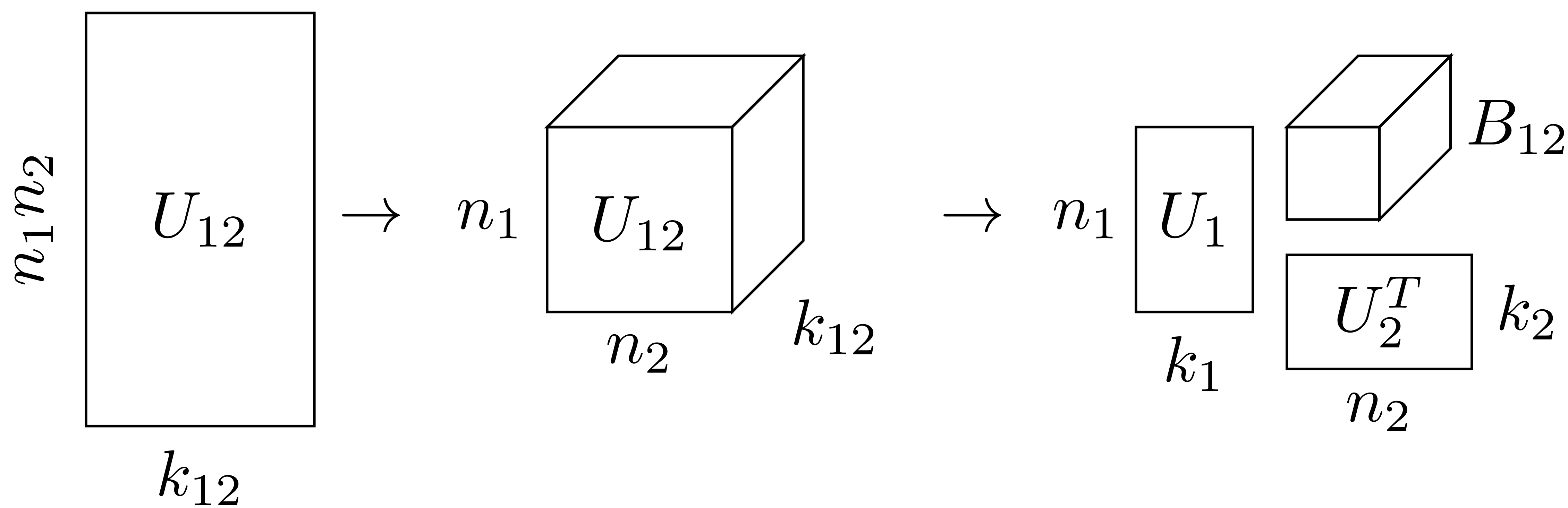
$$X - n_1 \times n_2 \times n_3 \times n_4 \text{ tensor}$$





$$X^{(1,2)} = U_{12} B_{1234}^T U_{34}^T$$

$$U_{12} = (U_1 \otimes U_2) B_{12}$$



Hierarchical Tucker Format

- We don't need to store the matrices U_t when t is not a leaf node
  - We only need to store U_t for the leaves, B_t for the internal nodes
  - We don't need to store the (full) tensor

Hierarchical Tucker Format

- Storage $\leq dNK + (d-2)K^3 + K^2$ where $K = \max_{t \in T} k_t$, $N = \max_{i=1,\ldots,d} n_i$

- Compare to N^d storage for the full tensor
- Effectively breaking the curse of dimensionality when $K \ll N$, $d > 3$

www.sam.math.ethz.ch/NLAgroup/htucker_toolbox.html

Hierarchical Tucker Format

- Truncation from a N^d array to the Hierarchical Tucker format can be performed in $O(dN^{d+1})$ time by using SVDs + a hierarchical construction
    - More details in the hTucker toolbox
- Cannot be applied when missing entries of the tensor

[1] A. Uschmajew, B. Vandereycken. The geometry of algorithms using hierarchical tensors. 2012

Quotient Manifold Structure

- The authors in [1] study the differential geometric structure of the *non*-orthogonal HTucker format
- We go beyond their analysis with the orthogonalized HTucker format + analysis to derive a computationally efficient optimization scheme for interpolation

Quotient Manifold Structure

- When we constrain U_t, B_t to be orthogonal (except at the root), the group action

$$\theta : M \times A \to M$$

$$\theta_{\{A_t\}_{t \in T}}(U_t, B_t) = (U_t A_t, (A_{t_l}^T \otimes A_{t_r}^T) B_t A_t)$$

induces a quotient manifold structure on the parameter matrices M

Optimization

- By characterizing the horizontal and vertical spaces of M induced by this action, and by our choice of orthogonality constraints, we have a *Riemannian manifold* over which we can optimize
- The Riemannian metric is the standard inner product

Singular Value Regularization

- Currently we are only trying to minimize the energy misfit between our model + the data we have (as well as choosing a rank for our underlying model)
  - Our assumption is that the underlying tensor has *quickly*-decaying singular values in different matricizations

Singular Value Regularization

- The *Gramian* matrix G_t associated to the node t is the $k_t \times k_t$ symmetric positive semidefinite matrix which satisfies

$$\lambda_i(G_t) = \sigma_i(X^{(t)})^2$$

Singular Value Regularization

- The Gramian matrices for every node can be computed recursively via

$$G_{t_{\mathrm{root}}} = 1$$

$$G_{t_r} = (B_t^{(k,k_l)})^H (I_{k_l} \otimes G_t) B_t^{(k,k_l)}$$

$$G_{t_l} = (B_t^{(k,k_r)})^H (I_{k_r} \otimes G_t) B_t^{(k,k_r)}$$

Regularized Problem

$$\min_{x=(U_t,B_t)} \|A\phi(x) - D\|_2^2 + \sum_t \alpha_t \|G_t\|_*$$

$$\alpha_t \geq 0 \quad U_t^T U_t = I_{k_t}, (B^{(1,2)})^H B^{(1,2)} = I_{k_t}$$

Since G_t is spd, $\|G_t\|_* = \mathrm{tr}(G_t)$, this simplifies to

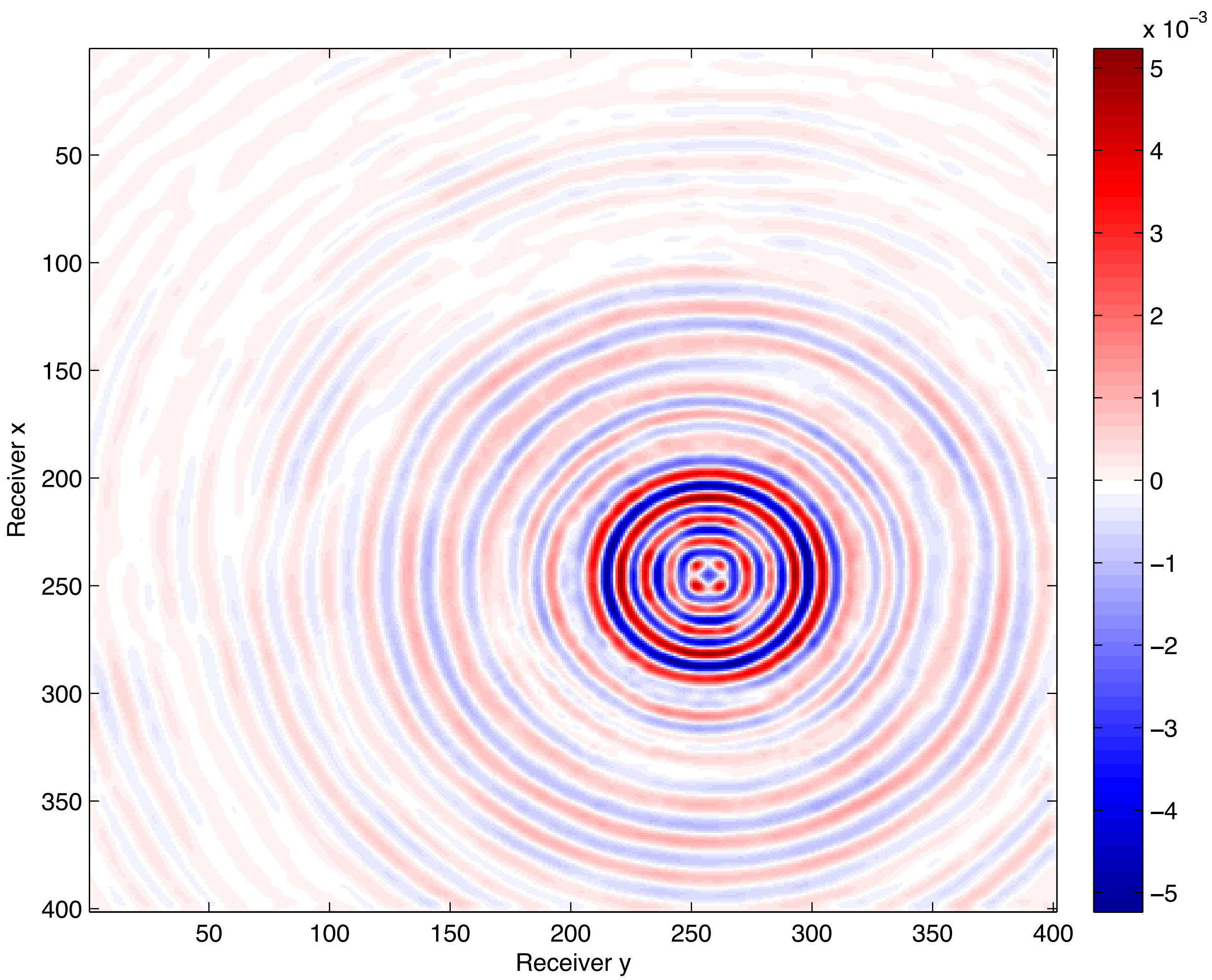
$$\min_{x=(U_t,B_t)} \|A\phi(x) - D\|_2^2 + \sum_t \alpha_t \mathrm{tr}(G_t)$$

Regularization

- This objective is differentiable
  - We already worked out the derivative of the tensor expansion function previously
  - It is straightforward to derive the total derivative $\frac{\partial G_t}{\partial B_{t'}}$ and compute it efficiently

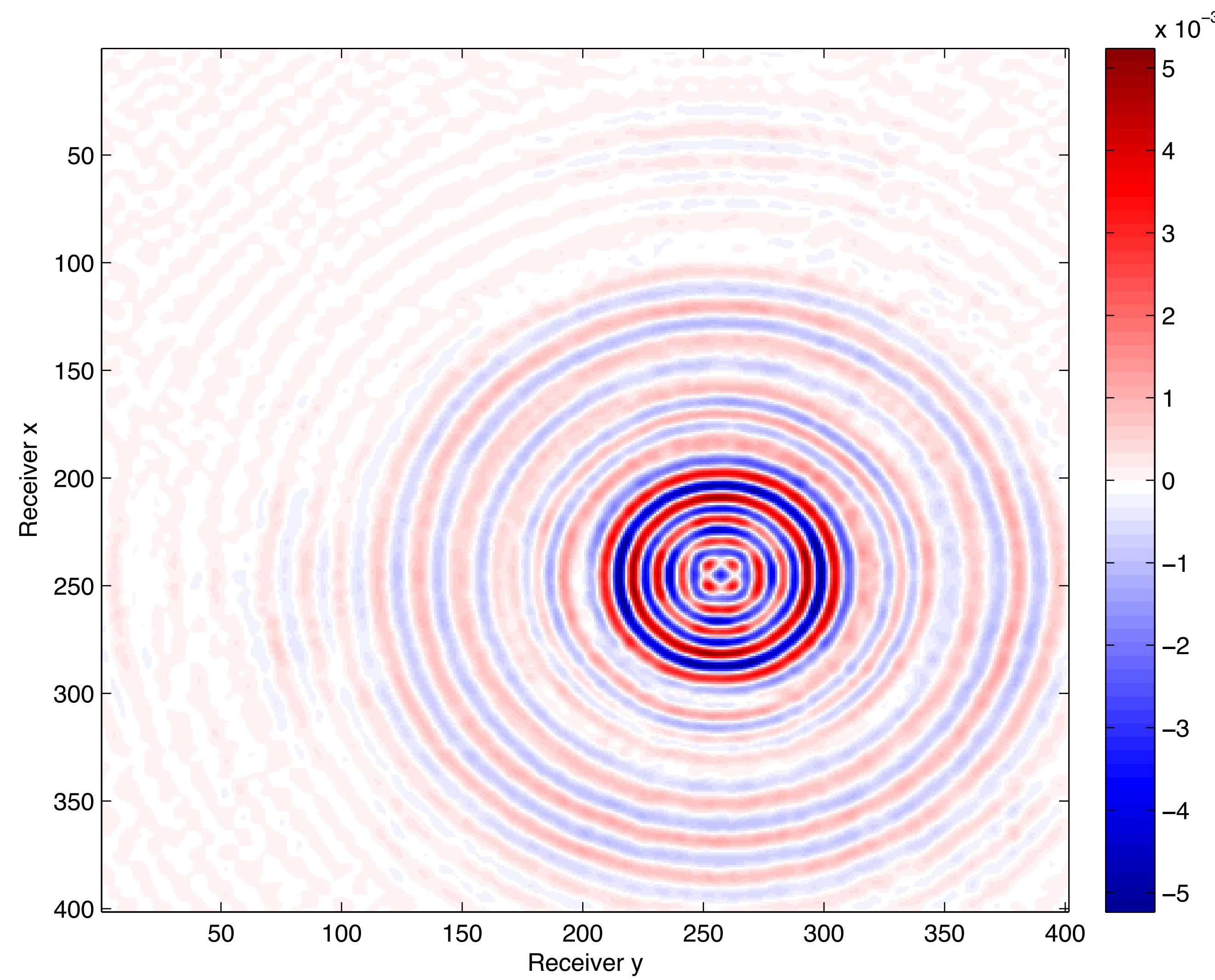
Hierarchical Tucker Interpolant

Reconstruction from 200 shots -> 6400 shots



Known data

(Src x, Src y) = (63,66)

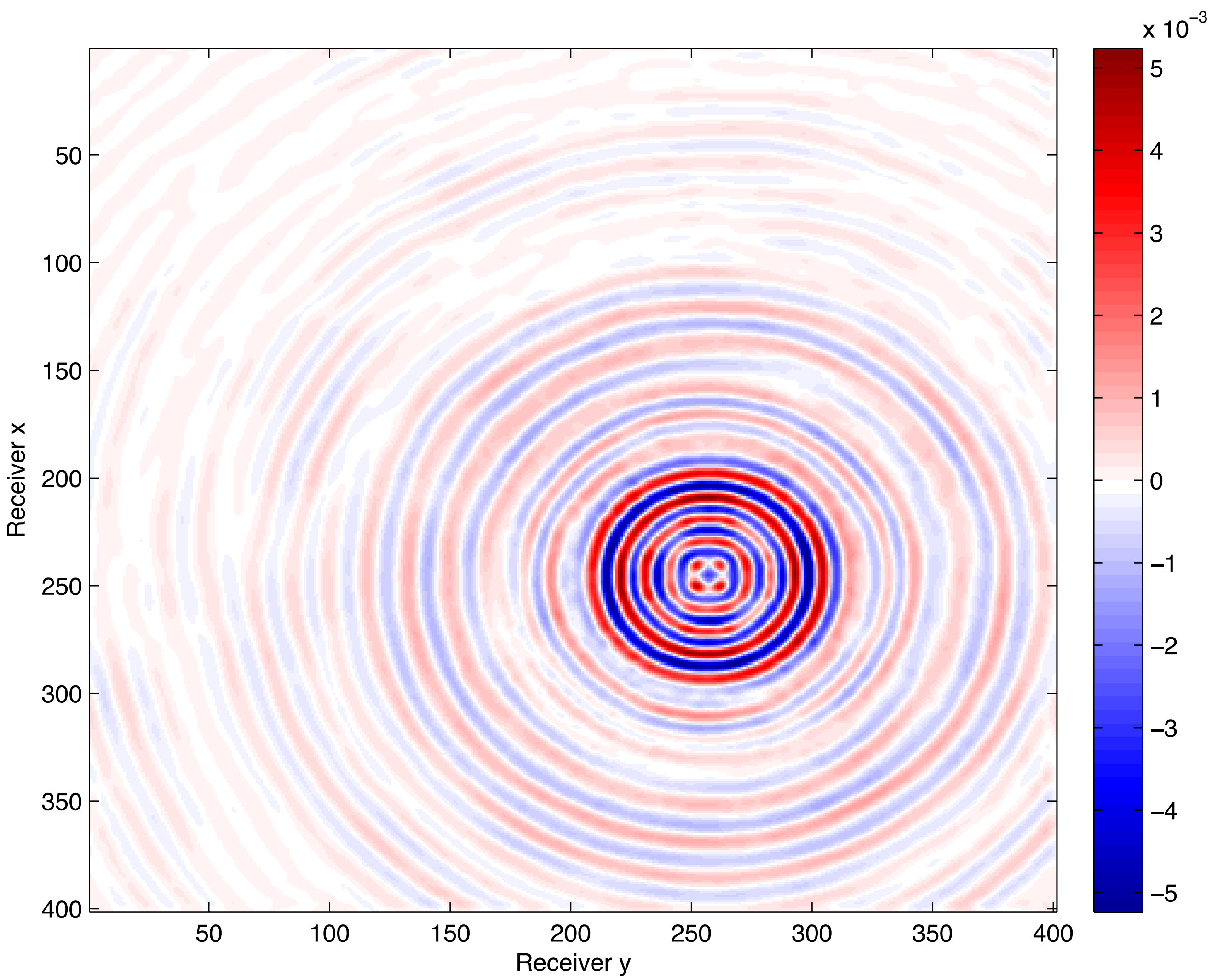


Interpolated data -

SNR 13.2 dB

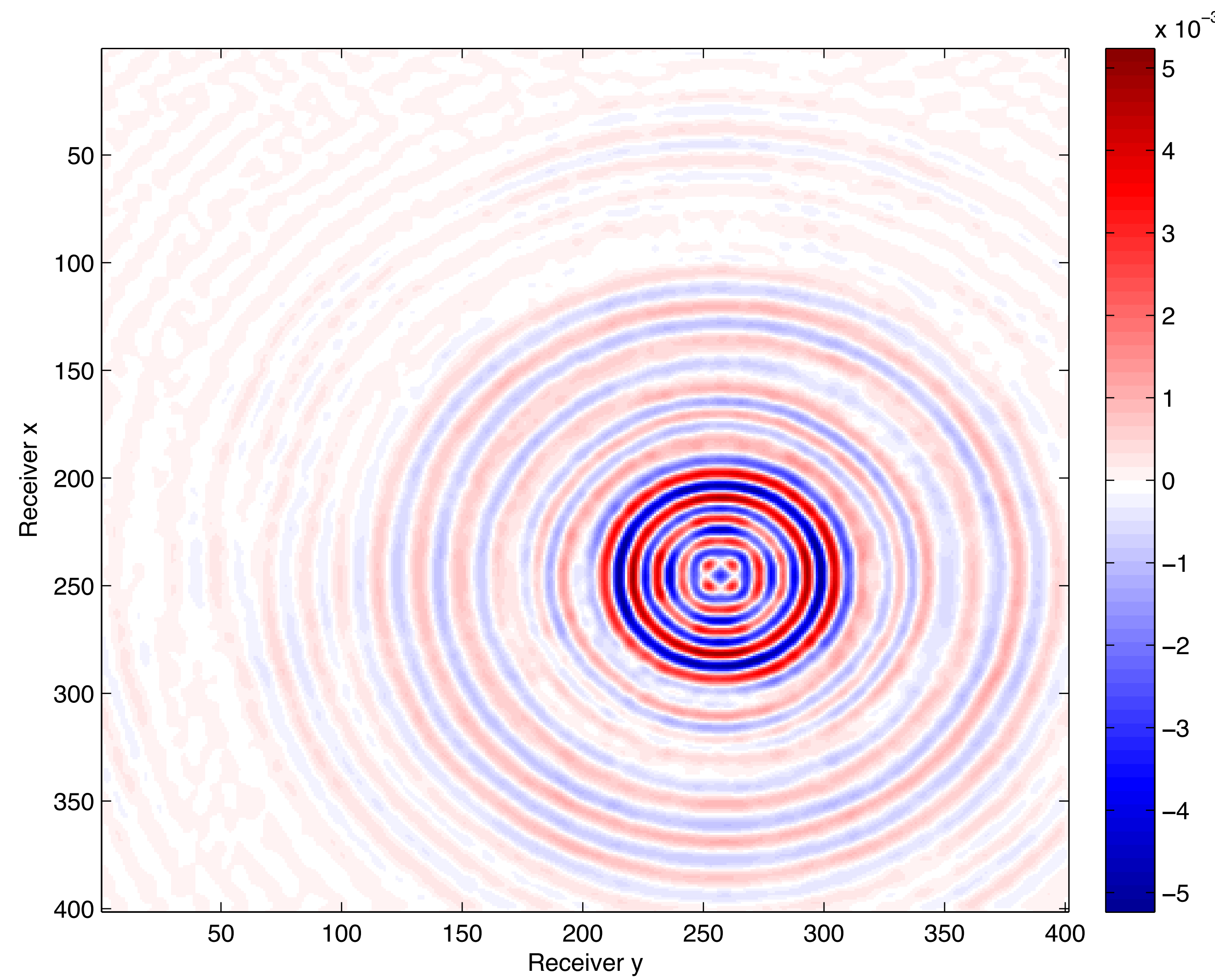
HTuck Interpolant - Regularized

Reconstruction from 200 shots -> 6400 shots



Known data

(Src x, Src y) = (63,66)

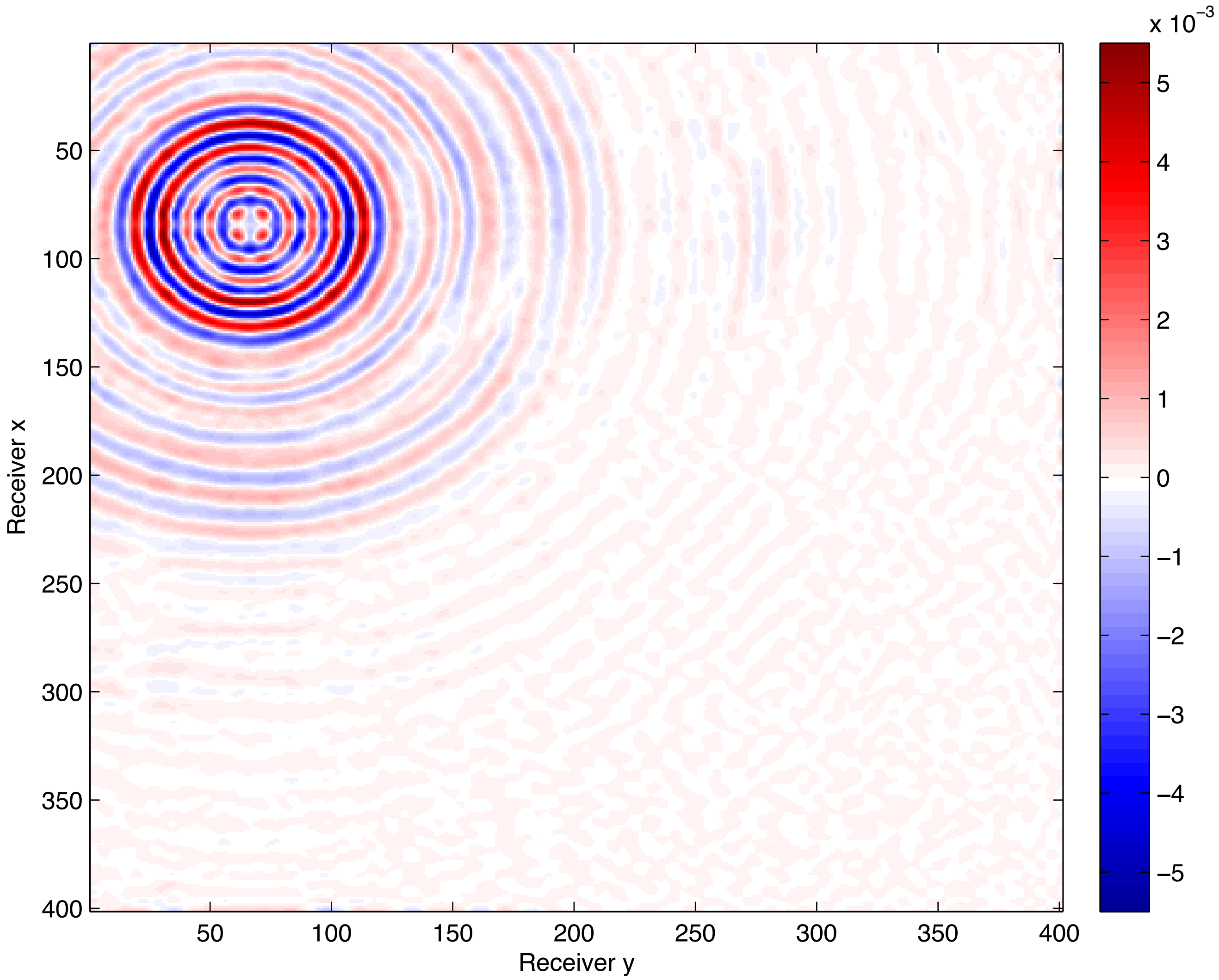


Interpolated data -

SNR 15 dB

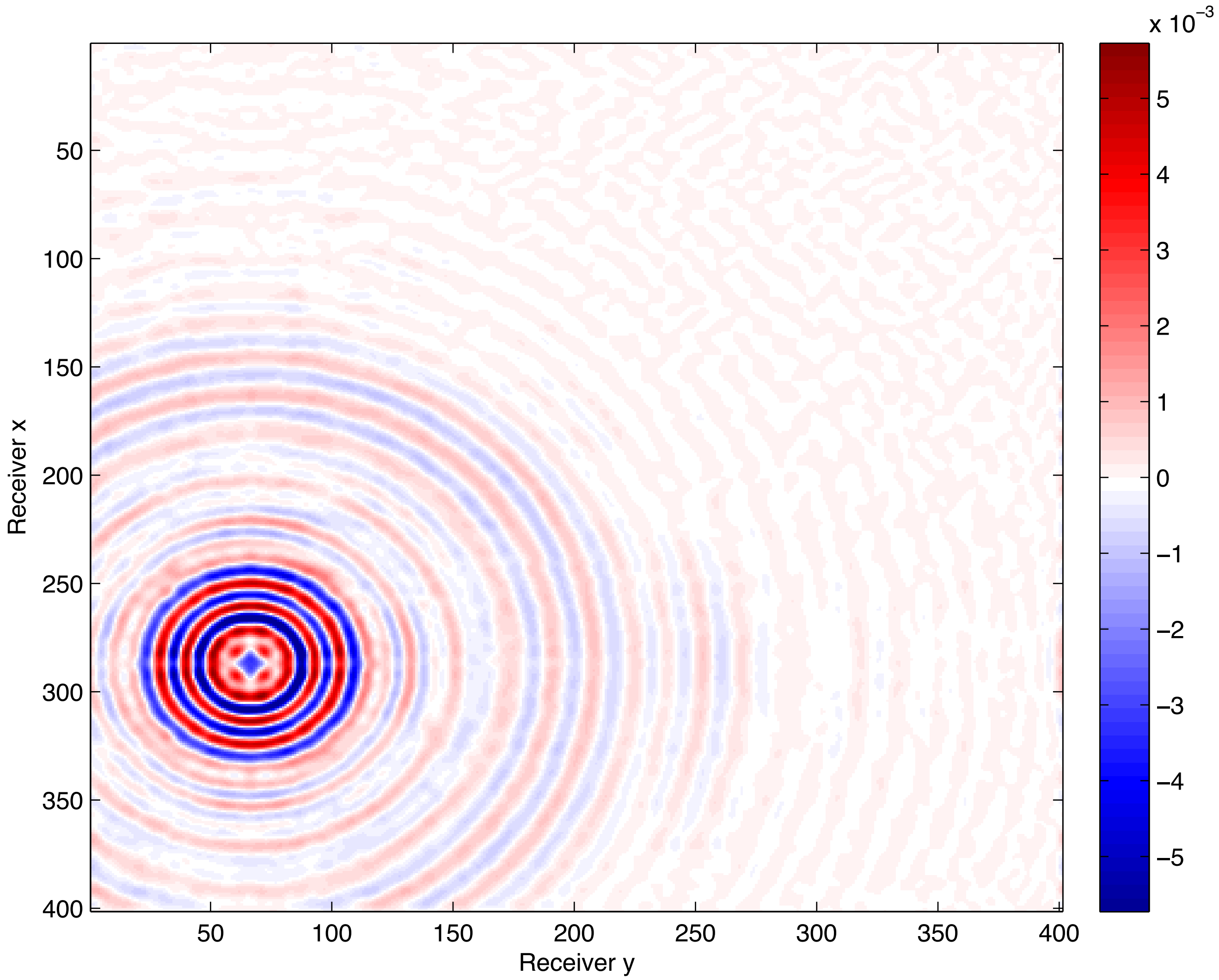
Hierarchical Tucker Interpolant

Reconstruction from 200 shots -> 6400 shots



Interpolated data

(Src x, Src y) = (22,18)

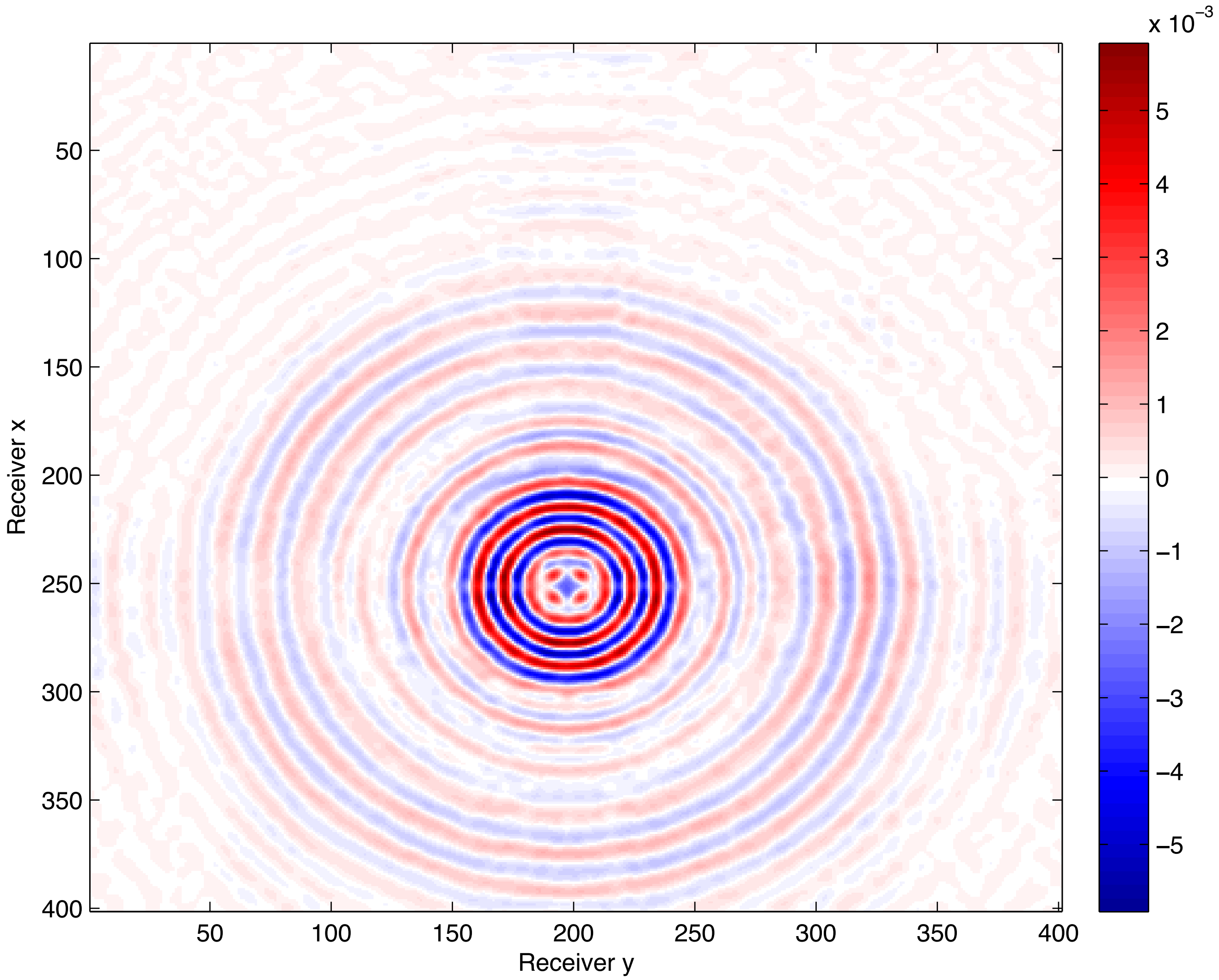


Interpolated data

(Src x, Src y) = (73,18)

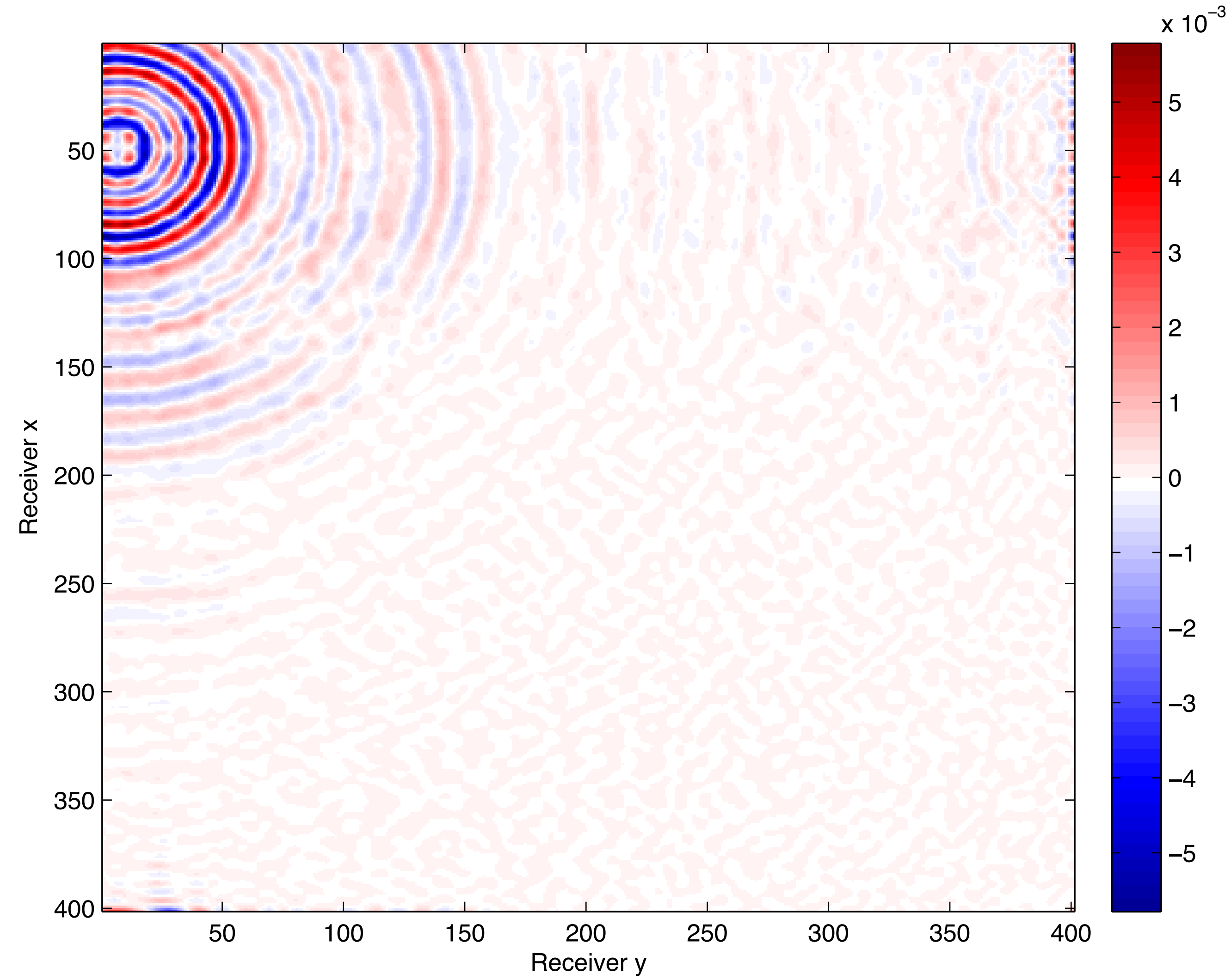
Hierarchical Tucker Interpolant

Reconstruction from 200 shots -> 6400 shots



Interpolated data

(Src x, Src y) = (64,51)

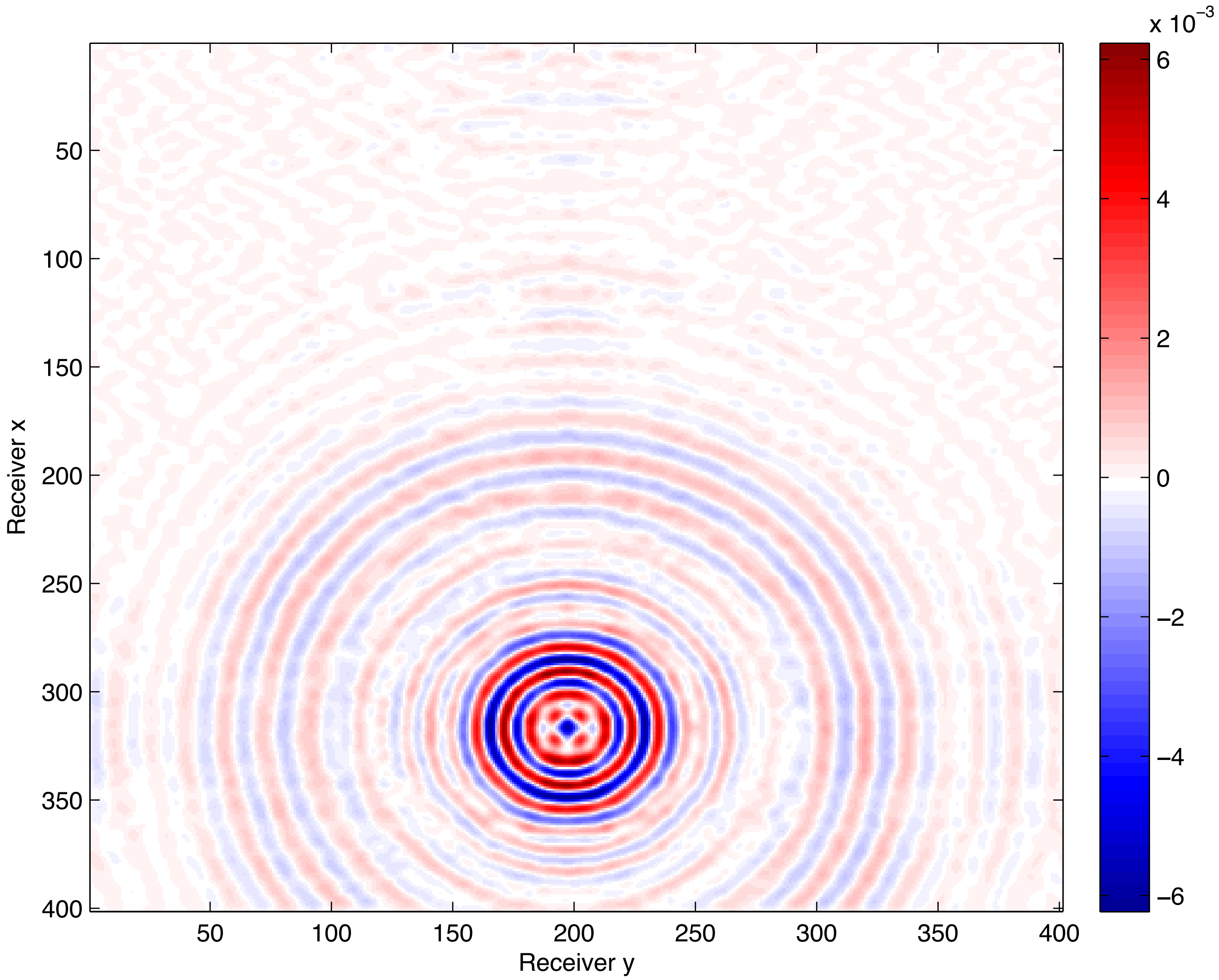


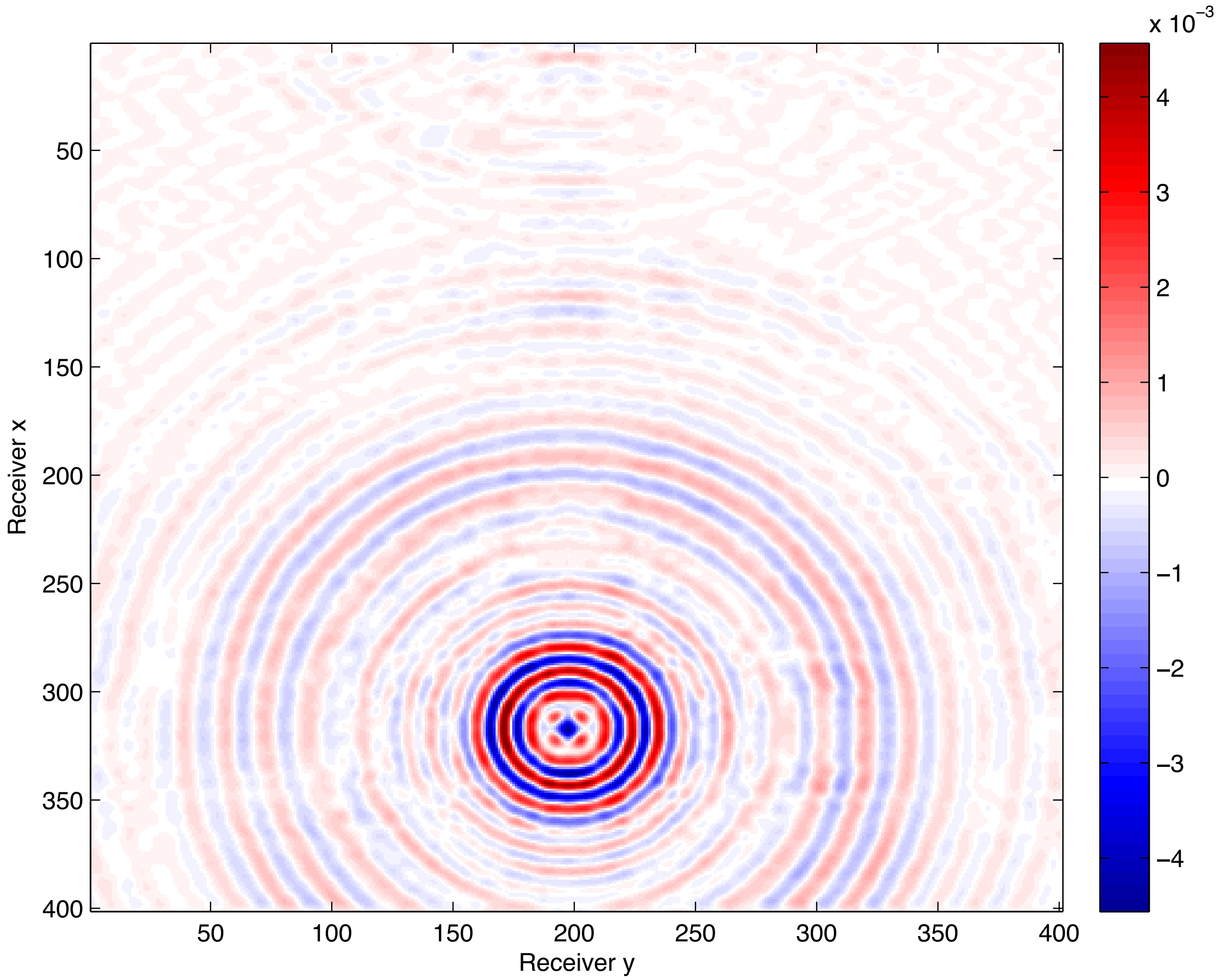
Interpolated data

(Src x, Src y) = (13,3)

Hierarchical Tucker Interpolant

Reconstruction from 200 shots -> 6400 shots





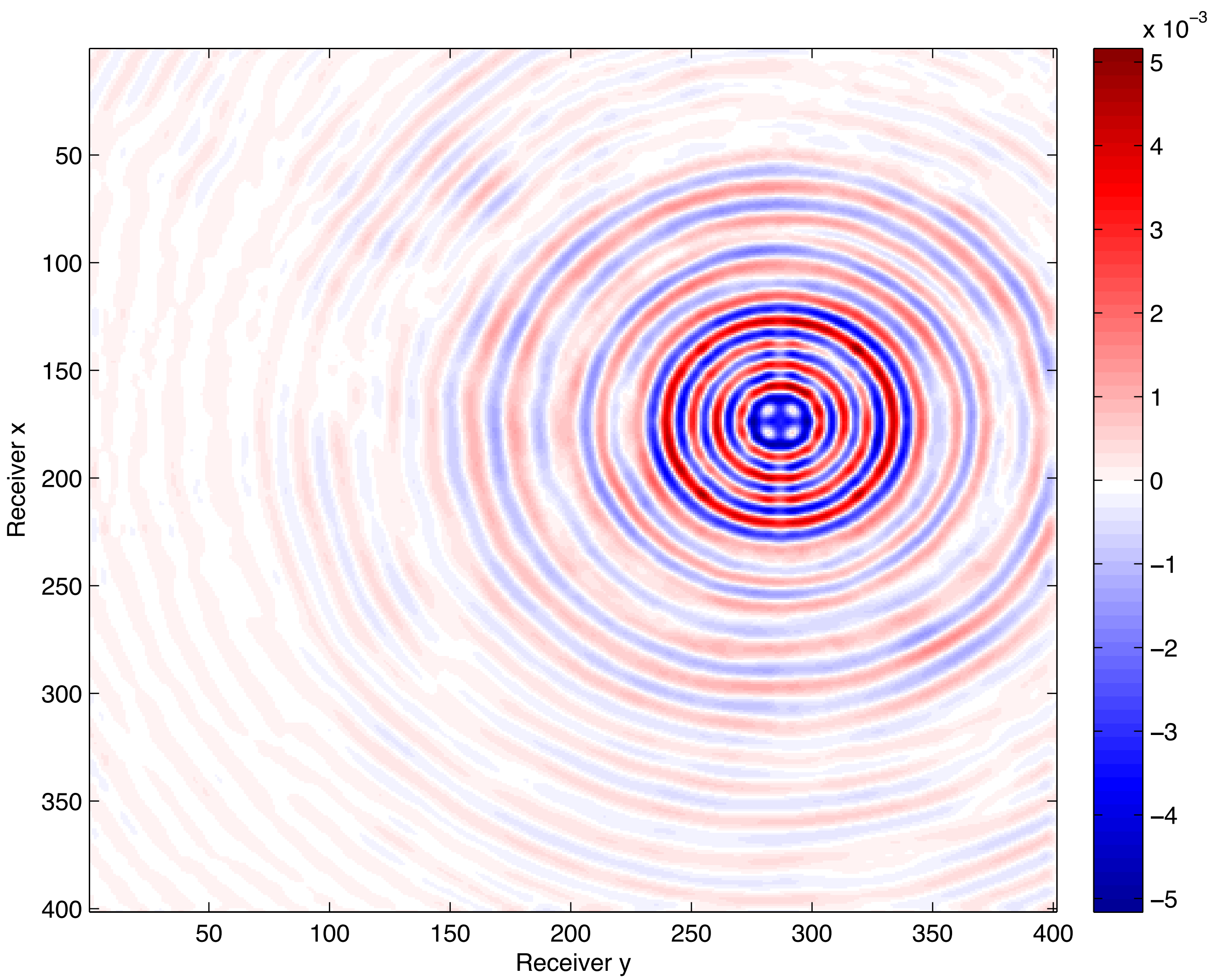
No Regularization

Regularization

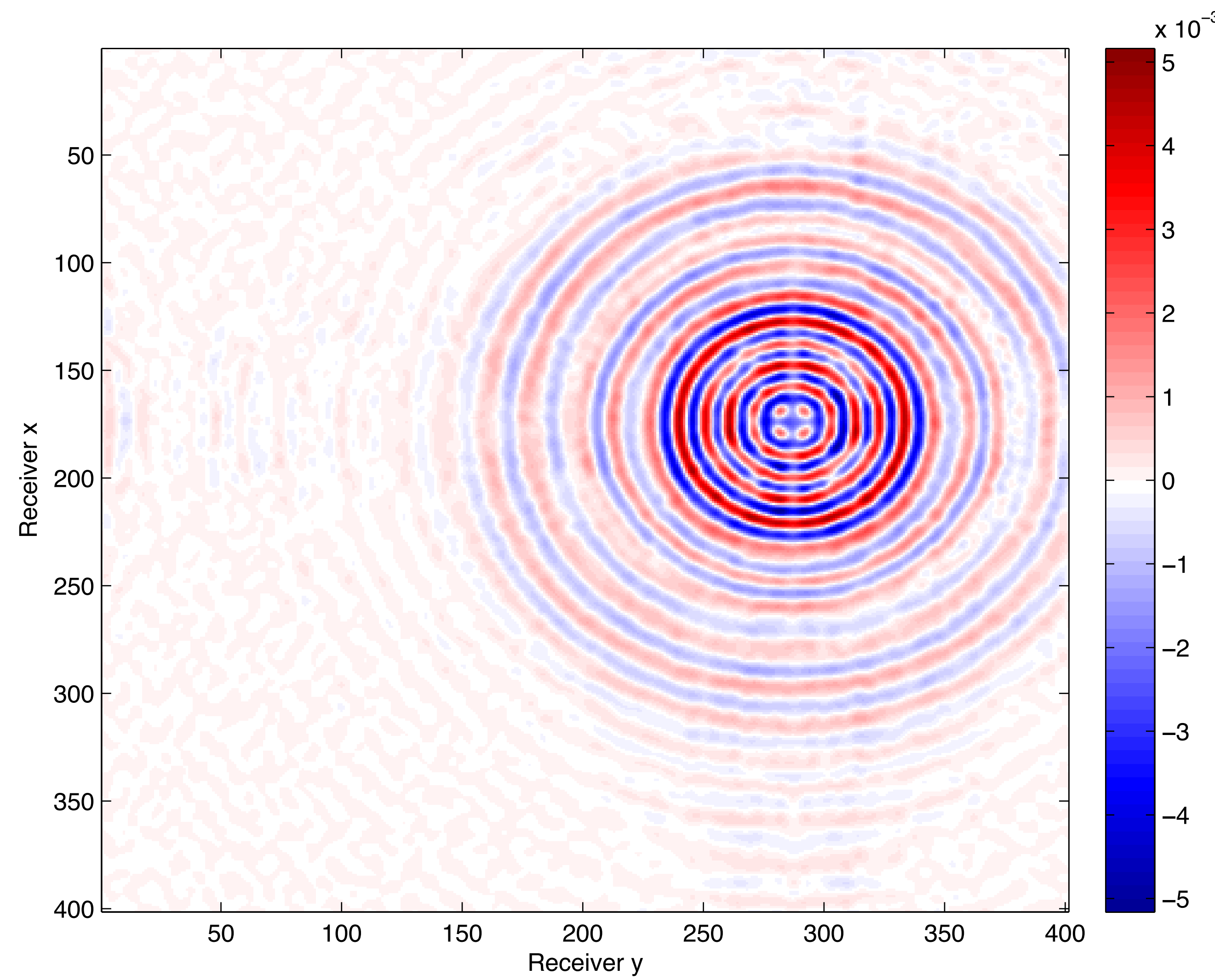
(Src x, Src y) = (81, 51)

HTucker & Jellyfish Shot Reconstruction

3 shots removed + reconstructed



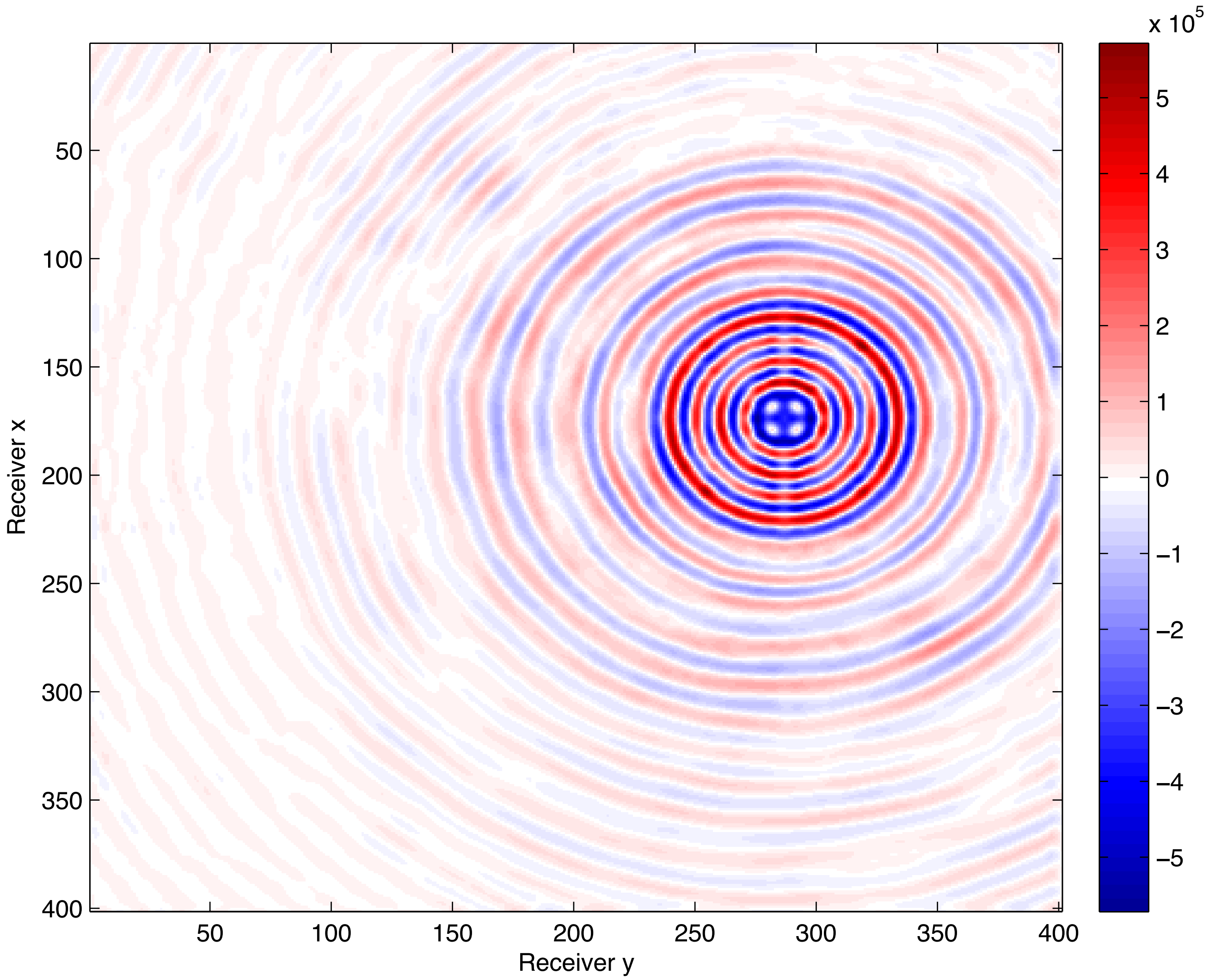
True data
(Src x, Src y) = (45,73)



HTucker - SNR 8.46 dB

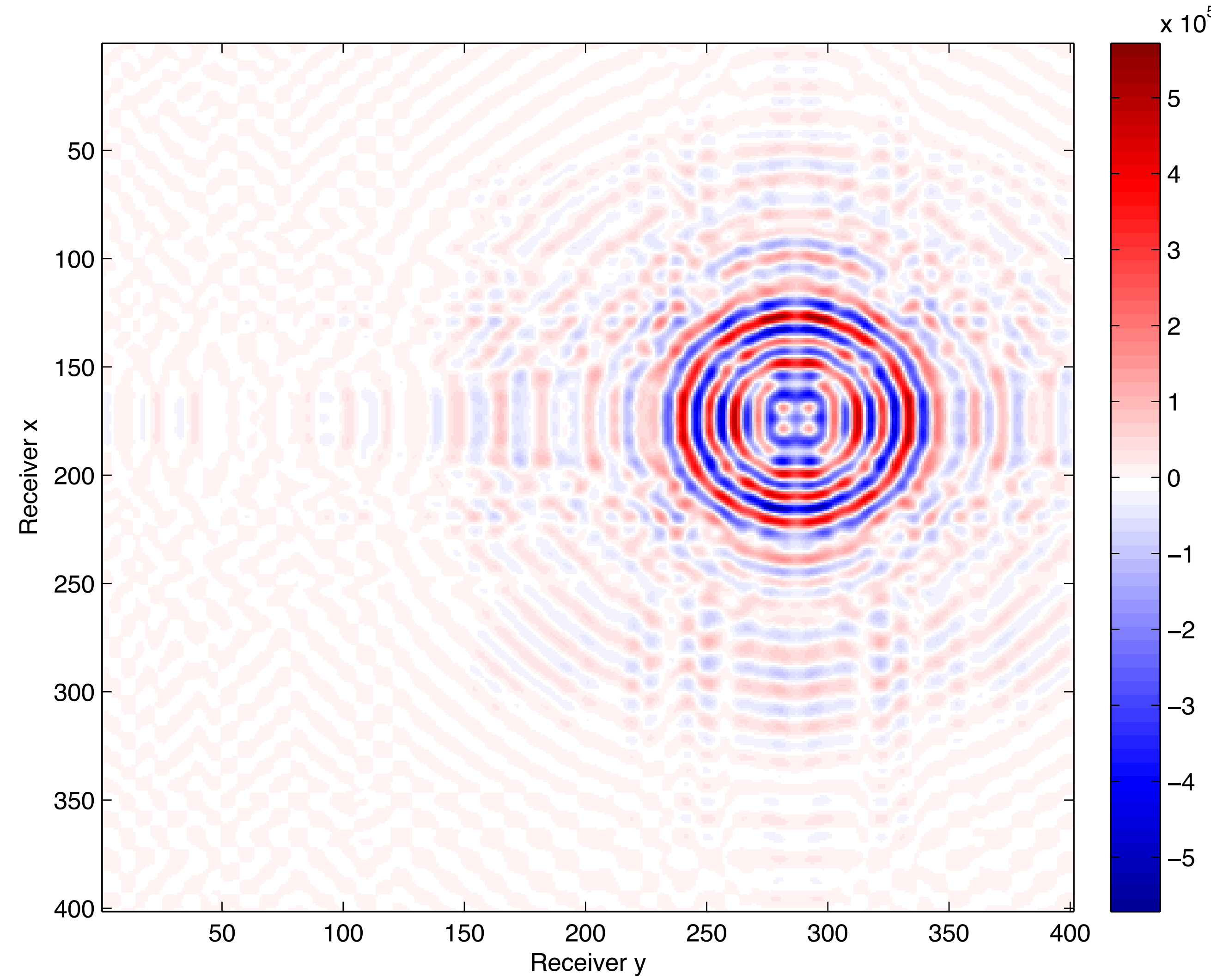
HTucker & Jellyfish Shot Reconstruction

3 shots removed + reconstructed



True data

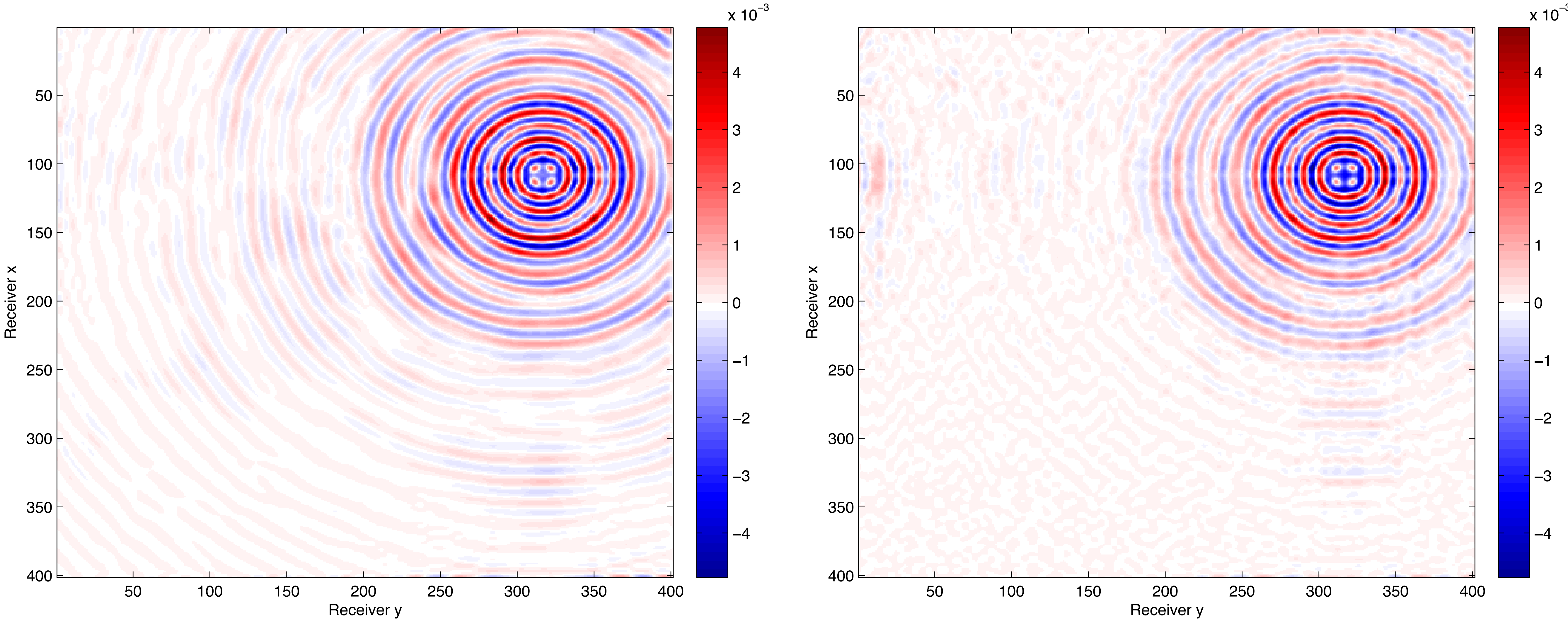
(Src x, Src y) = (45,73)



Jellyfish - SNR 3.86 dB

HTucker & Jellyfish Shot Reconstruction

3 shots removed + reconstructed

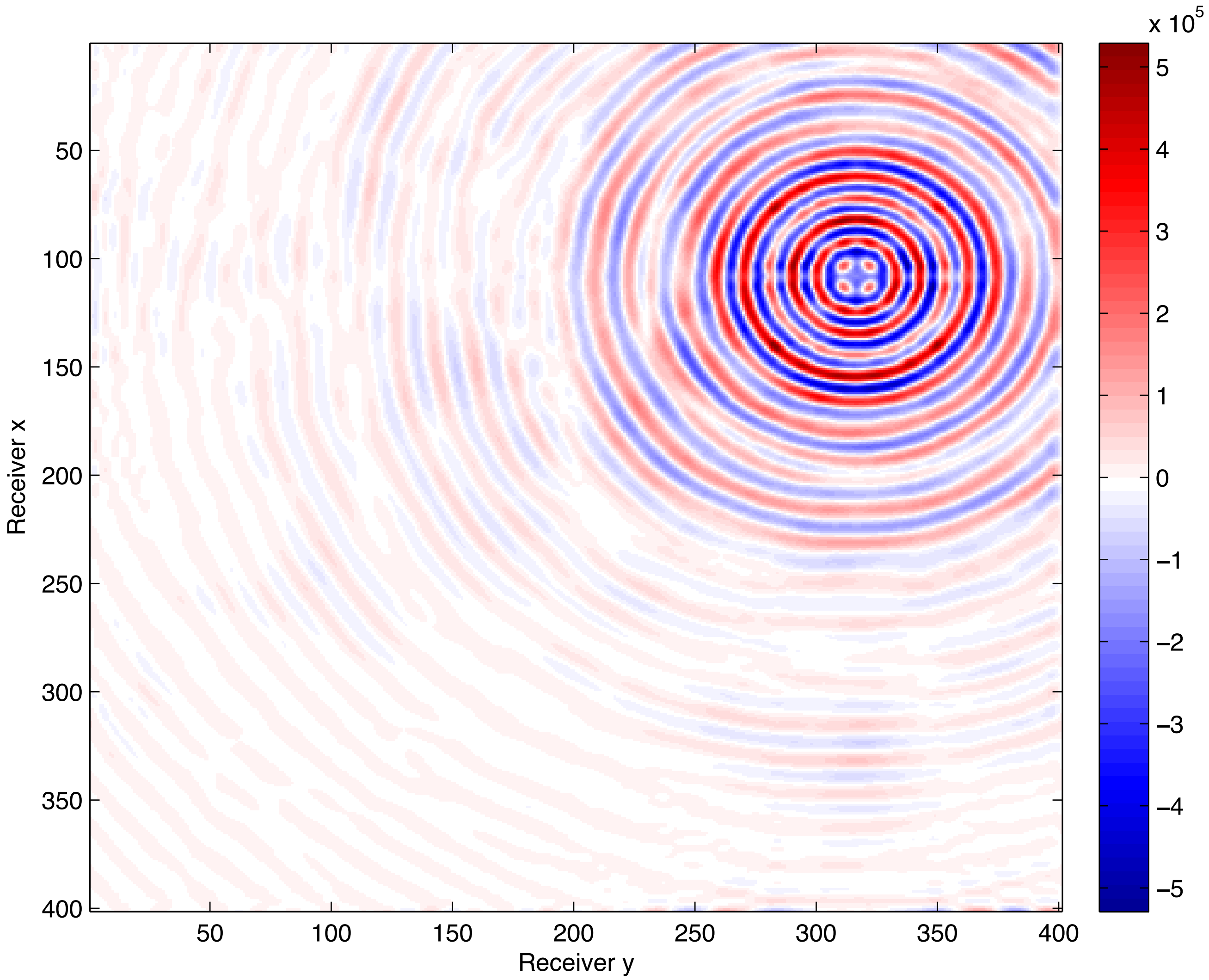


True data
(Src x, Src y) = (28,81)

HTucker - SNR 7.98 dB

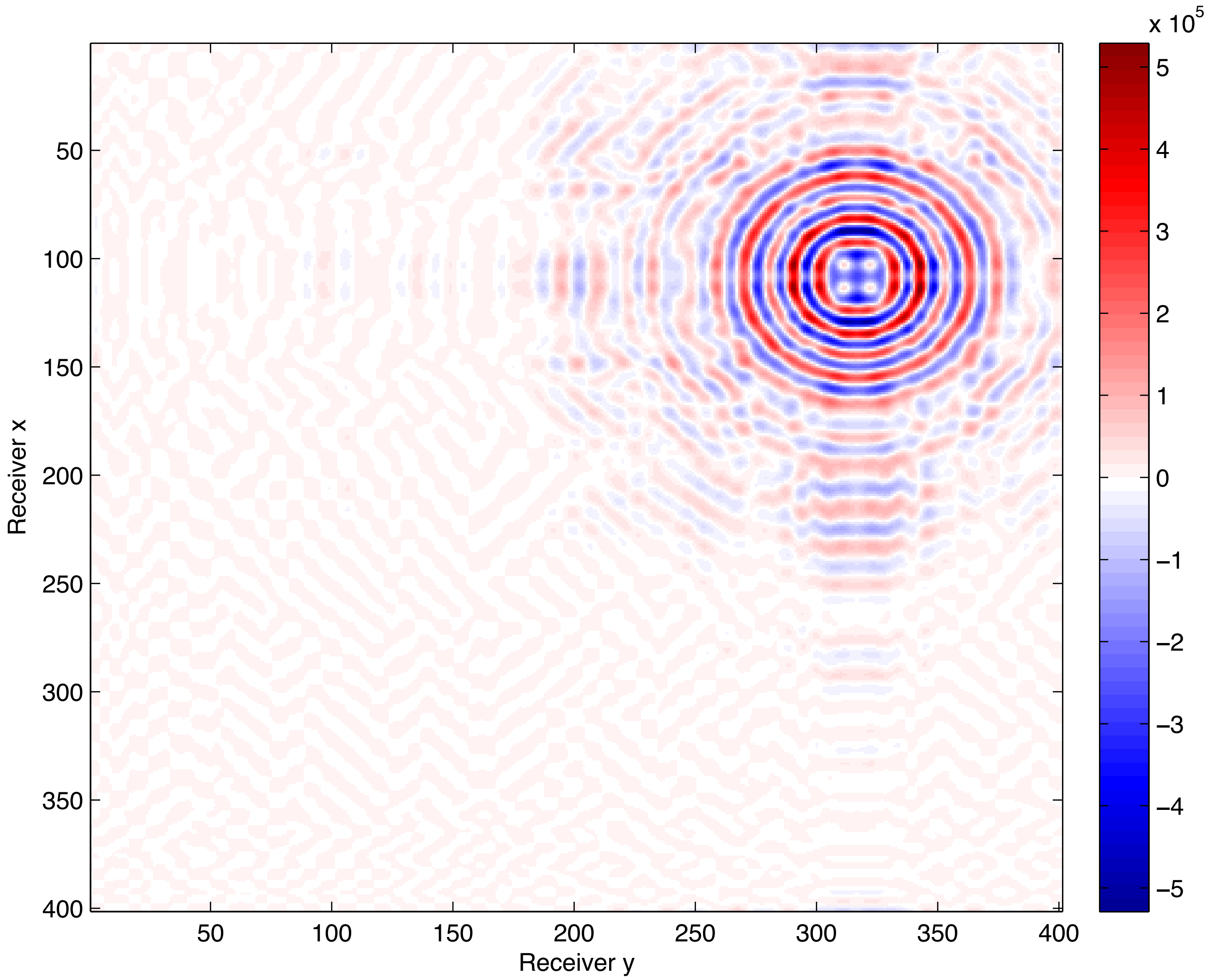
HTucker & Jellyfish Shot Reconstruction

3 shots removed + reconstructed



True data

(Src x, Src y) = (28,81)



Jellyfish - SNR 3.89 dB