Regularizing waveform inversion by projections onto convex sets

Bas Peters
Joint work with Brendan Smithyman and Mathias Louboutin

FWI workshop Natal, September 2015

SLIM
University of British Columbia

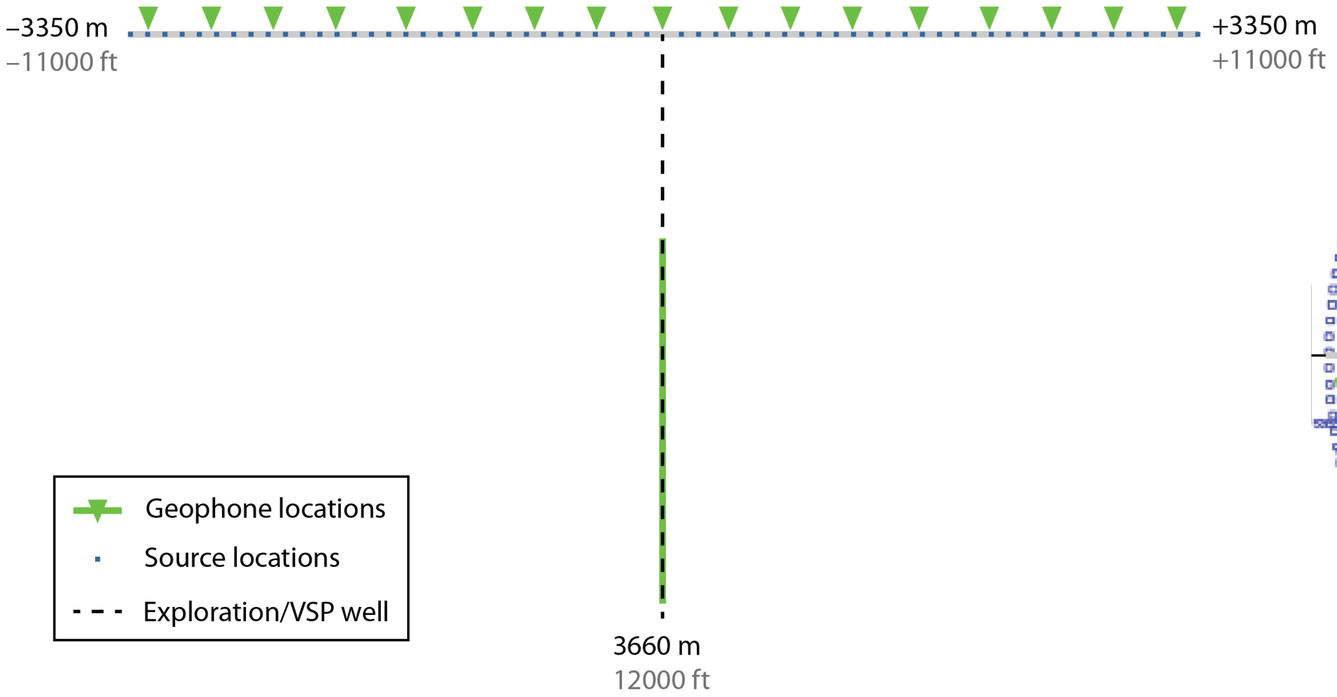
Bas Peters, Zhilong Fang, Brendan Smithyman, and Felix J. Herrmann, "Regularizing waveform inversion by projections onto convex sets –- application to the 2D Chevron 2014 synthetic blind-test dataset".

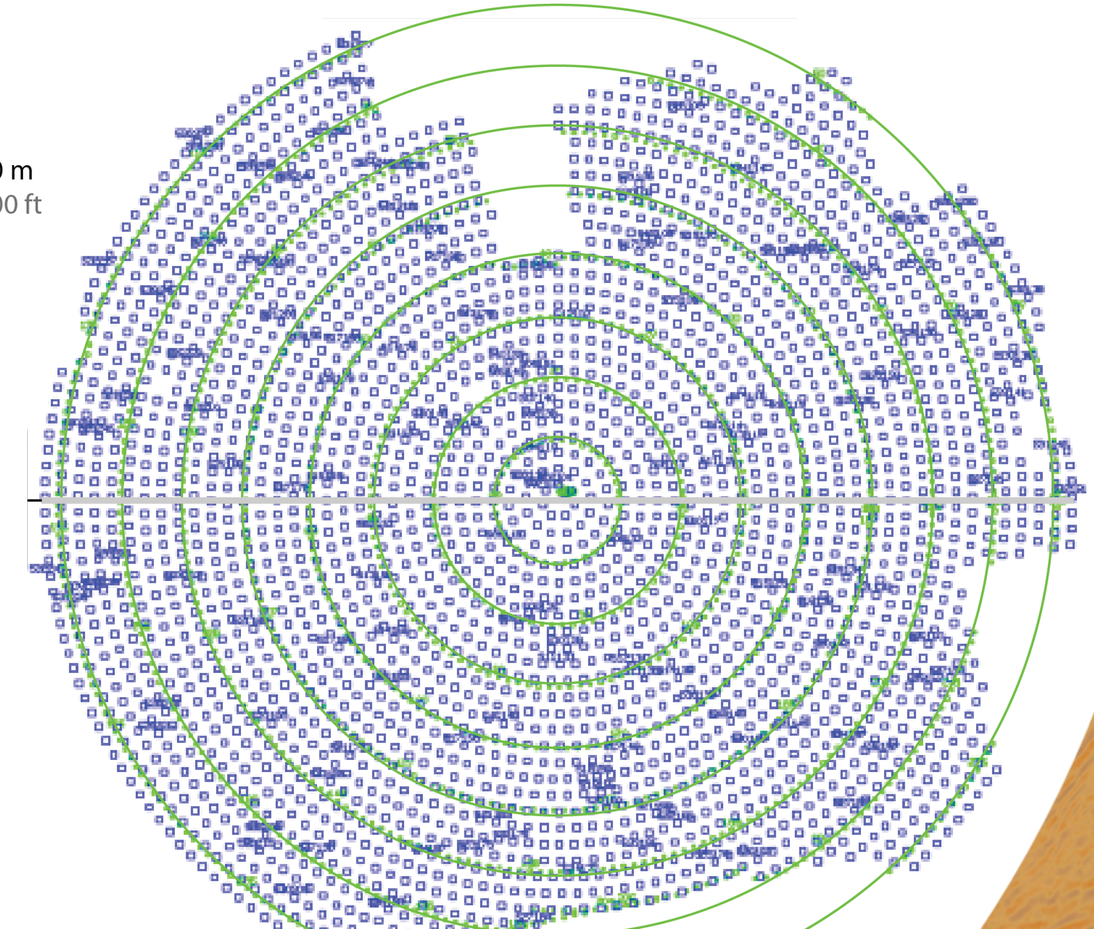


Motivation

Land data set with surface sources and surface & well receivers

Constant density acoustic inversion

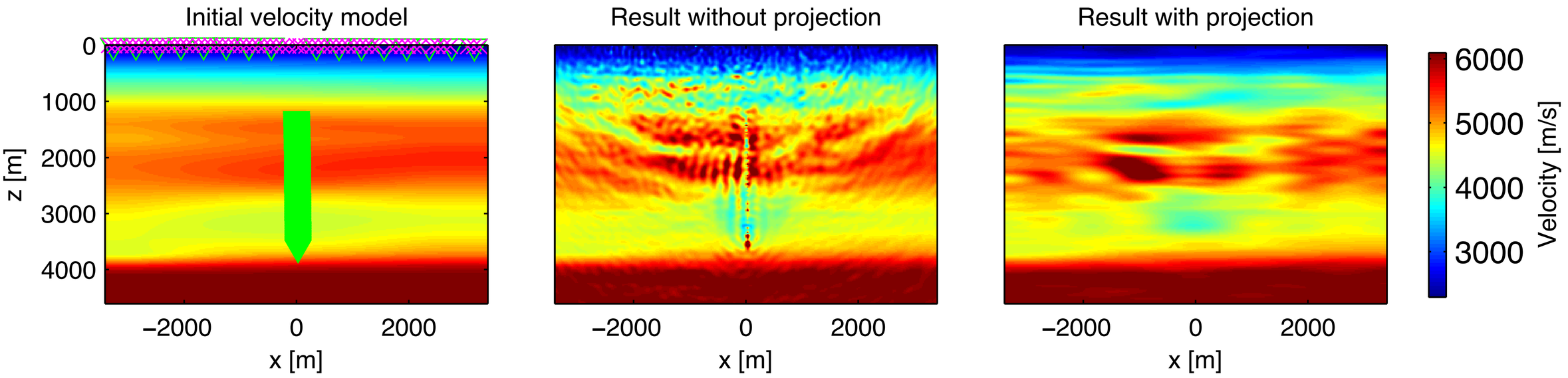




Motivation

Land data set with surface sources and surface & well receivers

Constant density acoustic inversion (2D slice)



Fore challenging problems, some regularization is required

A few regularization strategies

Objective function: $f(\mathbf{m})$ (differentiable, time or frequency)

Tikhonov / quadratic: $$\phi(\mathbf{m}) = f(\mathbf{m}) + \frac{\alpha}{2}\|R_1\mathbf{m}\|^2 + \frac{\beta}{2}\|R_2\mathbf{m}\|^2$$

Gradient filtering: $$\mathbf{m}_{k+1} = \mathbf{m}_k - \gamma F \nabla_{\mathbf{m}} f(\mathbf{m})$$

Constrained formulation: $$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

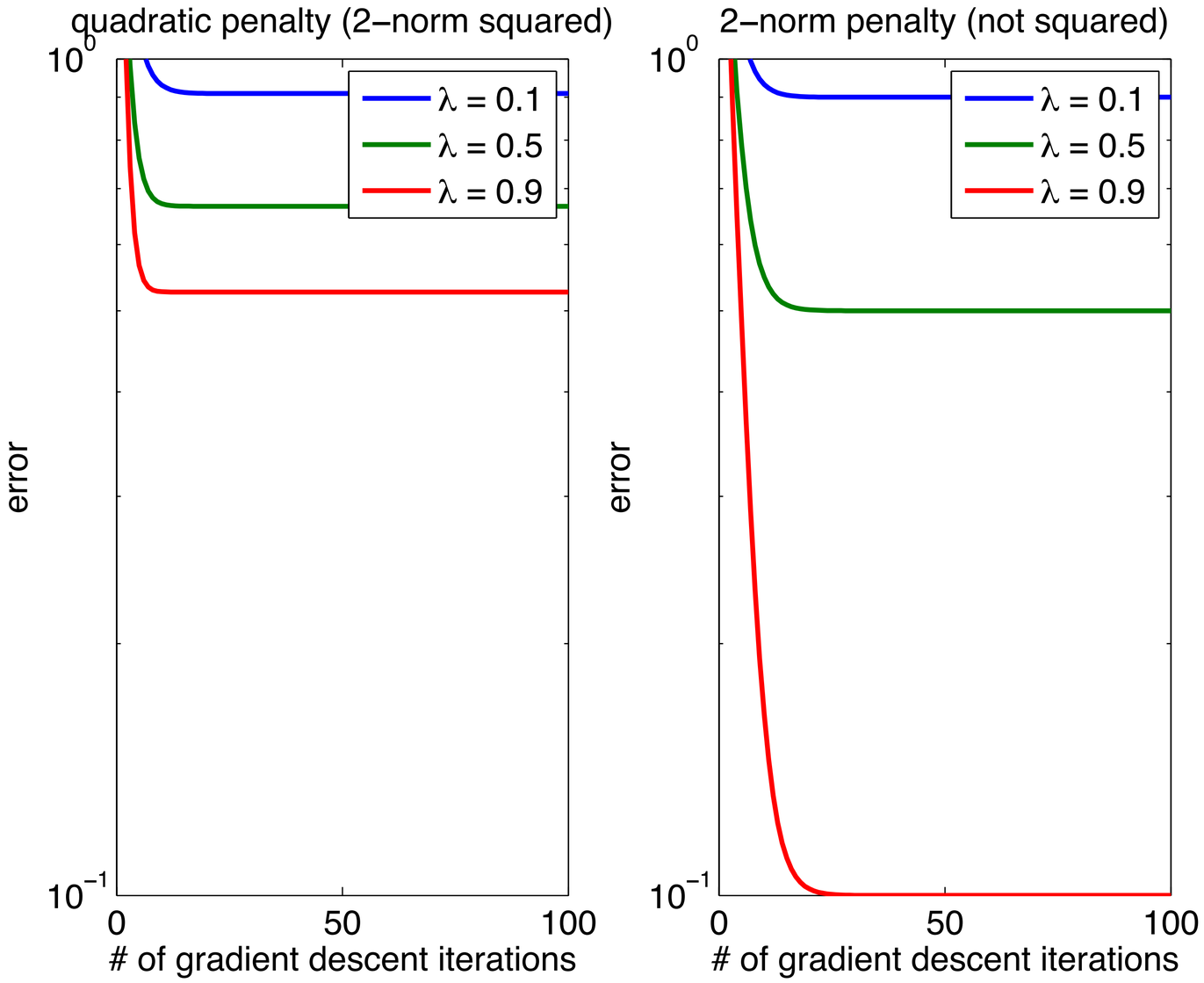
A few regularization strategies

Tikhonov / quadratic: $$\phi(\mathbf{m}) = f(\mathbf{m}) + \frac{\alpha}{2}\|R_1\mathbf{m}\|^2 + \frac{\beta}{2}\|R_2\mathbf{m}\|^2$$

Potential problems:

- squared norm is not an exact penalty
- difficult/costly to determine penalty-parameters
- potentially ill-conditioned Hessian
- may not be obvious which constrained problem is solved for a given penalty parameter

A few regularization strategies



exact versus non-exact penalty

Toy problem:

$$\min_x \frac{1}{2}\|x-1\|_2^2 \quad \text{s.t.} \quad x = 2$$

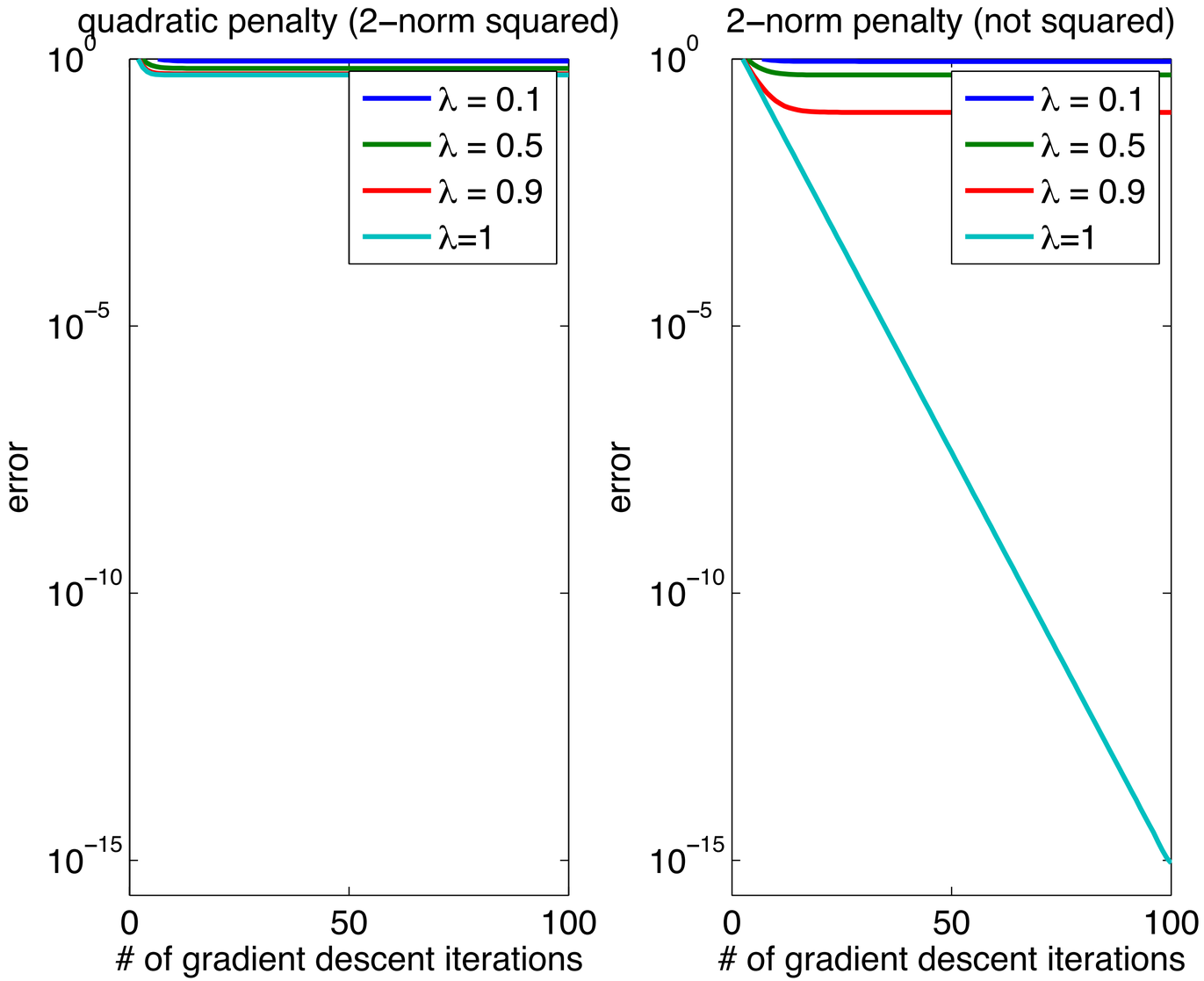
Quadratic-penalty:

$$\min_x \frac{1}{2}\|x-1\|_2^2 + \lambda\|x-2\|_2^2$$

2-norm penalty:

$$\min_x \frac{1}{2}\|x-1\|_2^2 + \lambda\|x-2\|_2$$

A few regularization strategies



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A few regularization strategies

Gradient filtering: $\mathbf{m}_{k+1} = \mathbf{m}_k - \gamma F \nabla_{\mathbf{m}} f(\mathbf{m})$

If the gradient filter F is the inverse Hessian, this is just Newton's method

Can work if F is definite positive

Potential problems:

- filtered gradient is not a gradient of the objective anymore
  - F can modify the descent direction
- no obvious generalization to include multiple filters

A few regularization strategies

Constrained formulation: $\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$

"find a model which satisfies all pieces of prior info simultaneously"

- constraints can be satisfied at every iteration
- feasible part of the objective function is unmodified
- works with gradient/quasi-Newton/Newton-type methods
- can define more than two constraint-sets

- no weights or other parameters required, just define the sets

Prior information as convex sets

Projection (Euclidean, minimum-distance projection):

$$\mathcal{P}_{\mathcal{C}}(\mathbf{m}) = \arg\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{m}\|_2 \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

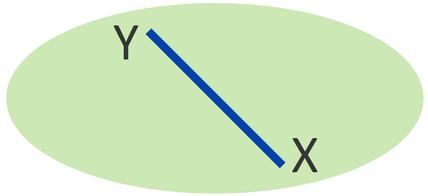
- the closest element in the intersection (definition)

Important property:

$$\mathcal{P}_{\mathcal{C}}(\mathbf{m}) = \mathcal{P}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}(\mathbf{m}))$$

Convex sets : some properties

- Convex set :
  - there is a linear path contained in the set between every pair of the set
  - every point is linearly reachable from another point

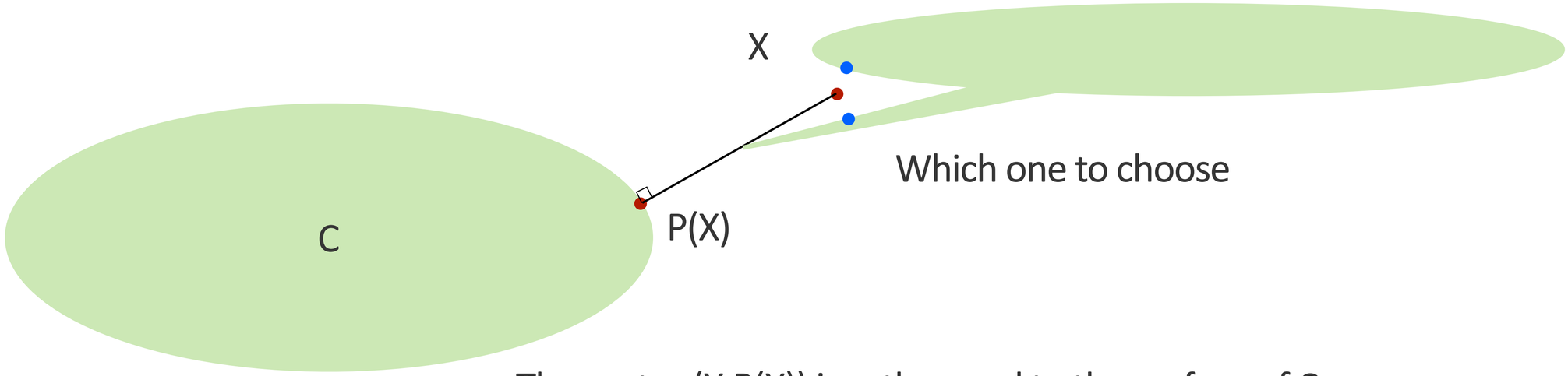


Convex

Non convex

- Projection onto a convex set is unique

Projection onto a convex set



The vector (X,P(X)) is orthogonal to the surface of C

P(X) is the closest point of X in C

Prior information as convex sets

example 1: (spatially varying) bound constraints:

$$\mathcal{C}_1 \equiv \{\mathbf{m} \,|\, \mathbf{b}_l \leq \mathbf{m} \leq \mathbf{b}_u\}$$

can include reference models as:

$$\mathbf{b}_l = \mathbf{m}_{\text{ref}} - \delta\mathbf{m}$$

Projector: (element-wise)

$$\mathcal{P}_{\mathcal{C}_1}(\mathbf{m}) = \text{median}\{\mathbf{b}_l, \mathbf{m}, \mathbf{b}_u\}$$

Prior information as convex sets

example 2: minimum smoothness of the model:

$$\mathcal{C}_2 \equiv \{\mathbf{m} \mid E^* F^* (I - S) F E \mathbf{m} = 0\}$$

"the 2D spatial Fourier-transform of the mirror-extended model is contained within an ellipse"

$$E \in \mathbb{R}^{4N \times N} \quad \text{Mirror-extention}$$

$$\mathbf{m} \in \mathbb{R}^{N} \quad \text{medium parameters}$$

$$F \in \mathbb{C}^{N \times N} \quad \text{DFT matrix}$$

$$S \in \mathbb{R}^{N \times N} \quad \text{Selection matrix (diagonal), 'filter coefficients'}$$

Prior information as convex sets

example 2: minimum smoothness of the model:

$$\mathcal{C}_2 \equiv \{\mathbf{m} \mid E^* F^* (I - S) F E \mathbf{m} = 0\}$$

1. 2D mirror extension of the model (to avoid periodic boundaries)
2. 2D DFT
3. Remove coefficients outside ellipse (highest spatial frequencies)
4. 2D inverse DFT

ellipse takes directional varying smoothness (geology) into account

Prior information as convex sets

example 2: minimum smoothness of the model:

$$\mathcal{C}_2 \equiv \{\mathbf{m} \mid E^* F^* (I - S) F E \mathbf{m} = 0\}$$

- Choose initial ellipse based on the lowest frequency band and the smoothness of the start model.
- Adapt to different frequency bands by stretching the ellipse based on a formula like: $d\dfrac{f_{\max}}{v_{\min}}$

Projector: $\mathcal{P}_{\mathcal{C}_2}(\mathbf{m}) = E^* F^* S F E \mathbf{m}$

Bas Peters, Zhilong Fang, Brendan Smithyman, and Felix J. Herrmann, "Regularizing waveform inversion by projections onto convex sets –- application to the 2D Chevron 2014 synthetic blind-test dataset".

Algorithmic development

$$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

$\mathcal{C}_1 \bigcap \mathcal{C}_2$ is convex if $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex

We would like the model to be in $\mathcal{C}_1 \bigcap \mathcal{C}_2$ at every iteration

One possibility:

$$\min_{\mathbf{m}} f(\mathbf{m}) + \iota_{\mathcal{C}_1}(\mathbf{m}) + \iota_{\mathcal{C}_2}(\mathbf{m})$$

$$\iota_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C}, \\ +\infty & \text{if } x \notin \mathcal{C}. \end{cases}$$

Algorithmic development

$$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

$$\min_{\mathbf{m}} f(\mathbf{m}) + \iota_{\mathcal{C}_1}(\mathbf{m}) + \iota_{\mathcal{C}_2}(\mathbf{m}) \quad \rightarrow \text{ not differentiable}$$

Can use forward-backward splitting / proximal-gradient algorithms.

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Algorithmic development

$$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

Project onto an intersection of convex sets:

- sometimes known analytically
- otherwise compute numerically; Dykstra's algorithm is used in this work

Dykstra splitting

Toy example:
find projection onto intersection of a circle and a square

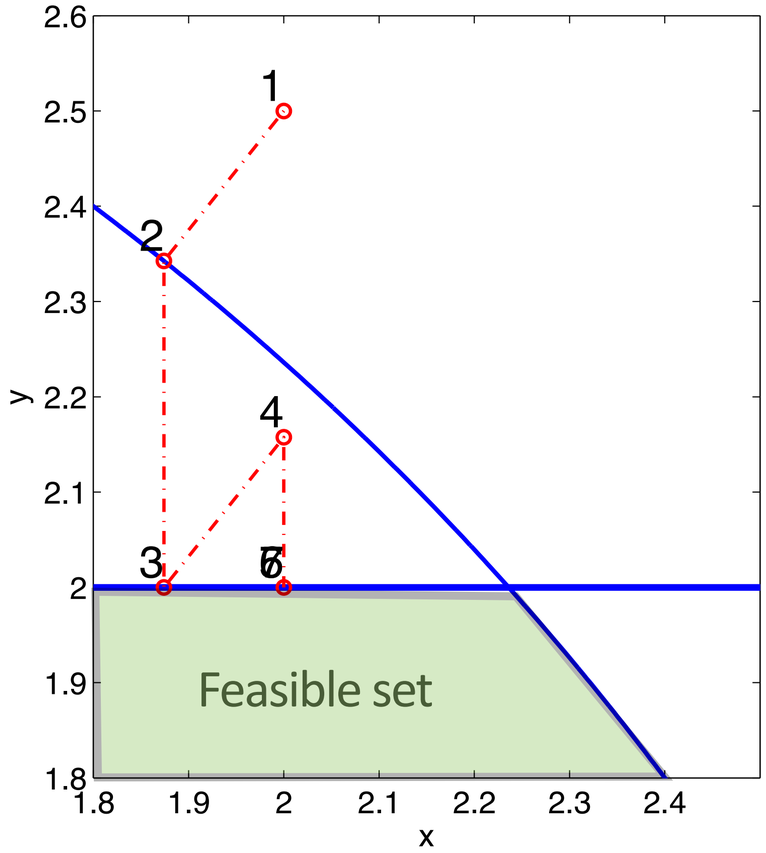
Algorithm 1 Dykstra.

$x_0 = \mathbf{m},\ p_0 = 0,\ q_0 = 0$

For $k = 0, 1, \ldots$

$$y_k = \mathscr{P}_{C_1}(x_k + p_k)$$
$$p_{k+1} = x_k + p_k - y_k$$
$$x_{k+1} = \mathscr{P}_{C_2}(y_k + q_k)$$
$$q_{k+1} = y_k + q_k - x_{k+1}$$

End



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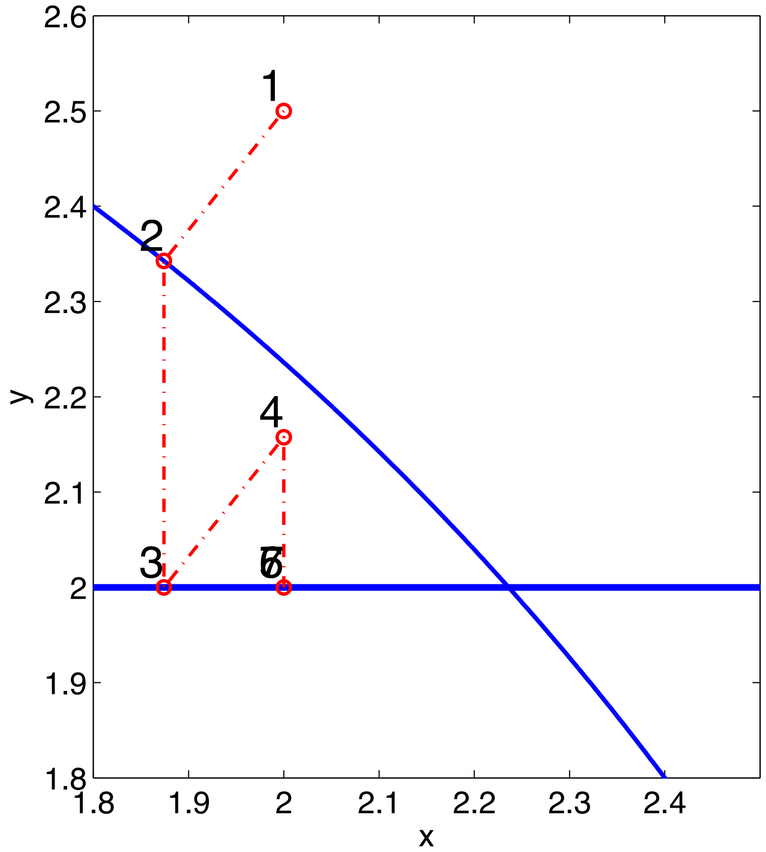

only need projection onto each set separately

Dykstra splitting

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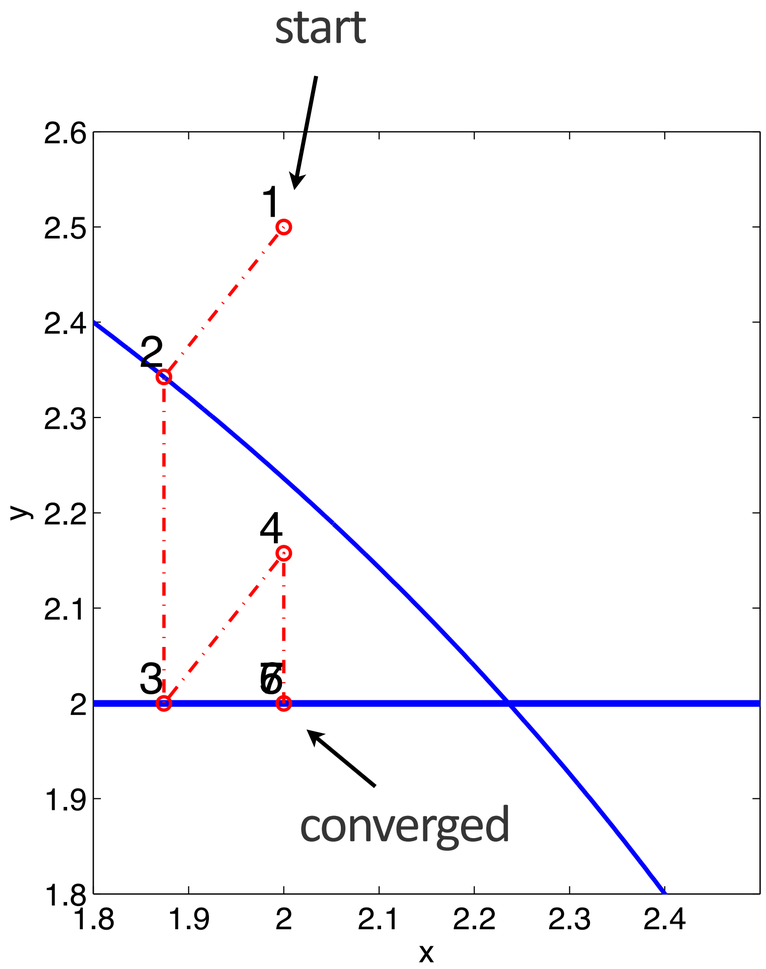
End



Dykstra splitting

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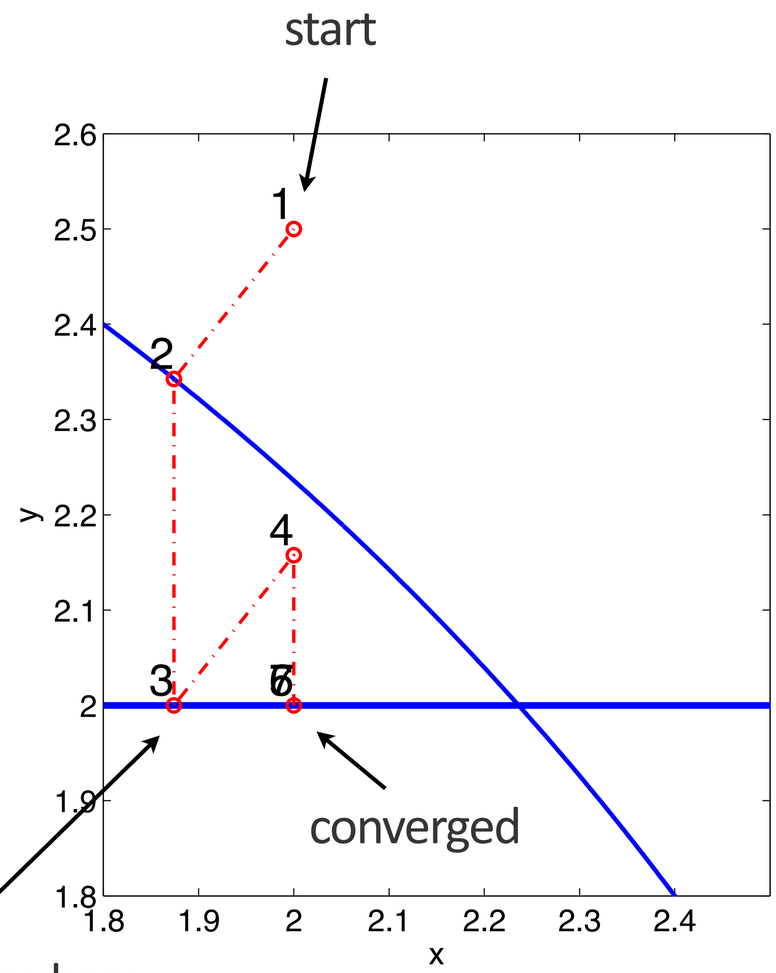

POCS would converge here,

feasible point, not the projection onto

Dykstra splitting

Projection-onto-convex-sets (POCS) solves the convex feasibility problem:

$$\text{find} \quad x \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

Dykstra's algorithm solves:

$$\min_x \iota_{\mathcal{C}_1}(x) + \iota_{\mathcal{C}_2}(x) + \frac{1}{2}\|x - y\|^2$$

with indicator function:

$$\iota_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C}, \\ +\infty & \text{if } x \notin \mathcal{C}. \end{cases}$$

Dykstra splitting

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is equivalent to:

$$\min_x \frac{1}{2}\|x - y\|^2 \quad \text{s.t.} \quad x \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

Dykstra splitting

Projection-onto-convex-sets (POCS):

$$\text{find} \quad x \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

find any point in the intersection, may be the closest point

Dykstra's algorithm solves:

$$\min_x \iota_{\mathcal{C}_1}(x) + \iota_{\mathcal{C}_2}(x) + \frac{1}{2}\|x-y\|^2$$

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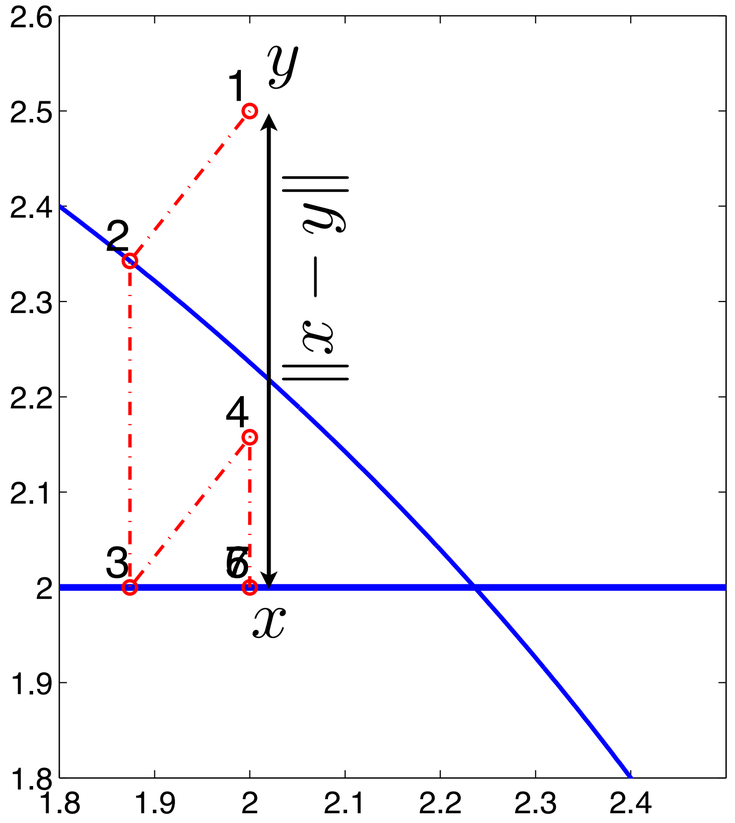

Algorithmic development

$$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

Projected-gradient: $\mathbf{m}_{k+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{m}_k - \gamma \nabla_{\mathbf{m}} f(\mathbf{m}_k))$

Algorithmic development

$$\min_{\mathbf{m}} f(\mathbf{m}) \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

Projected-gradient: $\mathbf{m}_{k+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{m}_k - \gamma \nabla_{\mathbf{m}} f(\mathbf{m}_k))$

Can this simply be accelerated using Hessian approximation $B(\mathbf{m}_k)$?

$$\mathbf{m}_{k+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{m}_k - \gamma B(\mathbf{m}_k)^{-1} \nabla_{\mathbf{m}} f(\mathbf{m}_k))$$

Algorithmic development

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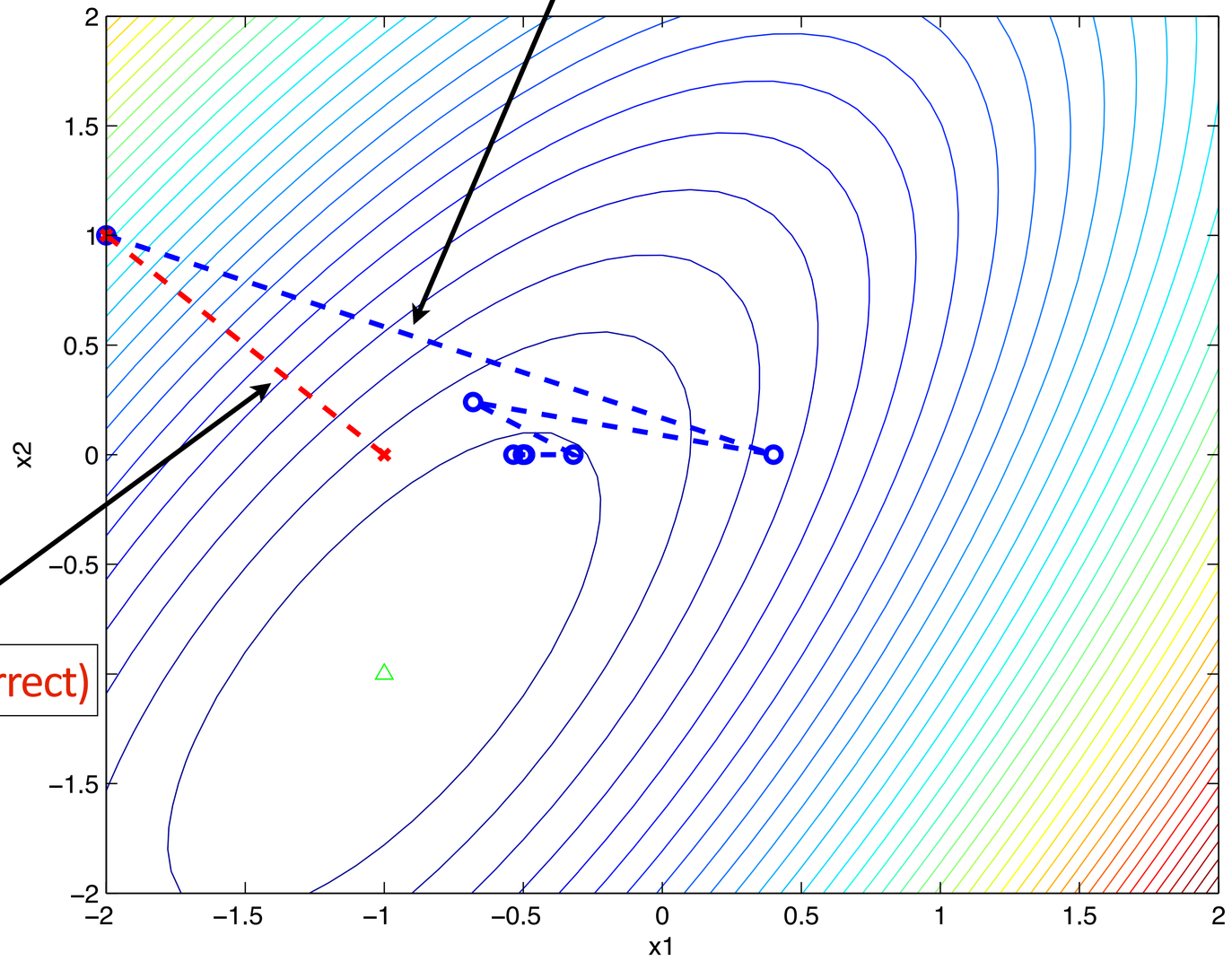
~~$\mathbf{m}_{k+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{m}_k - \gamma B(\mathbf{m}_k)^{-1} \nabla_{\mathbf{m}} f(\mathbf{m}_k))$~~

Generally not, when using the Euclidean projection and general $B(\mathbf{m}_k)$

$$\min_{\mathbf{x}} \mathbf{x}^* A\mathbf{x} - \mathbf{x}^*\mathbf{b} \quad \text{s.t.} \quad x_2 \geq 0$$



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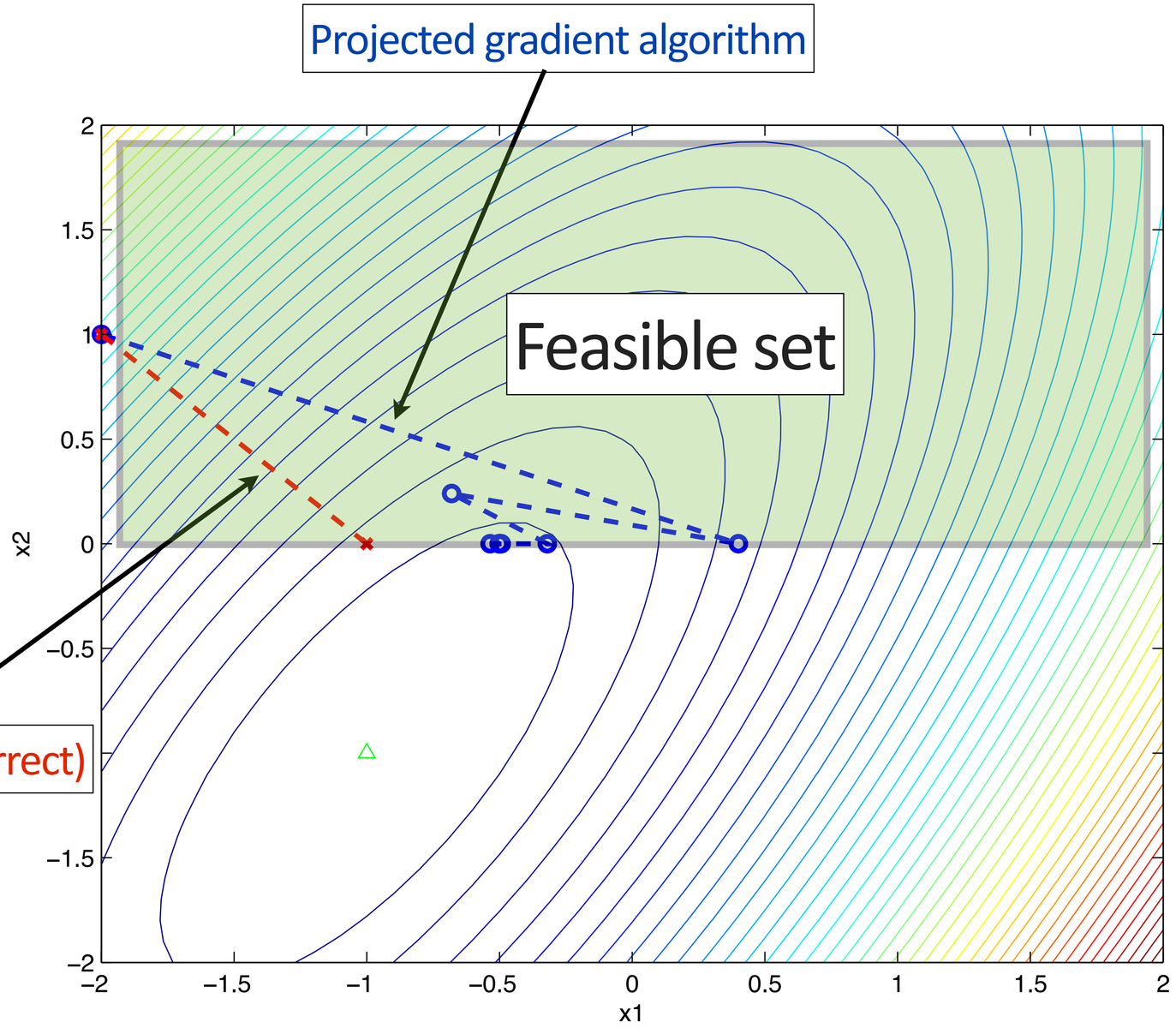

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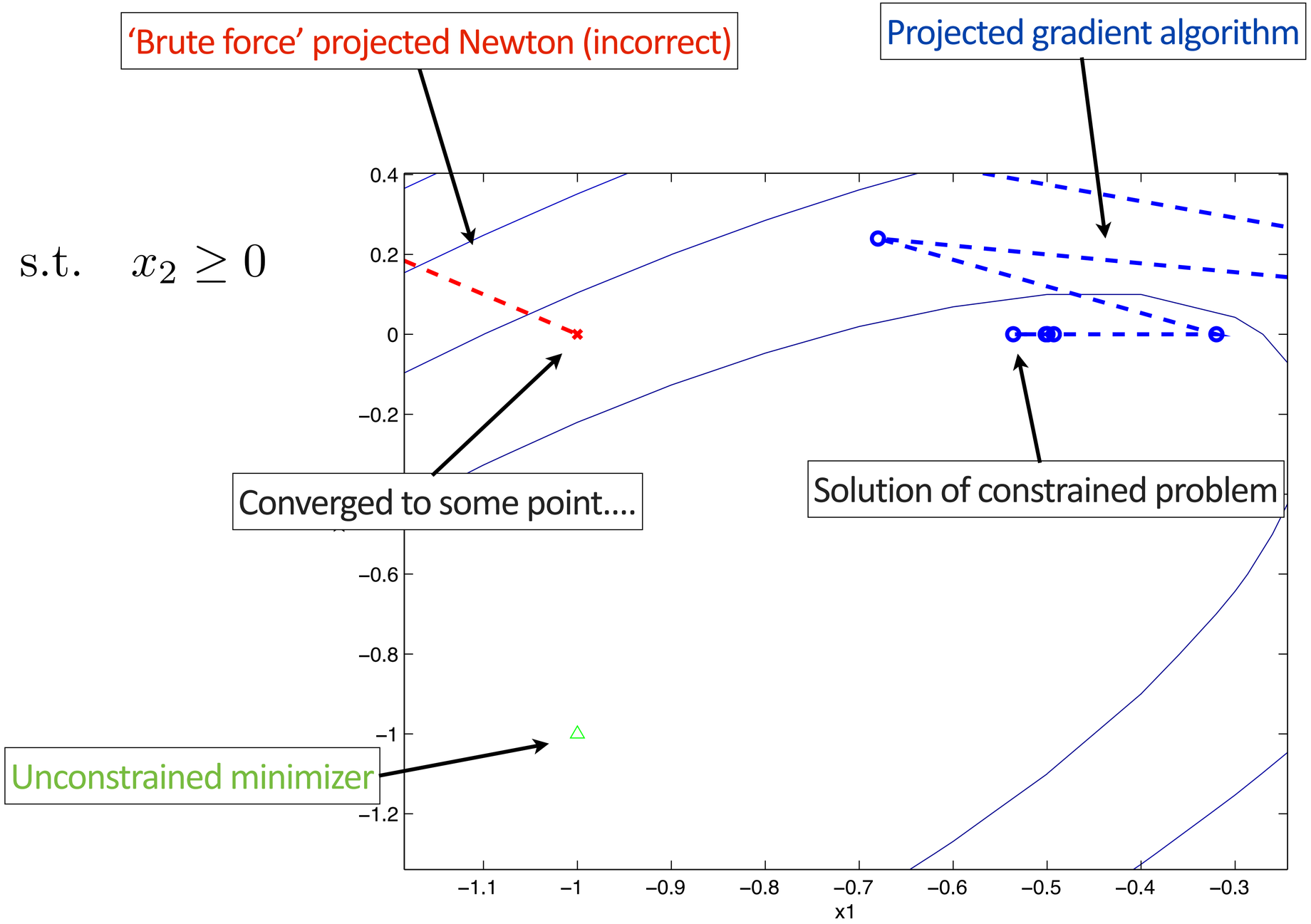


Ernie Esser, Tristan van Leeuwen, Aleksandr Y. Aravkin, and Felix J. Herrmann, "A scaled gradient projection method for total variation regularized full waveform inversion". 2014

Algorithmic development

Projected-gradient: $\mathbf{m}_{k+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{m}_k - \gamma \nabla_{\mathbf{m}} f(\mathbf{m}_k))$

Projected Quasi-Newton [M. Schmidt et. al., 2009]

- solves quadratic sub-problem with constraints using the spectral projected-gradient algorithm (inexactly)
- L-BFGS Hessian

Projected Newton-type:

- solves quadratic sub-problem with constraints
- efficient if approximate Hessian is 'easy to invert'

Algorithmic development

Projected Newton-type:

- solves quadratic sub-problem with constraints:

$$Q(\mathbf{m}) = f(\mathbf{m}_k) + (\mathbf{m}-\mathbf{m}_k)^* \nabla_{\mathbf{m}} f(\mathbf{m}_k) + (\mathbf{m}-\mathbf{m}_k)^* B_k (\mathbf{m}-\mathbf{m}_k)$$

$$\mathbf{m}_{k+1} = \min_{\mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2} Q(\mathbf{m})$$

- efficient if approximate Hessian is ‘easy to invert’ (factored Hessian, sparse & well conditioned, diagonal)

Multiple algorithms can solve the constrained sub-problem

We use Alternating Direction Method of Multipliers (ADMM)

Algorithmic development

Projected Newton-type:

- solves quadratic sub-problem with constraints:

$$\mathbf{m}_{k+1} = \min_{\mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2} Q(\mathbf{m})$$

- can be reformulated as: [M. Schmidt et. al., 2011]

$$\mathbf{y}_k = \mathbf{m}_k - B_k^{-1} \nabla_{\mathbf{m}} f(\mathbf{m}_k)$$

(unconstrained Newton-step)

$$\mathbf{m}_{k+1} = \min_{\mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2} \frac{1}{2} \|\mathbf{y}_k - \mathbf{m}\|_{B_k}^2$$

(projection w.r.t. metric induced by the approximate Hessian)

Workflow summary

1. Define convex feasible sets, possibly velocity & frequency dependent
2. Set up Dykstra's algorithm for projection onto intersections of sets
3. Set up an algorithm to solve the quadratic sub-problem with constraints (ADMM)
4. Solve waveform inversion problem using the a Projected Newton-type algorithm

Algorithm

Projected quasi-Newton (PQN) version:

At every iteration of PQN:

- solve PDE's
- solve quadratic problem with constraints using SPG
- at every iteration of SPG:
  - solve projection problem onto an intersection of convex sets using Dykstra's algorithm
  - at every iteration of Dykstra's algorithm:
    - compute projections on each set separately

Algorithm

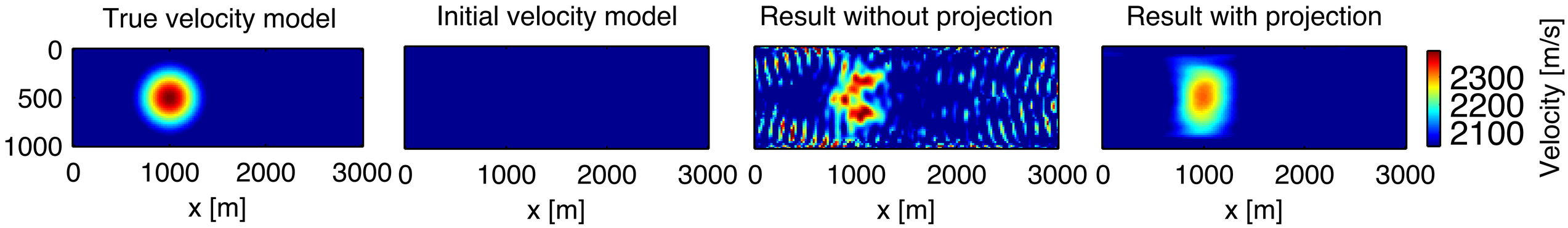
Projected Newton-type version:

At every iteration of projected Newton-type:

- solve PDE's
- Solve quadratic problem with constraints using ADMM
- at every iteration of ADMM:
  - invert Hessian (possibly iteratively)
  - solve projection problem onto an intersection of convex sets using Dykstra's algorithm
  - at every iteration of Dykstra's algorithm:
    - compute projections on each set separately

Example 1 - FWI with a lot of noise

- Sources and receivers at top & bottom of the domain
- 10 Hz data
- $\|\text{noise}\|_2 / \|\text{signal}\|_2 = 0.3$
- used bound constraints and minimum smoothness constraints

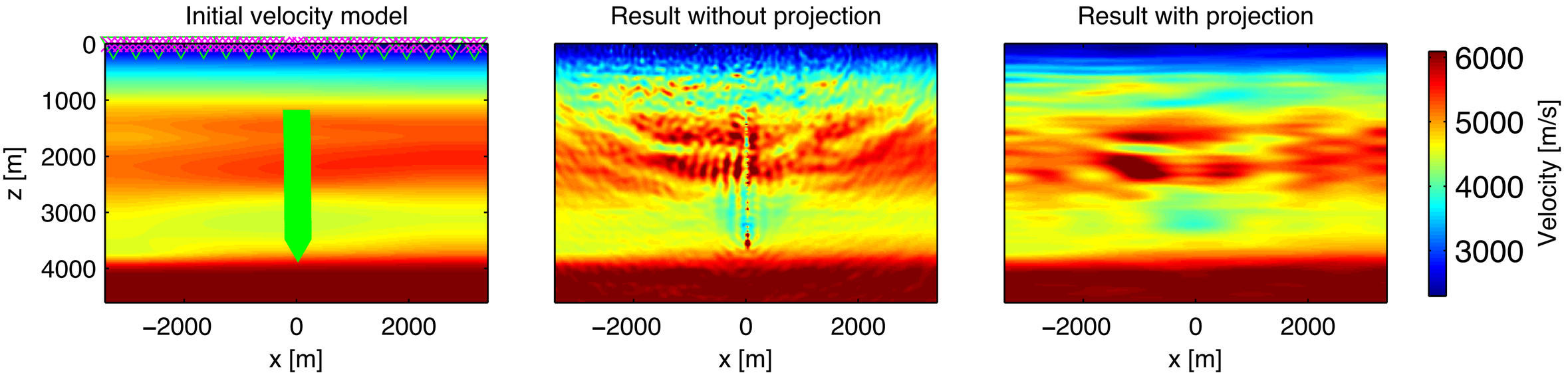


acts as an image-domain noise filter in this case

Example 2 - FWI on a real land dataset

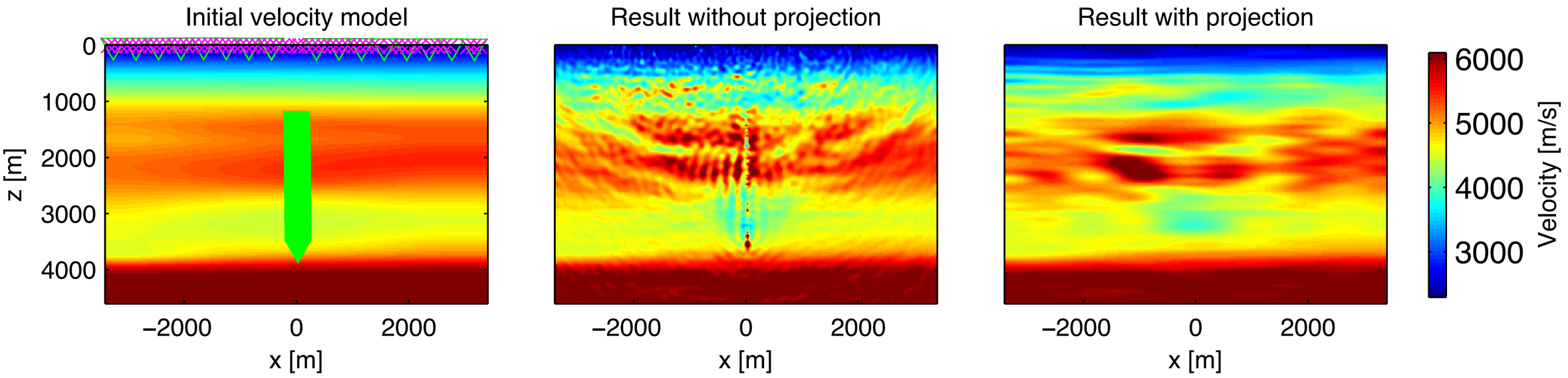
Land data set with surface sources and surface & well receivers

Constant density acoustic inversion (2D slice)



Example 2 - FWI on a real land dataset

- Bound constraints
- Minimum smoothness constraints



Example 3 – WRI

Chevron blind-test elastic data set, work by Zhilong Fang

Acoustic constant density inversion (result in progress, March 2015)

Setup :

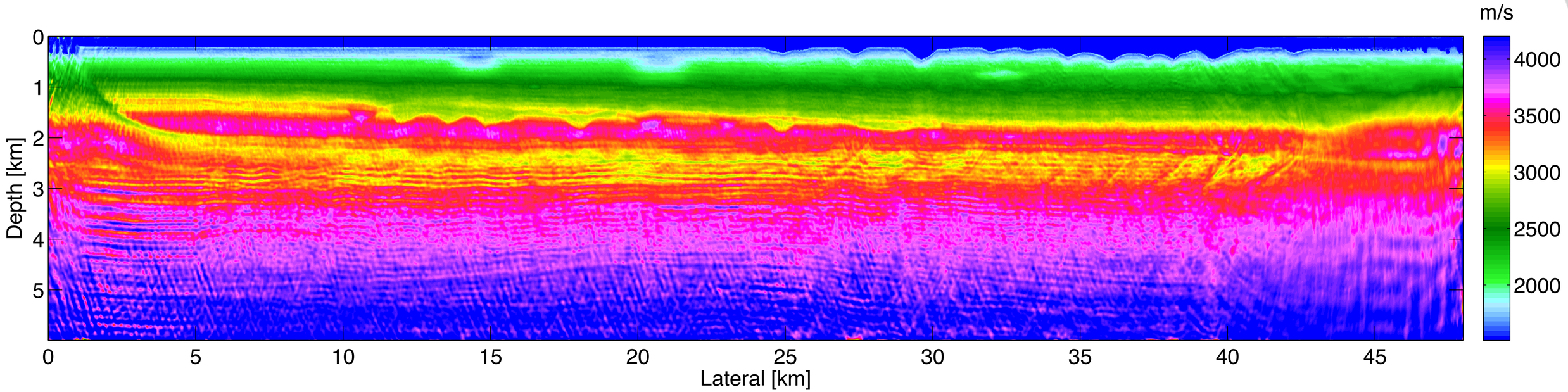
- 341 receivers per sources, 8km offset
- 3Hz-21Hz, 18 frequency bands
- 300 randomly selected sources with redraw

- 3 pass
- 10 iterations of Gauss-Newton per frequency band

Example 3 – WRI

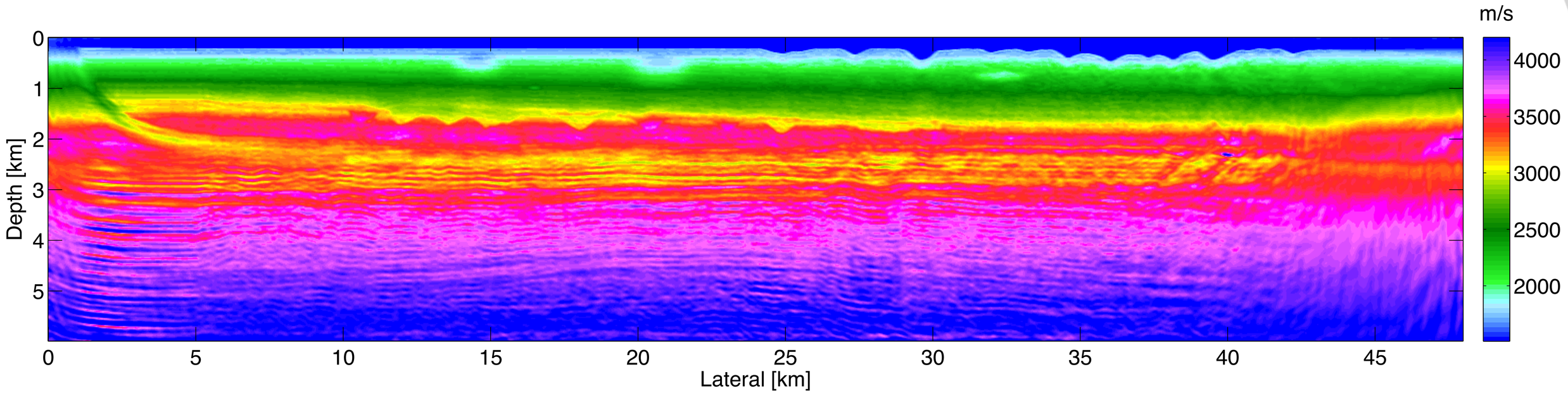
Chevron blind-test elastic data set, work by Zhilong Fang

Acoustic constant density inversion (result in progress, March 2015)



Without minimum smoothness constraint

Example 3 - WRI



Using minimum smoothness constraint

Application to time-domain (adjoint-state)

- Memory efficient method
  - 5 to 20 times more memory efficient than usual FWI via randomization techniques
  - Less i/o or computationally more efficient (No extra computation during back-propagation)
- Use projection onto convex set
  - Bound constraints
  - Minimum smoothness to remove most of the method related artifact
- Method won't be explained (confidentiality)

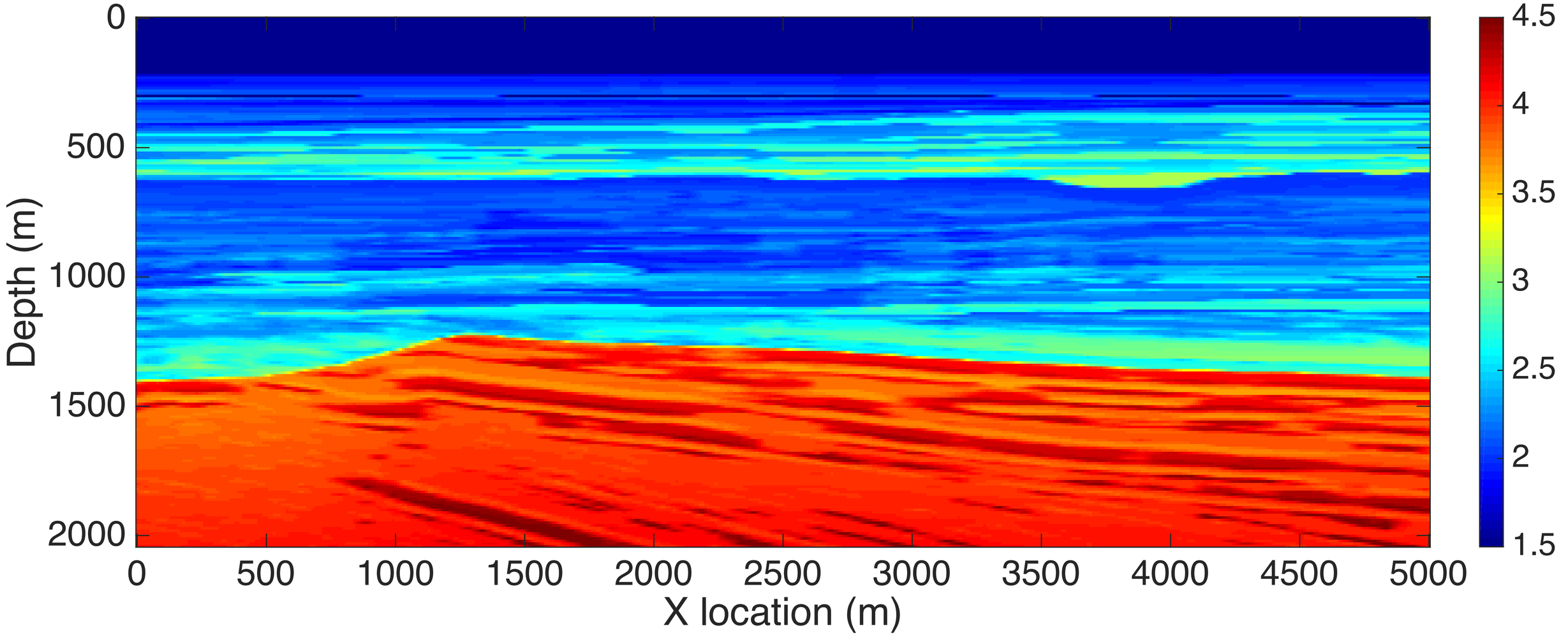
BG Compass 2D

- Data
  - Ricker wavelet at 15Hz, 2.4s recording
  - 51 sources at 100m interval
  - 201 receivers at 25m interval

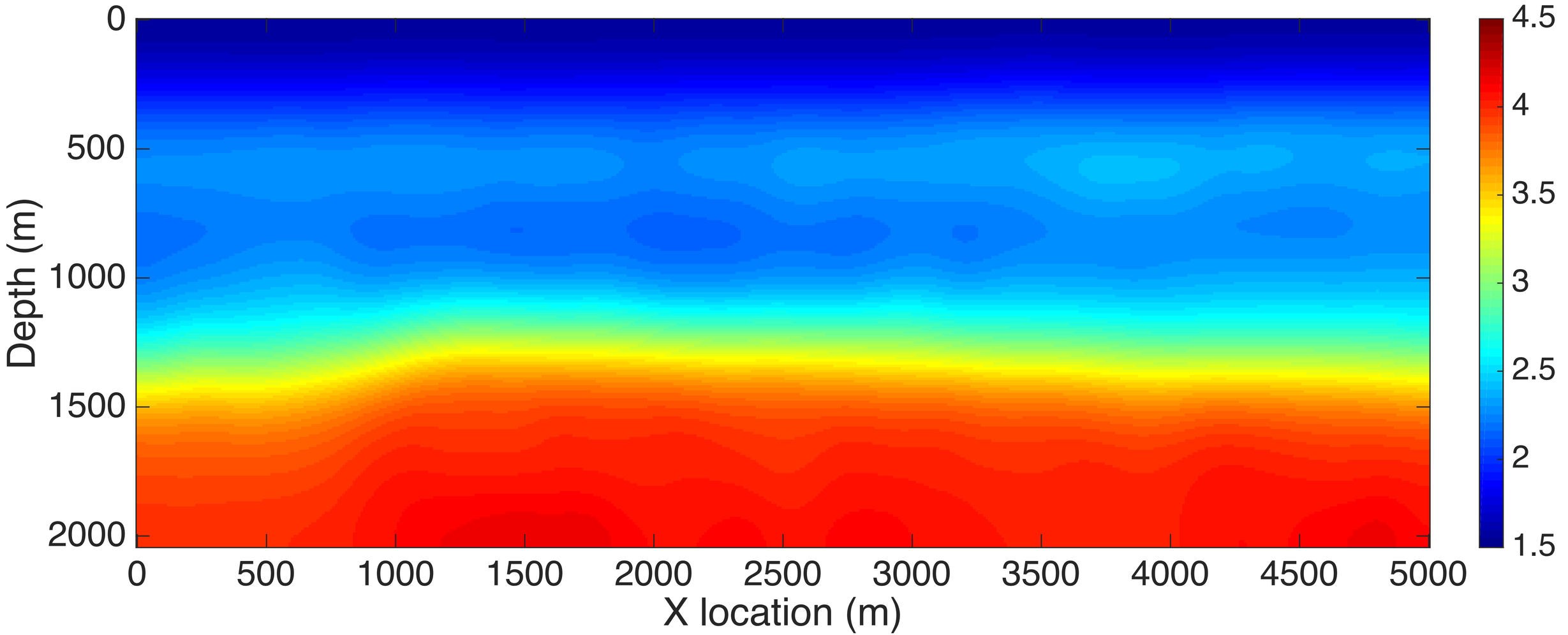
- Acoustic modelling and inversion

- 10 PQN iterations

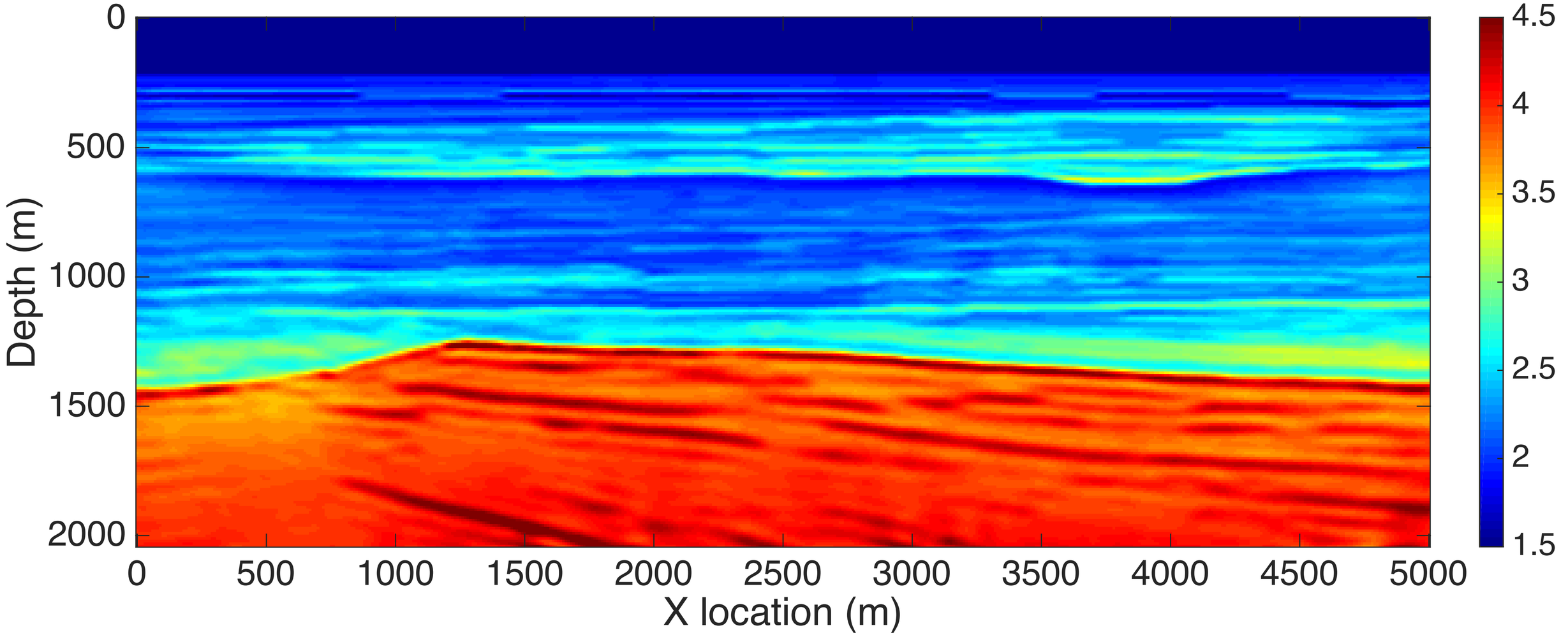
True velocity



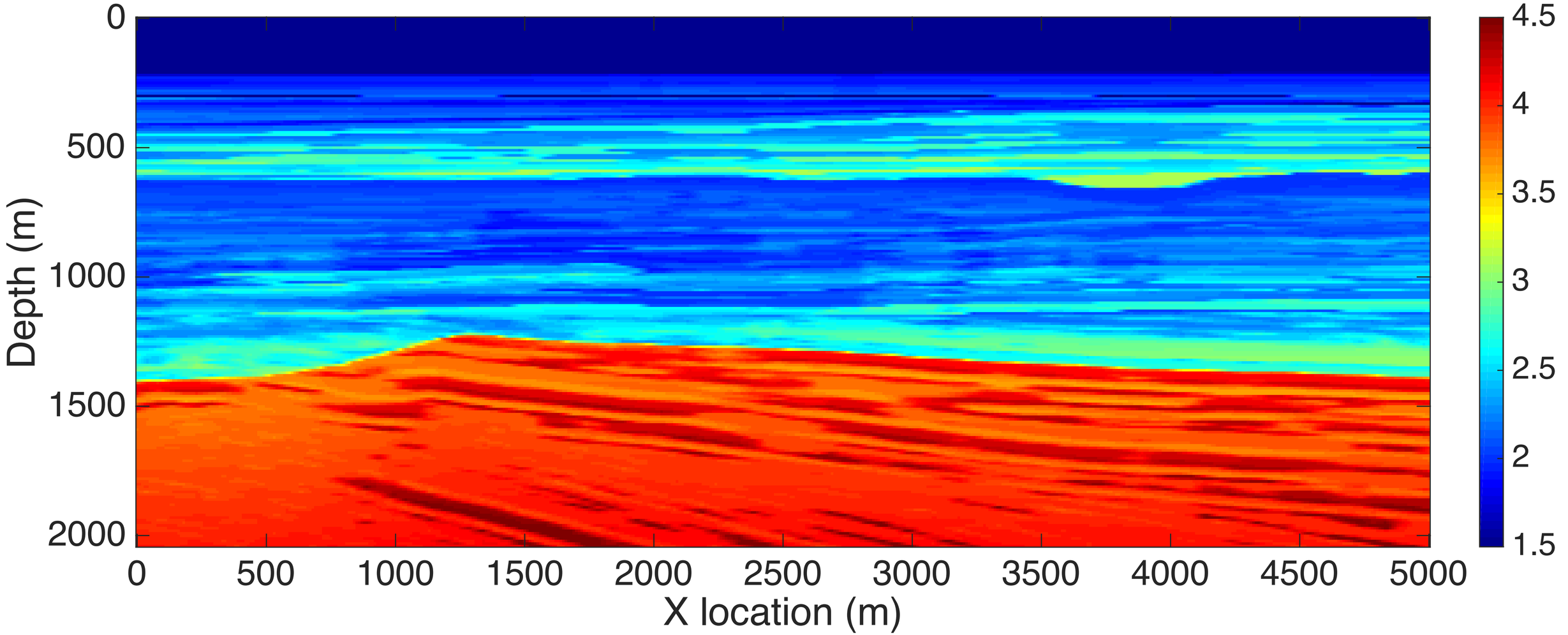
Good initial model



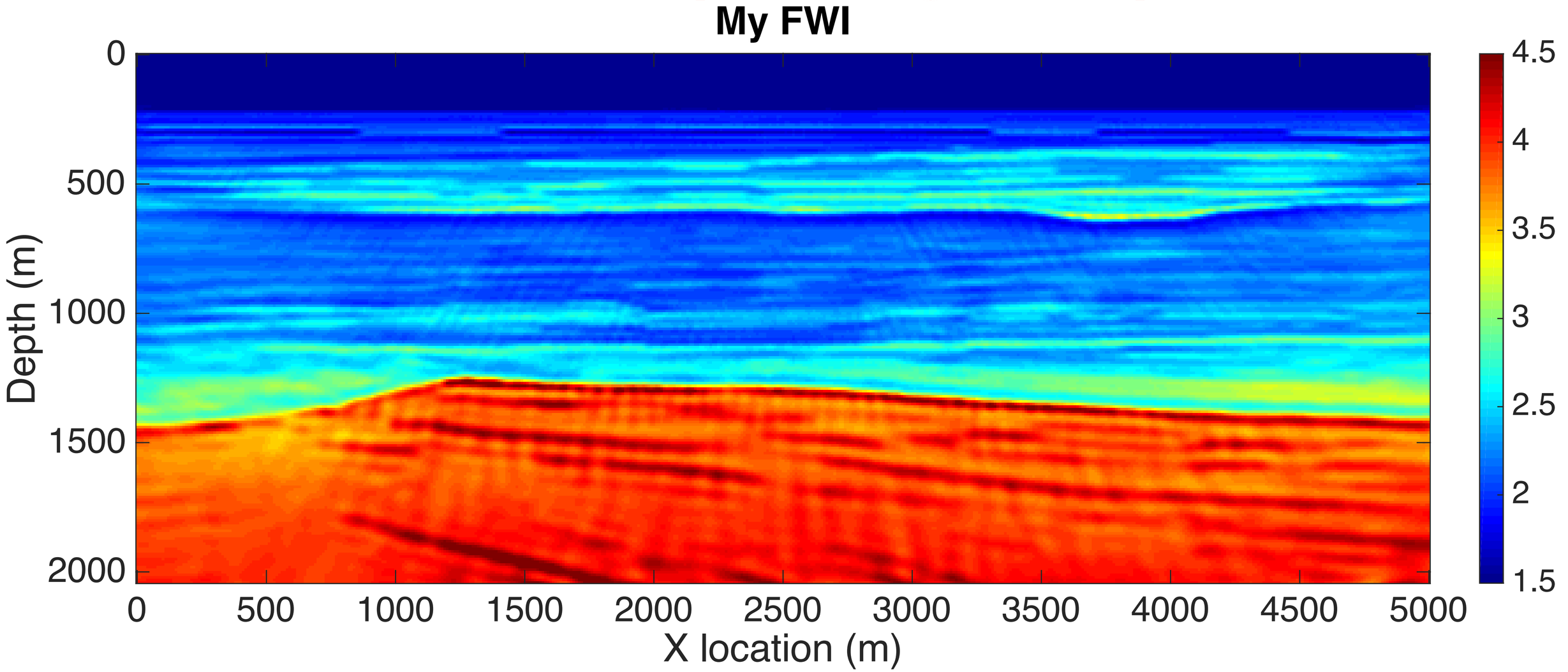
FWI

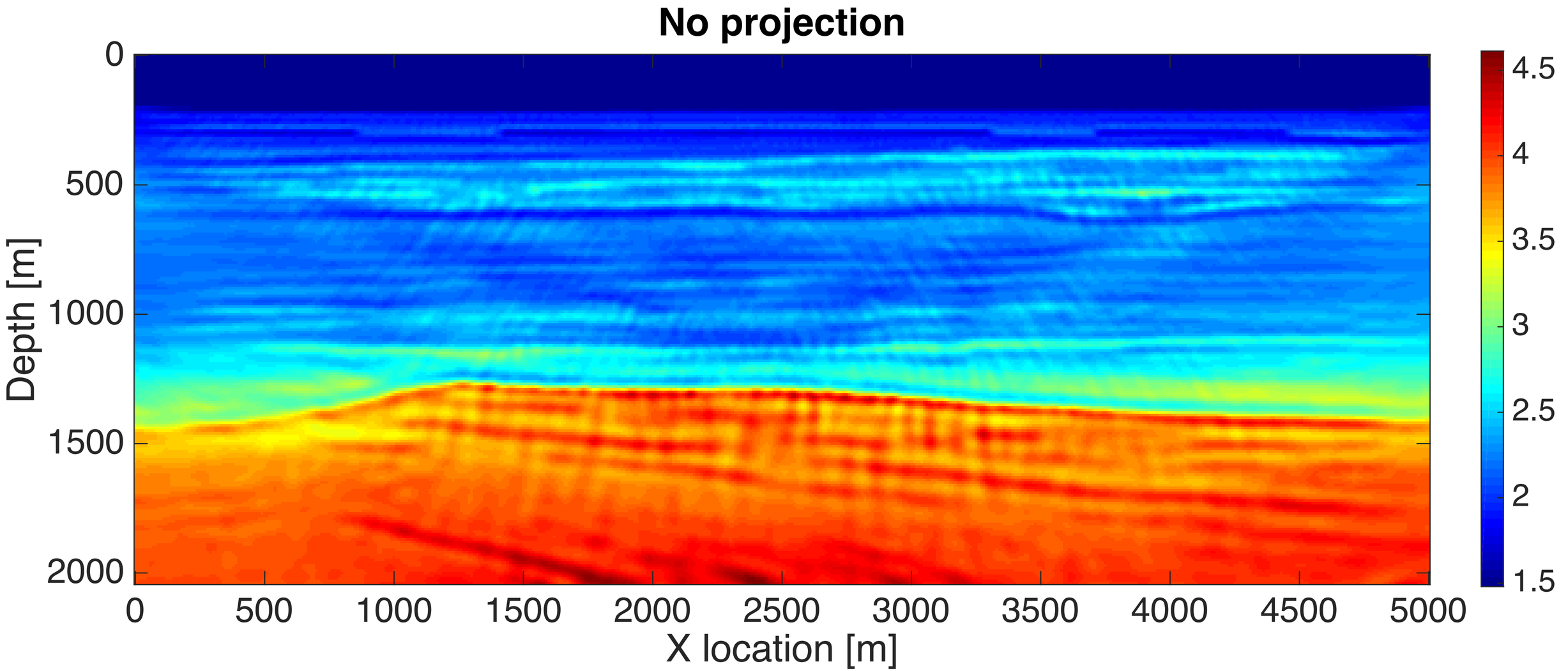

0
500
1000
1500
2000
Depth (m)
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
X location (m)
4.5
4
3.5
3
2.5
2
1.5

True velocity

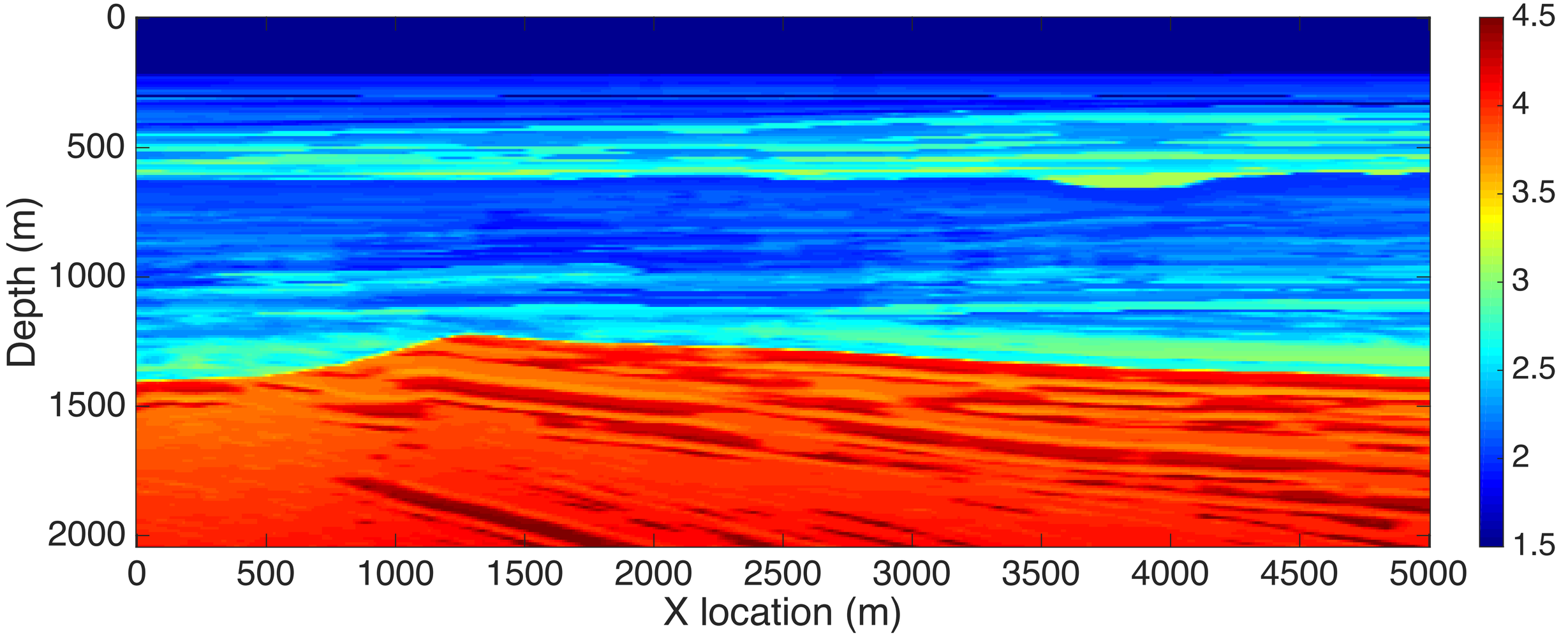


16 times less memory usage than FWI.
Same computational time (Everything in RAM)




No projection
Depth [m]
0
500
1000
1500
2000
X location [m]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
4.5
4
3.5
3
2.5
2
1.5

True velocity



Extended adjoint state (different randomization)

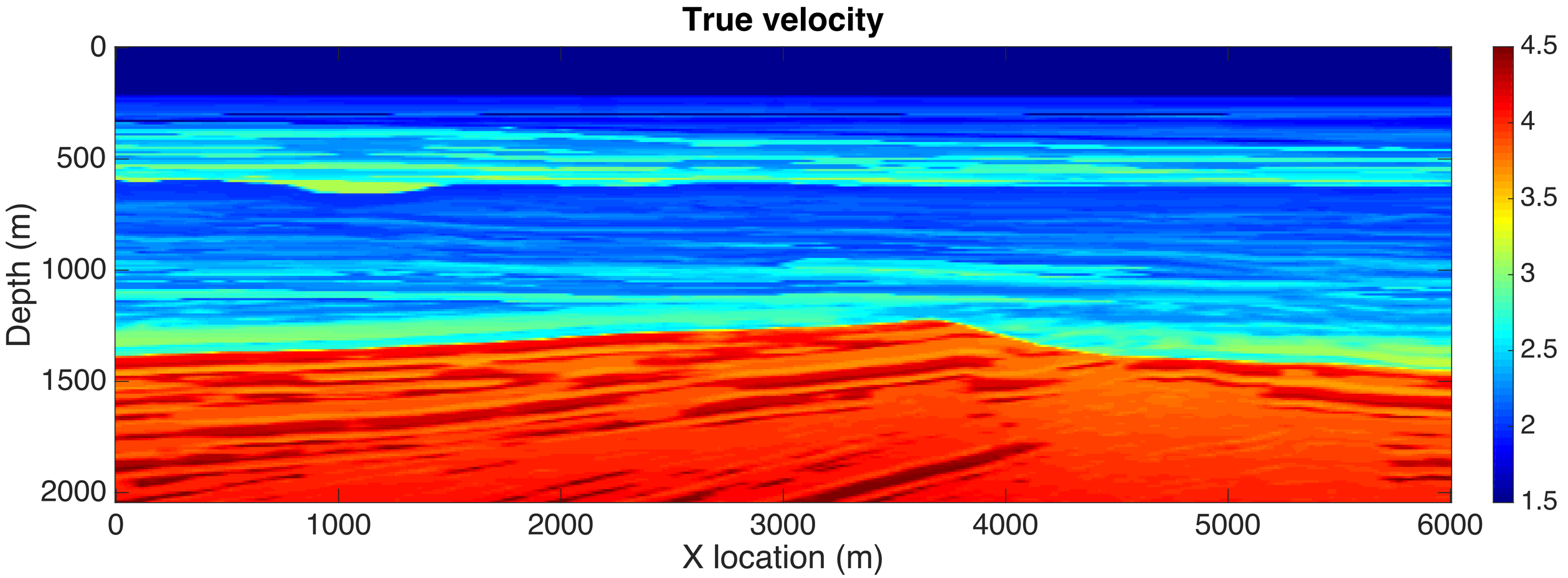
- Same memory efficiency
- Same computational efficiency
- Same projection
- Extended method, uses more information for the same cost

BG Compass 2D (wider)

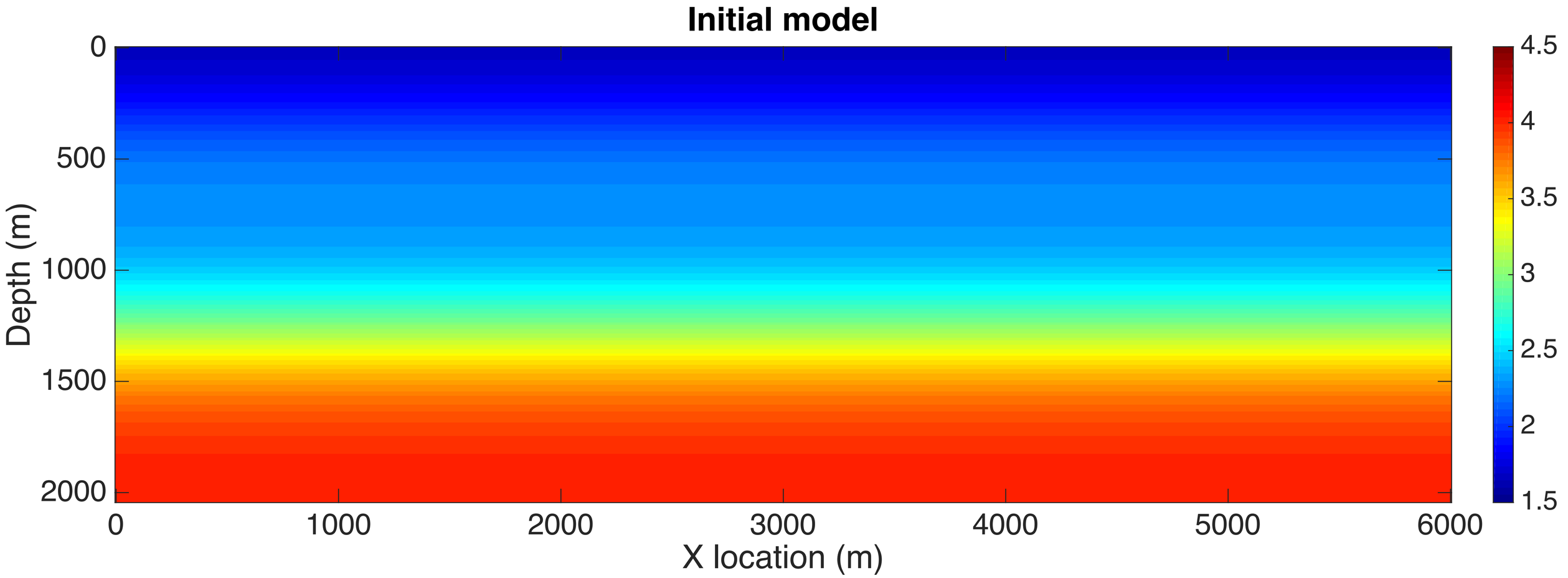
- Data
  - Ricker wavelet at 15Hz, 2.4s recording
  - 61 sources at 100m interval
  - 241 receivers at 25m interval

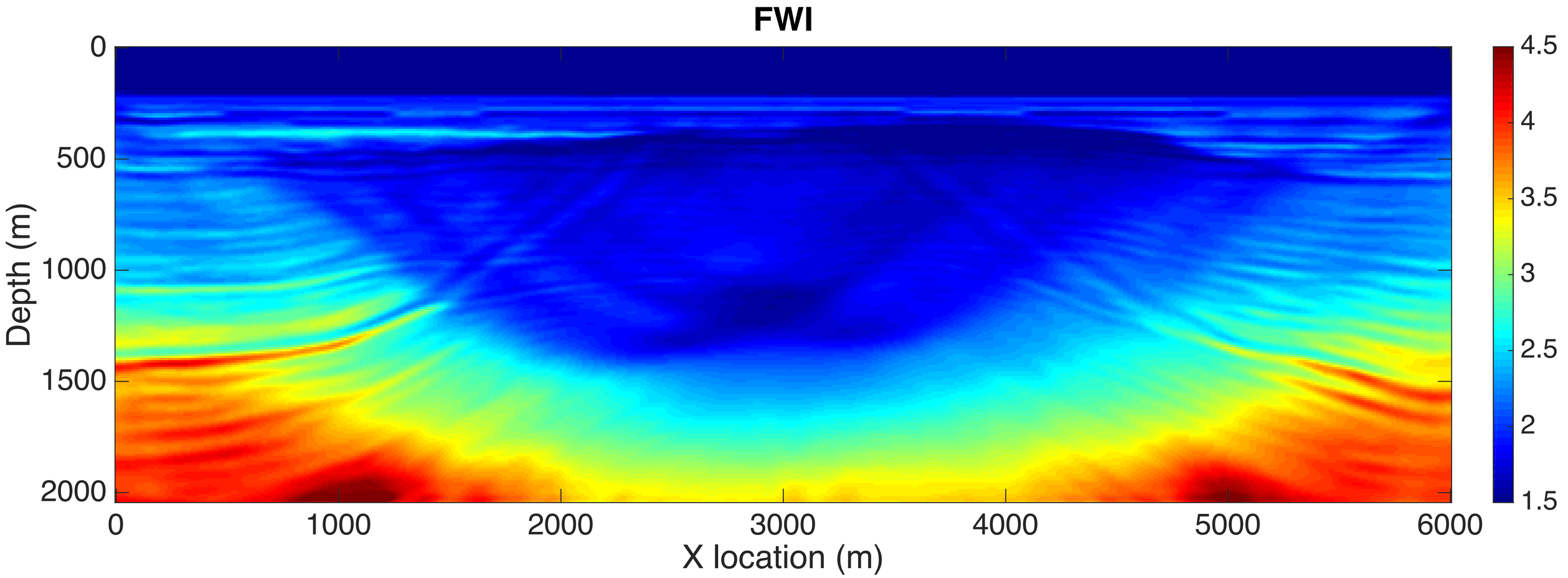
- Acoustic modelling and inversion

- 40 PQN iterations

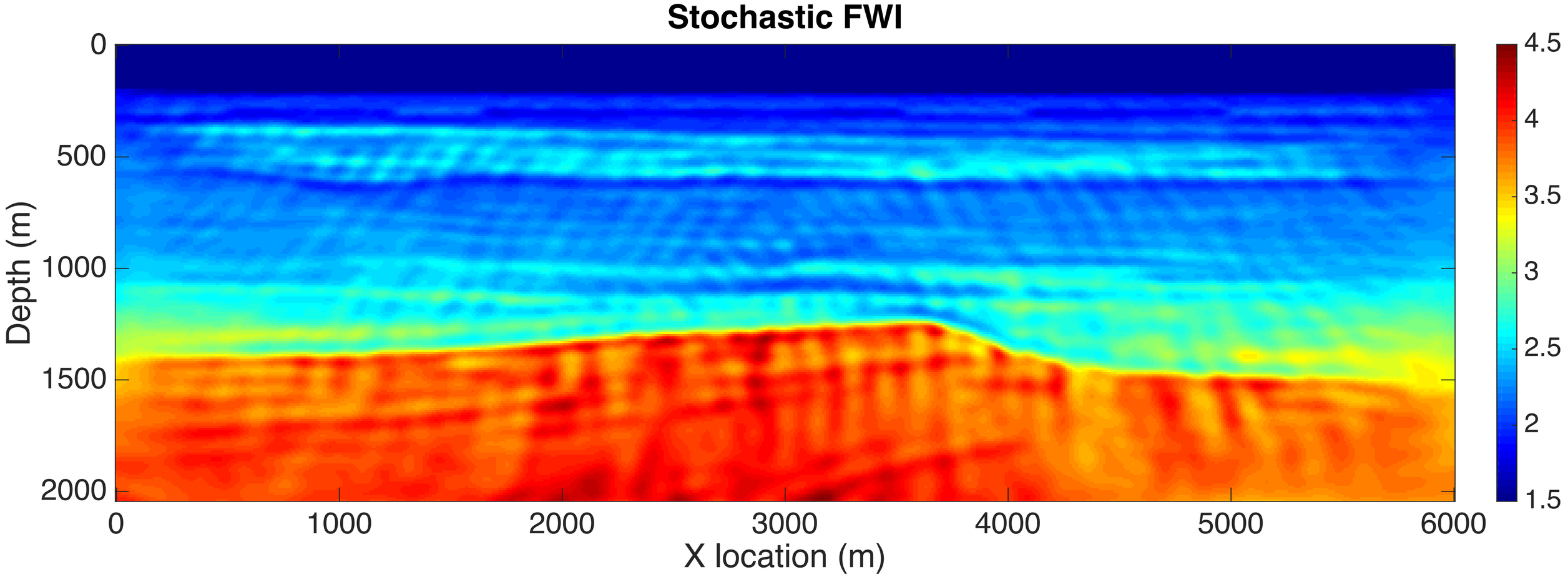

True velocity
Depth (m)
0
500
1000
1500
2000
0
1000
2000
3000
4000
5000
6000
X location (m)
4.5
4
3.5
3
2.5
2
1.5

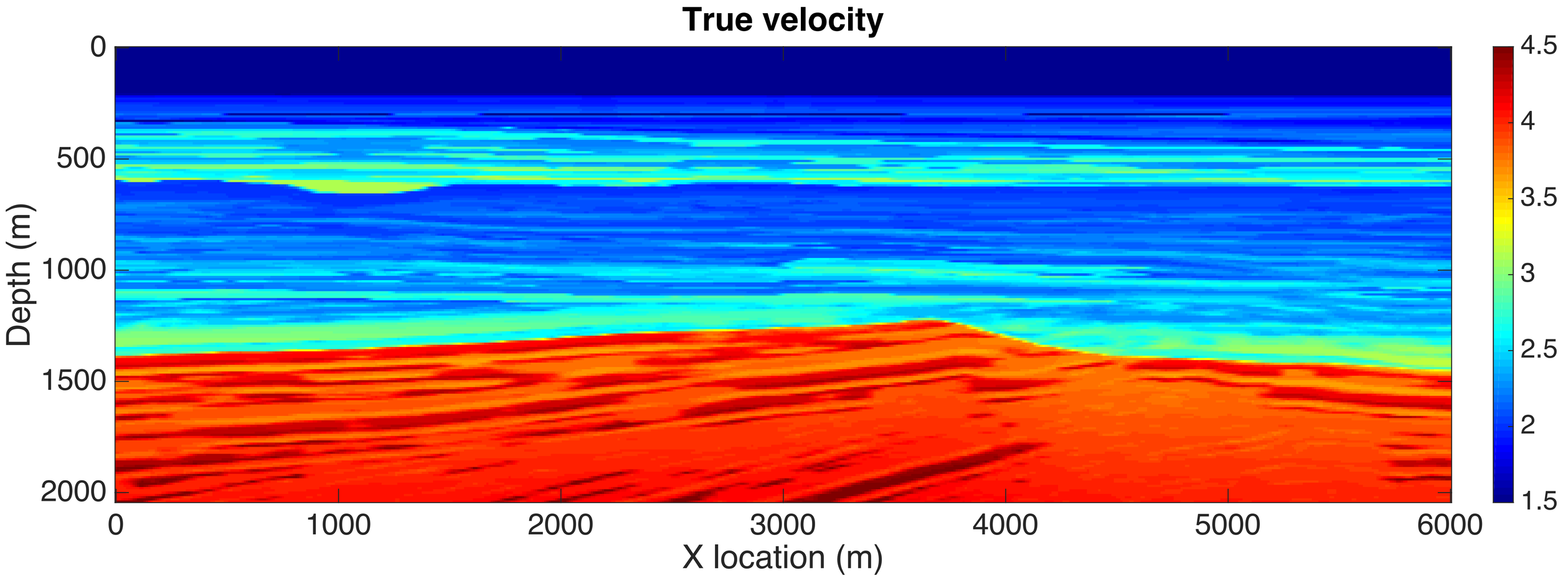
“Bad” initial model



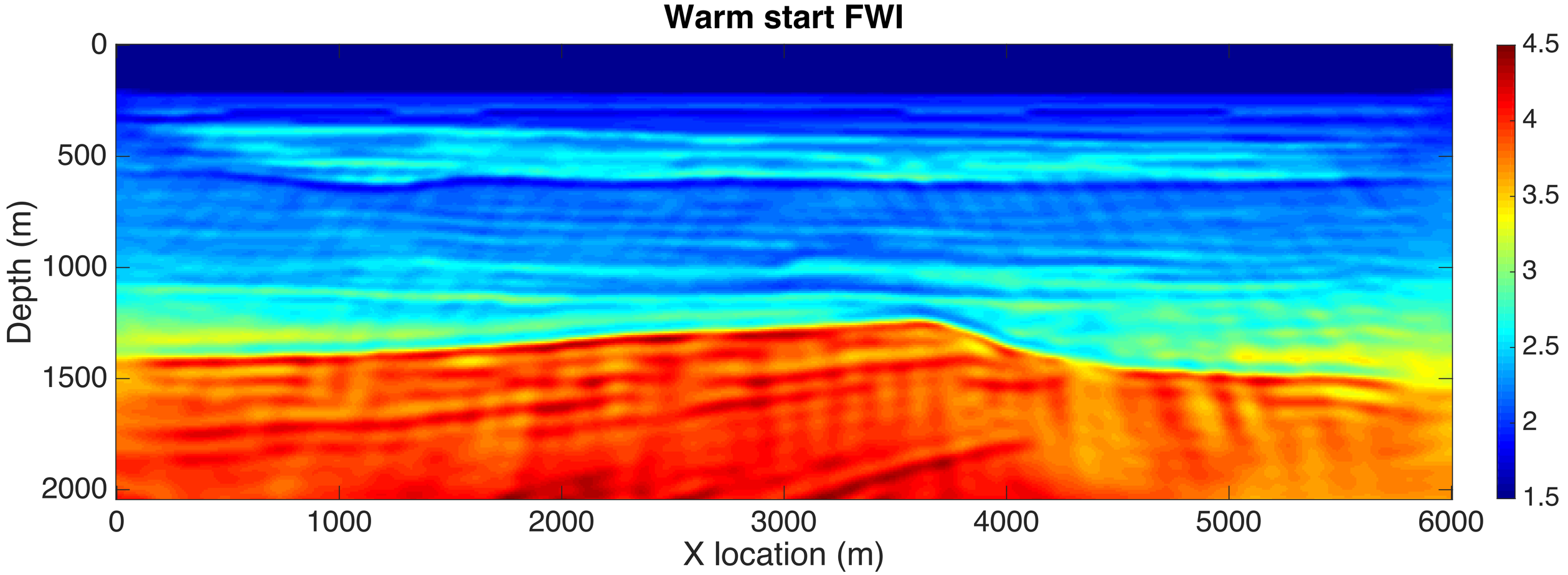

FWI
Depth (m)
0
500
1000
1500
2000
0
1000
2000
3000
4000
5000
6000
X location (m)
4.5
4
3.5
3
2.5
2
1.5

16 times less memory usage than FWI.
Same computational time (Everything in RAM)




True velocity
Depth (m)
0
500
1000
1500
2000
0
1000
2000
3000
4000
5000
6000
X location (m)
4.5
4
3.5
3
2.5
2
1.5

FWI with previous result as initial model
20 PQN iterations



Related geophysical work

- **[A. Baumstein, 2013]** . This work attempts to find the projection onto an intersection using POCS, for different constraints. Includes preconditioner in the Projected-gradient algorithm. May not converge.
- **[E. Esser et. al., 2014; 2015]** (UBC Tech report; this EAGE). Similar philosophy/ideas & problem formulation, different constraints and algorithms.
- **[B. Peters, B. R. Smithyman & F.J. Herrmann, 2015]** (UBC Tech report) projected quasi-Newton based version of this presentation.
- **[B. R. Smithyman, B. Peters & F.J. Herrmann, 2015]** (this EAGE). About the land dataset, uses projected quasi-Newton.
- **[S. Becker et. al., 2015]**. (this EAGE) Also uses projected quasi-Newton, for projections onto a single set. (similar)
- **[B. Peters, Z. Fang, B. R. Smithyman & F.J. Herrmann, 2015]** (submitted to SEG 2015 conference). About the Chevron blind-test dataset (2014). Projected Newton-type using ADMM.

Summary & conclusions

- Can combine different regularization approaches as:

$$\min_{\mathbf{m}} f(\mathbf{m}) + \frac{\alpha}{2}\|R_1\mathbf{m}\|_2^2 + \frac{\beta}{2}\|R_2\mathbf{m}\|_2^2 \quad \text{s.t.} \quad \mathbf{m} \in \mathcal{C}_1 \bigcap \mathcal{C}_2$$

- Developed flexible and extendable framework for including constraints for any differentiable objective.
- Works with various optimization algorithms.
- Requires no extra PDE-solves.
- Easy to use, prior information translates into constraints directly, without penalty parameters.

Outlook

- Add more constraint sets with their projectors
  - What sets?

- Non convex sets?
  - How to set it up
  - What sets?

Acknowledgements

PhD students and Postdocs at SLIM

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9. Schmidt, Mark, Dongmin Kim, and Suvrit Sra. "Projected Newton-type methods in machine learning." (2011).

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9. Bas Peters, Zhilong Fang, Brendan Smithyman, Felix J. Herrmann. Regularizing waveform inversion by projections onto convex sets —- application to the 2D Chevron 2014 synthetic blind-test dataset. (submitted to the SEG conference). 2015. https://www.slim.eos.ubc.ca/Publications/Private/Conferences/SEG/2015/peters2015SEGrwi/peters2015SEGrwi.html

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ADMM

Problem:

$$\min_{\mathbf{m},\mathbf{z}} f(\mathbf{m}) + g(\mathbf{z}) \quad \text{s.t.} \quad A\mathbf{m} + B\mathbf{z} = \mathbf{c}$$

Form augmented-Lagrangian:

$$L_\rho(\mathbf{m}, \mathbf{z}, \mathbf{v}) = f(\mathbf{m}) + g(\mathbf{z}) + \mathbf{v}^*(A\mathbf{m} + B\mathbf{z} - \mathbf{c}) + \frac{\rho}{2}\|A\mathbf{m} + B\mathbf{z} - \mathbf{c}\|_2^2$$

using dual variable $\mathbf{v}$ and scalar penalty ρ

combines advantages of 'dual ascend' and 'method of multipliers'

ADMM

$$L_\rho(\mathbf{m},\mathbf{z},\mathbf{v}) = f(\mathbf{m}) + g(\mathbf{z}) + \mathbf{v}^*(A\mathbf{m} + B\mathbf{z} - \mathbf{c}) + \frac{\rho}{2}\|A\mathbf{m} + B\mathbf{z} - \mathbf{c}\|_2^2$$

iterations:

$$\begin{aligned}
\mathbf{m}^{k+1} &= \arg\min_{\mathbf{m}} L_\rho(\mathbf{m},\mathbf{z}^k,\mathbf{v}^k)\\
\mathbf{z}^{k+1} &= \arg\min_{\mathbf{z}} L_\rho(\mathbf{m}^{k+1},\mathbf{z},\mathbf{v}^k)\\
\mathbf{v}^{k+1} &= \mathbf{v}^k + \rho(A\mathbf{m}^{k+1} + B\mathbf{z}^{k+1} - \mathbf{c})
\end{aligned}$$

ADMM

$$L_\rho(\mathbf{m},\mathbf{z},\mathbf{v}) = f(\mathbf{m}) + g(\mathbf{z}) + \mathbf{v}^*(A\mathbf{m} + B\mathbf{z} - \mathbf{c}) + \frac{\rho}{2}\|A\mathbf{m} + B\mathbf{z} - \mathbf{c}\|_2^2$$

iterations in scaled form ($\mathbf{u} = \mathbf{v}/\rho$) and after some rewriting:

$$\begin{aligned}
\mathbf{m}^{k+1} &= \arg\min_{\mathbf{m}} \left(f(\mathbf{m}) + \frac{\rho}{2}\|A\mathbf{m} + B\mathbf{z}^k - \mathbf{c} + \mathbf{u}^k\|_2^2 \right) \\
\mathbf{z}^{k+1} &= \arg\min_{\mathbf{z}} \left(g(\mathbf{z}) + \frac{\rho}{2}\|A\mathbf{m}^{k+1} + B\mathbf{z} - \mathbf{c} + \mathbf{u}^k\|_2^2 \right) \\
\mathbf{u}^{k+1} &= \mathbf{u}^k + A\mathbf{m}^{k+1} + B\mathbf{z}^{k+1} - \mathbf{c})
\end{aligned}$$

ADMM applied to Projected-Newton subproblem:

$$\mathbf{y}^k = \mathbf{m}^k - (B^k)^{-1}\nabla_{\mathbf{m}} f(\mathbf{m}^k)$$ (unconstrained Newton-step)

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m}\in\mathcal{C}_1\bigcap\mathcal{C}_2} \frac{1}{2}\|\mathbf{y}^k - \mathbf{m}\|^2_{B^k}$$ (projection w.r.t. metric induced by the approximate Hessian)

$$= \arg\min_{\mathbf{m}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k(\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{m})$$

$$= \arg\min_{\mathbf{m},\mathbf{z}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k(\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{m} = \mathbf{z}$$

ADMM applied to Projected-Newton subproblem:

ADMM solves:

$$\min_{\mathbf{m},\mathbf{z}} f(\mathbf{m}) + g(\mathbf{z}) \quad \text{s.t.} \quad A\mathbf{m} + B\mathbf{z} = \mathbf{c}$$

projected-Newton subproblem (reformulated):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m},\mathbf{z}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k (\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{m} = \mathbf{z}$$

identify: $A = I, B = -I, \mathbf{c} = 0$

$$f(\mathbf{m}) = \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k (\mathbf{y}^k - \mathbf{m})$$

$$g(\mathbf{z}) = \iota_{\mathcal{C}}(\mathbf{z})$$

ADMM applied to Projected-Newton subproblem:

Projected-Newton subproblem (reformulated):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m},\mathbf{z}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k (\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{m} = \mathbf{z}$$

Iterations in scaled form (general):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m}} \left(f(\mathbf{m}) + \frac{\rho}{2} \| A\mathbf{m} + B\mathbf{z}^k - \mathbf{c} + \mathbf{u}^k \|_2^2 \right)$$

$$\mathbf{z}^{k+1} = \arg\min_{\mathbf{z}} \left(g(\mathbf{z}) + \frac{\rho}{2} \| A\mathbf{m}^{k+1} + B\mathbf{z} - \mathbf{c} + \mathbf{u}^k \|_2^2 \right)$$

$$\mathbf{u}^{k+1} = \mathbf{u}^k + A\mathbf{m}^{k+1} + B\mathbf{z}^{k+1} - \mathbf{c})$$

ADMM applied to Projected-Newton subproblem:

Projected-Newton subproblem (reformulated):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m},\mathbf{z}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k (\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{m} = \mathbf{z}$$

Iterations in scaled form (problem specific):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m}} \left(f(\mathbf{m}) + \frac{\rho}{2} \|\mathbf{m} - \mathbf{z}^k + \mathbf{u}^k\|_2^2 \right)$$

$$\mathbf{z}^{k+1} = \arg\min_{\mathbf{z}} \left(g(\mathbf{z}) + \frac{\rho}{2} \|\mathbf{m}^{k+1} - \mathbf{z} + \mathbf{u}^k\|_2^2 \right)$$

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \mathbf{m}^{k+1} - \mathbf{z}^{k+1}$$

ADMM applied to Projected-Newton subproblem:

Projected-Newton subproblem (reformulated):

$$\mathbf{m}^{k+1} = \arg\min_{\mathbf{m},\mathbf{z}} \frac{1}{2}(\mathbf{y}^k - \mathbf{m})^* B^k (\mathbf{y}^k - \mathbf{m}) + \iota_{\mathcal{C}}(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{m} = \mathbf{z}$$

Iterations in scaled form (problem specific):

$$\begin{aligned}
\mathbf{m}^{k+1} &= \left(B^k + \rho I\right)^{-1}\left(B^k \mathbf{y}^k - \rho(\mathbf{u} - \mathbf{z})\right) \\
\mathbf{z}^{k+1} &= \mathbf{prox}_{(\rho,\mathbf{g})}(\mathbf{m}^{k+1} + \mathbf{u}^k) = \Pi_{\mathcal{C}}(\mathbf{m}^{k+1} + \mathbf{u}^k) \\
\mathbf{u}^{k+1} &= \mathbf{u}^k + \mathbf{m}^{k+1} - \mathbf{z}^{k+1}
\end{aligned}$$