AMP Medical & Seismic

Sparsity Promoting Formulations and Algorithms for FWI

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Full Waveform Inversion

- **The Full Waveform Inversion (FWI) problem is to estimate subsurface velocity parameters for which solutions to the corresponding Helmholtz PDE best match data from source experiments.**

$$\mathbf{H}_\omega[\mathbf{m}]\mathbf{u} = [\omega^2\mathbf{m} + \nabla^2]\mathbf{u}$$

- **Problems are very large: billions of variables and terabytes of data.**

- **FWI is typically formulated as a Nonlinear Least Squares (NLLS) problem**

Single source monochromatic:

$$\min_{\mathbf{m},\mathbf{u}} \quad \frac{1}{2}\|\mathbf{P}\mathbf{u} - \mathbf{d}_\omega\|_2^2 \quad \text{subject to} \quad \mathbf{H}_\omega[\mathbf{m}]\mathbf{u} = \mathbf{q}_\omega$$

Variable	Type	Dimension	Description
$\mathbf{m}$	$\mathbb{R}$	$n_x n_z$	Model (slowness squared)
$\mathbf{H}_\omega[\mathbf{m}]$	$\mathbb{C}$	$n_x n_z \times n_x n_z$	Discrete Helmholz with boundary
$\mathbf{P}$	$\mathbb{R}$	$n_r \times n_x n_z$	Sampling operator
$\mathbf{d}_\omega$	$\mathbb{C}$	n_r	Data vector
$\mathbf{q}_\omega$	$\mathbb{C}$	$n_x n_z$	Source
$\mathbf{u}$	$\mathbb{C}$	$n_x n_z$	Wavefield

$$\min_{\mathbf{m}} \quad \phi(\mathbf{m}) := \frac{1}{2}\|\mathbf{P}\mathbf{H}_\omega[\mathbf{m}]^{-\mathbf{1}}\mathbf{q}_\omega - \mathbf{d}_\omega\|_2^2$$

Evaluating the gradient: just PDE solves

- Adjoint formulation using the Lagrangian

$$\mathcal{L}(\mathbf{v},\mathbf{u},\mathbf{m}) := \frac{1}{2}\|\mathbf{P}\mathbf{u}-\mathbf{d}\|_2^2 + \mathbf{v}^*(\mathbf{H}[\mathbf{m}]\mathbf{u}-\mathbf{q})$$

- Gradient of the Lagrangian:

$$\begin{aligned}
\partial_{\mathbf{v}}\mathcal{L} &= \mathbf{H}[\mathbf{m}]\mathbf{u}-\mathbf{q} \\
\partial_{\mathbf{u}}\mathcal{L} &= \mathbf{P}^T(\mathbf{P}\mathbf{u}-\mathbf{d}) + \mathbf{H}[\mathbf{m}]^*\mathbf{v} \\
\partial_{\mathbf{m}_i}\mathcal{L} &= \mathbf{v}^*\frac{\partial \mathbf{H}[\mathbf{m}]}{\partial \mathbf{m}_i}\mathbf{u}
\end{aligned}$$

- Evaluate last term at particular $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$

$$\begin{aligned}
\bar{\mathbf{u}} &= \mathbf{H}[\mathbf{m}]^{-1}\mathbf{q} \\
\bar{\mathbf{v}} &= -\mathbf{H}[\mathbf{m}]^{-*}\mathbf{P}^T(\mathbf{P}\bar{\mathbf{u}}-\mathbf{d})
\end{aligned}$$

Multi-source, single-frequency FWI

$$\min_{\mathbf{m},\mathbf{U}} \quad \frac{1}{2}\|\mathcal{P}_f(\mathbf{U}) - \mathbf{D}_{\boldsymbol{\omega}}\|_F^2 \quad \text{subject to} \quad \mathbf{H}_{\boldsymbol{\omega}}[\mathbf{m}]\mathbf{U} = \mathbf{Q}_{\boldsymbol{\omega}}$$

Variable	Type	Dimension	Description
$\mathbf{H}_{\boldsymbol{\omega}}[\mathbf{m}]$	$\mathbb{C}$	$n_x n_z \times n_x n_z$	Discrete Helmholz with boundary for ω
$\mathbf{D}_{\boldsymbol{\omega}}$	$\mathbb{C}$	$n_r \times n_s$	Data vector for ω
$\mathcal{P}_f$	$\mathbb{R}$	$n_x n_z \times n_s \to n_r \times n_s$	Sampling operator
$\mathbf{Q}_{\boldsymbol{\omega}}$	$\mathbb{C}$	$n_x n_z \times n_s$	Source for frequency ω
$\mathbf{U}_{\boldsymbol{\omega}}$	$\mathbb{C}$	$n_x n_z \times n_s$	Wavefield for frequency ω

$$\min_{\mathbf{m}} \quad \phi(\mathbf{m}) := \frac{1}{2}\|\mathcal{P}_f(\mathbf{H}_{\boldsymbol{\omega}}[\mathbf{m}]^{-1}\mathbf{Q}_{\boldsymbol{\omega}}) - \mathbf{D}_{\boldsymbol{\omega}}\|_F^2$$

Multi-source, multi-frequency FWI

$$\min_{\mathbf{m},\mathbf{U}} \quad \frac{1}{2}\|\mathcal{P}(\mathbf{U}) - \mathbf{D}\|_F^2 \quad \text{subject to} \quad \mathbf{H}[\mathbf{m}]\mathbf{U} = \mathbf{Q}$$

Variable	Type	Dimension	Description
$\mathbf{H}[\mathbf{m}]$	$\mathbb{C}$	$n_f(n_x n_z \times n_x n_z)$	$\text{diag}[\mathbf{H}_{\omega_1}[\mathbf{m}], \ldots, \mathbf{H}_{\omega_{n_f}}[\mathbf{m}]]$
$\mathbf{D}$	$\mathbb{C}$	$n_f(n_r \times n_s)$	$\text{stack}[\mathbf{D}_{\omega_1}, \ldots, \mathbf{D}_{\omega_{n_f}}]$
$\mathcal{P}$	$\mathbb{R}$	$n_f(n_x n_z \times n_s) \rightarrow n_f(n_r \times n_s)$	Applies $\mathcal{P}_f$ to each frequency
$\mathbf{Q}$	$\mathbb{C}$	$n_f(n_x n_z \times n_s)$	$\text{stack}[\mathbf{Q}_{\omega_1}, \ldots, \mathbf{Q}_{\omega_{n_f}}]$
$\mathbf{U}$	$\mathbb{C}$	$n_f(n_x n_z \times n_s)$	$\text{stack}[\mathbf{U}_{\omega_1}, \ldots, \mathbf{U}_{\omega_{n_f}}]$

$$\min_{\mathbf{m}} \quad \phi(\mathbf{m}) := \frac{1}{2}\|\underbrace{\mathcal{P}(\mathbf{H}[\mathbf{m}]^{-1}\mathbf{Q})}_{\mathcal{F}[\mathbf{m},\mathbf{Q}]} - \mathbf{D}\|_F^2$$

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Difficulties with NLLS

- The size of FWI requires algorithms that reduce computation time, e.g. by working on reduced data volumes.

- In addition to size, there are problems with the NLLS formulation:

  1) Local minima (missing low frequency information, model misspecification, cycle skipping)

  2) Insufficient data (multiple models fit the same data)

  3) Inadequate data (data not in the range of modeling operator)

  4) Sensitivity - small changes in data yield large changes in the model estimate

- Here we focus on sparse formulations to address some of these problems.

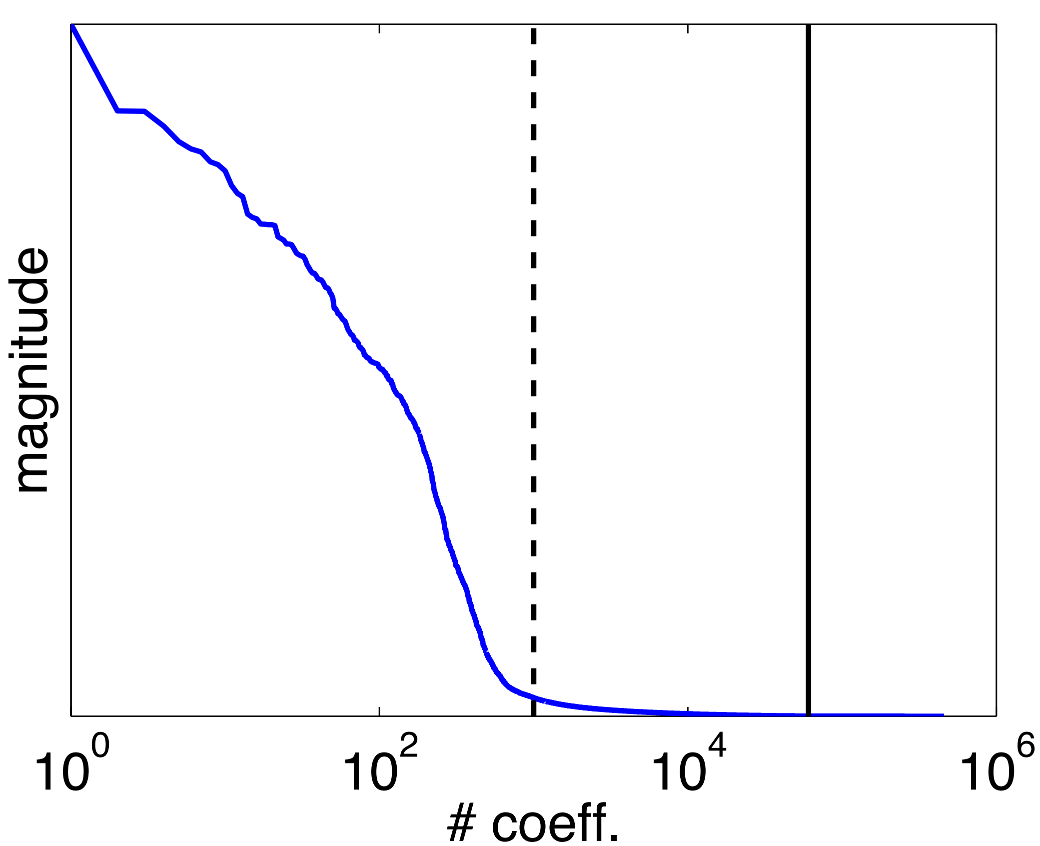
[Virieux ’09; Symes ’09; Symes ’08]

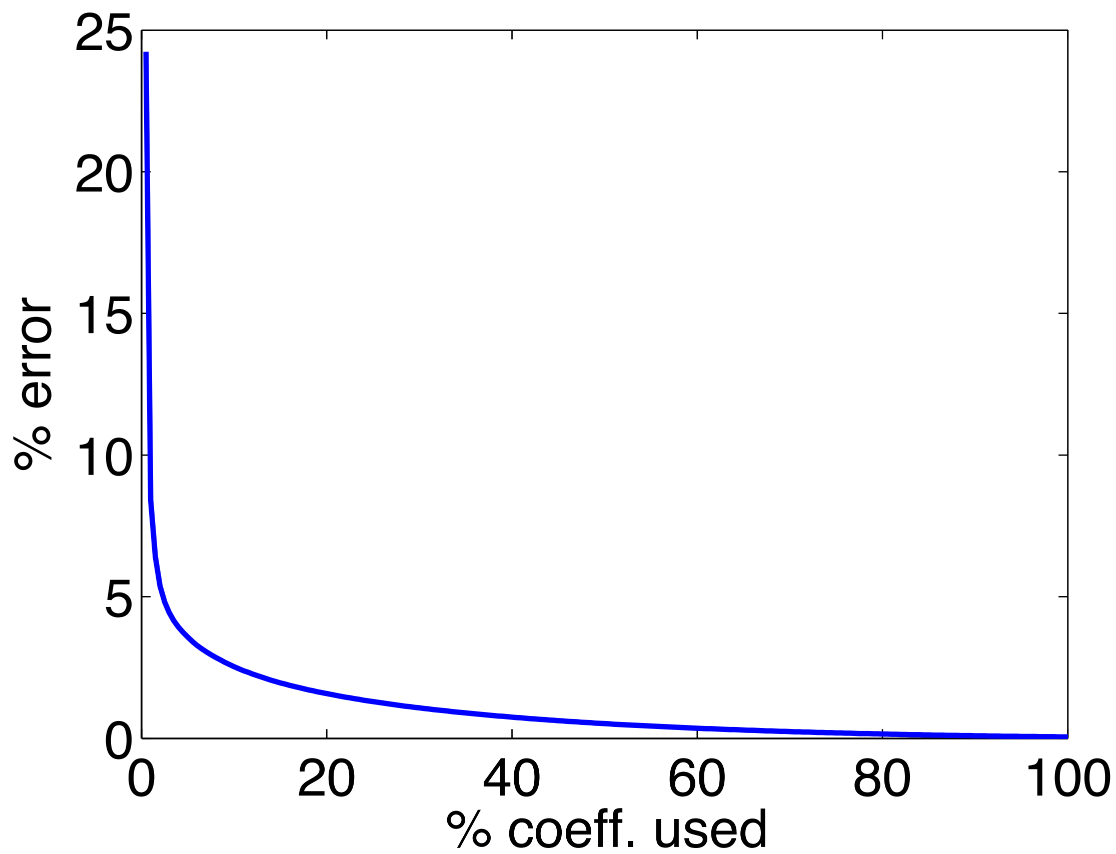
Compressibility in Curvelets

- **Velocity models are compressible in Curvelets.**

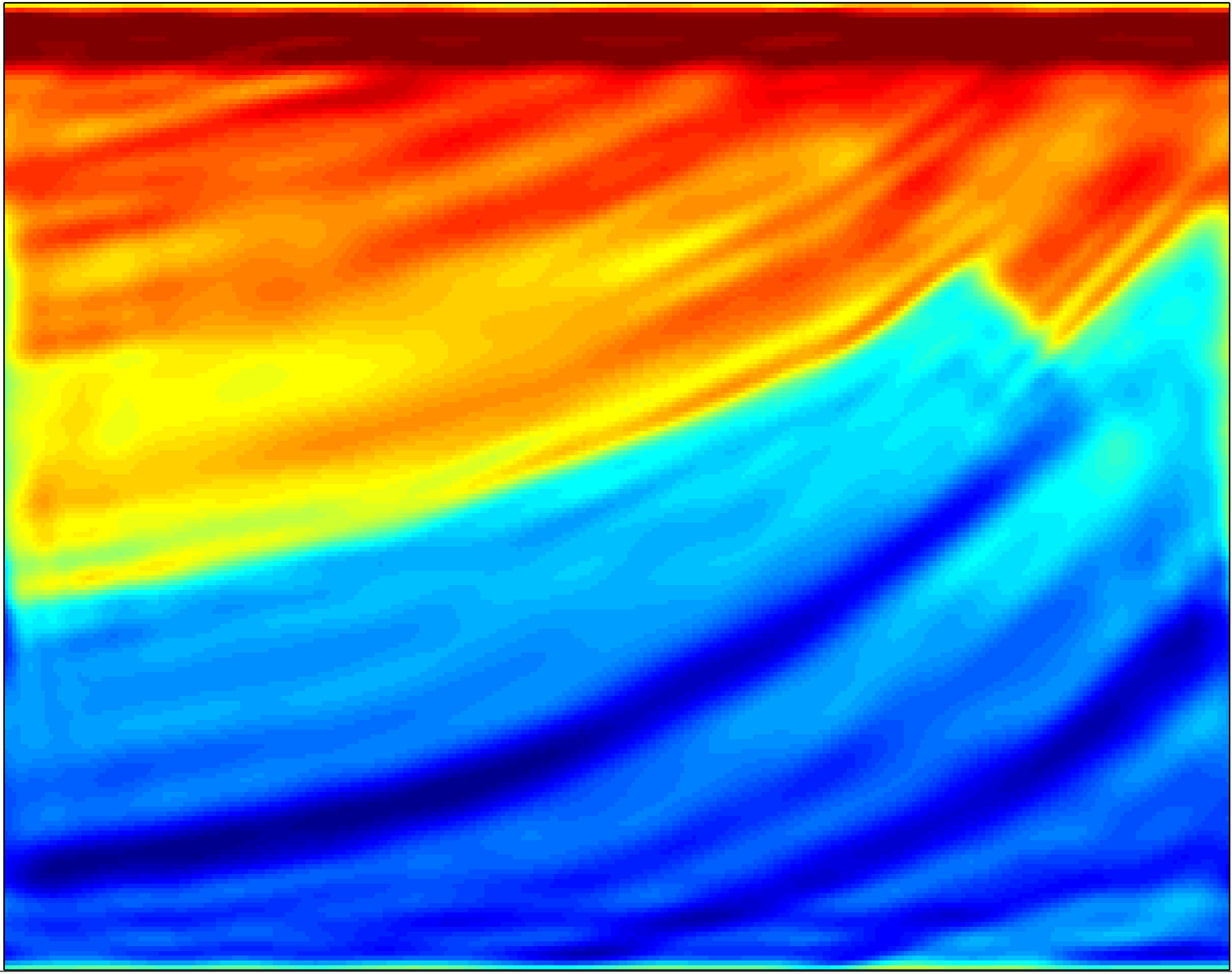
- **Geophysical images are layered, and may me modeled as objects with edges. Curvelets provide sparse representations for such images.**

[Candes ’00]

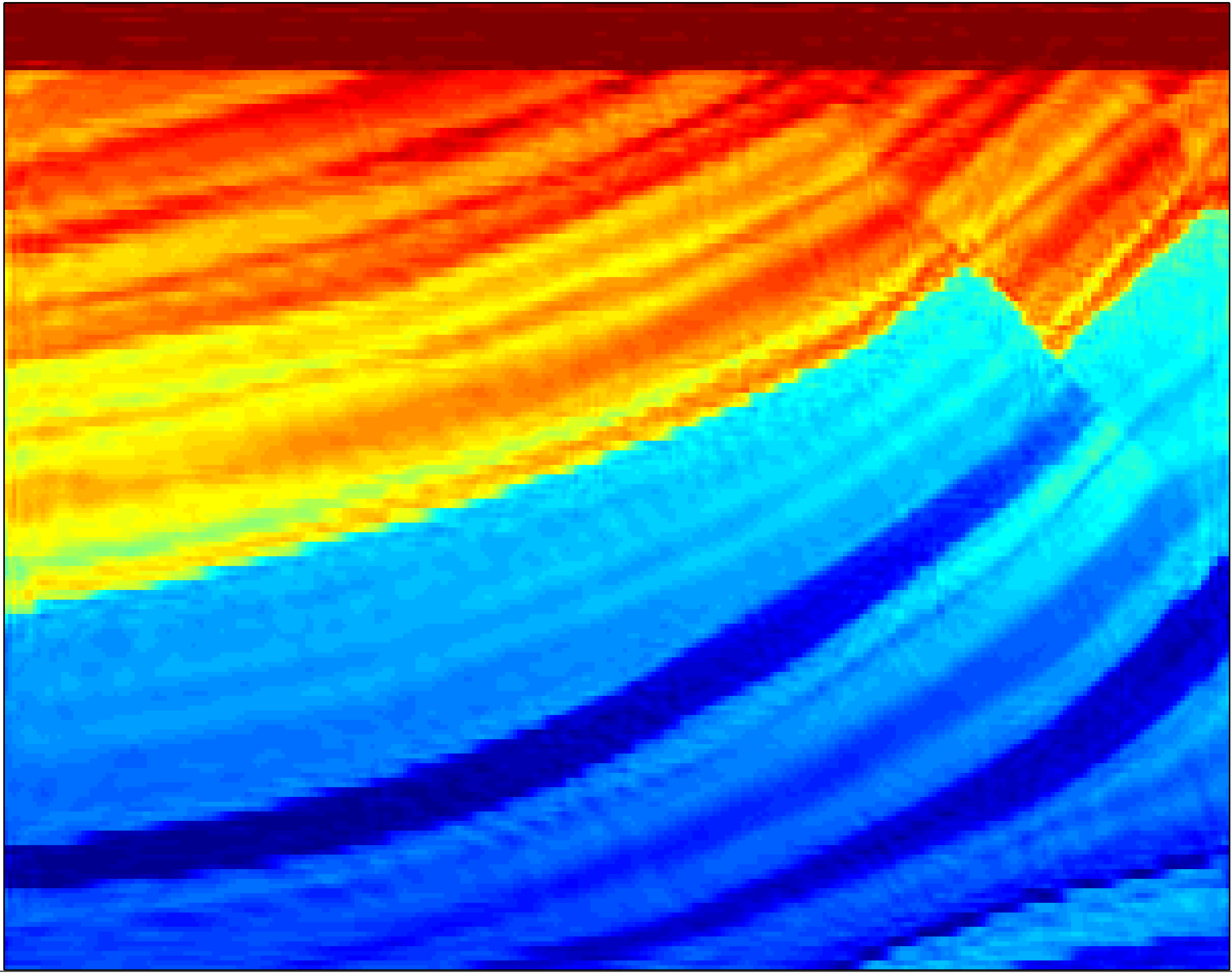




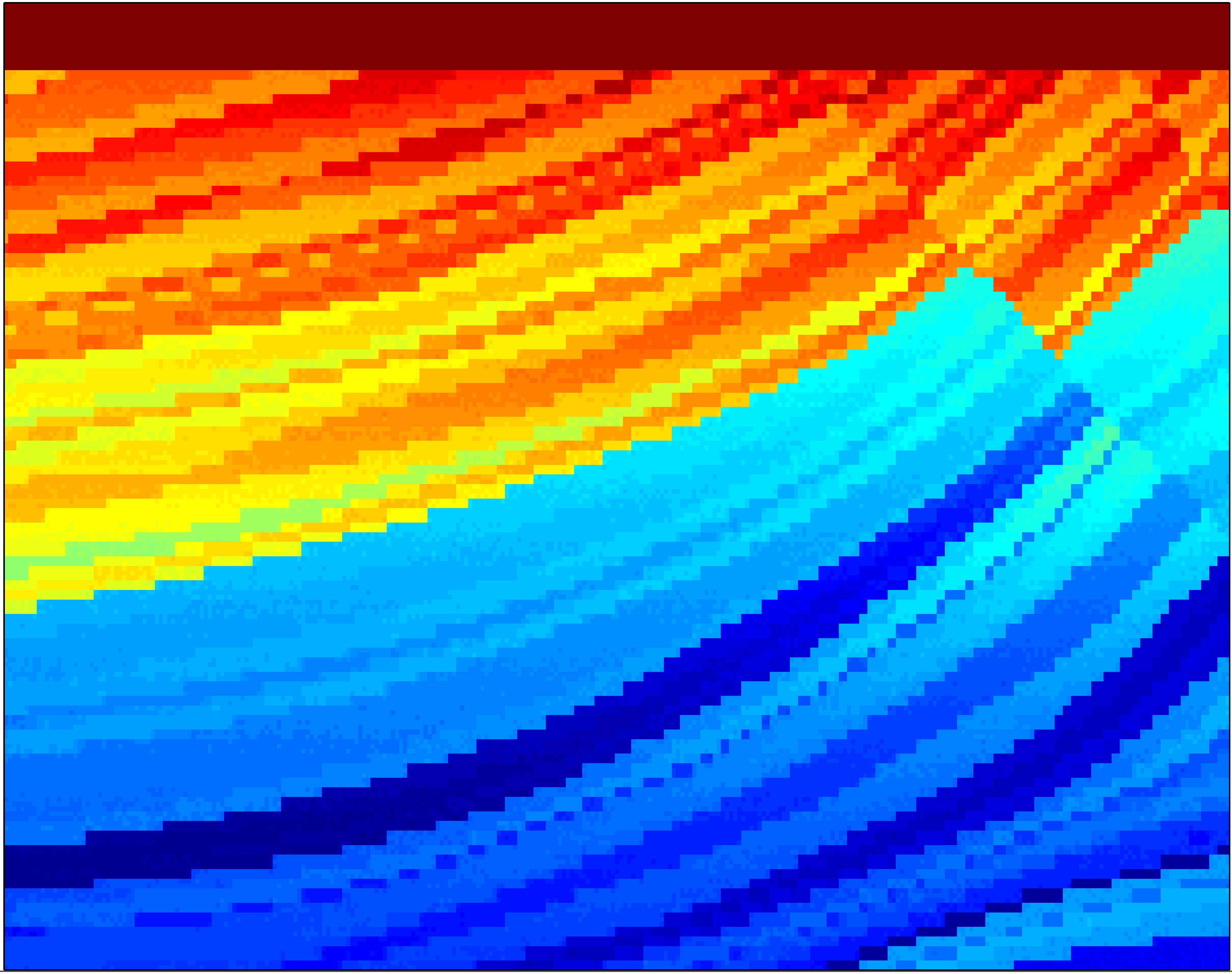
1% of coeff.



5% of coeff.



50% of coeff.



FWI: Sparsity Regularization

Sparsity-promoting formulations:

1: "QP" $$\min_{\mathbf{x}} \|\mathbf{D} - \mathcal{F}[\mathcal{C}^*\mathbf{x};\mathbf{Q}]\|_F^2 + \lambda\|\mathbf{x}\|_1$$

2: "Lasso" $$\min_{\mathbf{x}} \|\mathbf{D} - \mathcal{F}[\mathcal{C}^*\mathbf{x};\mathbf{Q}]\|_F^2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

3: "BPDN" $$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \|\mathbf{D} - \mathcal{F}[\mathcal{C}^*\mathbf{x};\mathbf{Q}]\|_F^2 \leq \sigma$$

BPDN formulation looks promising from a scientific standpoint, but Lasso formulation is easier to optimize.

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Algorithms I

For now we focus on the nonlinear LASSO formulation:

$$\min_{\mathbf{x}} \|\mathbf{D} - \mathcal{F}[\mathcal{C}^*\mathbf{x}; \mathbf{Q}]\|_F^2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

A Limited Memory Projected Quasi-Newton method has recently been proposed for optimization problems of the form

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in \mathbf{C}$$

[Schmidt et al. '09]

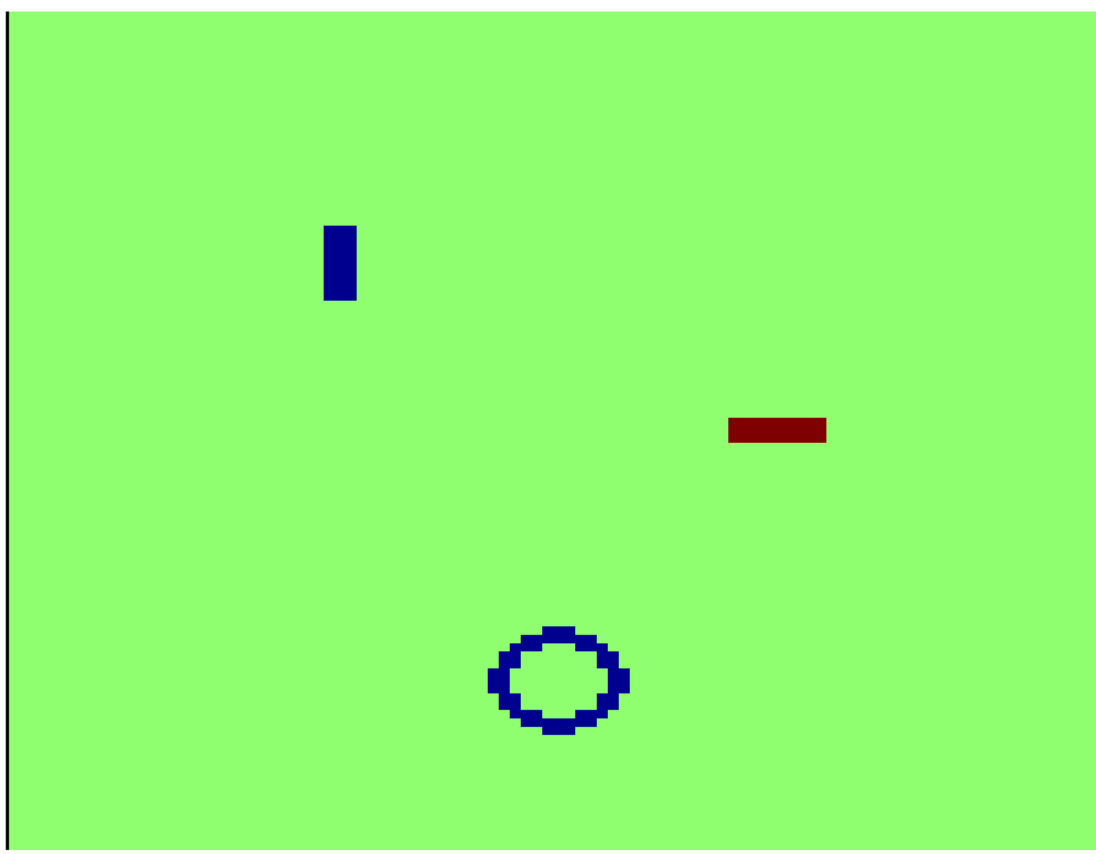
Matlab code is available from
http://www.cs.ubc.ca/~schmidtm/Software/PQN.html

Proof of Concept

- We consider a model that is sparse in physical domain: sparse perturbation of constant background velocity (2km/s)
- Cross-well setting, 101 sources and receivers in vertical wells 800 m. apart
- 9 pt. discretization of Helmholtz operator with absorbing boundary; 10 m. spacing on grid
- Sample of Frequencies [5.0, 6.0, 11.5, 14.0, 15.5, 17.5, 23.5] Hz
- We consider full inversion, and subsampling with 5 sim. shots

Geometric Setup

True Model

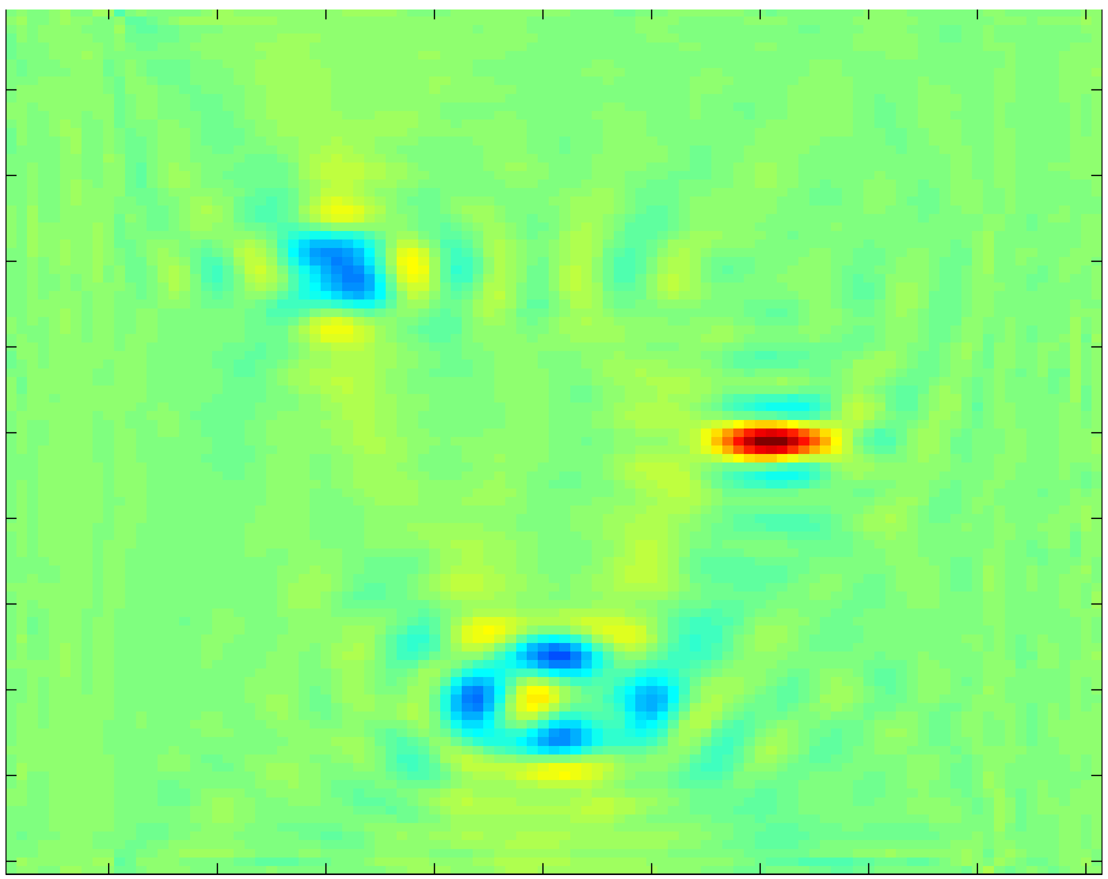


True L1-norm: 5.7

Initial Model

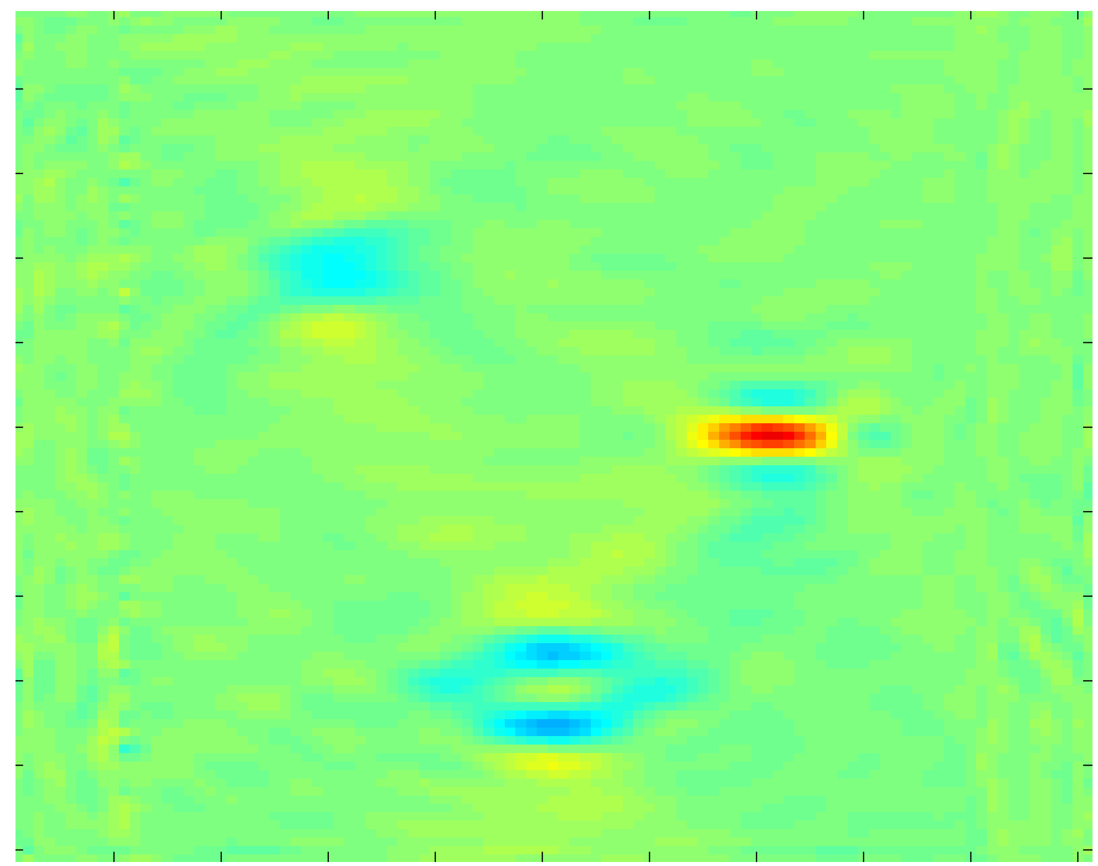
Least Squares Results:

Full model, LBFGS (500)



L1-norm: 19.2

5 Shots, LBFGS (200)



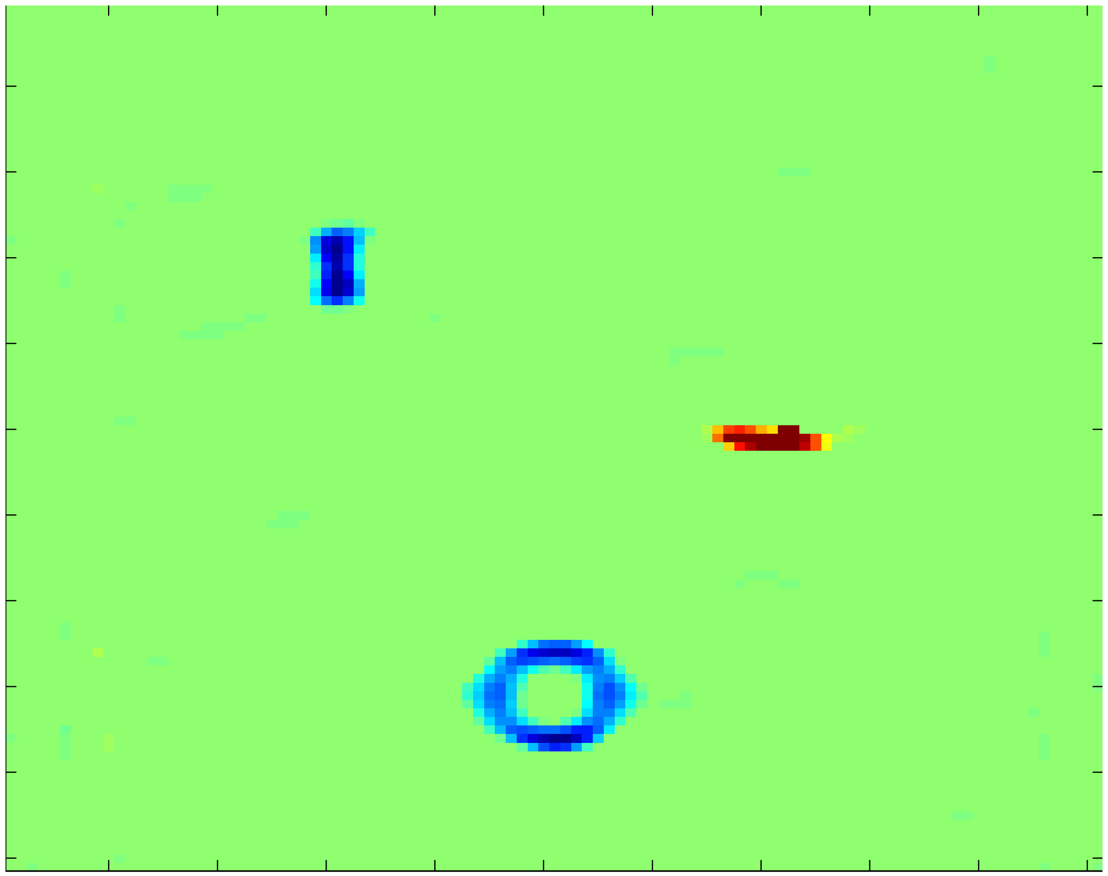
L1-norm: 22.7

Lasso Results

Lasso formulation

$$\min_{\mathbf{m}} \quad \|\mathbf{D} - \mathcal{F}[\mathbf{m_0} + \mathbf{m}; \mathbf{Q}]\|_F^2$$
$$\text{s.t.} \quad \|\mathbf{m}\|_1 \leq \tau$$

5 Shots, SPG (400)



L1-norm: 5.7

Marmoussi Example

- We consider a subset of the Marmoussi model
- 151 shots, 301 receivers
- 9 pt. discretization of Helmholtz operator with absorbing boundary; 10 m. spacing on grid
- Sample of Frequencies [5.0, 6.0, 11.5, 14.0, 15.5, 17.5, 23.5] Hz
- We consider subsampling with 5 sim. shots

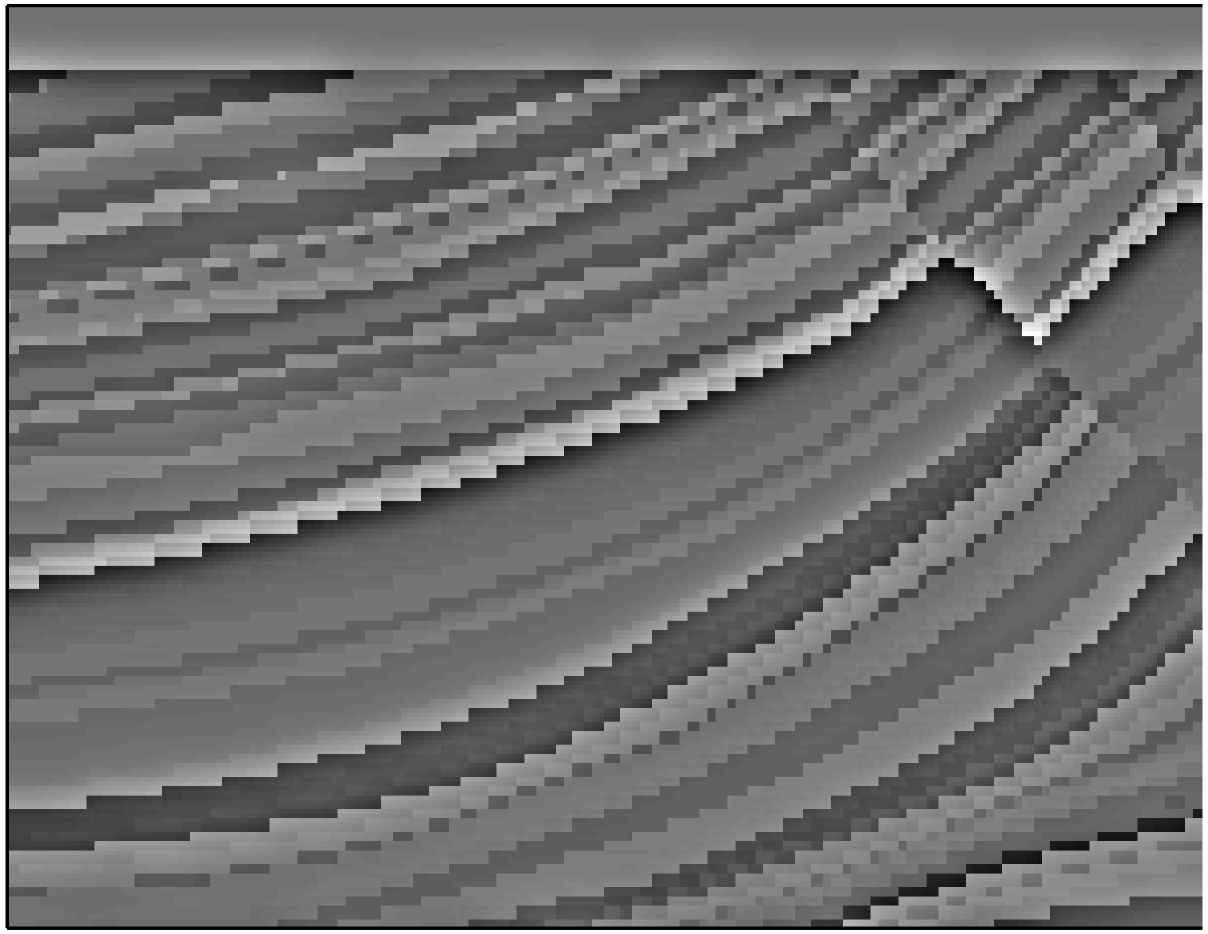
Curvelet Example

CURVELET LASSO FORMULATION

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \mathcal{F}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$

$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

TRUE REFLECTIVITY

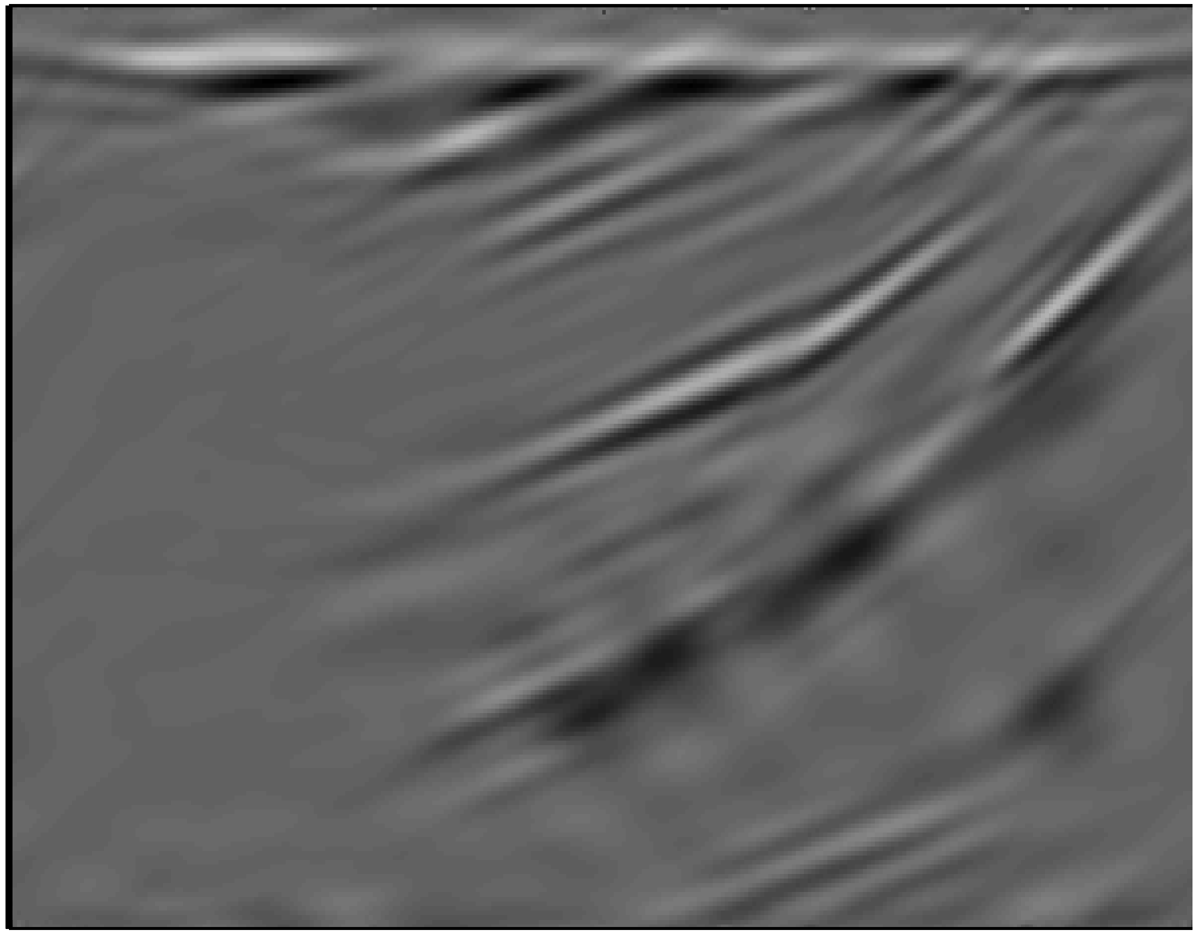


Curvelet Results

CURVELET LASSO FORMULATION

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \mathcal{F}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$
$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 30$

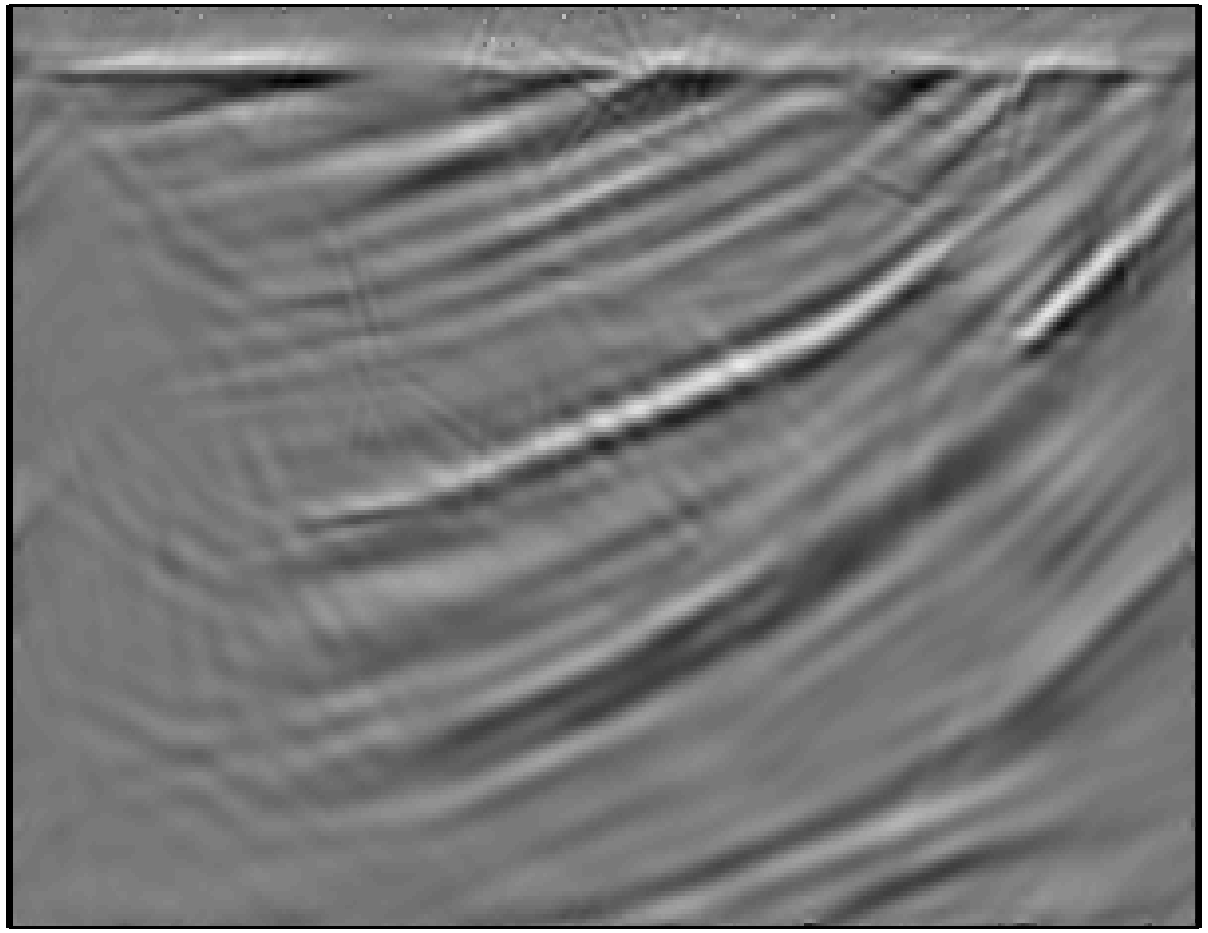


Curvelet Results

Curvelet Lasso formulation

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$
$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 100$



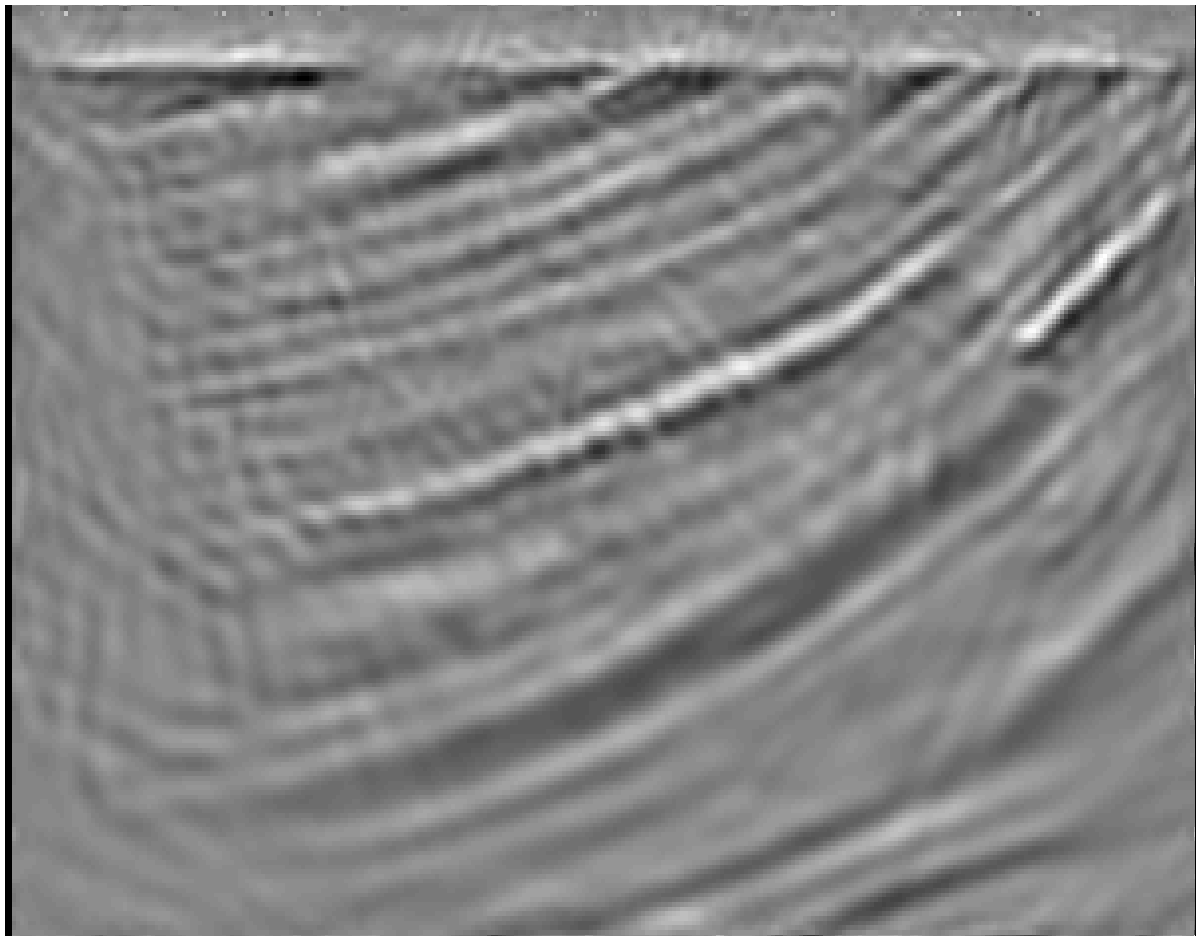
Curvelet Results

Curvelet Lasso formulation

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$

$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 170$



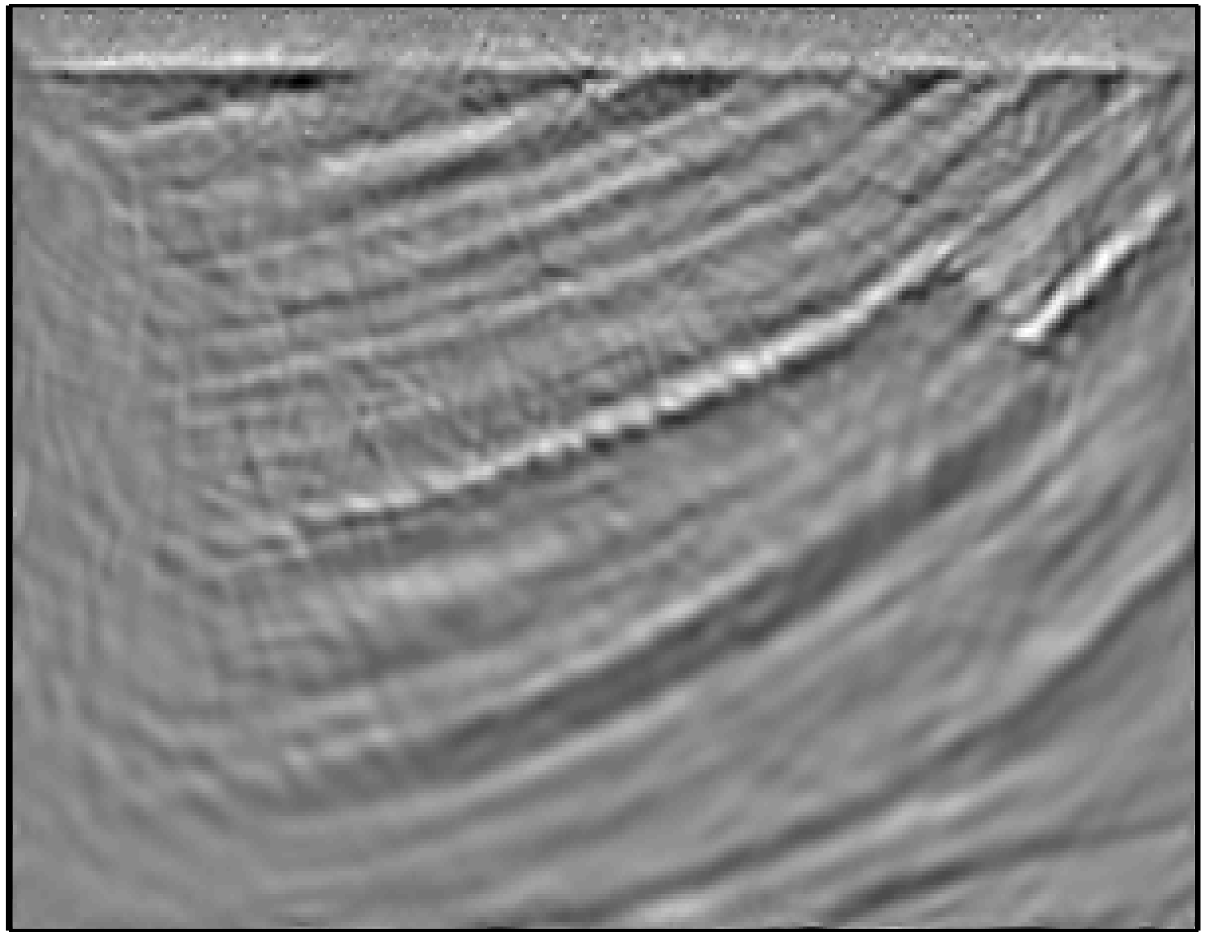
Curvelet Results

Curvelet Lasso formulation

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$

$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 250$



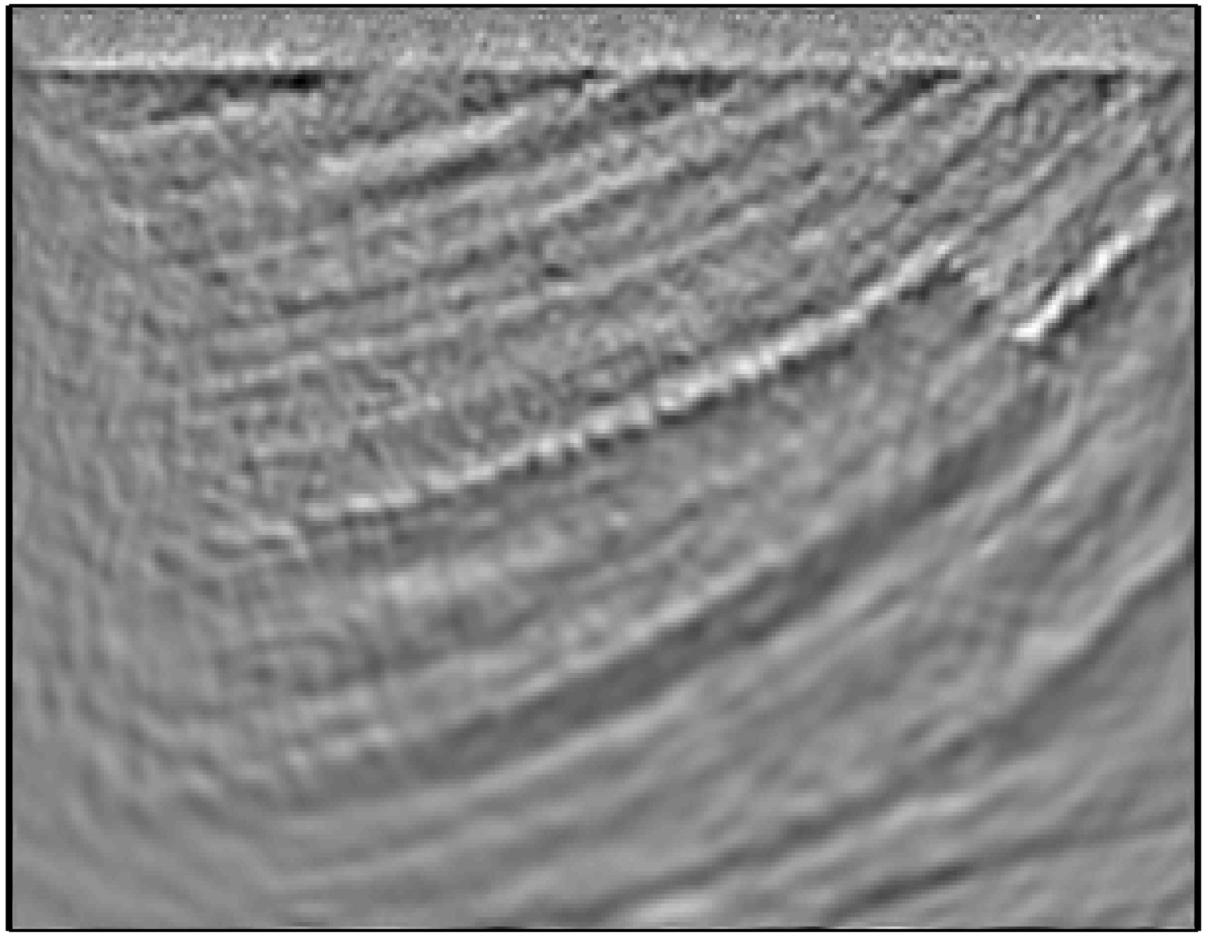
Curvelet Results

Curvelet Lasso formulation

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$

$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 400$



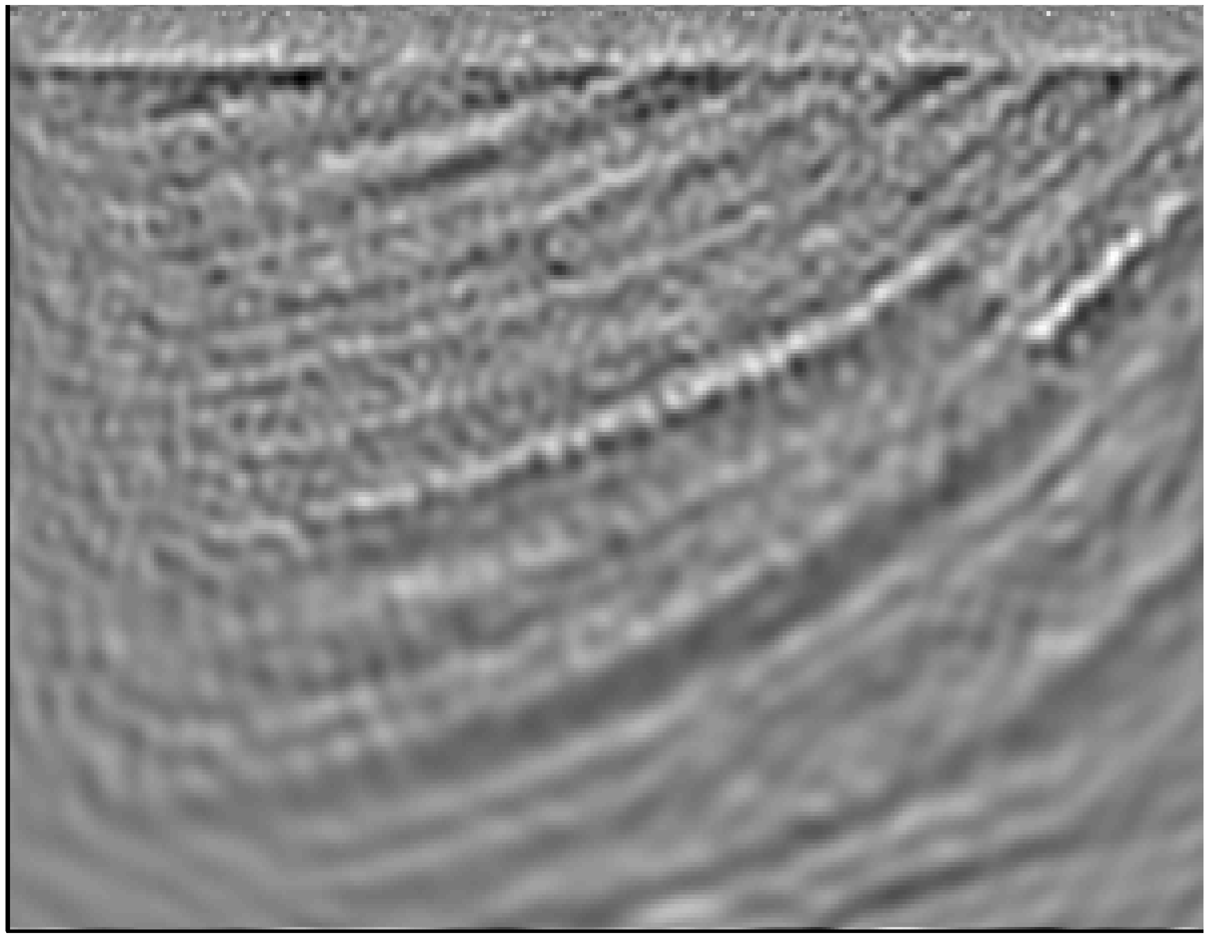
Curvelet Results

Curvelet Lasso formulation

$$\min_{\mathbf{x}} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{m_0} + \overbrace{C^*\mathbf{x}}^{\mathbf{m}}; \mathbf{Q}]\|_F^2$$

$$\text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

$\tau = 760$

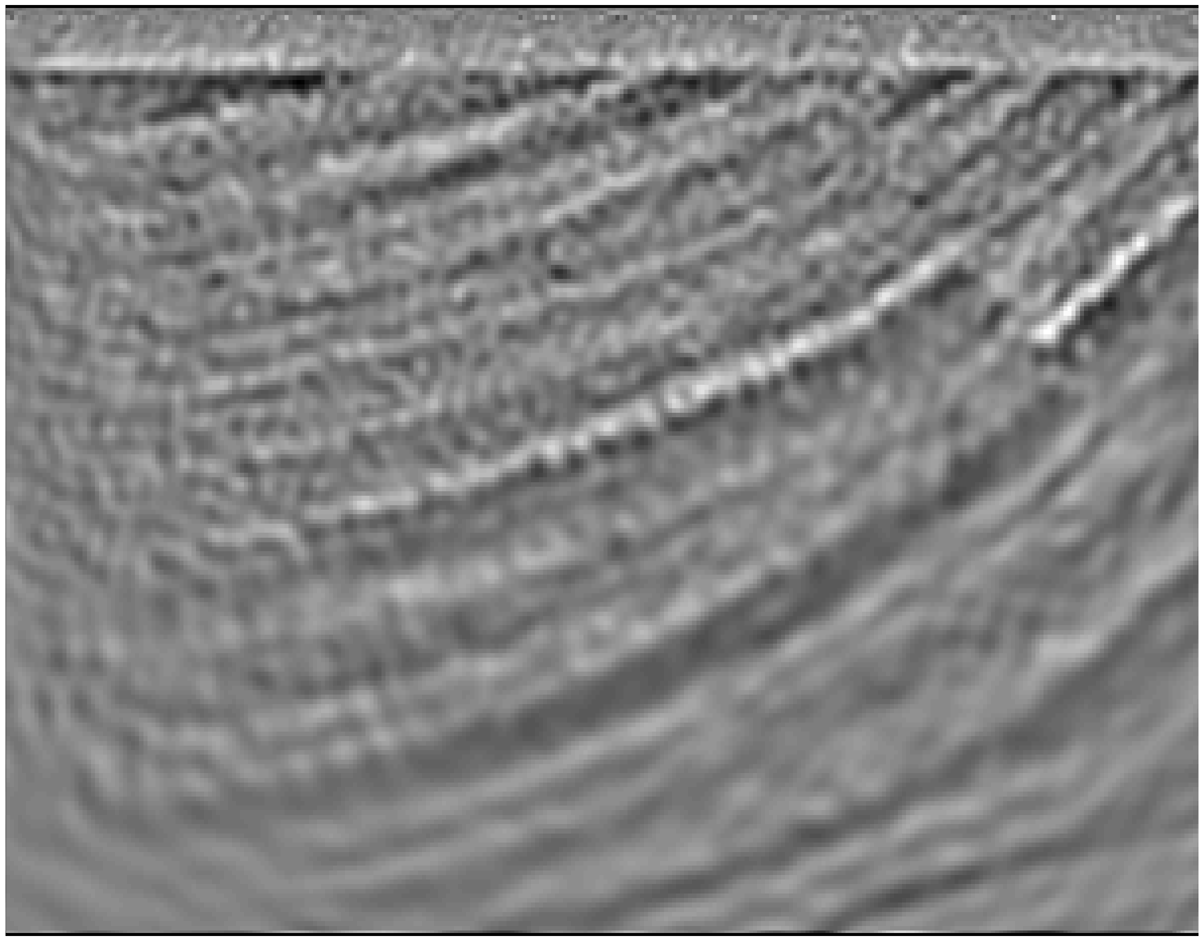


Curvelet Results

STANDARD FWI

$$\min_{\mathbf{m}} \quad \|\mathbf{D} - \mathcal{F}[\mathbf{m_0} + \mathbf{m}; \mathbf{Q}]\|_F^2$$

LBFGS



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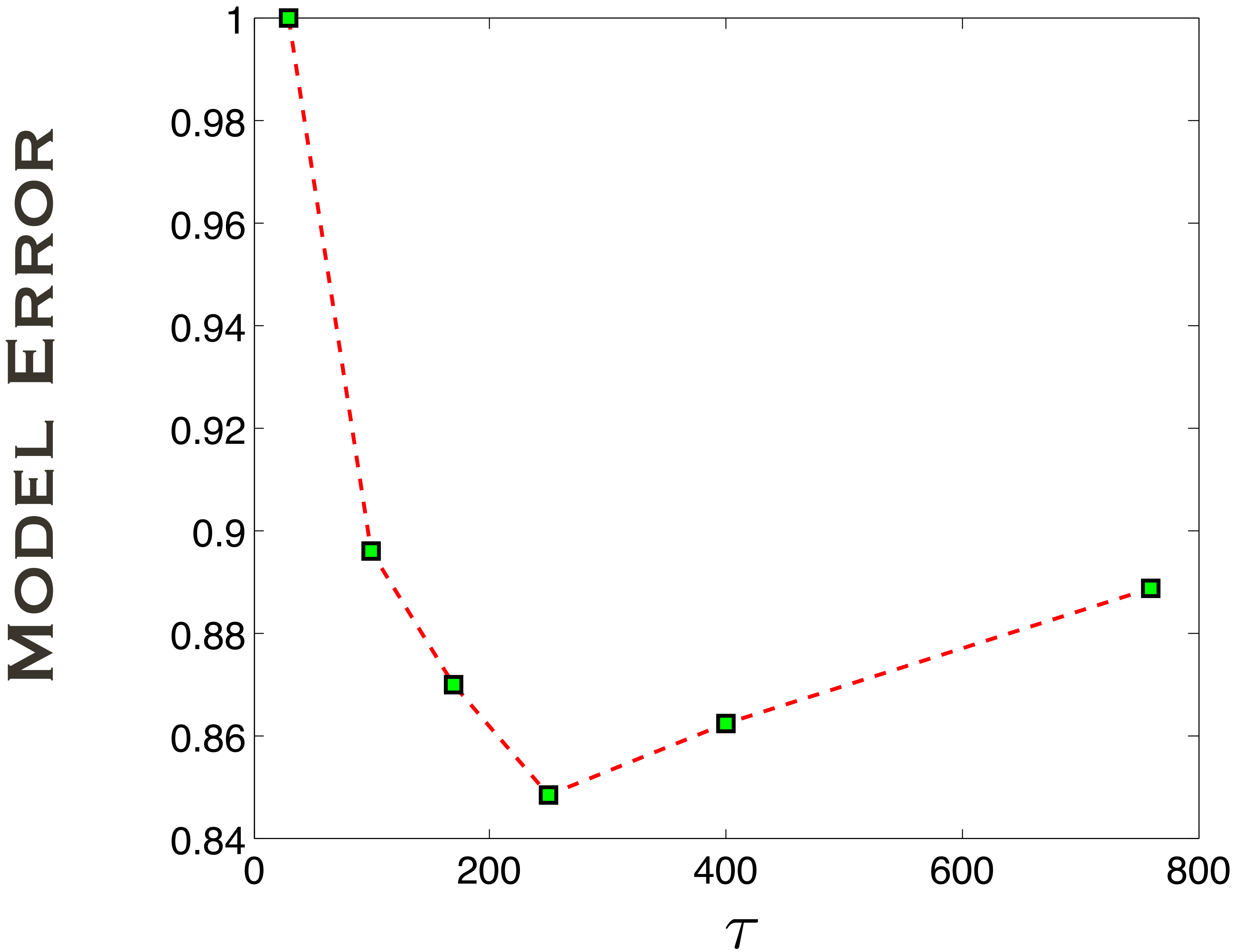
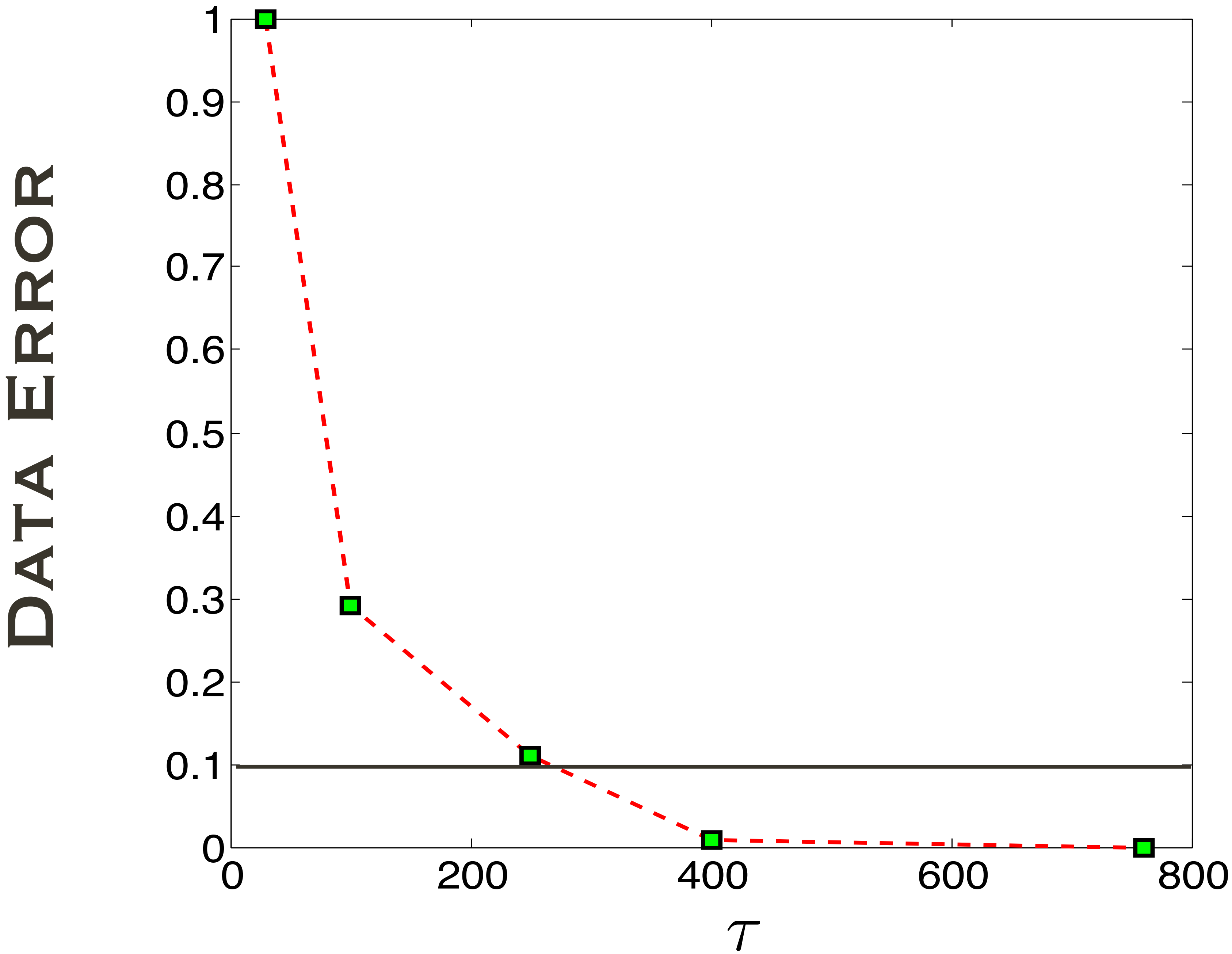
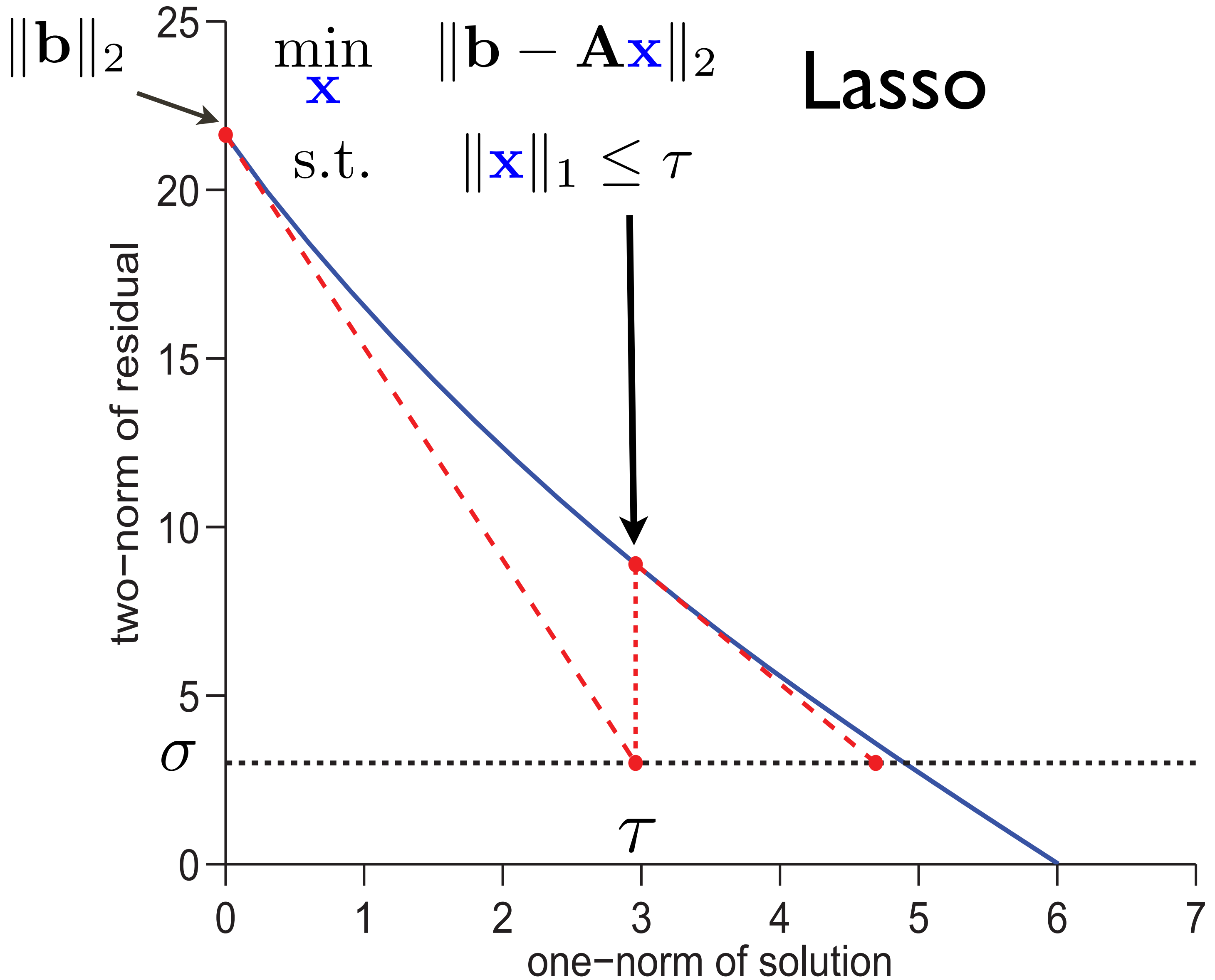
Model Error vs. Tau



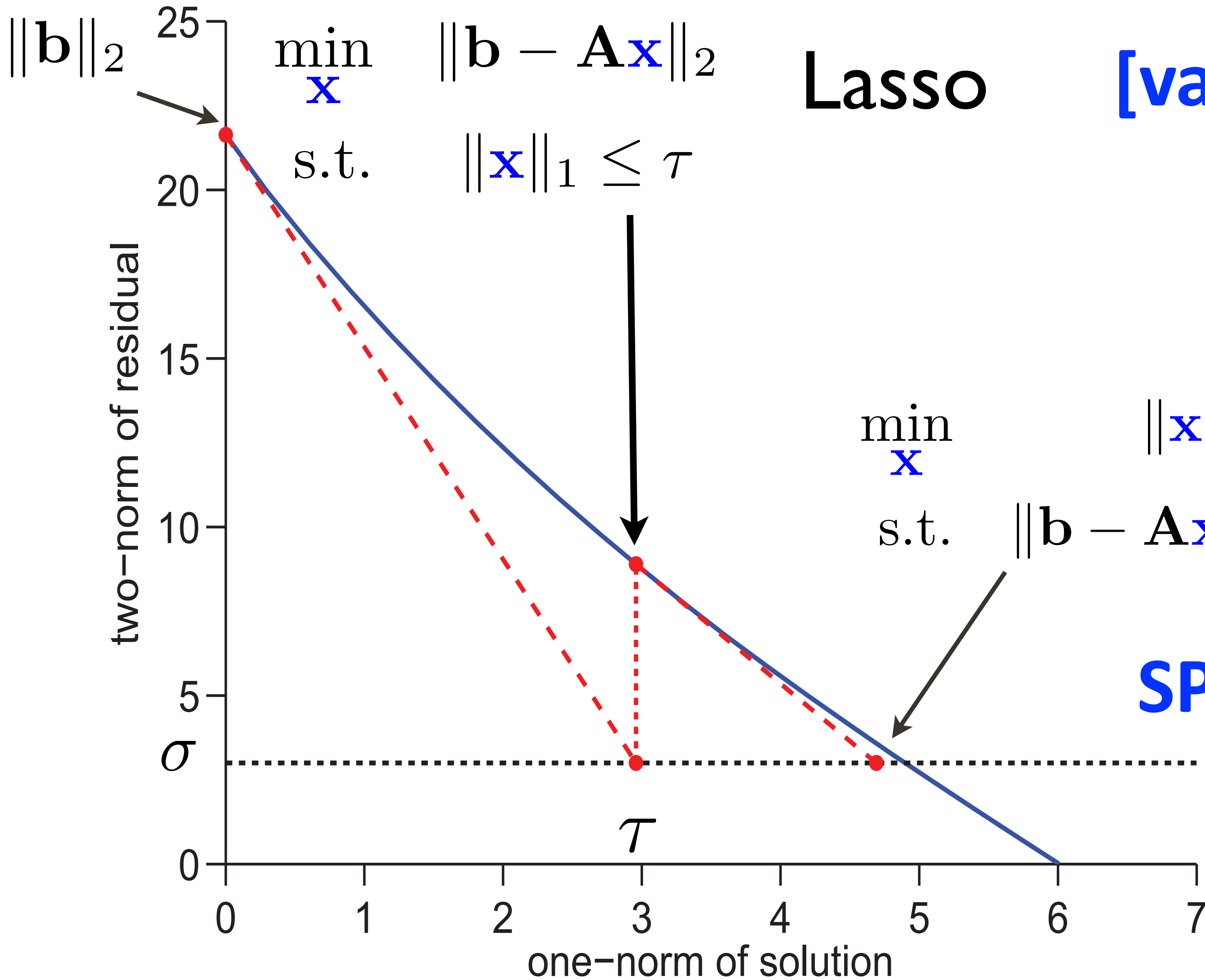
Data Error vs. Tau



Pareto Trade-Off Curve



Basis Pursuit Denoise



$$\min_{\mathbf{x}} \quad \|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau$$

Lasso

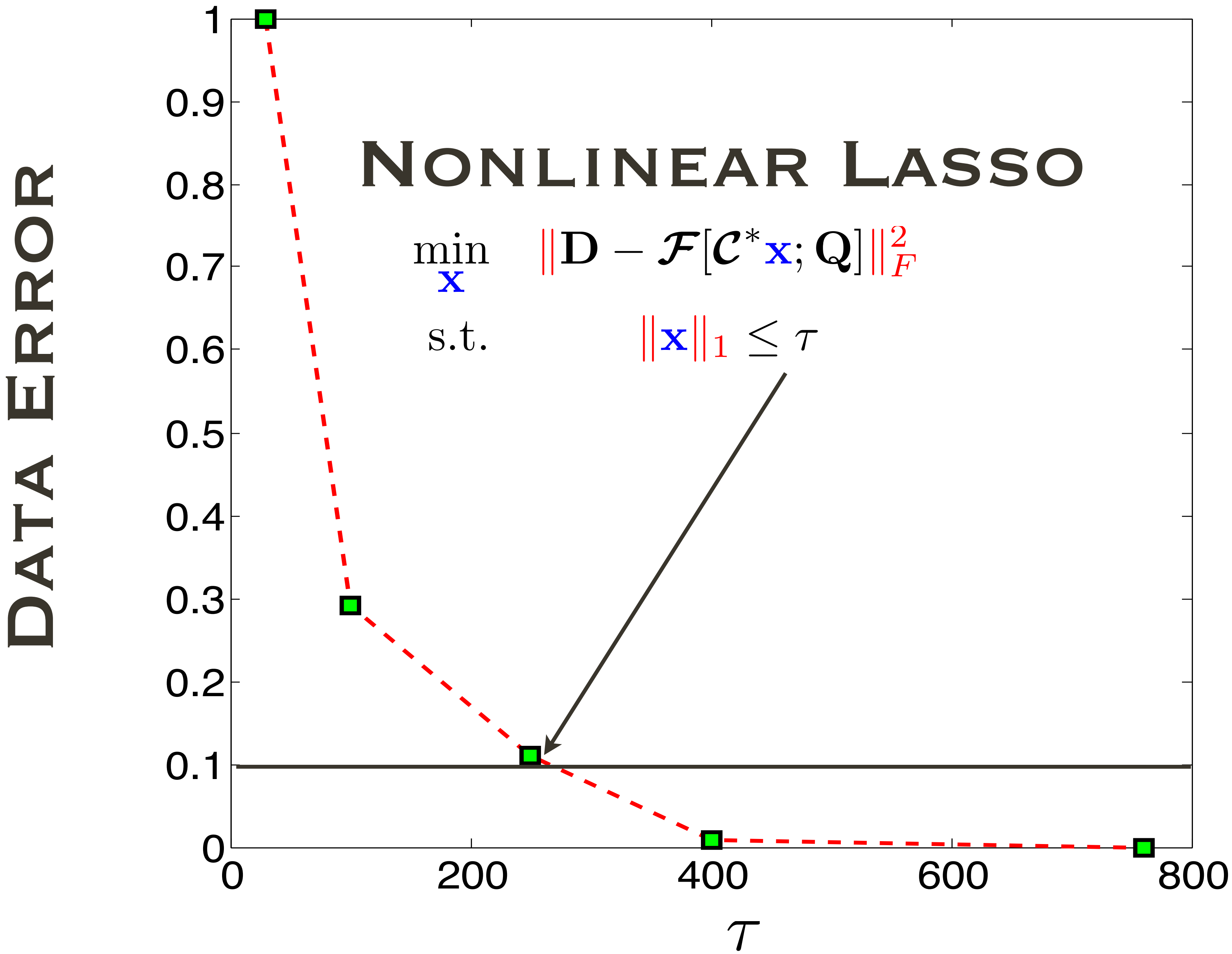
[van den Berg ’08]

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2 \leq \sigma$$

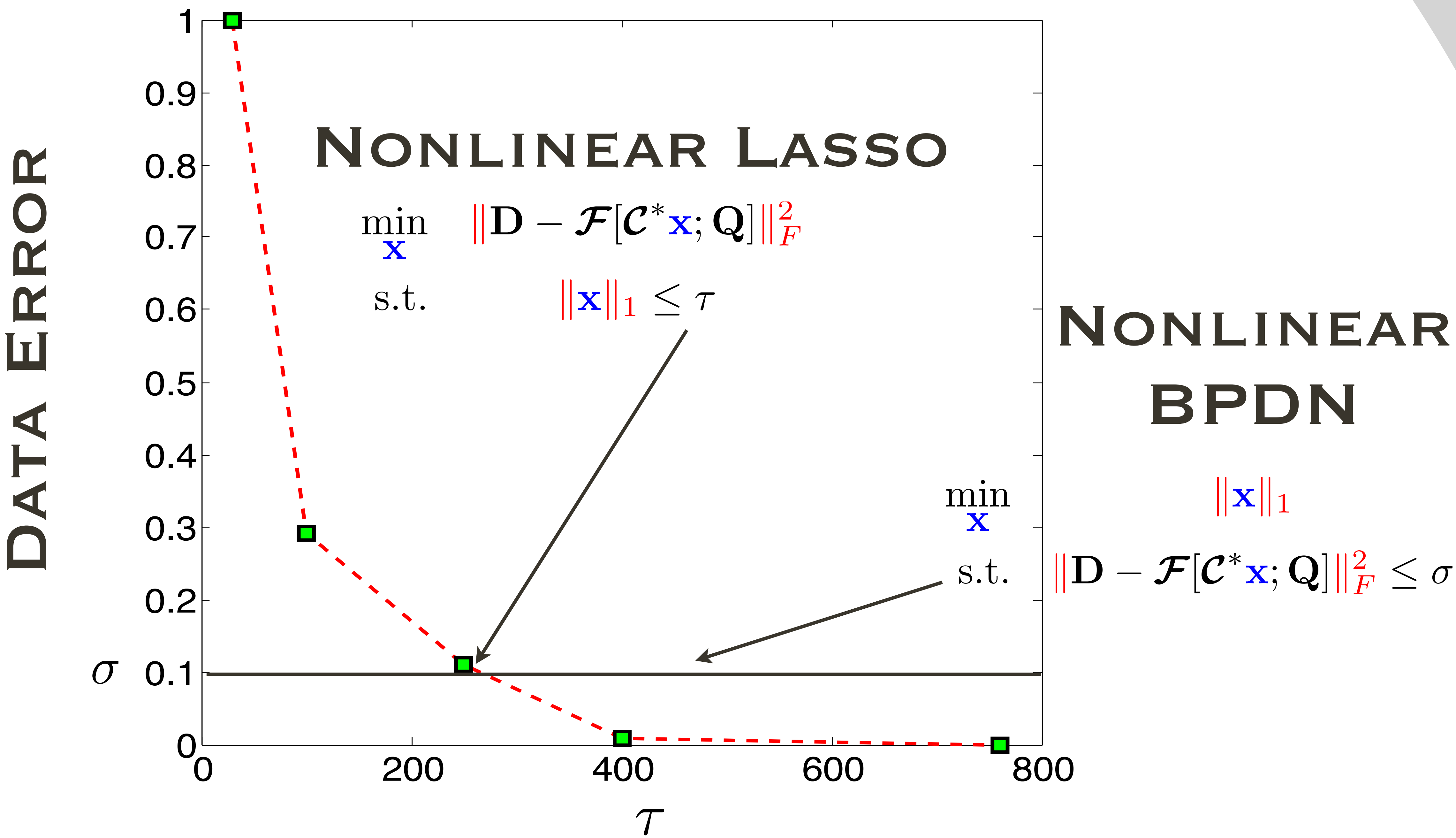
BPDN

SPGL1 Algorithm

Nonlinear Lasso



Nonlinear BPDN



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Conclusions

- Exploiting sparsity is a promising direction for modeling/regularization of FWI
- Preliminary results are promising: we can improve recovery from insufficient data with sparsity promotion.
- Understanding trade-off between NONLINEAR least-squares and model sparsity is our current focus in this work.

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