Uncertainty Quantification for Wavefield-Reconstruction Inversion

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Motivation

![Motivation Diagram]

\[ \rho(m) \]

Distribution of model

\[ \rho(d) \]

Distribution of data
## Motivation

<table>
<thead>
<tr>
<th>FWI</th>
<th>WRI</th>
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<td>Strongly nonlinear</td>
<td>Less nonlinear</td>
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<td>Dense Gauss-Newton Hessian</td>
<td>Approximately diagonal Hessian</td>
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Motivation

Gauss-Newton Hessian of FWI  Hessian of WRI

[R. Gerhard Pratt, Changsoo Shin and G.J. Hicks, 1998]
[van Leeuwen, T and Herrmann, F J, 2013]
Goals

- Set up a reasonable distribution for the model given observed data.
- Derive a practical method to calculate/estimate this distribution.
- Generate different statistical parameters of the model to quantify the uncertainty.
Full-waveform inversion

Original problem:

\[
\begin{align*}
\text{minimize} \quad & \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 \\
\text{subject to} \quad & A_{k,l}(m) u_{k,l} = q_{k,l},
\end{align*}
\]

where,

- \( u_{k,l} \) – Wavefield of the \( k \)th shot at \( l \)th frequency
- \( d_{k,l} \) – Observed data of the \( k \)th shot at \( l \)th frequency
- \( q_{k,l} \) – Source of the \( k \)th shot at \( l \)th frequency
- \( A_{k,l} \) – Helmholtz of the \( k \)th shot at \( l \)th frequency
- \( P_k \) – Receiver projection operator of the \( k \)th shot
- \( m \) – Squared-slowness
Wavefield-Reconstruction Inversion (WRI)

Joint optimization problem:

$$\min_{u,m} \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 + \lambda^2 \| A_{k,l}(m) u_{k,l} - q_{k,l} \|^2_2$$

Eliminating $u$ using variable projection:

$$\bar{u} = \arg \min_u \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 + \lambda^2 \| A_{k,l}(m) u_{k,l} - q_{k,l} \|^2_2$$
WRI vs FWI

Larger # of degrees of freedom

“more convex”
Simple test
FWI data with initial model

Receiver

Noise
True
Helm
WRI data with initial model
Statistical FWI vs WRI

Full-waveform inversion:

\[
\rho_{\text{post}}(m) \propto \exp \left( - \left\| PA(m)^{-1} q - d_{\text{obs}} \right\|^2_{\Sigma_{\text{noise}}^{-1}} - \left\| m - m_{\text{prior}} \right\|^2_{\Sigma_{\text{prior}}^{-1}} \right)
\]

Wavefield-reconstruction inversion:

\[
\rho_{\text{post}}(m, u) \propto \exp \left( - \left\| Pu - d_{\text{obs}} \right\|^2_{\Sigma_{\text{noise}}^{-1}} - \lambda^2 \left\| A(m)u - q \right\|^2_{\Sigma_{\text{pde}}^{-1}} - \left\| m - m_{\text{prior}} \right\|^2_{\Sigma_{\text{prior}}^{-1}} \right)
\]
Posterior distribution of WRI

Marginal distribution of \( m \):

\[
\rho_{\text{post}}(m) \propto \int \rho_{\text{post}}(m, u) \, du \\
= (2\pi)^{N_u/2} |\Sigma_u|^{1/2} \rho_{\text{post}}(m, \bar{u}(m))
\]

Here

\[
\begin{pmatrix}
\lambda \Sigma_{\text{pde}}^{-1/2} A \\
\Sigma_{\text{noise}}^{-1/2} P
\end{pmatrix}
\bar{u}(m) =
\begin{pmatrix}
\lambda \Sigma_{\text{pde}}^{-1/2} q \\
\Sigma_{\text{noise}}^{-1/2} d_{\text{obs}}
\end{pmatrix}
\]

\[
|\Sigma_u| = \text{det}((\lambda^2 A^T \Sigma_{\text{pde}}^{-1} A + P^T \Sigma_{\text{noise}}^{-1} P)^{-1}) \quad \text{Huge computational cost !!!}
\]
Posterior distribution of WRI

Approximate the marginal distribution:

\[ \rho_{\text{post}}(m) \propto \int \rho_{\text{post}}(m, u) du \]

\[ = (2\pi)^{N_u/2} |\Sigma_u|^{1/2} \rho_{\text{post}}(m, \bar{u}(m)) \]

\[ \approx C \rho_{\text{post}}(m, \bar{u}(m)) \]

\[ \propto \rho_{\text{post}}(m, \bar{u}(m)) \]
Marginal distribution vs Approximate distribution

Joint distribution

Marginal distribution

vs Approximate distribution
Quantify the uncertainty

Goal: Quantify the uncertainty based on the posterior distribution $\rho_{\text{post}}(\mathbf{m})$

Solution:
- Integrate the posterior distribution

Huge computational cost!!!
Quantify the uncertainty

Goal: Quantify the uncertainty based on the posterior distribution $\rho_{\text{post}}(m)$

Solution:
- Integrate the posterior distribution
- McMC method to sample the posterior distribution
Newton Type McMC: \( \tilde{\pi}_k(m) \sim \mathcal{N}(m_k - H^{-1}_k g_k, H^{-1}_k) \)

Computational cost:
1) Low rank approximation of the Hessian.
2) Number of PDE solvers \( \sim \) Number of samples.
Quantify the uncertainty

Goal: Quantify the uncertainty based on the posterior distribution $\rho_{post}(\mathbf{m})$

Solution:
- Integrate the posterior distribution
- McMC method to sample the posterior distribution
  - Advantage: the true uncertainty can be quantified
  - Disadvantage: Huge computational cost
Quantify the uncertainty

Goal: Quantify the uncertainty based on the posterior distribution $\rho_{\text{post}}(m)$

Solution:

- Integrate the posterior distribution
- McMC method to sample the posterior distribution
  - Advantage: the true uncertainty can be quantified
  - Disadvantage: Huge computational cost
- Use an approximate distribution to quantify the uncertainty
Quadratic approximation

\[ f(m) \approx f(m_{\text{MAP}}) + g^T (m - m_{\text{MAP}}) + \frac{1}{2} (m - m_{\text{MAP}})^T H (m - m_{\text{MAP}}) := \bar{f}(m) \]
Hessian of FWI

Full Hessian of FWI:

\[ H = H_{GN} + H_2 \]

Gauss-Newton Hessian of FWI:

\[ H_{GN} = G^T A^{-T} P^T P A^{-1} G \]

where

\[ G = \frac{\partial A(m) u}{\partial m} \text{ sparse} \]
Hessian of WRI:

\[
H = \lambda^2 (G^T G - G^T (I + \lambda^{-2} A^{-T} P^T P A^{-1})^{-1} G)
\]

[van Leeuwen, T and Herrmann, F J, 2013]
Quadratic approximation

\[ f(\mathbf{m}_{MAP} + \alpha \mathbf{d}_m) \]

dm – Different random directions
Quantify the uncertainty

Goal: Quantify the uncertainty based on the posterior distribution $\rho_{\text{post}}(m)$

Solution:

• Use an approximate distribution to quantify the uncertainty.
  ‣ Quantify the uncertainty by estimating the diagonal part of the inverse of the Hessian.

Sparse, no additional PDE solves are required!
Diagonal approximation vs true Hessian

Diagonal approximation

Diagonal part of the true Hessian
Diagonal approximation vs true Hessian – random realizations

Diagonal approximation

\[ H^{-1/2}_a r \]

True Hessian

\[ H^{-1/2}_t r \]

\[ r \sim \mathcal{N}(0, I) \]
Workflow
– uncertainty quantification

1. Solve the deterministic WRI problem to obtain the MAP estimate.
2. Compute the Hessian at the MAP estimate and generate the Gaussian distribution.
3. Quantify the uncertainty of the model.

No additional PDE solves needed.
Numerical experiment

Model size: 500m x 2000m
Source spacing: 80m
Receiver spacing: 20m
Fixed spread 2km
Frequency: 10-30 Hz

Standard deviation of data noise: 0.5
Standard deviation of pde: 0.5
lambda: 1
Simple model

a) Maximum a posteriori estimate

b) The standard deviation
Posterior distribution

Maximum a posterior
Posterior distribution

Distribution at $x = 1000m$, $z = 50m$

Distribution at $x = 1000m$, $z = 200m$

Distribution at $x = 1000m$, $z = 400m$
Confidence intervals

Maximum a posterior
Confidence intervals

(a) x = 500m

(b) x = 1000m

(c) x = 1700m

Velocity [km/s]

Depth [m]
Confidence intervals

Five random realizations of data:

\[ d_i = F(m_t) + \epsilon, \epsilon \sim \mathcal{N}(0, \sigma^2 I) \]

Inversion results corresponds to these data:

\[ d_i \rightarrow m_i \]
Confidence intervals

(a) x = 500m

(b) x = 1000m

(c) x = 1700m

Velocity [km/s] vs. Depth [m] for different results at various positions.
**BG model**

**Model size:** 2000m x 4500m  
**Source spacing:** 50m  
**Receiver spacing:** 10m  
**Fixed spread 4.5km**  
**Frequency:** 5~31 Hz

**Standard deviation of data noise:** 0.5  
**Standard deviation of pde:** 0.5  
**lambda:** 1
BG model

a) Maximum a posterior estimate

b) The standard deviation
Confidence intervals

MAP

Depth [m]

Lateral [m]

km/s

Maximum a posterior

Confidence intervals
Confidence intervals

(a) $x = 1000\text{m}$

(b) $x = 2500\text{m}$

(c) $x = 3500\text{m}$
Confidence intervals

(a) x = 1000m

(b) x = 2500m

(c) x = 3500m
Posterior distribution

Maximum a posterior
Posterior distribution vs Prior distribution

(a) Distribution at $x = 2500m$, $z = 200m$

(b) Distribution at $x = 2500m$, $z = 1000m$

(c) Distribution at $x = 2500m$, $z = 1800m$
Summary

• We can approximate the marginal distribution in a computationally feasible manner
  ‣ No extra PDE solves are required

• The approximate diagonal Hessian is a good approximation of the true Hessian to quantify the uncertainty.

• The results of inverting noisy data still lie in the confidence intervals
  ‣ This gives us confidence in our confidence intervals
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Thank you for your attention !!