Source estimation for wavefield-reconstruction inversion

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Motivation

Data $\equiv$ Source wavelet $\ast$ Velocity model
Motivation

Data + Correct source wavelet → Correct gradient
Motivation

Data + Wrong source wavelet → Wrong gradient
Chevron blind test data
— Wavefield-reconstruction inversion with source estimation
Full-waveform inversion

Original problem:

$$\text{minimize}_{u,m} \sum_{k,l} \left\| P_k u_{k,l} - d_{k,l} \right\|^2_2$$

subject to $A_{k,l}(m)u_{k,l} = q_{k,l}$,

where,

- $u_{k,l}$ – Wavefield of the $k$th shot at $l$th frequency
- $d_{k,l}$ – Observed data of the $k$th shot at $l$th frequency
- $q_{k,l}$ – Source of the $k$th shot at $l$th frequency
- $A_{k,l}$ – Helmholtz of the $k$th shot at $l$th frequency
- $P_k$ – Receiver projection operator of the $k$th shot
- $m$ – Squared-slowness

[Tarantola, 1984]
[J. Virieux and S. Operto, 2009]
Full-waveform inversion

Reduced/adjoint-state method:

\[
\text{minimize } \sum_{k,l} \| P_k A_{k,l}(m)^{-1} q_{k,l} - d_{k,l} \|_2^2
\]

with the gradient given by

\[
g = \sum_{k,l} u_{k,l}^* \frac{\partial A_{k,l}^*}{\partial m} v_{k,l}
\]

\[
u_{k,l} = A_{k,l}(m)^{-1} q_{k,l}
\]

\[
v_{k,l} = A_{k,l}^{-*}(m) P_k^* r_{k,l}
\]

\[
r_{k,l} = P_k A_{k,l}(m)^{-1} q_{k,l} - d_{k,l}
\]

2 PDE solves are required!
Wavefield-reconstruction inversion

Joint optimization problem:

$$\minimize_{u, m} \sum_{k,l} \|P_k u_{k,l} - d_{k,l}\|_2^2 + \lambda^2 \|A_{k,l}(m) u_{k,l} - q_{k,l}\|_2^2$$

Eliminating $u$ w/ variable projection:

$$\bar{u} = \arg \min_u \sum_{k,l} \|P_k u_{k,l} - d_{k,l}\|_2^2 + \lambda^2 \|A_{k,l}(m) u_{k,l} - q_{k,l}\|_2^2$$
Wavefield-reconstruction inversion

Corresponds to solving the following augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l} \\
\mathbf{P}_k
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_{k,l} \\
\mathbf{d}_{k,l}
\end{pmatrix}
= \begin{pmatrix}
\lambda q_{k,l} \\
\mathbf{d}_{k,l}
\end{pmatrix}
\]

with the gradient

\[
g = \sum_{k,l} \mathbf{u}_{k,l}^* \frac{\partial A_{k,l}^*}{\partial \mathbf{m}} \mathbf{v}_{k,l}
\]

\[
\mathbf{v}_{k,l} = A_{k,l}(\mathbf{m}) \mathbf{u}_{k,l} - q_{k,l}
\]

1 augmented system solves is required!
FWI vs WRI

Result FWI

Result WRI, $\lambda = 1$

[van Leeuwen, T and Herrmann, F J, 2013]
[Peters, B, Herrmann, F J and van Leeuwen, T, 2014]
WRI with source estimation

Triple parameters optimization problem:

\[
\min_{u,m,\alpha} \sum_{k,l} \left\| P_k u_{k,l} - d_{k,l} \right\|^2_2 + \lambda^2 \left\| A_{k,l}(m) u_{k,l} - \alpha_{k,l} e_{k,l} \right\|^2_2
\]
WRI with source estimation

Triple parameters optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 + \lambda^2 \| A_{k,l}(m)u_{k,l} - \alpha_{k,l}e_{k,l} \|^2_2 \\
\text{subject to} \quad & u_{k,l}, m, \alpha
\end{aligned}
$$

Eliminate $u$ and $\alpha$ jointly w/ variable projection:

$$
[u, \alpha] = \arg \min_{u, \alpha} \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 + \lambda^2 \| A_{k,l}(m)u_{k,l} - \alpha_{k,l}e_{k,l} \|^2_2
$$
WRI with source estimation

Corresponds to solving the following augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l} & -\lambda e_{k,l}
\end{pmatrix}
\begin{pmatrix}
\bar{u}_{k,l}
\end{pmatrix}
= \begin{pmatrix}
0
\end{pmatrix}
\]

Cf. original augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l}
\end{pmatrix}
\begin{pmatrix}
\bar{u}_{k,l}
\end{pmatrix}
= \begin{pmatrix}
\lambda q_{k,l}
\end{pmatrix}
\]

Full column rank!
No additional computational cost!
WRI with source estimation

\[
\begin{pmatrix}
\lambda A & -\lambda e \\
\text{P} & 0 \\
\end{pmatrix}
\begin{pmatrix}
\bar{u} \\
\bar{\alpha} \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\text{d}_{\text{obs}} \\
\end{pmatrix}
\]
Synthetic example

True Model

Initial Model
Gradient comparison

Gradient with true source wavelet

Gradient with wrong source wavelet
Gradient comparison

Gradient with true source wavelet  Gradient with estimated source wavelet
BG model

Modeling information:
- Model size: 2000m x 4500m
- Source spacing: 50m
- Receiver spacing: 10m
- Fixed spread 4.5km
- Frequency: 2~31 Hz

Inversion information:
- Optimization Solver: Gauss-Newton
- Iterations per frequency band: 21
- Batch size: 15
Source wavelet

Source Wavelet

Spectrum
Initial model

Depth [m]

Lateral [m]

km/s
First gradient comparison

Gradient with true source wavelet

Gradient with estimated source wavelet
True Model

Depth [m]

Lateral [m]

km/s
Result with true source wavelet
Result with estimated source wavelet
Relative model-error comparison

![Graph showing Relative Model Error comparison between True Source Wavelet and Estimated Source Wavelet. The x-axis represents iterations ranging from 0 to 250, and the y-axis represents the relative model error ranging from 0.84 to 1.02. The graph illustrates the reduction in error as iterations progress, with the True Source Wavelet showing a more steady decline compared to the Estimated Source Wavelet.]
Source wavelet comparison

- Phase
- Amplitude

True Source Wavelet
Estimated Source Wavelet
Initial Source Wavelet
Chevron blind test data

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Chevron blind test data
Chevron blind test data

Source wavelet

Amplitude spectrum
Chevron blind test data

Data-set information:
1. 1600 shots: ds = 25 m, Source depth = 15 m;
2. 321 hydrophone recs/shot: dr = 25 m, Receiver depth = 15 m;
3. Maximum offset = 8000 m;
4. Record time = 8.0 s, sample rate 4 ms;
5. Vp water = constant = 1510 m/s;
6. With free surface multiples present in the data;
Chevron blind test data

**Inversion strategy:**

1. Frequency domain WRI with Source estimation;
3. Batch sizes of random frequency subsets: 3, 6, 10, 10, 15;
4. Batch size of random source subsets: 300;
5. Optimization solver: l-BFGS with 20 iterations per frequency band;
6. 4 passes of WRI at frequency 3-11 Hz and 1 pass to 19 Hz;
7. Grid size: 20m for 3-11Hz and 12m for 3-19Hz;
8. No pre-processing !!!
Data comparison
— 3 Hz Data of 800th shot

![Graph showing frequency response with Receiver [km] on the x-axis and Real Part on the y-axis. The graph includes a line labeled $D_0$.](image)
Data comparison
— 3 Hz Data of 800th shot

\[ \lambda = 1e3 \]
Initial model

8 km

Depth [km]
0 1 2 3 4 5

Lateral [km]
0 10 20 30 40

1500 2000 2500 3000 3500 4000

8 km
Inversion result
Source wavelet comparison

![Graphs showing amplitude and phase comparisons between true and estimated wavelets.](image)
Model update
Kirchhoff migration
—Initial model

Depth [km]

Lateral [km]
Kirchhoff migration
—Inversion result

Lateral [km]

Depth [km]
Well-log comparison
Well-log comparison

Depth [m]
1000 1500 2000 2500

Velocity [m/s]
2200 2400 2600 2800 3000 3200 3400 3600 3800 4000

Well log
Initial
Inversion Result
Conclusions

1. Using the variable projection method, we can estimate the source wavelet for the WRI.
   • Synthetic BG model

2. Source estimation enhances the robustness of WRI for field seismic data.
   • Chevron blind test data
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