Source estimation for wavefield-reconstruction inversion

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Motivation

Data

Source wavelet

Velocity model
Motivation

Data + Correct source wavelet → Correct gradient
Motivation

Data + Wrong source wavelet → Wrong gradient
Chevron blind test data
— Wavefield-reconstruction inversion with source estimation
Full-waveform inversion

Original problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|^2_2 \\
\text{subject to} & \quad A_{k,l}(m) u_{k,l} = q_{k,l},
\end{align*}
\]

where,

- \( u_{k,l} \) – Wavefield of the \( k \)th shot at \( l \)th frequency
- \( d_{k,l} \) – Observed data of the \( k \)th shot at \( l \)th frequency
- \( q_{k,l} \) – Source of the \( k \)th shot at \( l \)th frequency
- \( A_{k,l} \) – Helmholtz of the \( k \)th shot at \( l \)th frequency
- \( P_k \) – Receiver projection operator of the \( k \)th shot
- \( m \) – Squared-slowness
Full-waveform inversion

Reduced/adjoint-state method:

$$\text{minimize } \sum_{k,l} \| P_k A_{k,l}(m)^{-1} q_{k,l} - d_{k,l} \|_2^2$$

with the gradient given by

$$g = \sum_{k,l} u_{k,l}^* \frac{\partial A_{k,l}^*}{\partial m} v_{k,l}$$

$$u_{k,l} = A_{k,l}(m)^{-1} q_{k,l}$$

$$v_{k,l} = A_{k,l}^*(m) P_k^* r_{k,l}$$

$$r_{k,l} = P_k A_{k,l}(m)^{-1} q_{k,l} - d_{k,l}$$

2 PDE solves are required!
Wavefield-reconstruction inversion

Joint optimization problem:

$$\min_{\mathbf{u}, \mathbf{m}} \sum_{k,l} \| \mathbf{P}_k \mathbf{u}_{k,l} - \mathbf{d}_{k,l} \|^2_2 + \lambda^2 \| \mathbf{A}_{k,l}(\mathbf{m}) \mathbf{u}_{k,l} - \mathbf{q}_{k,l} \|^2_2$$

Eliminating $\mathbf{u}$ w/ variable projection:

$$\bar{\mathbf{u}} = \arg \min_{\mathbf{u}} \sum_{k,l} \| \mathbf{P}_k \mathbf{u}_{k,l} - \mathbf{d}_{k,l} \|^2_2 + \lambda^2 \| \mathbf{A}_{k,l}(\mathbf{m}) \mathbf{u}_{k,l} - \mathbf{q}_{k,l} \|^2_2$$
Wavefield-reconstruction inversion

Corresponds to solving the following augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l} \\
\mathbf{P}_k
\end{pmatrix}
\begin{pmatrix}
\bar{u}_{k,l} \\
\bar{d}_{k,l}
\end{pmatrix}
= 
\begin{pmatrix}
\lambda q_{k,l} \\
\mathbf{d}_{k,l}
\end{pmatrix}
\]

with the gradient

\[
g = \sum_{k,l} \bar{u}_{k,l}^* \frac{\partial A_{k,l}}{\partial \mathbf{m}} \bar{v}_{k,l}
\]

1 augmented system solves is required!

\[
\bar{v}_{k,l} = A_{k,l}(\mathbf{m}) \bar{u}_{k,l} - q_{k,l}
\]
True & initial model

[van Leeuwen, T and Herrmann, F J, 2013]
[Peters, B, Herrmann, F J and van Leeuwen, T, 2014]
FWI vs WRI

Result FWI

Result WRI, $\lambda = 1$

[van Leeuwen, T and Herrmann, F J, 2013]
[Peters, B, Herrmann, F J and van Leeuwen, T, 2014]
WRI with source estimation

Triple parameters optimization problem:

\[
\minimize_{\mathbf{u}, \mathbf{m}, \alpha} \sum_{k,l} \| \mathbf{P}_k \mathbf{u}_{k,l} - \mathbf{d}_{k,l} \|_2^2 + \lambda^2 \| \mathbf{A}_{k,l}(\mathbf{m}) \mathbf{u}_{k,l} - \alpha_{k,l} \mathbf{e}_{k,l} \|_2^2
\]
WRI with source estimation

Triple parameters optimization problem:

\[
\text{minimize} \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|_2^2 + \lambda^2 \| A_{k,l}(m) u_{k,l} - \alpha_{k,l} e_{k,l} \|_2^2
\]

Eliminate \( u \) and \( \alpha \) jointly w/ variable projection:

\[
[\overline{u}, \overline{\alpha}] = \arg \min_{u, \alpha} \sum_{k,l} \| P_k u_{k,l} - d_{k,l} \|_2^2 + \lambda^2 \| A_{k,l}(m) u_{k,l} - \alpha_{k,l} e_{k,l} \|_2^2
\]
WRI with source estimation

Corresponds to solving the following augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l} & -\lambda e_{k,l} \\
P_{k} & 0
\end{pmatrix}
\begin{pmatrix}
\bar{u}_{k,l} \\
\bar{c}_{k,l}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
d_{k,l}
\end{pmatrix}
\]

Cf. original augmented system:

\[
\begin{pmatrix}
\lambda A_{k,l} \\
P_{k}
\end{pmatrix}
\bar{u}_{k,l}
= 
\begin{pmatrix}
\lambda q_{k,l} \\
d_{k,l}
\end{pmatrix}
\]

Full column rank!
No additional computational cost!
WRI with source estimation

\[
\begin{pmatrix}
\lambda & A \\
P & -\lambda e
\end{pmatrix}
\begin{pmatrix}
\bar{u} \\
\bar{\alpha}
\end{pmatrix}
=
\begin{pmatrix}
d_{\text{obs}} \\
0
\end{pmatrix}
\]

\( \bar{\alpha} \) and

\( \bar{u} \)
Synthetic example

True Model

Initial Model
Gradient comparison

Gradient with true source wavelet

Gradient with wrong source wavelet
Gradient comparison

Gradient with true source wavelet

Gradient with estimated source wavelet
BG model

Modeling information:
- Model size: 2000m x 4500m
- Source spacing: 50m
- Receiver spacing: 10m
- Fixed spread 4.5km
- Frequency: 2~31 Hz

Inversion information:
- Optimization Solver: Gauss-Newton
- Iterations per frequency band: 21
- Batch size: 15
Source wavelet

Source Wavelet

Spectrum
Initial model

Depth [m]

Lateral [m]

km/s
First gradient comparison

Gradient with true source wavelet

Gradient with estimated source wavelet
True Model

![Diagram representing a True Model with depth and lateral coordinates. The depth ranges from 0 to 2000 m, and the lateral distance ranges from 0 to 4000 m. The velocity scale ranges from 1.5 km/s to 4.5 km/s.]
Result with true source wavelet
Result with estimated source wavelet
Relative model-error comparison

![Graph showing relative model error comparison between True Source Wavelet and Estimated Source Wavelet over iterations. The graph plots relative model error against iteration number. The error decreases as the iterations progress, with the Estimated Source Wavelet generally closer to the True Source Wavelet than the other line.]
Source wavelet comparison

Phase

Frequency

Amplitude

-3 -2 -1 0 1 2 3 4 5

0 20 40 60 80 100 120 140 160

5 10 15 20 25 30

True Source Wavelet
Estimated Source Wavelet
Initial Source Wavelet

5 10 15 20 25 30

0 50 100 150 200

True Source Wavelet
Estimated Source Wavelet
Initial Source Wavelet

Phase (rad)

Frequency

Amplitude
Chevron blind test data

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Chevron blind test data
Chevron blind test data

Source wavelet

Amplitude spectrum
Chevron blind test data

Data-set information:

1. 1600 shots: \( ds = 25 \) m, Source depth \( = 15 \) m;
2. 321 hydrophone recs/shot: \( dr = 25 \) m, Receiver depth \( = 15 \) m;
3. Maximum offset \( = 8000 \) m;
4. Record time \( = 8.0 \) s, sample rate \( = 4 \) ms;
5. \( V_p \) water \( = \) constant \( = 1510 \) m/s;
6. With free surface multiples present in the data;
Inversion strategy:

1. Frequency domain WRI with Source estimation;
3. Batch sizes of random frequency subsets: 3, 6, 10, 10, 15;
4. Batch size of random source subsets: 300;
5. Optimization solver: l-BFGS with 20 iterations per frequency band;
6. 4 passes of WRI at frequency 3-11 Hz and 1 pass to 19 Hz;
7. Grid size: 20m for 3-11Hz and 12m for 3-19Hz;
8. No pre-processing !!!
Data comparison
— 3 Hz Data of 800th shot

![Graph showing real part of data comparison over receiver distance.]
Data comparison
— 3 Hz Data of 800th shot

\[ \lambda = 1e3 \]
Initial model
Source wavelet comparison

Amplitude

Phase

True Wavelet
Estimated Wavelet

Frequency [Hz]

Amplitude

Phase

Frequency [Hz]
Model update
Kirchhoff migration
—Initial model
Kirchhoff migration
—Inversion result
Well-log comparison
Well-log comparison

![Graph showing depth vs. velocity comparison between well log, initial result, and inversion result. The graph displays a trend in velocity over depth, with the well log in red, initial result in blue, and inversion result marked with an asterisk.](image)
Conclusions

1. Using the variable projection method, we can estimate the source wavelet for the WRI.
   - Synthetic BG model

2. Source estimation enhances the robustness of WRI for field seismic data.
   - Chevron blind test data
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