Solution of Time-Harmonic Wave Equation for Full-Waveform Inversion

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Frequency Domain Full Waveform Inversion Overview

Full Waveform Inversion - Overview

Define the misfit function as

$$\min_{m \in \mathcal{A}} \Phi(m) = \sum_{i}^{n_s} \left\| d^i - P_r u^i \right\|_W .$$

for some norm W (usually L^2 with some regularization), and the gradient of the reduced formulation

$$\nabla \Phi(m) = \sum_{i}^{n_s} \mathcal{R} \left\{ (u^i)^T \left[\frac{\partial A}{\partial m} \right]^T w^i \right\} .$$

- d^i observed data
- u^i (approximated) computed data
- m Earth parameters; what we are trying to invert!

Forward Modelling - Overview

Forward problem:

$$u^i = A^{-1}(m)q^i$$

(computation of $\Phi(m)$)

Backward problem:

$$w^i = A^{-H}(m)P_r^H(d^i - u^i).$$

(computation of $\nabla\Phi(m)$)

Both require a PDE solve, computed with an (sufficiently large) accuracy ε.

- $A(m)$ operator governing the physics of Earth
- d^i, u^i observed and computed data
- P_r restricts the computed data to the receivers
- q^i source

Frugal FWI Overview

Gradient-Descent with Errors

Let

$$\nabla\tilde{\Phi}(m_k) = \nabla\Phi(m_k) + \boldsymbol{e_k}$$

for some **error** e_k. Then, for **strongly convex** problems:

$$\Phi(m_k) - \Phi(m_*) < a_k(\Phi(m_0) - \Phi(m_*))$$

$$a_k = max\left\{c^k, \|\boldsymbol{e_k}\|_2^2\right\}, \qquad 0 \leq c \leq 1$$

where c is the condition number of the problem.

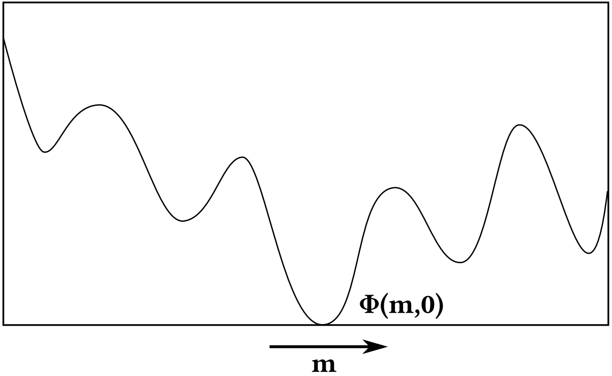
[Friedlander and Schmidt, 2012]

Relaxing the Physics - Approximating u^i and w^i

$$\Phi(m) = \sum_{i}^{n_s} \left\| d^i - P_r \boldsymbol{u^i} \right\|_W$$

$$\boldsymbol{u^i} \approx A^{-1}(m) q^i$$

$$\frac{\left\| A(m)\boldsymbol{u^i} - q^i \right\|_2}{\left\| q^i \right\|_2} < \varepsilon$$



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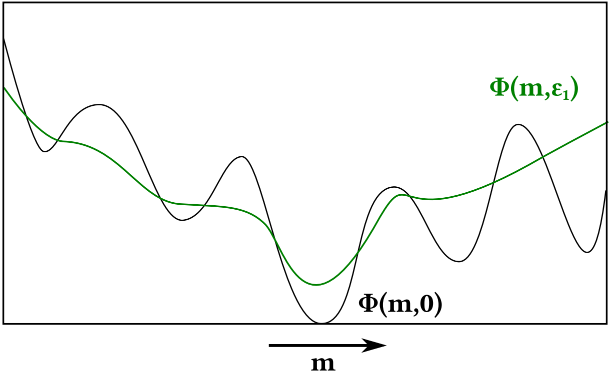

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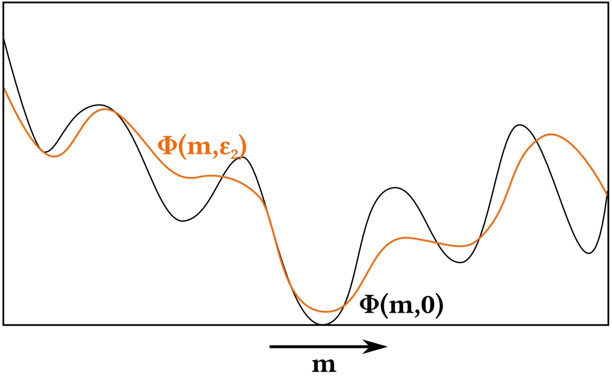


We hope that "large" ε_k can **"convexify"** Φ
and that $\Phi(m, \varepsilon_k)$ **"converges"** to $\Phi(m, 0)$

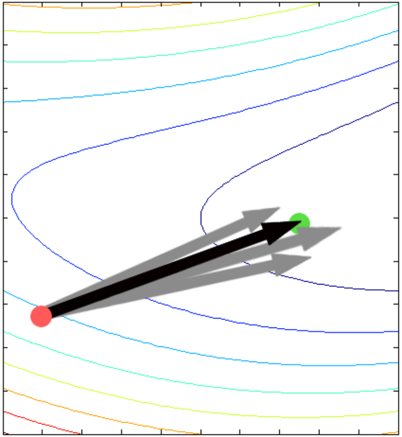
Choosing ε_k - Approximating u^i and w^i

$$|\Phi(m, \alpha^k \epsilon) - \Phi(m, \boldsymbol{\alpha^{k+1}\epsilon})| \leq \boldsymbol{\eta}\Phi(m, \boldsymbol{\alpha^{k+1}\epsilon})$$

$$\boldsymbol{u^i} \approx A^{-1}(m) q^i$$

$$\frac{\left\|A(m)\boldsymbol{u^i} - q^i\right\|_2}{\left\|q^i\right\|_2} < \boldsymbol{\alpha^{k+1}\epsilon}$$

(use the final tolerance to compute w^i)



Chosen Parameters

- $\alpha = 0.5$
- $\epsilon = 10^{-2}$
- $\eta = 5 \times 10^{-2}$

[Herrmann et al., 2013]

Frugal FWI - Source Subsampling

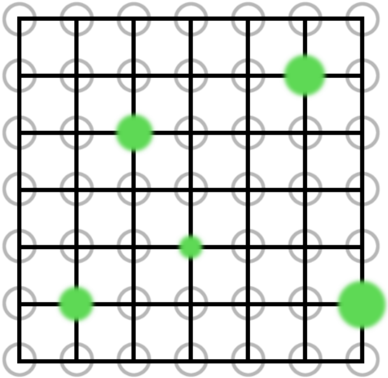
$$\tilde{\Phi}(m) = \sum_{i \in \mathcal{I}_k}^{b_k} \left\| d^i - u^i \right\|_W$$

$$\mathcal{I}_k \subset \{1, 2, ..., n_s\}, \mathcal{I}_k^{\#} = b_k$$

$\mathcal{I}_k$ is chosen at random *without replacement*. The expected error is given by

$$\|e_k\|_2 \propto \sqrt{\frac{1}{b_k} - \frac{1}{n_s}}$$

$$b_k \sim \min\left\{ \left(\epsilon^k + \frac{1}{n_s} \right)^{-1}, n_s \right\}.$$



[Friedlander and Schmidt, 2012] and [Herrmann et al., 2013]

CGMN, CRMN & Kaczmarz Sweep

Kaczmarz (Double) Sweep - Overview

$$u_{i+1} = u_i + \frac{\gamma(q_i - a_i^H u_i)a_i}{\|a_i\|_2^2}$$

- q_i ith element of q
- a_i ith row of A as a column vector
- γ relaxation parameter $\in (0, 2)$

Kaczmarz sweeps **guarantees convergence** in a finite (possibly **large**) number of steps.

Kaczmarz (Double) Sweep - Overview

$$Q = Q_1 Q_2 \dots Q_N Q_N Q_{N-1} \dots Q_1 \qquad Q_i = I - \frac{\gamma}{\|a_i\|_2^2} a_i a_i^H$$

$$u := Qu + Rq \qquad \Longrightarrow \qquad (I - Q)u = Rq$$

- Q is symmetric positive definite
- We can use **CG** to solve this system
- Neither Q nor R need to be computed in practice
- Equivalent to using **CG on the normal equations**, preconditioned by SSOR

1 (CGMN).

$p_0 = r_0 = dkswp(A, u_0, b, \gamma) - u_0;$
while *not converged* **do**
  $q_k = p_k - dkswp(A, p_k, 0, \gamma);$
  $\alpha_k = \langle r_k, r_k \rangle \,/\, \langle p_k, q_k \rangle;$
  $u_{k+1} = u_k + \alpha_k r_k;$
  $r_{k+1} = r_k - \alpha_k q_k;$
  $\beta_k = \langle r_{k+1}, r_{k+1} \rangle \,/\, \langle r_k, r_k \rangle;$
  $p_{k+1} = r_k + \beta_k p_k;$
  $k = k + 1;$
end while

- Very low memory cost
- Very simple implementation
- Suitable for any matrix A (even nonsquare)

On CG:[Hestenes and Stiefel, 1952], on CGMN: [Björck and Elfving, 1979]

2 (CRMN).

while *not converged* **do**

$\quad Ar_k := r_k - dkswp(A, r_k, 0, \gamma)$;

$\quad \beta_k = \langle r_k, Ar_k \rangle \, / \, \langle r_{k-1}, Ar_{k-1} \rangle$;

$\quad p_k = r_k + \beta_k p_{k-1}$;

$\quad Ap_k = Ar_k + \beta_k Ap_{k-1}$;

$\quad \alpha_k = \langle r_k, Ar_k \rangle \, / \, \langle Ap_k, Ap_k \rangle$;

$\quad u_{k+1} = u_k + \alpha_k r_k$;

$\quad r_{k+1} = r_k - \alpha_k q_k$;

$\quad k = k + 1$;

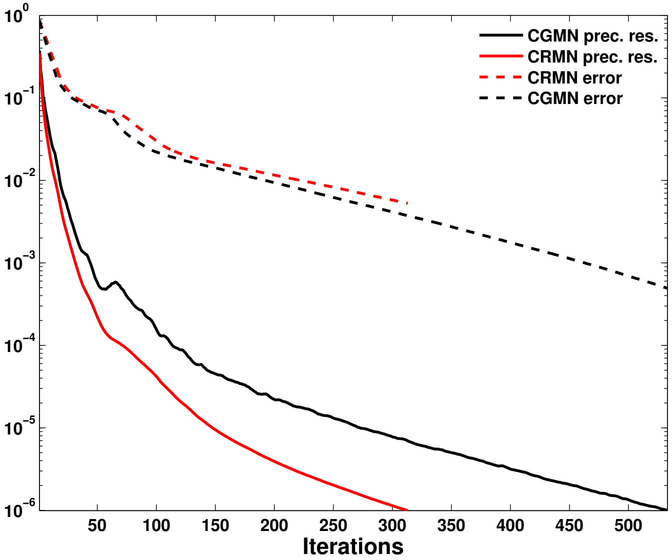
end while

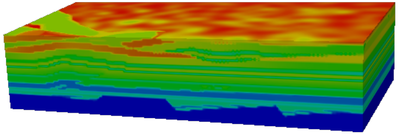
- Very **low memory cost**
- One **extra** vector storage
- One **extra** inner product
- **Minimal residual** properties

On CR: [Stiefel, 1955], comparisons:[Fong and Saunders, 2012, Eiermann and Ernst, 2001]

Numerical Experiment

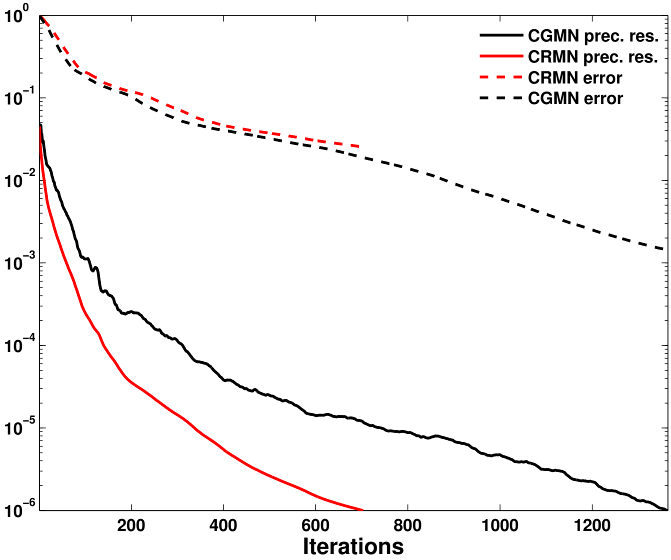
Forward Modeling - SEG/EAGE Overthrust

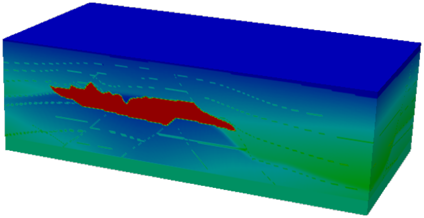




- $20.1 \times 20.1 \times 4.7\ km^3$
- $100m$ grid spacing
- $\mathcal{O}(1.9 \times 10^6)$ points
- 3Hz, $n_\lambda = 7.2$
- $v_{min} = 2179m/s$,
- $v_{max} = 6000m/s$
- PML: 15 points

Forward Modeling - SEG/EAGE Salt Dome





- $4 \times 4 \times 1.2\ km^3$
- $20m$ grid spacing
- $\mathcal{O}(2.5 \times 10^6)$ points
- 3Hz, $n_\lambda = 22.7$
- $v_{min} = 1365m/s$
- $v_{max} = 4991m/s$
- PML: 15 points

The True Stopping Criterion

CGMN/CRMN stops when $\frac{\|r_j\|_2}{\|r_0\|_2} < \alpha^{k+1}\epsilon$

For k satisfying

$$\left|\Phi(m, \alpha^k \epsilon) - \Phi(m, \alpha^{k+1}\epsilon)\right| \leq \eta \Phi(m, \alpha^{k+1}\epsilon)$$

Chosen Parameters

- $\alpha = 0.5$
- $\epsilon = 10^{-2}$
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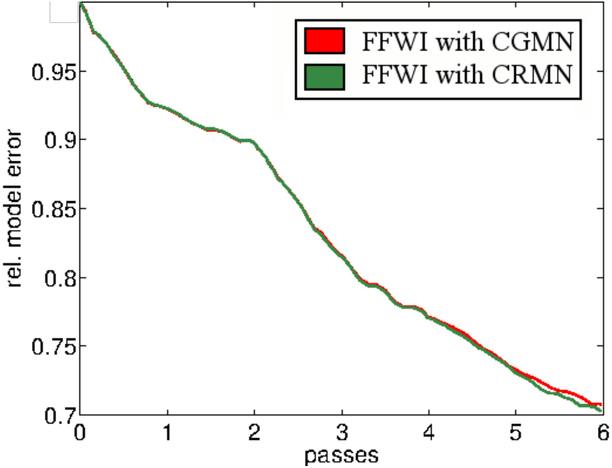
FFWI-CGMN vs. FFWI-CRMN - Overthrust



	FFWI with CGMN	FFWI with CRMN	Speedup
$4\ Hz$	23,403	19,846	18%
$6\ Hz$	30,189	24,387	24%
$8\ Hz$	34,724	26,265	32%

Table: *Total number of iterations of CGMN and CRMN during the inversion for each frequency slice*

Conclusions & Future Work

Conclusions & Future Work

- Smaller error computed by CGMN does not bring any improvement to Frugal FWI
- CRMN seems to be a feasible option
- Does the performance gain of CRMN grow with the frequency?
- Does the same result hold for other kind of PDEs?
- Does this behaviour holds for other models?
- Does this behaviour holds for other heuristics?

Questions?

Björck, Å. and Elfving, T. (1979).
Accelerated projection methods for computing pseudoinverse solutions of systems of linear equations.
BIT Numerical Mathematics, 19(2):145–163.

Eiermann, M. and Ernst, O. G. (2001).
Geometric aspects in the theory of Krylov subspace methods.
Acta Numerica, 10(10):251–312.

Fong, D. and Saunders, M. (2012).
CG versus MINRES: An empirical comparison.
SQU Journal for Science, 17:1:44–62.

Friedlander, M. P. and Schmidt, M. (2012).
Hybrid deterministic-stochastic methods for data fitting.
SIAM Journal on Scientific Computing, 34(3):A1380–A1405.

Gordon, D. and Gordon, R. (2005).

Component-averaged row projections: A robust, block-parallel scheme for sparse linear systems.
SIAM J. Scientific Computing, 27(3):1092–1117.

Gordon, D. and Gordon, R. (2010).
CARP-CG: A robust and efficient parallel solver for linear systems, applied to strongly convection dominated pdes.
Parallel Computing, 36(9):495–515.

Gordon, D. and Gordon, R. (2012).
Parallel solution of high frequency Helmholtz equations using high order finite difference schemes.
Applied Mathematics and Computation, 218(21):10737–10754.

Herrmann, F. J., Calvert, A. J., Hanlon, I., Javanmehri, M., Kumar, R., van Leeuwen, T., Li, X., Smithyman, B., Takougang, E. T., and Wason, H. (2013).
Frugal full-waveform inversion: from theory to a practical algorithm.
The Leading Edge, 32(9):1082–1092.

Hestenes, M. R. and Stiefel, E. (1952).

Methods of conjugate gradients for solving linear systems.
Journal of Research of the National Bureau of Standards, 49(6):409–436.

Operto, S., Virieux, J., Amestoy, P., L'Excellent, J.-Y., Giraud, L., and Ali, H. B. H. (2007).
3d finite-difference frequency-domain modeling of visco-acoustic wave propagation using a massively parallel direct solver: A feasibility study.
Geophysics, 72(5):SM195–SM211.

Stiefel, E. (1955).
Relaxationsmethoden bester strategie zur losung linearer gleichungssystem.
Commentarii Mathematici Helvetici, 29(1):157–179.