Application of a convex phase retrieval method to blind seismic deconvolution

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Abstract

A classical strategy for blind seismic deconvolution is to first estimate the autocorrelation of the unknown source wavelet from the data and then recover the wavelet by assuming it has minimum phase. However, computing the minimum phase wavelet directly from the amplitude spectrum can be sensitive to even extremely small errors, especially in the coefficients close to zero. Since the minimum phase requirement follows from an assumption that the wavelet should be as impulsive as possible, we propose to directly estimate an impulsive wavelet by minimizing a weighted l_2 penalty subject to a constraint on its amplitude spectrum. This nonconvex model has the form of a phase retrieval problem, in this case recovering a signal given only estimates of the magnitudes of its Fourier coefficients. Following recent work on convex relaxations of phase retrieval problems, we propose a convex semidefinite program for computing an impulsive minimum phase wavelet whose amplitude spectrum is close to a given estimate, and we show that this can be robustly solved by a Douglas Rachford splitting method for convex optimization.

Introduction

A simplified model for seismic data is to represent it as a convolution of a stationary source wavelet with the primary reflectivity (Ulrych and Sacchi (2006)). A classical strategy for blind seismic deconvolution is to assume the unknown reflectivity is white, estimate the autocorrelation of the the unknown source wavelet from the data and then recover the wavelet by assuming it has minimum phase. One can then estimate a sparse reflectivity consistent with the data by for instance solving an l_1-minimization problem (Dossal and Mallat (2005); Santosa and Symes (1986); Gholami and Sacchi (2012)).

The minimum phase strategy is based on the assumption that the true source wavelet should be impulsive, having have most of its energy concentrated at the beginning. A minimum phase signal has the smallest group delay for a given magnitude response. However, computing the minimum phase wavelet directly from the amplitude spectrum can be sensitive to even extremely small errors, especially in the coefficients close to zero. The minimum phase signal corresponding to an incorrectly estimated amplitude spectrum might not be a sufficiently impulsive source wavelet. Moreover, it can be challenging to apply numerical techniques for estimating minimum phase signals that involve either finding and manipulating roots of a high order polynomial (Kabal (2011)) or applying the Discrete Hilbert Transform to the log amplitude spectrum (Damera-Venkata et al. (2000)).

Inspired by the notion of front loaded causal signals suggested by Lamoureux and Margrave (2007), we propose a robust method to recover an impulsive wavelet whose amplitude spectrum is similar to an estimate of the amplitude spectrum that we compute from the data. In particular, we propose minimizing a weighted l_2 penalty where the weights increase with time, subject to a constraint on the amplitude spectrum. This nonconvex model has the form of a phase retrieval problem, in this case recovering a signal given only estimates of the magnitudes of its Fourier coefficients and the assumption that most of its energy should be concentrated at the start. Following recent work on convex relaxations of phase retrieval problems (Candès et al. (2013); Demanet and Hand (2012); Lemaréchal and Oustry (1999)), we reformulate the model as a convex semidefinite program, and we show that this can be robustly solved by a Douglas Rachford splitting method for convex optimization similar to the method used by Demanet and Hand (2012). The convex relaxation involves solving for a positive semidefinite matrix representing a lifted, higher dimensional version of the source wavelet. Empirically, we find that this method is always able to find a rank one minimizer when the true wavelet is impulsive. The minimizer also corresponds to a minimum phase signal, so altogether we propose a convex semidefinite program for computing an impulsive minimum phase wavelet whose amplitude spectrum is close to a given estimate.

Method

We suppose we have M redundant traces $f_j \in \mathbb{R}^N$ that can be modeled by

$$f_j = w * u_j + \eta_j \ , \ j = 1, \ldots, M,$$

where w denotes the source wavelet, which is assumed to be the same for all M traces, and u_j denotes different sparse reflectivity signals indexed by j. We also assume f_j is corrupted by white Gaussian noise η_j. Assuming u_j is approximately white and the noise is small, the Discrete Fourier Transform (DFT) of the autocorrelation of w should be well approximated up to a constant by the average of the DFTs of the autocorrelations of the f_j.

$$\frac{1}{M}\sum_{j=1}^{M} |\hat{f}_j|^2 = \frac{1}{M}\sum_{j=1}^{M} |\hat{w}|^2 |\hat{u}_j|^2 \approx c|\hat{w}|^2 \qquad \text{for some c} \ ,$$

where $\hat{w}$ denotes the DFT of w and $|\hat{w}|$ denotes the vector of absolute values of the DFT coefficients. Assuming $\|w\| = 1$, we can determine the constant c and estimate

$$|\hat{w}|^2 \approx \frac{\sum_{j=1}^{M} |\hat{f}_j|^2}{\text{mean}(\sum_{j=1}^{M} |\hat{f}_j|^2)} \ . \tag{1}$$

To improve this estimate when the data is noisy, we can use the assumption that the wavelet should be approximately band limited to estimate some of the noise statistics from the high frequency portion of the raw estimate of $|\hat{w}|$. Letting x_0 denote the noisy estimate of $|\hat{w}|$, we use the high frequency part to estimate the mean μ and standard deviation σ of the noise in x_0, then compute a denoised approximation by solving

$$\min_{x\geq 0} \|x\|_1 \qquad \text{such that} \qquad \|x - x_0 + \mu\| \leq \sigma\sqrt{N}\,. \tag{2}$$

As shown in Figure 1, if we generate synthetic data f_j by convolving a fixed wavelet w with randomly generated sparse u_i plus Gaussian noise, the relative error between the denoised estimate and the true amplitude spectrum gets smaller as the signal to noise ratio (SNR) increases or as the number of measurements increases. Let b denote an estimate of the square amplitude spectrum $|\hat{w}|^2$ normalized so that

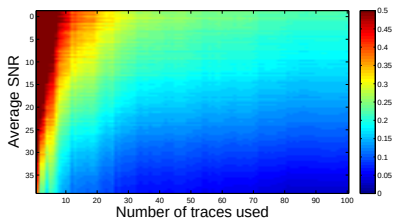

Figure 1 *Relative error vs SNR and number of measurements M*

$\|\sqrt{b}\| = \sqrt{N}$. A standard technique for estimating w is to construct $\hat{w} = \sqrt{b}e^{i\theta}$ where the phase θ is computed so that the corresponding w is minimum phase. This can theoretically be approximated by applying a Discrete Hilbert Transform (DHT) to the log amplitude spectrum $\log(\sqrt{b})$. The approximation used for example by Damera-Venkata et al. (2000) computes $\theta \approx -iF(sF^{-1}(\log(\sqrt{b})))$, where F denotes the DFT and $s_n = \begin{cases} 1 & 0 < n < \frac{N}{2} \\ -1 & \frac{N}{2} < n < N \\ 0 & \text{otherwise} \end{cases}$. This approximation improves as more frequency samples are used, which makes the DFT a better approximation to the Discrete Time Fourier Transform.

The denoising preprocessing tends to estimate b that incorrectly has too many zeros, which complicates using the DHT to reconstruct the phase. In fact, it is not possible for a nonzero minimum phase signal to have an amplitude spectrum that is zero on an interval (Lamoureux and Margrave (2007)). One could attempt to first shift the amplitude spectrum by a small constant, using $\sqrt{b} + d$ for some small d. However, Figure 2 shows that the estimated minimum phase signal can be quite sensitive to d and tends to be less impulsive for smaller d. As an alternative strategy, we propose solving

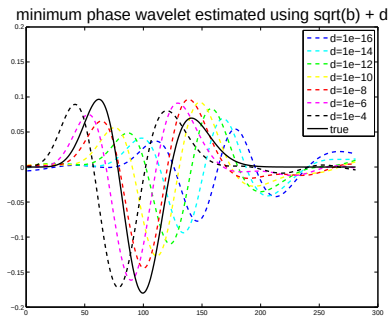

Figure 2 *Sensitivity of a minimum phase signal to small shifts in amplitude spectrum*

$$\min_{w} \sum_{n=1}^{N} n^2 w_n^2 \qquad \text{such that} \qquad \||\hat{w}|^2 - b\| \leq \varepsilon \quad \text{and} \quad \|w\| = 1 \tag{3}$$

to directly encourage the energy to be concentrated at the beginning. Suppose $q \in \mathbb{R}^N$ is front loaded in the sense of Lamoureux and Margrave (2007), which means for any $p \in \mathbb{R}^N$ with the same amplitude spectrum, $\sum_{n=1}^{m} |q_n|^2 \geq \sum_{n=1}^{m} |p_n|^2$ for $m = 1, \ldots, N$. This is true in particular if q is minimum phase. Then $\sum_{n=1}^{N} n^2 |q_n|^2 \leq \sum_{n=1}^{N} n^2 |p_n|^2$. Using the identity $n^2 = (N+1)^2 - \sum_{j=n}^{N} 2j+1$ and the fact that $\|q\| = \|p\|$ it follows that

$$\sum_{n=1}^{N} n^2 |q_n|^2 = (N+1)^2 \sum_{j=1}^{N} |q_j|^2 - \sum_{n=1}^{N} \sum_{j=n}^{N} (2j+1)|q_n|^2 = (N+1)^2 \sum_{j=1}^{N} |p_j|^2 - \sum_{j=1}^{N} (2j+1) \sum_{n=1}^{j} |q_n|^2$$
$$\leq (N+1)^2 \sum_{j=1}^{N} |p_j|^2 - \sum_{j=1}^{N} (2j+1) \sum_{n=1}^{j} |p_n|^2 = \sum_{n=1}^{N} n^2 |p_n|^2 \ ,$$

so q also minimizes the weighted l_2 penalty in (3) over all signals p that have the same amplitude spectrum.

The optimization problem in Equation 3 is a nonconvex formulation of a kind of phase retrieval problem to reconstruct the phase of $\hat{w}$, and it can be lifted to a convex model which can be robustly solved (Candès et al. (2013); Demanet and Hand (2012); Lemaréchal and Oustry (1999)). The key idea is to note that the measurements $b = |\hat{w}|^2$ are linear in ww^T. By lifting ww^T to a symmetric positive semidefinite (no negative eigenvalues) matrix $W \succeq 0$ and letting F be the DFT matrix, we can define a linear operator by $\mathscr{A}(W) = \mathrm{diag}(FWF^*)$ so that $\mathscr{A}(ww^T) = b$. The convex relaxation we propose to solve is

$$\min_{W \succeq 0, \mathrm{tr}(W)=1} \mathrm{tr}(CW) \qquad \text{such that} \qquad \|\mathscr{A}(W) - b\| \leq \varepsilon \ , \tag{4}$$

where tr denotes trace, and C is a diagonal matrix with $C_{nn} = n^2$, $n = 1, \ldots, N$. It is very beneficial to have a convex formulation of the problem. For convex problems, any local minimum is a global minimum, and there are a variety of efficient convex optimization methods that are guaranteed to converge to a solution. Similar to the algorithm used by Demanet and Hand (2012) for a closely related problem, we use the Douglas Rachford method to solve the convex relaxation (4). The iterations are given by

$$V^{k+1} = \Pi_{\|\mathscr{A}(\cdot)-b\| \leq \varepsilon}(2W^k - V^k) - W^k + V^k$$
$$W^{k+1} = \Pi_{\triangle}(V^{k+1} - \alpha C) \ ,$$

where the initialization can be arbitrary, $\alpha > 0$ and $\Pi_{\triangle}$ is the orthogonal projection onto symmetric positive semidefinite matrices with trace equal to one. The special structure of the measurement operator $\mathscr{A}$ can be used to efficiently compute the projection onto $\|\mathscr{A}(W) - b\| \leq \varepsilon$. When W is a rank one matrix, w can be recovered up to a sign by simply normalizing any nonzero row or column of W. More generally, we could recover w as the top normalized eigenvector. For all the examples tested, the method returned a rank one solution, in which case it should correspond to a minimizer of the nonconvex problem (3).

Results

A sample result is shown in Figure 3. Synthetic data $f_j \in \mathbb{R}^{1000}$, $j = 1, \ldots, 100$, was created by defining a ground truth wavelet w, then convolving it with randomly generated sparse u_j and adding Gaussian noise corresponding to a SNR of roughly 10. The amplitude spectrum of w was estimated using Equations 1 and 2. Then w was estimated from the rank one matrix W produced by applying the Douglas Rachford method to the convex relaxation (4). The required computational time was on the order of minutes for an average laptop. Empirically, this method is able to estimate impulsive wavelets up to a small shift, but it may do a poor job of estimating wavelets that are not not sufficiently impulsive.

Conclusions

The proposed convex model and method for computing an impulsive wavelet consistent with an estimate of its amplitude spectrum may be an interesting alternative to minimum phase source wavelet estimation

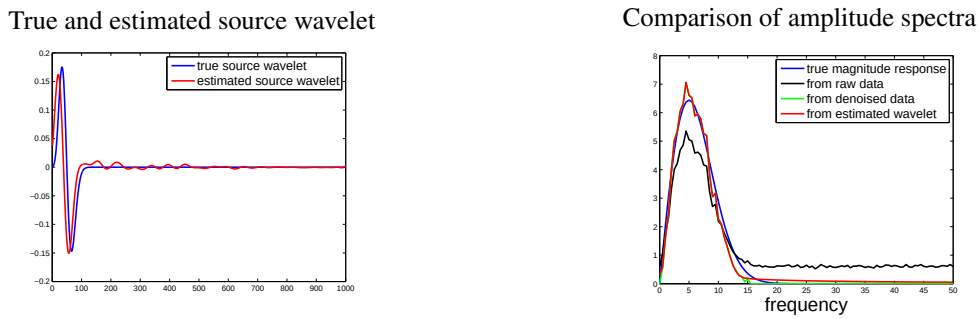

Figure 3 *True and recovered source wavelet with noise, (SNR* ≈ 10*), using Douglas Rachford to solve the convex relaxation (4) with* $\varepsilon = .001\|b\| \approx .2$

even though an application to real data would require significant preprocessing such as multiple removal and amplitude adjustments to bring the data in line with the simplified model considered here. Future work should include using the sparsity assumption on the reflectivity to help estimate source wavelets that are not minimum phase. The tool of lifting to convex semidefinite programs is also worth further study in the context of blind seismic deconvolution. Related lifting techniques have been used for special cases of blind deconvolution problems with known support in Ahmed et al. (2012) and sparse PCA in D'Aspremont et al. (2007). It would be interesting if these ideas could be extended to the kind of problem considered here where the data is a convolution of something impulsive with something sparse.

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